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MODIFIED SUBGRADIENT EXTRAGRADIENT RULE FOR VARIATIONAL INCLUSIONS SYSTEMS AND COUNTABLE PSEUDOCONTRACTIONS

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Abstract. In a real Hilbert space, let the VIP, SGVI, and CFPP stand for a pseudomonotone variational inequality problem, a system of general variational inclusions, and a common fixed-point problem of countable uniformly-Lipschitzian pseudocontraction operators and an asymptotically nonexpansive operator, respectively. In this paper, via a modified subgradient extragradient rule with line-search process, we design and analyze two iterative algorithms for finding a common solution of the CFPP, SGVI, and VIP. Some strong convergence theorems for the proposed algorithms are established under some suitable conditions. Our results improve and extend some corresponding results in the recent literature.
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1. Introduction

Variational inequalities play a vital role in the study of several fields, such as, management science, computer science, economic and financial engineer. Numerous real-world problems in these fields can be modeled into a variational inequality; see, e.g., [5, 8, 16, 18, 24]. Let $(H, \langle \cdot, \cdot \rangle)$ be a real Hilbert space with induced norm $\| \cdot \|$.

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Let the nonempty set C be convex and closed in H. Let P_C be the metric (nearest point) projection from H onto set C. Given an operator $A : H \to H$, consider the problem of finding a vector u in set C such that $\langle Au, v - u\rangle \geq 0$ for all $v \in C$. This is now called the classical variational inequality problem (VIP). We denote by $\mathrm{VI}(C, A)$ its solutions set. It is known that one of the most effective techniques for settling the VIP is the extragradient technique proposed by Korpelevich [12]. For recent results on Korpelevich's extragradient schemes, we refer to [2, 3, 6, 9, 12, 13, 14, 17, 19, 20, 22, 23, 25] and the references therein.

Let S be a mapping on H. We use the $\mathrm{Fix}(S)$ to denote the fixed-point set of S. $\to$ and $\rightharpoonup$ are borrowed to denote the strong convergence and weak convergence in H, respectively. Recall that S is said to be asymptotically nonexpansive if $\|T^n u - T^n v\| \leq (1 + \theta_n)\|u - v\|$ for all $u, v \in C$, $n \geq 1$, where $\{\theta_n\}_{n=1}^{\infty}$ is a real sequence in $[0, +\infty)$ with $\lim_{n\to\infty} \theta_n = 0$. In particular, if $\theta_n = 0$ for all $n \geq 1$, S is then said to be nonexpansive. Let $B_1, B_2 : C \to H$ be single-valued mappings and $F_1, F_2 : C \to 2^H$ be multi-valued mappings. In this paper, we consider the following system of general variational inclusions (SGVI), which aims to seek $(u^*, v^*) \in C \times C$ such that

$$\begin{cases} 0 \in \mu_2(B_2u^* + F_2v^*) + v^* - u^*, \\ 0 \in \mu_1(B_1v^* + F_1u^*) + u^* - v^*, \end{cases} \tag{1.1}$$

with constants $\mu_1, \mu_2 > 0$. Note that SGVI (1.1) can be transformed into a fixed-point problem in the following way.

Lemma 1.1 (see [1, Lemma 2]). *Suppose that both the mappings $F_1, F_2 : C \to 2^H$ are maximally monotone. For given $u^*, v^* \in C$, (u^*, v^*) is called a solution to GSVI (1.1) if and only if $u^* \in \mathrm{Fix}(G)$, where $\mathrm{Fix}(G)$ is the fixed point set of $G := J^{F_1}_{\mu_1}(I - \mu_1B_1)J^{F_2}_{\mu_2}(I - \mu_2B_2)$, and $v^* = J^{F_2}_{\mu_2}(I - \mu_2B_2)u^*$.*

In particular, when $F_1 = F_2 = \partial i_C$, where i_C is the indicator function of C given by $i_C(x) = 0\ \forall x \in C$ and $i_C(x) = \infty$ for all $x \notin C$, then Lemma 1.1 reduces to the following.

For given $u^*, v^* \in C$, (u^*, v^*) is called a solution to the following system of variational inequalities

$$\begin{cases} \forall u \in C,\ \langle \mu_2B_2u^* + v^* - u^*, v - v^*\rangle \geq 0, \\ \forall v \in C,\ \langle \mu_1B_1v^* + u^* - v^*, u - u^*\rangle \geq 0, \end{cases}$$

if and only if $u^* \in \mathrm{Fix}(G)$, where $\mathrm{Fix}(G)$ is the fixed point set of

$$G := P_C(I - \mu_1B_1)P_C(I - \mu_2B_2), \text{ and } v^* = P_C(I - \mu_2B_2)u^*.$$

Suppose that two mappings B_1, B_2 are α-inverse-strongly monotone and β-inverse-strongly monotone, respectively. Let $f : C \to C$ be a contraction with coefficient $\delta \in [0, 1)$, and let $F : C \to H$ be κ-Lipschitzian and η-strongly monotone with constants $\kappa, \eta > 0$ such that $\delta < \tau := 1 - \sqrt{1 - \rho(2\eta - \rho\kappa^2)} \in (0, 1]$ for $\rho \in \left(0, \frac{2\eta}{\kappa^2}\right)$. Let $S : C \to C$ be an asymptotically nonexpansive mapping with a sequence $\{\theta_n\}$. Let $\{S_l\}_{l=1}^{\infty}$ be a countable family of ς-uniformly Lipschitzian pseudocontractive self-mappings on C with $\Omega := \bigcap_{l=0}^{\infty} \mathrm{Fix}(S_l) \cap \mathrm{Fix}(G) \neq \emptyset$, where $S_0 := S$ and $G := P_C(I - \mu_1B_1)P_C(I - \mu_2B_2)$ for $\mu_1 \in (0, 2\alpha)$ and $\mu_2 \in (0, 2\beta)$. Recently, Ceng and

