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ON SOME OPEN QUESTIONS ABOUT MULTIVALUED ENRICHED INTERPOLATIVE KANNAN AND ĆIRIĆ-REICH-RUS TYPE CONTRACTIONS IN A BANACH SPACE

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Abstract. This paper delves into the examination of some unresolved inquiries regarding Kannan and Ćirić-Reich-Rus contractions in a Banach space by employing the notion of enriched contractions. We introduce multivalued enriched interpolative versions of these results to address those questions. In this context, additionally, we introduce a novel extension of the Banach contraction principle and substantiate its validity through an example in an infinite-dimensional normed space.
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1. Introduction and preliminaries

In this paper, we attempt to address some open questions asked in literature e.g. in [6, 18, 19, 22, 23, 26, 32] about the Kannan and Ćirić-Reich-Rus (CRR) type contractions. Questions like dropping some stronger conditions in those works [18, 32], generalizing them further using enriched contractions [6, 22], establishing their multivalued analogues [18, 19] and studying their common fixed points [23] have posed sufficient mathematical interest and challenge to the researchers. Drawing inspiration from these findings, our present study focuses on establishing multivalued counterparts of interpolative Kannan and CRR contractions through the utilization of enriched contractions. Subsequently, we examine various fixed point outcomes in this particular context and ultimately present a fresh generalization of the Banach contraction principle (BCP) by extending Theorem 2.4 of [6]. The majority of our findings have been derived under the assumption that the images of the multivalued map under investigation are compact.

In 1969, Nadler [34] introduced an intriguing set of fixed point results for multivalued maps by using the concept of measuring distance between two arbitrary sets. The method of measuring distance between two sets was introduced long before by Pompeiu and it was used and made famous by Hausdorff. For details in this regard, we refer to the wonderful discussion in the paper by Berinde and Păcurar [5]. These findings have since captured significant attention in the field [21, 37, 42].

Consider a complete metric space (X, d), where X is a set and d is a metric defined on X. Let $BC(X)$ denote the collection of all nonempty closed and bounded subsets of X. Then for $X, Y \in BC(X)$, define the map $H : BC(X) \times BC(X) \to [0, \infty)$ by

$$H(X, Y) = \max\{\sup_{y \in Y} D(y, X), \sup_{x \in X} D(x, Y)\},$$

where $D(x, Y) = \inf_{y \in Y} d(x, y)$. Consequently, $(BC(X), H)$ is a metric space induced by d.

$x \in X$ is a fixed point of the multivalued mapping $\Phi : X \to BC(X)$ if $x \in \Phi x$. The set of all fixed points of Φ is denoted by $Fix(\Phi)$. If $\Phi, \Psi : X \to BC(X)$ be two multivalued mappings, then $\omega \in X$ is said to be a common fixed point of Φ and Ψ if $\omega \in \Phi\omega \cap \Psi\omega$.

Remark 1.1. (1) In the metric space $(BC(X), H)$, a fixed point υ of Φ exists if and only if the distance between υ and $\Phi\upsilon$ is zero, denoted by $D(\upsilon, \Phi\upsilon) = 0$.

(2) If A and B are two closed sets with a non-empty intersection ($A \cap B \neq \emptyset$), then we have $D(A, B) = 0$.

(3) D exhibits continuity property; indeed, if $x_n \to x$ as $n \to \infty$, then $D(x_n, A) \to D(x, A)$ as $n \to \infty$ for any $A \subseteq X$.

For some recent important fixed point results of multivalued maps, we refer to the work of Ćirić [16] and Klim and Wardowski [31].

The subsequent two lemmas hold significance for what follows.

Lemma 1.2. [14, 17] *Consider the metric space (X, d), and let A, B be elements of the set $BC(X)$. Then*

(1) $D(x, B) \leq d(x, y)$ for any $y \in B$ and $x \in X$;

(2) $D(x, B) \leq H(A, B)$ for any $x \in X$.

Lemma 1.3. [34] *Suppose $A, B \in BC(X)$, and let $x \in X$. Then, for any $s > 0$, there exists $y \in Y$ such that*

$$d(x, y) \leq H(A, B) + s.$$

It is possible that there does not exist a point $y \in Y$ satisfying

$$d(x, y) \leq H(A, B).$$

If Y is compact, then we can find a point y such that $d(x, y) \leq H(A, B)$.