Wen [3] proposed the hybrid extragradient-like implicit method for finding an element in Ω, that is, for any initial $x_1 \in C$, $\{x_l\}$ is the sequence constructed by

$$\begin{cases} u_l = \beta_l x_l + (1-\beta_l) S_l u_l, \\ v_l = P_C(u_l - \mu_2 B_2 u_l), \\ y_l = P_C(v_l - \mu_1 B_1 v_l), \\ x_{l+1} = P_C[\alpha_l f(x_l) + (I - \alpha_l \rho F) S^l y_l] \quad \forall l \geq 1, \end{cases}$$

with $\{\alpha_l\}, \{\beta_l\} \subset (0,1]$. Under some appropriate assumptions, it was proven in [3] that $\{x_l\}$ converges to an element $x^* \in \Omega$ strongly (in norm). Very recently, Reich et al. [22] suggested the modified projection-type method for solving the pseudomonotone VIP with uniform continuity mapping A. Let $\{\alpha_l\} \subset (0,1)$ and $f : C \to C$ be contractive with constant $\delta \in [0,1)$.

Algorithm 1.1 (see [22]).

Initial step: Given any initial $x_1 \in C$, let $\nu > 0,\ \ell \in (0,1),\ \lambda \in \left(0, \frac{1}{\nu}\right)$.

Iterative steps: Given the current iterate x_l, compute x_{l+1} below:

Step 1. Calculate $y_l = P_C(x_l - \lambda A x_l)$ and $\varepsilon_\lambda(x_l) := x_l - y_l$. If $\varepsilon_\lambda(x_l) = 0$, then stop; x_l is a solution of $\mathrm{VI}(C, A)$. Otherwise,

Step 2. Calculate $w_l = x_l - \tau_l \varepsilon_\lambda(x_l)$, where $\tau_l := \ell^{j_l}$ and j_l is the smallest nonnegative integer j s.t. $\frac{\nu}{2}\|\varepsilon_\lambda(x_l)\|^2 \geq \langle A x_l - A(x_l - \ell^j \varepsilon_\lambda(x_l)), \varepsilon_\lambda(x_l)\rangle$.

Step 3. Calculate $x_{l+1} = \alpha_l f(x_l) + (1-\alpha_l) P_{C_l}(x_l)$, where $C_l := \{x \in C : \bar{h}_l(x_l) \leq 0\}$ and $\bar{h}_l(x) = \langle A w_l, x - x_l\rangle + \frac{\tau_l}{2\lambda}\|\varepsilon_\lambda(x_l)\|^2$.

Again set $l := l + 1$ and go to Step 1.

2. Preliminaries

Let $(H, \langle \cdot, \cdot \rangle)$ be a real Hilbert space with induced norm $\|\cdot\|$. Let the nonempty set C be convex and closed in H. $\|u + v\|^2 \leq \|u\|^2 + 2\langle v, u + v\rangle$ for all $u, v \in H$ is trivial. Given a sequence $\{v_k\} \subset H$, we let $v_k \to v$ (resp., $v_k \rightharpoonup v$) present the strong (resp., weak) convergence of $\{v_k\}$ to v. An operator $S : C \to H$ is said to be

(a) L-Lipschitz continuous or L-Lipschitzian if $\|Su - Sv\| \leq L\|u - v\|$ for all $u, v \in C$, where $L0$ is a positive real number;

(b) pseudocontractive if $\langle Su - Sv, u - v\rangle \leq \|u - v\|^2$ for all $u, v \in C$;

(c) pseudomonotone if $\langle Su, v - u\rangle \geq 0 \Rightarrow \langle Sv, v - u\rangle \geq 0$ for all $u, v \in C$;

(d) α-strongly monotone if $\langle Su - Sv, u - v\rangle \geq \alpha\|u - v\|^2$ for all $u, v \in C$, where α is a positive real number;

(e) β-inverse-strongly monotone if $\langle Su - Sv, u - v\rangle \geq \beta\|Su - Sv\|^2$ for all $u, v \in C$; where β is a positive real number;

(f) sequentially weakly continuous if, for all $\{v_k\} \subset C$, $v_k \rightharpoonup v \Rightarrow Sv_k \rightharpoonup Sv$.

It is clear that each monotone mapping is pseudomonotone, but the converse is not true. It is known that, for all $u \in H$, there exists (nearest point) $P_C u \in C$ with $\|u - P_C u\| \leq \|u - v\|$ for all $v \in C$. P_C is said to be a metric (or nearest point) projection of H onto C. The following facts are trivial.

(a) $\langle u - v, P_C u - P_C v\rangle \geq \|P_C u - P_C v\|^2$ for all $u, v \in H$;

(b) $w = P_C u \Leftrightarrow \langle u - w, v - w\rangle \leq 0$ for all$u \in H, v \in C$;

(c) $\|u - P_C u\|^2 + \|v - P_C u\|^2 \leq \|u - v\|^2$ for all $u \in H, v \in C$;

(d) $\|u\|^2 - \|v\|^2 - 2\langle u - v, v\rangle = \|u - v\|^2$ for all $u, v \in H$;

(e) $s\|u\|^2+(1-s)\|v\|^2-s(1-s)\|u-v\|^2 = \|su+(1-s)v\|^2$ for all $u, v \in H, s \in [0,1]$.

Let $\{S_l\}_{l=1}^{\infty}$ be a sequence of continuous pseudocontractive self-mappings on C. Then $\{S_l\}_{l=1}^{\infty}$ is said to be a countable family of ς-uniformly Lipschitzian pseudocontractive self-mappings on C ([3]) if there exists a constant $\varsigma > 0$ such that each S_l is ς-Lipschitz continuous.

The following d tools are useful in the convergence analysis of the proposed algorithms.

Proposition 2.1 (see [21]). *Let C be a nonempty, convex, and closed set of in a Hilbert space H. Let $\{S_k\}_{k=1}^{\infty}$ be a countable family of self-mappings on C such that $\sum_{k=1}^{\infty}\sup\{\|S_kx - S_{k+1}x\| : x \in C\} < \infty$. Then, for each $y \in C$, $\{S_ky\}$ converges strongly to some point of C. Moreover, let $\tilde{S}$, a self-mapping on C, be defined by $\tilde{S}y = \lim_{k\to\infty} S_ky$ for all $y \in C$. Then $\lim_{k\to\infty}\sup\{\|\tilde{S}x - S_kx\| : x \in C\} = 0$.*

Proposition 2.2 (see [7]). *Let C be a nonempty, convex, and closed set in a Hilbert space H, and let $T : C \to C$ be a continuous and strong pseudocontraction mapping. Then, T has a unique fixed point in C.*

Let $\Gamma : C \to 2^H$ be a multi-valued operator with $\Gamma y \neq \emptyset$ for all $y \in C$. Then Γ is called monotone if, for all $x, y \in C$, $u \in \Gamma x$ and $v \in \Gamma y$ imply $\langle x - y, u - v\rangle \geq 0$. A monotone operator $\Gamma : C \to 2^H$ is said to be maximal if its graph $\mathrm{Gph}(\Gamma)$ is not properly contained in the graph of any other monotone operator. It is known that a monotone operator Γ is maximal if and only if $(I + r\Gamma)C = H$ for each $r > 0$. If $\Gamma : C \to 2^H$ is maximal monotone, we define the resolvent $J_r^{\Gamma} : H \to C$ of Γ by $J_r^{\Gamma} = (I + r\Gamma)^{-1}$. It is easy to check that $\mathrm{Fix}(J_r^{\Gamma}) = \Gamma^{-1}0 = \{x \in C : 0 \in \Gamma x\}$.

Proposition 2.3 (see [1, Lemma 1]). *Let $\Gamma : C \to 2^H$ be a maximal monotone operator. Then, for any given $r > 0$, $\langle x - y, J_r^{\Gamma}x - J_r^{\Gamma}y\rangle \geq \|J_r^{\Gamma}x - J_r^{\Gamma}y\|^2$ for all $x, y \in H$.*

Lemma 2.1 (see [1, Lemma 4]). *Let the mapping $B : C \to H$ be γ-inverse-strongly monotone. Then, for a given $\lambda \geq 0$,*

$$\|(I - \lambda B)u - (I - \lambda B)v\|^2 + \lambda(2\gamma - \lambda)\|Bu - Bv\|^2 \leq \|u - v\|^2, \quad \forall u, v \in C.$$

In particular, if $0 \leq \lambda \leq 2\gamma$, then $I - \lambda B$ is nonexpansive.

Lemma 2.2 (see [1, Lemma 5]). *Suppose that $F_1, F_2 : C \to 2^H$ are two maximal monotone operators. Let the mappings $B_1, B_2 : C \to H$ be α-inverse-strongly monotone and β-inverse-strongly monotone, respectively. Let the mapping $G : C \to C$ be defined as $G := J_{\mu_1}^{F_1}(I - \mu_1 B_1)J_{\mu_2}^{F_2}(I - \mu_2 B_2)$. If $0 \leq \mu_1 \leq 2\alpha$ and $0 \leq \mu_2 \leq 2\beta$, then $G : C \to C$ is nonexpansive.*

The following lemma is trivial.

Lemma 2.3. *Let $A : C \to H$ be pseudomonotone and continuous. Then $u \in C$ is a solution to the VIP $\langle Au, v - u\rangle \geq 0$ for all $v \in C$ if and only if $\langle Av, v - u\rangle \geq 0$ for all $v \in C$.*

Lemma 2.4 (see [25]). *Let $\{a_l\}$ be a sequence of nonnegative numbers satisfying the conditions: $a_{l+1} \leq \lambda_l\gamma_l + (1 - \lambda_l)a_l$ for all $l \geq 1$, where $\{\lambda_l\}$ and $\{\gamma_l\}$ are sequences of real numbers with $\{\lambda_l\} \subset [0,1]$ and $\sum_{l=1}^{\infty}\lambda_l = \infty$, and $\limsup_{l\to\infty}\gamma_l \leq 0$ or $\sum_{l=1}^{\infty}|\lambda_l\gamma_l| < \infty$. Then $\lim_{l\to\infty} a_l = 0$.*

Lemma 2.5 (see [15]). *Let H_1 and H_2 be two real Hilbert spaces. Suppose that $A : H_1 \to H_2$ is uniformly continuous on bounded subsets of H_1 and M is a bounded subset of H_1. Then, $A(M)$ is bounded.*

Lemma 2.6 (see [10]). *Let h be a real-valued function on H and define $K := \{x \in C : h(x) \leq 0\}$. If K is nonempty and h is Lipschitz continuous on C with modulus $\theta > 0$, then $\operatorname{dist}(x, K) \geq \theta^{-1} \max\{h(x), 0\}$ for all $x \in C$, where $\operatorname{dist}(x, K)$ denotes the distance of x to K.*

Lemma 2.7 (see [4]). *Let X be a Banach space which admits a weakly continuous duality mapping, C be a convex and closed subset of X, and $T : C \to C$ be an asymptotically nonexpansive mapping with $\operatorname{Fix}(T) \neq \emptyset$. Then $I - T$ is demiclosed at zero.*

The following lemmas are vital to the convergence analysis of the proposed algorithms.