The concept of H-continuity is of immense interest in multivalued analysis as discussed next.

Definition 1.4. [24] In a metric space (X, d), a multivalued map $\Phi : X \to BC(X)$ is said to be H-continuous at a point x_0 if, for each sequence $x_n \subset X$ such that $\lim_{n\to\infty} d(x_n, x_0) = 0$, it follows that $\lim_{n\to\infty} H(\Phi x_n, \Phi x_0) = 0$. (i.e., if $x_n \to x_0$, then $\Phi x_n \to \Phi x_0$ as $n \to \infty$).

Definition 1.5. [34] A multivalued map $\Phi : X \to BC(X)$ is termed as a multivalued contraction if $H(\Phi x, \Phi y) \leq \kappa d(x, y)$ holds for all $x, y \in X$, where $\kappa \in [0, 1)$.

Remark 1.6. (1) When a multivalued mapping Φ is continuous at every point in its domain X, it is referred to as being H-continuous on X.
(2) If Φ is a multivalued contraction, it implies that Φ is H-continuous. It can be proved that the sequential definition of H-continuity (as defined in 1.4) is equivalent to usual definition of continuity of a multivalued map.

In [32], the authors defined the concept of multivalued interpolative Kannan (MVIK)-type contractions.

Definition 1.7. [32] Let $(X, ||\cdot||)$ be a normed linear space with the induced metric $d(x, y) = ||x - y||$ for all $x, y \in X$. A mapping $\Phi : X \to BC(X)$ is said to be a multivalued interpolative Kannan (MVIK)-type contraction if there exists $\sigma \in [0, 1)$ and $\alpha \in (0, 1)$ such that

$$H(\Phi x, \Phi y) \leq \sigma \big[D(x, \Phi x)\big]^{\alpha} \big[D(y, \Phi y)\big]^{1-\alpha}$$

for all $x, y \in X$.

Enriched contraction for single-valued mappings in a normed linear space was introduced by Berinde and Păcurar [6]. In [1], Abbas et al. introduced the notion of enriched multivalued contractions.

Definition 1.8. [1] Let $(X, ||\cdot||)$ be a normed linear space. A mapping $\Phi : X \to BC(X)$ is called enriched multivalued contraction if there exists $k \in [0, \infty)$ and $\sigma \in [0, k + 1)$ such that

$$H(kx + \Phi x, ky + \Phi y) \leq \sigma ||x - y||$$

for all $x, y \in X$.

Remark 1.9. [1] Let M be a convex subset of a normed space X and consider the mapping $\Phi : M \to BC(M)$. Then, for $\lambda \in (0, 1)$, the mapping $\Phi_\lambda : M \to BC(M)$ defined by

$$\begin{aligned} \Phi_\lambda &= (1 - \lambda)x + \lambda \Phi x \\ &= \{(1 - \lambda)x + \lambda y : y \in \Phi x\} \end{aligned}$$

signifies the translation of the set $\lambda \Phi x$ by the vector $(1 - \lambda)x$. It is easy to see that $Fix(\Phi_\lambda) = Fix(\Phi)$.

Indeed, if $x \in \Phi x$, then $x = (1 - \lambda)x + \lambda x \in \Phi_\lambda x$. On the other hand, if $x \in \Phi_\lambda x$, then for some $y \in \Phi x$, we have $x = (1 - \lambda)x + \lambda y$, which subsequently implies that $x \in \Phi x$.

Remark 1.10. It is easy to see that if $k = 0$, in Definition 1.8, then Φ is a multivalued contraction and hence H-continuous. Thus, enriched multivalued contraction is a proper generalization of a multivalued contraction.