Lemma 2.8 (see [14]). *Let $\{\Gamma_m\}$ be a sequence of real numbers that does not decrease at infinity in the sense that there exists a subsequence $\{\Gamma_{m_k}\}$ of $\{\Gamma_m\}$ which satisfies $\Gamma_{m_k} < \Gamma_{m_k+1}$ for each integer $k \geq 1$. Let $\{\phi(m)\}_{m \geq m_0}$ be the sequence of integers formulated by $\phi(m) = \max\{k \leq m : \Gamma_k < \Gamma_{k+1}\}$, where integer $m_0 \geq 1$ such that $\{k \leq m_0 : \Gamma_k < \Gamma_{k+1}\} \neq \emptyset$. Then (i) $\phi(m_0) \leq \phi(m_0 + 1) \leq \cdots$ and $\phi(m) \to \infty$; and (ii) $\Gamma_{\phi(m)} \leq \Gamma_{\phi(m)+1}$ and $\Gamma_m \leq \Gamma_{\phi(m)+1}$ $\forall m \geq m_0$.*

Lemma 2.9 (see [25]). *Let $\lambda \in (0, 1]$ and $T : C \to C$ be nonexpansive. Let $T^\lambda : C \to H$ be formulated as $T^\lambda u := (I - \lambda\rho F)Tu$ for all $u \in C$ with $F : C \to H$ being κ-Lipschitzian and η-strongly monotone. Then T^λ is a contraction provided $0 < \rho < \frac{2\eta}{\kappa^2}$.*

3. Main Results

In this section, we always assume that the following conditions hold.

$A : H \to H$ is pseudomonotone and uniformly continuous on C such that $\|Az\| \leq \liminf_{n\to\infty} \|Av_n\|$ for each $\{v_n\} \subset C$ with $v_n \rightharpoonup z$.

$F_1, F_2 : C \to 2^H$ are two maximal monotone operators, and $B_1, B_2 : C \to H$ are α-inverse-strongly monotone and β-inverse-strongly monotone, respectively.

$f : C \to H$ is a contraction with constant $\delta \in [0, 1)$ and $F : C \to H$ is η-strongly monotone and κ-Lipschitzian with $\delta < \tau := 1 - \sqrt{1 - \rho(2\eta - \rho\kappa^2)}$ for $\rho \in \left(0, \frac{2\eta}{\kappa^2}\right)$.

$\{S_n\}_{n=1}^\infty$ is a countable family of ς-uniformly Lipschitzian pseudocontractive self-mappings on C and $S : C \to C$ is an asymptotically nonexpansive self-mapping with a sequence $\{\theta_n\}$.

$\Omega = \bigcap_{n=0}^\infty \operatorname{Fix}(S_n) \cap \operatorname{Fix}(G) \cap \operatorname{VI}(C, A) \neq \emptyset$ with $S_0 := S$, and $\operatorname{Fix}(G)$ is the fixed point set of $G = J_{\mu_1}^{F_1}(I - \mu_1 B_1) J_{\mu_2}^{F_2}(I - \mu_2 B_2)$ for $0 < \mu_1 < 2\alpha$ and $0 < \mu_2 < 2\beta$.

$\sum_{n=1}^\infty \sup_{x\in D} \|S_n x - S_{n+1} x\| < \infty$ for any bounded subset D of C and

$\operatorname{Fix}(\tilde{S}) = \bigcap_{n=1}^\infty \operatorname{Fix}(S_n)$ where $\tilde{S} : C \to C$ is defined as $\tilde{S}x = \lim_{n\to\infty} S_n x$ for all $x \in C$.

$\{\sigma_n\}, \{\alpha_n\} \subset (0, 1]$ and $\{\beta_n\} \subset [0, 1]$ such that

(i) $\sum_{n=1}^\infty \alpha_n = \infty$, $\lim_{n\to\infty} \alpha_n = 0$, and $\lim_{n\to\infty} \theta_n/\alpha_n = 0$;

(ii) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$; and

(iii) $\limsup_{n\to\infty} \sigma_n < 1$.

Algorithm 3.1.

Initial step: Set $\nu > 0$, $\ell \in (0,1)$, and $\lambda \in \left(0, \frac{1}{\nu}\right)$. Choose an initial x_1 in set C arbitrarily.

Iterative steps: Given the current iterate x_n, compute x_{n+1} below:

Step 1. Calculate $u_n = \sigma_n x_n + (1-\sigma_n)S_n u_n$,

$$y_n = P_C(u_n - \lambda A u_n) \text{ and } \varepsilon_\lambda(u_n) := u_n - y_n.$$

Step 2. Calculate $t_n = u_n - \tau_n \varepsilon_\lambda(u_n)$, where $\tau_n := \ell^{j_n}$ and j_n is the smallest nonnegative integer j satisfying

$$2\langle Au_n - A(u_n - \ell^j \varepsilon_\lambda(u_n)), u_n - y_n\rangle \le \nu \|\varepsilon_\lambda(u_n)\|^2. \tag{3.1}$$

Step 3. Calculate $z_n = P_{C_n}(u_n)$, where $C_n := \{u \in C : \bar{h}_n(u) \le 0\}$ and

$$\bar{h}_n(u) = \frac{2\lambda\langle At_n, u - u_n\rangle + \tau_n\|\varepsilon_\lambda(u_n)\|^2}{2\lambda}.$$

Step 4. Calculate $v_n = J_{\mu_2}^{F_2}(z_n - \mu_2 B_2 z_n)$, $w_n = J_{\mu_1}^{F_1}(v_n - \mu_1 B_1 v_n)$ and

$$x_{n+1} = \beta_n x_n + (1-\beta_n)P_C[(I - \alpha_n \rho F)S^n w_n + \alpha_n f(x_n)]. \tag{3.2}$$

Set $n := n+1$ and return to Step 1.

Lemma 3.1. *Armijo-type search rule (3.1) is well specified, and the relation holds:* $\lambda^{-1}\|\varepsilon_\lambda(u_n)\|^2 \le \langle \varepsilon_\lambda(u_n), Au_n\rangle$.

Proof. On account of $\ell \in (0,1)$ and uniform continuity of A on C, we can easily see that $\lim_{j\to\infty}\langle Au_n - A(u_n - \ell^j\varepsilon_\lambda(u_n)), \varepsilon_\lambda(u_n)\rangle = 0$. If $\varepsilon_\lambda(u_n) = 0$, one has $j_n = 0$. Otherwise, from $\varepsilon_\lambda(u_n) \ne 0$, it follows that $\exists j_n \ge 0$ verifying (3.1). Hence, the firm nonexpansivity of P_C ensures $\langle u - P_C v, u - v\rangle \ge \|u - P_C v\|^2$ for all $u \in C, v \in H$. Setting $v = u_n - \lambda A u_n$ and $u = u_n$, one obtains

$$\lambda\langle u_n - P_C(u_n - \lambda Au_n), Au_n\rangle \ge \|u_n - P_C(u_n - \lambda Au_n)\|^2.$$

Thus the relation holds. □

Lemma 3.2. *Let $\bar{h}_n$ be the function constructed in (3.2). Then, $\bar{h}_n(\upsilon) \le 0$ for all $\upsilon \in \Omega$. Additionally, if $\varepsilon_\lambda(u_n) \ne 0$, then $\bar{h}_n(u_n) > 0$.*

Proof. Since it is clear that the latter claim of Lemma 3.2 holds, it is sufficient to demonstrate the former claim. Indeed, choose a fixed $\upsilon \in \Omega$ arbitrarily. By Lemma 2.3, one has $\langle At_n, t_n - \upsilon\rangle \ge 0$. Hence

$$\bar{h}_n(\upsilon) = \langle At_n, t_n - u_n\rangle + \langle At_n, \upsilon - t_n\rangle + \frac{\tau_n}{2\lambda}\|\varepsilon_\lambda(u_n)\|^2 \le -\tau_n\langle At_n, \varepsilon_\lambda(u_n)\rangle + \frac{\tau_n}{2\lambda}\|\varepsilon_\lambda(u_n)\|^2.$$

Meanwhile, from (3.1) it follows that $\nu\|\varepsilon_\lambda(u_n)\|^2 \ge 2\langle Au_n - At_n, \varepsilon_\lambda(u_n)\rangle$. Thus, by Lemma 3.1,

$$\langle At_n, \varepsilon_\lambda(u_n)\rangle \ge -\frac{\nu}{2}\|\varepsilon_\lambda(u_n)\|^2 + \langle \varepsilon_\lambda(u_n), Au_n\rangle \ge \left(-\frac{\nu}{2} + \frac{1}{\lambda}\right)\|\varepsilon_\lambda(u_n)\|^2,$$

so $\bar{h}_n(\upsilon) \le -\frac{\tau_n}{2}(\frac{1}{\lambda} - \nu)\|\varepsilon_\lambda(u_n)\|^2$. □

Lemma 3.3. *Let $\{u_n\}, \{x_n\}, \{y_n\}$, and $\{z_n\}$ be the bounded sequences constructed in Algorithm 3.1. Let $x_n - x_{n+1} \to 0$, $x_n - S_n u_n \to 0$, $u_n - y_n \to 0$, $x_n - z_n \to 0$ and*

$x_n - Gz_n \to 0$. *If* $S^n x_n - S^{n+1}x_n \to 0$ *and* $\exists\{x_{n_i}\} \subset \{x_n\}$ *such that* $x_{n_i} \rightharpoonup z \in C$, *then* $z \in \Omega$.

Proof. From Algorithm 3.1, we see that $\|u_n - x_n\| \le \|S_n u_n - x_n\|$. In view of $S_n u_n - x_n \to 0$ as $n \to \infty$, we have

$$\lim_{n\to\infty} \|u_n - x_n\| = 0. \tag{3.3}$$

Putting $q_n := \alpha_n f(x_n) + (I - \alpha_n \rho F)S^n w_n$, we see that $x_{n+1} = \beta_n x_n + (1-\beta_n)P_C(q_n)$ and $q_n - S^n w_n = \alpha_n f(x_n) - \alpha_n \rho F S^n w_n$. Hence,

$$(1-\beta_n)\|x_n - S^n w_n\| \le \|x_n - x_{n+1}\| + \alpha_n\|f(x_n)\| + \alpha_n\|\rho F S^n w_n\|.$$

The nonexpansivity of G ensures the boundedness of $\{w_n\}$. Observe that $x_n - x_{n+1} \to 0$, $\alpha_n \to 0$, and $\liminf_{n\to\infty}(1-\beta_n) > 0$. By the boundedness of $\{x_n\}, \{w_n\}$ we obtain $\lim_{n\to\infty}\|x_n - S^n w_n\| = 0$.

We claim that $\lim_{n\to\infty}\|x_n - Sx_n\| = 0$. In fact, using the asymptotical nonexpansivity of S, one deduces that

$$\begin{aligned}\|x_n - Sx_n\| &\le \|x_n - S^n z_n\| + \|S^n z_n - S^n x_n\| + \|S^n x_n - S^{n+1}x_n\| \\ &\quad + \|S^{n+1}x_n - S^{n+1}z_n\| + \|S^{n+1}z_n - Sx_n\| \\ &\le (2+\theta_1)\|x_n - S^n z_n\| + (2+\theta_n+\theta_{n+1})\|z_n - x_n\| \\ &\quad + \|S^n x_n - S^{n+1}x_n\|.\end{aligned}$$

Since $x_n - z_n \to 0$, $x_n - S^n z_n \to 0$, and $S^n x_n - S^{n+1}x_n \to 0$, we obtain

$$\lim_{n\to\infty}\|x_n - Sx_n\| = 0. \tag{3.4}$$

Also, we claim that $\lim_{n\to\infty}\|x_n - \bar{S}x_n\| = 0$, where $\bar{S} := (2I - \tilde{S})^{-1}$. From $u_n = \sigma_n x_n + (1-\sigma_n)S_n u_n$ and $x_n - u_n \to 0$, we have $(1-\sigma_n)\|S_n u_n - u_n\| \le \|x_n - u_n\| \to 0$ $(n \to \infty)$, which together with $0 < \liminf_{n\to\infty}(1-\sigma_n)$, yields $\lim_{n\to\infty}\|S_n u_n - u_n\| = 0$. Since $\{S_n\}_{n=1}^\infty$ is ς-uniformly Lipschitzian on C, we deduce from $x_n - u_n \to 0$ and $S_n u_n - u_n \to 0$ that

$$\begin{aligned}\|S_n x_n - x_n\| &\le \|S_n x_n - S_n u_n\| + \|S_n u_n - u_n\| + \|u_n - x_n\| \\ &\le (\varsigma+1)\|u_n - x_n\| + \|S_n u_n - u_n\| \to 0 \ (n\to\infty).\end{aligned}$$