New extensions of Kannan's [29, 30] and Reich's theorems [38, 40] were recently proved by Debnath and Srivastava [25]. Kannan's theorem possesses huge historical importance because of the fact that unlike the BCP, Kannan maps are not necessarily continuous. Kannan's result had important consequences regarding metric completeness [43]. The CRR type contractions follow similar approach and generalize both

Banach and Kannan's results. The study of fixed points at points of discontinuity of the map is a popular subject of study from application point of view [35]. The interpolative contractions work somewhat like a bridge between the continuous and discontinuous mappings. As such, study of interpolative contractions gained immense popularity in recent times e.g. [3, 24, 33].

Further, for information regarding terminology of the mappings in fixed point iterative methods, we refer to the work of Berinde et al. [11].

As a final introductory remark, we would like to mention that symbols and notations in this paper are as per the classical monograph in fixed point theory by Rus et al. [42]. This could result in increased amount of similarity, however using non-standard symbols is not a good practice for the beauty and tradition of mathematics. Thus, as suggested by the learned referee(s), we have gone with all the standard symbols and notations and we believe that this should not be treated as similarity because standard mathematical symbols, notations, definitions and results should not be, and in many cases, cannot be altered.

2. Multivalued enriched interpolative Kannan and Ćirić-Reich-Rus contractions

In this segment, we extend and generalize two famous results on fixed point theory - one due to Kannan and the other due to Ćirić-Reich-Rus to their multivalued analogues using enriched interpolative contractions.

In this connection, we find it obligatory to mention honorarily the very recent important and interesting contributions on enriched Kannan and enriched Ćirić-Reich-Rus contractions and their generalizations, e.g., the works of Anjum and Abbas [2], Berinde and Păcurar [7, 8, 9, 10], Debnath [20].

First, we discuss the result due to Kannan. In the proof, we make use of a result from [32].

Definition 2.1. Let $(X, ||\cdot||)$ be a normed linear space with the induced metric $d(x,y) = ||x-y||$ for all $x, y \in X$. A mapping $\Phi : X \to BC(X)$ is said to be a multivalued enriched interpolative Kannan contraction (MEIKC) if there exists $k \in [0,\infty)$, $\sigma, \alpha \in (0,1)$ such that

$$H(kx + \Phi x, ky + \Phi y) \leq \sigma \big[D(x, \Phi x)\big]^{\alpha} \big[D(y, \Phi y)\big]^{1-\alpha}$$

for all $x, y \in X$.

Theorem 2.2. *Let $(X, ||\cdot||)$ be a Banach space. An MEIKC $\Phi : X \to BC(X)$ with the property that Φx is compact in $BC(X)$ for each $x \in X$, has a fixed point.*

Proof. Let $\lambda = \frac{1}{k+1}$. Then $\lambda \in (0,1]$ and $k = \frac{1}{\lambda} - 1$. Since Φ is an MEIKC, we have that

$$H(kx + \Phi x, ky + \Phi y) \leq \sigma \big[D(x, \Phi x)\big]^{\alpha} \big[D(y, \Phi y)\big]^{1-\alpha}$$
$$\implies H((\frac{1}{\lambda} - 1)x + \Phi x, (\frac{1}{\lambda} - 1)y + \Phi y) \leq \sigma \big[D(x, \Phi x)\big]^{\alpha} \big[D(y, \Phi y)\big]^{1-\alpha}$$
$$\implies H((1-\lambda)x + \lambda \Phi x, (1-\lambda)y + \lambda \Phi y) \leq \sigma\lambda \big[D(x, \Phi x)\big]^{\alpha} \big[D(y, \Phi y)\big]^{1-\alpha}$$
$$\implies H(\Phi_\lambda x, \Phi_\lambda y) \leq \sigma\lambda \big[D(x, \Phi x)\big]^{\alpha} \big[D(y, \Phi y)\big]^{1-\alpha}. \quad (2.1)$$

Further, we observe that

$$\begin{aligned} D(x, \Phi_\lambda x) &= \inf_y \{d\big(x, (1-\lambda)x + \lambda y\big) : y \in \Phi x\} \\ &= \inf_y \{\lambda d(x, y) : y \in \Phi x\} \\ &= \lambda \inf_y \{d(x, y) : y \in \Phi x\} \\ &= D(x, \Phi x). \end{aligned} \quad (2.2)$$