It is clear that $\tilde{S} : C \to C$ is pseudocontractive and ς-Lipschitzian, where $\tilde{S}x = \lim_{n\to\infty} S_n x$ for all x in set C. We show that $\lim_{n\to\infty}\|\tilde{S}x_n - x_n\| = 0$. Using the boundedness of $\{x_n\}$ and putting $D = \overline{\mathrm{conv}}\{x_n : n \ge 1\}$ (the closed convex hull of the set $\{x_n : n \ge 1\}$), we have $\sum_{n=1}^\infty \sup_{x\in D}\|S_n x - S_{n+1}x\| < \infty$. By Proposition 2.1, we have $\lim_{n\to\infty}\sup_{x\in D}\|S_n x - \tilde{S}x\| = 0$, which immediately arrives at $\lim_{n\to\infty}\|S_n x_n - \tilde{S}x_n\| = 0$. Consequently, $\|x_n - \tilde{S}x_n\| \le \|x_n - S_n x_n\| + \|S_n x_n - \tilde{S}x_n\| \to 0$ as $n \to \infty$.

Now, let us show that if we define $\bar{S} := (2I - \tilde{S})^{-1}$, then $\bar{S} : C \to C$ is nonexpansive, $\mathrm{Fix}(\bar{S}) = \mathrm{Fix}(\tilde{S}) = \bigcap_{n=1}^\infty \mathrm{Fix}(S_n)$ and $\lim_{n\to\infty}\|x_n - \bar{S}x_n\| = 0$. In fact, it is known that $\bar{S}$ is nonexpansive and $\mathrm{Fix}(\bar{S}) = \mathrm{Fix}(\tilde{S}) = \bigcap_{n=1}^\infty \mathrm{Fix}(S_n)$ as a consequence of [11, Theorem 6]. From $x_n - \tilde{S}x_n \to 0$, it follows that

$$\|x_n - \bar{S}x_n\| \le \|\bar{S}^{-1}x_n - x_n\| = \|(2I - \tilde{S})x_n - x_n\| = \|x_n - \tilde{S}x_n\| \to 0 \ (n\to\infty). \tag{3.5}$$

Moreover, let us demonstrate that $\lim_{n\to\infty}\|x_n - Gx_n\| = 0$. From Lemma 2.2, we know that $G : C \to C$ is nonexpansive for $\mu_1 \in (0, 2\alpha)$ and $\mu_2 \in (0, 2\beta)$. By Algorithm 3.1, we have $w_n = Gz_n$. Using $\|Gx_n - x_n\| \le \|x_n - z_n\| + \|w_n - x_n\|$ and the hypotheses, $x_n - z_n \to 0$ and $x_n - w_n \to 0$, we obtain $\lim_{n\to\infty}\|Gx_n - x_n\| = 0$.

Next, let us demonstrate $z \in \mathrm{VI}(C, A)$. Indeed, $x_n - u_n \to 0$ and $x_{n_i} \rightharpoonup z$ yiled $u_{n_i} \rightharpoonup z$. Since C is convex and closed, from $\{u_n\} \subset C$ and $u_{n_i} \rightharpoonup z$, we see that $z \in C$. In what follows, we consider two cases. In the case of $Az = 0$, it is clear that $z \in \mathrm{VI}(C, A)$ because $\langle Az, y - z\rangle \ge 0$ for all y in C. In the case of $Az \ne 0$, it follows from $u_n - x_n \to 0$ and $x_{n_i} \rightharpoonup z$ that $u_{n_i} \rightharpoonup z$ as $i \to \infty$. Using the assumption on A, we have $0 < \|Az\| \le \liminf_{i\to\infty}\|Au_{n_i}\|$. This we assume that $\|Au_{n_i}\| \ne 0$ for all $i \ge 1$.

On the other hand, from $y_n = P_C(u_n - \lambda Au_n)$, one has $\langle u_n - \lambda Au_n - y_n, x - y_n\rangle \le 0$ for all x in C. Hence

$$\frac{1}{\lambda}\langle u_n - y_n, x - y_n\rangle + \langle Au_n, y_n - u_n\rangle \le \langle Au_n, x - u_n\rangle \quad \forall x \in C. \tag{3.6}$$

In the light of the uniform continuity of A on C, one knows that $\{Au_n\}$ is bounded (due to Lemma 2.5). From the boundedness of $\{y_n\}$ and $u_n - y_n \to 0$ (due to the hypothesis), we see from (3.6) that $\liminf_{i\to\infty}\langle Au_{n_i}, x - u_{n_i}\rangle \ge 0$ for all x in set C.

To prove that z is in $\mathrm{VI}(C, A)$, we now choose a sequence $\{\gamma_i\} \subset (0, 1)$ satisfying $\gamma_i \downarrow 0$ as $i \to \infty$. For each $i \ge 1$, we denote by l_i the smallest positive integer such that

$$\langle Au_{n_j}, x - u_{n_j}\rangle + \gamma_i \ge 0 \quad \forall j \ge l_i. \tag{3.7}$$

Because $\{\gamma_i\}$ is decreasing, it is readily known that $\{l_i\}$ is increasing. Note that $Au_{l_i} \ne 0$ for all $i \ge 1$ (due to $\{Au_{l_i}\} \subset \{Au_{n_i}\}$). Then $\upsilon_{l_i} = \frac{Au_{l_i}}{\|Au_{l_i}\|^2}$, and $\langle Au_{l_i}, \upsilon_{l_i}\rangle = 1$ for all $i \ge 1$. Using (3.7), one has $\langle Au_{l_i}, x + \gamma_i\upsilon_{l_i} - u_{l_i}\rangle \ge 0$ for all $i \ge 1$. From the pseudo-monotonicity of A, one has $\langle A(x + \gamma_i\upsilon_{l_i}), x + \gamma_i\upsilon_{l_i} - u_{l_i}\rangle \ge 0$ for all $i \ge 1$. This immediately arrives at

$$\langle Ax, x - u_{l_i}\rangle \ge \langle Ax - A(x + \gamma_i\upsilon_{l_i}), x + \gamma_i\upsilon_{l_i} - u_{l_i}\rangle - \gamma_i\langle Ax, \upsilon_{l_i}\rangle \quad \forall i \ge 1. \tag{3.8}$$

We claim that $\lim\limits_{i\to\infty}\gamma_i\upsilon_{l_i} = 0$. In fact, from $x_{n_i} \rightharpoonup z \in C$ and $u_n - x_n \to 0$, we obtain $u_{n_i} \rightharpoonup z$. Note that $\{u_{l_i}\} \subset \{u_{n_i}\}$ and $\gamma_i \downarrow 0$ as $i \to \infty$. Thus

$$0 \le \limsup_{i\to\infty}\|\gamma_i\upsilon_{l_i}\| = \limsup_{i\to\infty}\frac{\gamma_i}{\|Au_{l_i}\|} \le \frac{\limsup\limits_{i\to\infty}\gamma_i}{\liminf\limits_{i\to\infty}\|Au_{n_i}\|} = 0.$$

Hence $\gamma_i\upsilon_{l_i} \to 0$ as $i \to \infty$. Letting $i \to \infty$, we deduce that the right-hand side of (3.8) tends to zero by the uniform continuity of A, the boundedness of $\{u_{l_i}\}, \{\upsilon_{l_i}\}$, and the limit $\lim\limits_{i\to\infty}\gamma_i\upsilon_{l_i} = 0$. Therefore,

$$\langle Ax, x - z\rangle = \liminf_{i\to\infty}\langle Ax, x - u_{l_i}\rangle \ge 0 \ \forall x \in C.$$

Using Lemma 2.3, one has $z \in \mathrm{VI}(C, A)$.

Lastly, we claim that $z \in \Omega$. In fact, (3.4) yields $x_{n_i} - Sx_{n_i} \to 0$. By Lemma 2.7, one knows that $I - S$ is demiclosed at zero. From $x_{n_i} \rightharpoonup z$, it follows that $(I - S)z = 0$, i.e., $z \in \mathrm{Fix}(S)$. Note that $\bar{S} : C \to C$ is nonexpansive and (3.5) yields

$x_{n_i} - \bar{S}x_{n_i} \to 0$. By Lemma 2.7, one knows that $I - \bar{S}$ is demiclosed at zero. In the same way, we has $z \in \mathrm{Fix}(\bar{S}) = \mathrm{Fix}(\tilde{S}) = \bigcap_{n=1}^{\infty} \mathrm{Fix}(S_n)$. Besides, let us claim that $z \in \mathrm{Fix}(G)$. Actually, by Lemma 2.7, we deduce that $I - G$ is demiclosed at zero. Thus, from $x_n - Gx_n \to 0$ and $x_{n_i} \rightharpoonup z$, $(I-G)z = 0$, i.e., $z \in \mathrm{Fix}(G)$. Accordingly, $z \in \bigcap_{n=0}^{\infty} \mathrm{Fix}(S_n) \cap \mathrm{Fix}(G) \cap \mathrm{VI}(C, A) = \Omega$ with $S_0 := S$. This completes the proof. □

Lemma 3.4. *Let $\{u_n\}$ be the sequence constructed in Algorithm 3.1. Then,*

$$\lim_{n\to\infty} \tau_n \|\varepsilon_\lambda(u_n)\|^2 = 0 \ \Rightarrow \ \lim_{n\to\infty} \|\varepsilon_\lambda(u_n)\| = 0. \tag{3.9}$$

Proof. We demonstrate that $\limsup_{n\to\infty} \|\varepsilon_\lambda(u_n)\| = 0$. On the contrary, we suppose that $\limsup_{n\to\infty} \|\varepsilon_\lambda(u_n)\| = k > 0$. Then, there exists $\{n_p\} \subset \{n\}$ such that $\lim_{p\to\infty} \|\varepsilon_\lambda(u_{n_p})\| = k > 0$. Note that $\lim_{p\to\infty} \tau_{n_p} \|\varepsilon_\lambda(u_{n_p})\|^2 = 0$. In the case of $\liminf_{p\to\infty} \tau_{n_p} > 0$, we assume that there exists $\zeta > 0$ with $\tau_{n_p} \geq \zeta > 0$ for all $p \geq 1$. It follows that

$$|\varepsilon_\lambda(u_{n_p})\|^2 \leq \frac{1}{\zeta} \cdot \tau_{n_p} \|\varepsilon_\lambda(u_{n_p})\|^2,$$

which immediately attains

$$0 < k^2 = \lim_{p\to\infty} \|\varepsilon_\lambda(u_{n_p})\|^2 \leq \lim_{p\to\infty} \left\{ \frac{1}{\zeta} \cdot \tau_{n_p} \|\varepsilon_\lambda(u_{n_p})\|^2 \right\} = 0. \tag{3.10}$$

This reaches at a contradiction. In the case of $\liminf_{p\to\infty} \tau_{n_p} = 0$, there exists a subsequence of $\{\tau_{n_p}\}$, still denoted by $\{\tau_{n_p}\}$, such that $\lim_{p\to\infty} \tau_{n_p} = 0$. Put

$$\tilde{t}_{n_p} := \frac{1}{\ell} \tau_{n_p} y_{n_p} + \left(1 - \frac{1}{\ell} \tau_{n_p}\right) u_{n_p} = u_{n_p} - \frac{1}{\ell} \tau_{n_p} (u_{n_p} - y_{n_p}).$$