Thus, from (2.1) and (2.2), we obtain

$$\begin{aligned} H(\Phi_\lambda x, \Phi_\lambda y) &\leq \sigma\lambda \frac{\big[D(x, \Phi_\lambda x)\big]^{\alpha}}{\lambda^{\alpha}} \frac{\big[D(y, \Phi_\lambda y)\big]^{1-\alpha}}{\lambda^{1-\alpha}} \\ &= \sigma \big[D(x, \Phi_\lambda x)\big]^{\alpha} \big[D(y, \Phi_\lambda y)\big]^{1-\alpha}. \end{aligned}$$

Therefore, Φ_λ is a multivalued interpolative Kannan (MVIK)-type contraction as defined in [32]. From Theorem 1 of [32], we conclude that Φ_λ has a fixed point. Remark 1.9 ensures that Φ has a fixed point as well. □

The example next validates the above theorem.

Example 2.3. Consider $X = \mathbb{R}$ endowed with the usual norm $||x|| = |x|$ for all $x \in \mathbb{R}$. Then $(X, ||\cdot||)$ is a Banach space. Fix $k \in [0, \infty)$ and define $\Phi : X \to BC(X)$ by

$$\Phi x = \begin{cases} \{-kx\}, & \text{if } x \in (-\infty, 3) \\ \{x, x+3\}, & \text{if } x \geq 3. \end{cases}$$

Then Φx is compact for each $x \in X$. Further, if $x, y \notin Fix(\Phi)$, then $x, y \in (-\infty, 3) \setminus \{0\}$, because $Fix(\Phi) = \{0\} \cup [3, \infty)$.

We have that if $x, y \in X \setminus Fix(\Phi)$, then

$$\begin{aligned} H(kx + \Phi x, ky + \Phi y) &= H(\{kx + (-kx)\}, \{ky + (-ky)\}) \\ &= (\{0\}, \{0\}) \\ &= 0 \\ &\leq \sigma \big[D(x, \Phi x)\big]^{\alpha} \big[D(y, \Phi y)\big]^{1-\alpha}, \end{aligned}$$

for any $\sigma \in (0,1)$ and $\alpha \in (0,1)$.

Thus Φ is a MEIKC which has infinitely many fixed points.

Next, we present an extension of the famous CRR type contraction to its multivalued enriched interpolative analogue. In its proof, we make use of a result in [22].

Definition 2.4. Let $(X, ||\cdot||)$ be a normed linear space with the metric $d(x,y) = ||x-y||$ for all $x, y \in X$. A mapping $\Phi : X \to BC(X)$ is said to be a multivalued enriched interpolative Ciric-Reich-Rus (MEICRR) contraction if there exists $k \in [0,\infty)$, $\sigma, A, B \in (0,1)$ with $A+B<1$ such that

$$H(kx+\Phi x, ky+\Phi y) \leq \sigma\big[d(x,y)\big]^B\big[D(x,\Phi x)\big]^A\big[D(y,\Phi y)\big]^{1-A-B}$$

for all $x, y \in X \setminus Fix(\Phi)$.

Theorem 2.5. *Let $(X, ||\cdot||)$ be a Banach space. A MEICRR contraction $\Phi : X \to BC(X)$ with the property that Φx is compact in $BC(X)$ for each $x \in X$, has a fixed point.*

Proof. Let $\lambda = \frac{1}{k+1}$. Then $\lambda \in (0,1]$ and $k = \frac{1}{\lambda} - 1$. Since Φ is an MEIRRC, for all $x, y \in X \setminus Fix(\Phi)$, there exist $\sigma, A, B \in (0,1)$ with $A+B<1$ such that

$$\begin{aligned}
&H(kx+\Phi x, ky+\Phi y) \leq \sigma\big[d(x,y)\big]^B\big[D(x,\Phi x)\big]^A\big[D(y,\Phi y)\big]^{1-A-B}\\
\Longrightarrow\ &H((\frac{1}{\lambda}-1)x+\Phi x, (\frac{1}{\lambda}-1)y+\Phi y) \leq \sigma\big[d(x,y)\big]^B\big[D(x,\Phi x)\big]^A\big[D(y,\Phi y)\big]^{1-A-B}\\
\Longrightarrow\ &H((1-\lambda)x+\lambda\Phi x, (1-\lambda)y+\lambda\Phi y) \leq \sigma\lambda\big[d(x,y)\big]^B\big[D(x,\Phi x)\big]^A\big[D(y,\Phi y)\big]^{1-A-B}\\
\Longrightarrow\ &H(\Phi_\lambda x, \Phi_\lambda y) \leq \sigma\lambda\big[d(x,y)\big]^B\big[D(x,\Phi x)\big]^A\big[D(y,\Phi y)\big]^{1-A-B}.
\end{aligned} \tag{2.3}$$

Further, from (2.2) of Theorem 2.2 we have that

$$D(x,\Phi_\lambda x) = \lambda D(x,\Phi x). \tag{2.4}$$

From (2.3) and (2.4), we have

$$\begin{aligned}
H(\Phi_\lambda x, \Phi_\lambda y) &\leq \sigma\lambda\big[d(x,y)\big]^B\frac{\big[D(x,\Phi x)\big]^A}{\lambda^A}\frac{\big[D(y,\Phi y)\big]^{1-A-B}}{\lambda^{1-A-B}}\\
&= \sigma\lambda^B\big[d(x,y)\big]^B[D(x,\Phi_\lambda x)]^A\big[D(y,\Phi_\lambda y)\big]^{1-A-B}\\
&\leq \sigma\big[d(x,y)\big]^B[D(x,\Phi_\lambda x)]^A\big[D(y,\Phi_\lambda y)\big]^{1-A-B} \text{ (for } \lambda \leq 1).
\end{aligned}$$

In Theorem 3 of the paper by Debnath and de La Sen [22], putting $s=1$ (i.e., assuming (Ω,δ) as a metric space) and $X = \Omega$, we conclude that Φ_λ is an interpolative RRC-type contraction and hence Φ_λ has a fixed point.

Since $Fix(\Phi_\lambda) = Fix(\Phi)$, we obtain that $Fix(\Phi) \neq \phi$. □

Next, we present another generalization of the Kannan contraction.

Definition 2.6. Let $(X, ||\cdot||)$ be a normed linear space with the metric $d(x,y) = ||x-y||$ for all $x, y \in X$. A mapping $\Phi : X \to BC(X)$ is said to be a (σ, A, B)-multivalued enriched interpolative Kannan ((σ, A, B)-MEIK) contraction if there exists $k \in [0,\infty)$, $\sigma, A, B \in (0,1)$ with $A+B<1$ such that

$$H(kx+\Phi x, ky+\Phi y) \leq \sigma\big[D(x,\Phi x)\big]^A\big[D(y,\Phi y)\big]^B$$

for all $x, y \in X \setminus Fix(\Phi)$.

Theorem 2.7. *Let $(X, ||\cdot||)$ be a Banach space. A (σ, A, B)-MEIKC $\Phi : X \to BC(X)$ with the property that Φx is compact in $BC(X)$ for each $x \in X$, has a fixed point.*

Proof. Let $\lambda = \frac{1}{k+1}$. Then $\lambda \in (0, 1]$ and $k = \frac{1}{\lambda} - 1$. Since Φ is a (σ, A, B)-MEIKC, for all $x, y \in X \setminus Fix(\Phi)$, there exist $\sigma, A, B \in (0, 1)$ with $A + B < 1$ such that

$$\begin{aligned}
&H(kx + \Phi x, ky + \Phi y) \leq \sigma\big[D(x, \Phi x)\big]^A\big[D(y, \Phi y)\big]^B \\
\implies &H((\frac{1}{\lambda} - 1)x + \Phi x, (\frac{1}{\lambda} - 1)y + \Phi y) \leq \sigma\big[D(x, \Phi x)\big]^A\big[D(y, \Phi y)\big]^B \\
\implies &H((1 - \lambda)x + \lambda\Phi x, (1 - \lambda)y + \lambda\Phi y) \leq \sigma\lambda\big[D(x, \Phi x)\big]^A\big[D(y, \Phi y)\big]^B \\
\implies &H(\Phi_\lambda x, \Phi_\lambda y) \leq \sigma\lambda\big[D(x, \Phi x)\big]^A\big[D(y, \Phi y)\big]^B. \qquad (2.5)
\end{aligned}$$

Further, we have that

$$D(x, \Phi_\lambda x) = \lambda D(x, \Phi x); D(y, \Phi_\lambda y) = \lambda D(y, \Phi y). \qquad (2.6)$$

From (2.5) and (2.6), we have that

$$\begin{aligned}
H(\Phi_\lambda x, \Phi_\lambda y) &\leq \sigma\lambda\frac{\big[D(x, \Phi_\lambda x)\big]^A}{\lambda^A}\frac{\big[D(y, \Phi_\lambda y)\big]^B}{\lambda^B} \\
&= \sigma\lambda^{1-(A+B)}[D(x, \Phi_\lambda x)]^A\big[D(y, \Phi_\lambda y)\big]^B \\
&\leq \sigma[D(x, \Phi_\lambda x)]^A\big[D(y, \Phi_\lambda y)\big]^B \text{ (for } \lambda^{1-(A+B)} \leq 1).
\end{aligned}$$

Hence Φ_λ is a (σ, α, β)-multivalued interpolative Kannan-type contraction as in the Corollary 1 of the paper [32].

Thus, from Corollary 1 of [32], we conclude that Φ_λ has a fixed point.

Since $Fix(\Phi_\lambda) = Fix(\Phi)$, we obtain that $Fix(\Phi) \neq \phi$. □

3. Common fixed points of generalized Kannan type contraction pairs

In this section we establish some common fixed point results for MEIKC pairs.

To motivate the study, consider a family $\mathcal{S}$ of self-maps on the set X. The assumption that each $\phi \in \mathcal{S}$ has a fixed point, does not guarantee the existence of a fixed point which is common to all such maps. e.g., if $X = [-1, 1]$ and the functions $\phi(x) = \frac{x+1}{2}, \psi(x) = \frac{x-1}{2}$ for $x \in [-1, 1]$, then 1 is the unique fixed point of ϕ and -1 is the unique fixed point of ψ and consequently they do not have a common fixed point. Thus, the research towards finding conditions on the family $\mathcal{S}$ that ascertains the existence of common fixed point becomes important.

X being a commuting family of mappings was considered to be one such condition. However, Boyce [15] and Huneke [27] independently discovered self-maps on $[-1, 1]$ which commute but do not admit common fixed point. As such, look out for other conditions on the family of mappings remains a well motivated area of research. In this section, we establish one such condition for generalized multivalued Kannan-type contractions.

Theorem 3.1. *Let $(X, ||\cdot||)$ be a Banach space with the metric $d(x,y) = ||x-y||$ for all $x, y \in X$ and $\Phi, \Psi : X \to BC(X)$ be a pair of multivalued mappings such that $\Phi x, \Psi x$ are compact for each $x \in X$. Suppose there exist $k \in [0,\infty)$, $\sigma \in [0,1)$, $A \in (0,1)$ such that*

$$H(kx + \Phi x, ky + \Psi y) \leq \sigma\big[D(x,\Phi x)\big]^A\big[D(y,\Psi y)\big]^{1-A}$$

for all $x \in X \setminus Fix(\Phi)$ and $y \in X \setminus Fix(\Psi)$. Then the pair (Φ, Ψ) has a CFP.

Proof. Let $\lambda = \frac{1}{k+1}$. Then $\lambda \in (0,1]$ and $k = \frac{1}{\lambda} - 1$. Since Φ, Ψ are MEIK contractions, for each pair of points $x \in X \setminus Fix(\Phi)$ and $y \in X \setminus Fix(\Psi)$, we have that

$$\begin{aligned} & H(kx + \Phi x, ky + \Psi y) \leq \sigma\big[D(x,\Phi x)\big]^\alpha\big[D(y,\Psi y)\big]^{1-\alpha} \\ \implies & H((\frac{1}{\lambda}-1)x + \Phi x, (\frac{1}{\lambda}-1)y + \Psi y) \leq \sigma\big[D(x,\Phi x)\big]^\alpha\big[D(y,\Psi y)\big]^{1-\alpha} \\ \implies & H((1-\lambda)x + \lambda\Phi x, (1-\lambda)y + \lambda\Psi y) \leq \sigma\lambda\big[D(x,\Phi x)\big]^\alpha\big[D(y,\Psi y)\big]^{1-\alpha} \\ \implies & H(\Phi_\lambda x, \Psi_\lambda y) \leq \sigma\lambda\big[D(x,\Phi x)\big]^\alpha\big[D(y,\Psi y)\big]^{1-\alpha}. \end{aligned}$$

Further, from inequality (2.2), we observe that $D(x,\Phi_\lambda x) = \lambda D(x,\Phi x)$ and $D(y,\Psi_\lambda y) = \lambda D(y,\Psi y)$.

Thus,

$$\begin{aligned} H(\Phi_\lambda x, \Psi_\lambda y) &\leq \sigma\lambda\frac{\big[D(x,\Phi_\lambda x)\big]^\alpha}{\lambda^\alpha}\frac{\big[D(y,\Psi_\lambda y)\big]^{1-\alpha}}{\lambda^{1-\alpha}} \\ &= \sigma\big[D(x,\Phi_\lambda x)\big]^\alpha\big[D(y,\Psi_\lambda y)\big]^{1-\alpha}. \end{aligned}$$

From Theorem 2 of [32], we conclude that $Fix(\Phi_\lambda) \cap Fix(\Psi_\lambda) \neq \phi$. Since $Fix(\Phi_\lambda) = Fix(\Phi)$ and $Fix(\Psi_\lambda) = Fix(\Psi)$, we obtain $Fix(\Phi) \cap Fix(\Psi) \neq \phi$. □

Next, we present a corollary to the above theorem whose proof is similar to the theorem and hence omitted.

Corollary 3.2. *Let $(X, ||\cdot||)$ be a Banach space with the induced metric $d(x,y) = ||x-y||$ for all $x, y \in X$ and $\Phi, \Psi : X \to BC(X)$ be a pair of multivalued mappings such that $\Phi x, \Psi x$ are compact for each $x \in X$. Suppose there exist $k \in [0,\infty)$, $\sigma \in [0,1)$, $A, B \in (0,1)$ with $A + B < 1$ satisfying*

$$H(kx + \Phi x, ky + \Psi y) \leq \sigma\big[D(x,\Phi x)\big]^A\big[D(y,\Psi y)\big]^B$$

for all $x \in X \setminus Fix(\Phi)$ and $y \in X \setminus Fix(\Psi)$. Then the pair (Φ, Ψ) has a CFP.

Below we present an example to validate Theorem 3.1.

Example 3.3. Consider $X = \mathbb{R}$ endowed with the usual norm $||x|| = |x|$ for all $x \in \mathbb{R}$. Then $(X, ||\cdot||)$ is a Banach space. Fix $k \in [0,\infty)$ and define $\Phi, \Psi : X \to BC(X)$ by

$$\Phi x = \begin{cases} \{-kx\}, & \text{if } x \in (-\infty, 3) \\ \{x, x+3\}, & \text{if } x \geq 3, \end{cases}$$

and

$$\Psi x = \begin{cases} \{-ky\}, & \text{if } y \in (-\infty, 5) \\ \{y, y^2+1\}, & \text{if } y \geq 5. \end{cases}$$

Then Φx, Ψx are compact for each $x \in X$. It is easy to observe that $Fix(\Phi) = \{0\} \cup [3, \infty)$ and $Fix(\Psi) = \{0\} \cup [5, \infty)$.

We have that if $x \in X \setminus Fix(\Phi)$ and $y \in X \setminus Fix(\Psi)$, then

$$\begin{aligned} H(kx + \Phi x, ky + \Psi y) &= (\{0\}, \{0\}) \\ &= 0 \\ &\leq \sigma\big[D(x, \Phi x)\big]^{\alpha}\big[D(y, \Psi y)\big]^{1-\alpha}, \end{aligned}$$

for any $\sigma \in (0,1)$ and $\alpha \in (0,1)$.

Thus from Theorem 3.1, we conclude that $Fix(\Phi) \cap Fix(\Psi) \neq \phi$. In fact, $Fix(\Phi) \cap Fix(\Psi) = \{0\} \cup [5, \infty)$.

4. A new generalization of BCP

In this section we establish a new generalization of the well known BCP using the concept of single-valued enriched contractions.

First we define a composition of the enriched contraction due to Berinde and Păcurar [6].

Definition 4.1. Let $(X, ||\cdot||)$ be a normed linear space with the metric $||x-y|| = d(x,y)$ for all $x, y \in X$. For a natural number $m \geq 2$, a mapping $\Gamma^m : X \to X$ is said to be an enriched contraction if there exist $k \in [0, \infty)$ and $\sigma \in [0, k+1)$ such that

$$d(kx + \Gamma^m x, ky + \Gamma^m y) \leq \sigma d(x, y)$$

for all $x, y \in X$.

Theorem 4.2. *Let $(X, ||\cdot||)$ be a normed linear space. For $m \geq 2$, let $\Gamma^m : X \to X$ be an enriched contraction. Then Γ has a unique fixed point.*

Proof. By Theorem 2.4 of [6], we know that Γ^m has a unique fixed point $\omega \in X$, i.e., $\Gamma^m \omega = \omega$.

We have that

$$\Gamma\omega = \Gamma(\Gamma^m \omega) = \Gamma^m(\Gamma\omega).$$

Hence, $\Gamma\omega$ is a fixed point of Γ^m.

By the uniqueness of fixed point of Γ^m, we conclude that $\Gamma\omega = \omega$, i.e., ω is the unique fixed point of Γ. □

Similar to the Example 2.2 of [19], we present below an example to validate Theorem 4.2 in the context of an infinite dimensional normed linear space.

Example 4.3. Consider $X = C[a,b]$, the collection of all continuous real functions defined on the compact interval $[a,b]$. X constitutes an infinite-dimensional Banach space with the norm

$$||x|| = \max_{t \in [a,b]} |u(t)|, u \in X.$$

Define $\Gamma : X \to X$ by $\Gamma(u)(t) = \int_a^t u(s)ds$.

Using elementary concepts of double integral, we have that

$$\Gamma^2(u)(t) = \int_a^t \int_a^s u(\tau)d\tau ds = \int_a^t (t-s)u(s)ds$$

and by induction we obtain

$$\Gamma^m(u)(t) = \frac{1}{(m-1)!}\int_a^t (t-s)^{m-1}u(s)ds.$$

Thus, we have

$$||\Gamma^m(u) - \Gamma^m(v)|| \leq \frac{(b-a)^{2m}}{m!}||u-v||$$

for all $u, v \in X$.

Select the values of m such that $\frac{(b-a)^{2m}}{m!} < 1$, and for these chosen values of m, it is ensured that all conditions of Theorem 4.2 are satisfied. We observe that Γ has a fixed point $u = 0 \in X$.

5. Conclusions and future work

(1) We introduced the class of MEIKC which includes the usual Banach contractions and some non-expansive mappings.
(2) We introduced MEICRR contraction in a Banach space and showed that such generalized contractions admit fixed points. The well known classical Ćirić-Reich-Rus contractions are particular cases of these contractions.
(3) Another generalization and extension of the famous Kannan's result has been presented by defining (σ, A, B)-MEIK contraction and by studying their fixed points.
(4) Problems like whether or not more than one mappings have the same fixed have been of historical importance in mathematics. We have addressed this by studying common fixed points of a pair of MEIK contraction.
(5) A new generalization of the Banach contraction principle has been established and validated with an infinite dimensional Banach space.
(6) The results have been validated with examples.
(7) Our results address several open questions in literature related to Kannan and Ćirić-Reich-Rus contractions in the context of multivalued mappings.
(8) Comparison and further extension of the generalized fixed point results established in this paper following the work of Rhoades [41] and Bailey [4] is a suggested future work.
(9) Dropping the compactness criteria of the images of the maps under study would be another important future work. An approach to drop this strong condition and still avail the multivalued fixed point results has been described in [19]. In this context, it would be interesting to compare our work with some influential relevant results in literature such as [12, 13, 26, 28, 36, 39].

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