From $\lim_{p\to\infty} \tau_{n_p} \|\varepsilon_\lambda(u_{n_p})\|^2 = 0$, we obtain

$$\lim_{p\to\infty} \|\tilde{t}_{n_p} - u_{n_p}\|^2 = \lim_{p\to\infty} \frac{1}{\ell^2} \tau_{n_p} \cdot \tau_{n_p} \|\varepsilon_\lambda(u_{n_p})\|^2 = 0. \tag{3.11}$$

Using the stepsize rule (3.1), one sees that $2\langle Au_{n_p} - A\tilde{t}_{n_p}, \varepsilon_\lambda(u_{n_p})\rangle > \nu \|\varepsilon_\lambda(u_{n_p})\|^2$. Since A is uniformly continuous on bounded subsets of C, (3.11) leads to $\lim_{p\to\infty} \|Au_{n_p} - A\tilde{t}_{n_p}\| = 0$, which implies $\lim_{p\to\infty} \|\varepsilon_\lambda(u_{n_p})\| = 0$. So, this reaches a contradiction. Consequently, $\varepsilon_\lambda(u_n) \to 0$ as $n \to \infty$. This completes the proof. □

Theorem 3.1. *Suppose that $\{x_n\}$ is the sequence constructed in Algorithm 3.1. Then $x_n \to x^* \in \Omega$ provided $S^n x_n - S^{n+1} x_n \to 0$, with $x^* \in \Omega$ being only a solution to the hierarchical inequality (HVI): $\langle (\mu F - f)x^*, y - x^*\rangle \geq 0$ for all $y \in \Omega$.*

Proof. First, in view of $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$ and $\lim_{n\to\infty} \frac{\theta_n}{\alpha_n} = 0$, we may assume, without loss of generality, that $\{\beta_n\} \subset [a, b] \subset (0, 1)$ and $\theta_n \leq \frac{\alpha_n(\tau - \delta)}{2}$ for all $n \geq 1$. By Lemma 2.9, one sees $\|P_\Omega(I - \rho F + f)u - P_\Omega(I - \rho F + f)v\| \leq [1 - (\tau - \delta)]\|u - v\|$ for all $u, v \in C$. This implies that $P_\Omega(I - \rho F + f)$ is contractive. Thus $P_\Omega(I - \rho F + f)$ has a unique fixed point in set C, say $x^* \in C$. That is, $\exists$ |

(solution)

$$x^* \in \Omega = \bigcap_{n=0}^{\infty} \mathrm{Fix}(S_n) \cap \mathrm{Fix}(G) \cap \mathrm{VI}(C, A)$$

of the HVI:

$$\langle (\rho F - f)x^*, y - x^* \rangle \geq 0 \quad \forall y \in \Omega. \tag{3.12}$$

Next we demonstrate the conclusion of the theorem. To this goal, we divide the remainder of the proof into several claims.

Claim 1. We assert that $\{x_n\}$ is bounded. Indeed, for

$$x^* \in \Omega = \bigcap_{n=0}^{\infty} \mathrm{Fix}(S_n) \cap \mathrm{Fix}(G) \cap \mathrm{VI}(C, A)$$

we have $S_n x^* = x^*,\ Gx^* = x^*$, and $P_C(x^* - \lambda A x^*) = x^*$. We observe that

$$\|z_n - x^*\|^2 = \|P_{C_n}(u_n) - x^*\|^2 \leq \|u_n - x^*\|^2 - \mathrm{dist}^2(u_n, C_n), \tag{3.13}$$

which hence leads to

$$\|z_n - x^*\| \leq \|u_n - x^*\| \quad \forall n \geq 1. \tag{3.14}$$

Using Lemma 2.2, one knows that $G = J_{\mu_1}^{F_1}(I - \mu_1 B_1) J_{\mu_2}^{F_2}(I - \mu_2 B_2)$ is nonexpansive for $\mu_1 \in (0, 2\alpha)$ and $\mu_2 \in (0, 2\beta)$. Since each $S_n : C \to C$ is a pseudocontraction mapping, we have $\|u_n - x^*\|^2 \leq \sigma_n \|x_n - x^*\| \|u_n - x^*\| + (1 - \sigma_n)\|u_n - x^*\|^2$, and hence $\|u_n - x^*\| \leq \|x_n - x^*\|$. This together with (3.14), yields

$$\|z_n - x^*\| \leq \|u_n - x^*\| \leq \|x_n - x^*\| \quad \forall n \geq 1 \tag{3.15}$$

Thus, using (3.15), we obtain from Lemma 2.9 that

$$\begin{aligned}
\|x_{n+1} - x^*\| &\leq \beta_n \|x_n - x^*\| + (1 - \beta_n)\|\alpha_n(f(x_n) - f(x^*)) + (I - \alpha_n \rho F)S^n w_n \\
&\quad - (I - \alpha_n \rho F)x^* + \alpha_n (f - \rho F)x^*\| \\
&\leq \beta_n \|x_n - x^*\| + (1 - \beta_n)[\alpha_n \delta \|x_n - x^*\| + (1 - \alpha_n \tau)(1 + \theta_n)\|z_n - x^*\| + \alpha_n \|(f - \rho F)x^*\|] \\
&\leq \left[1 - \frac{\alpha_n(1 - \beta_n)(\tau - \delta)}{2}\right] \|x_n - x^*\| + \frac{\alpha_n(1 - \beta_n)(\tau - \delta)}{2} \cdot \frac{2\|(f - \rho F)x^*\|}{\tau - \delta} \\
&\leq \max\left\{\|x_n - x^*\|, \frac{2\|(f - \rho F)x^*\|}{\tau - \delta}\right\}.
\end{aligned}$$

By induction, we have

$$\|x_n - x^*\| \leq \max\left\{\|x_1 - x^*\|, \frac{2\|(f - \rho F)x^*\|}{\tau - \delta}\right\} \ \forall n \geq 1.$$

Thus $\{x_n\}$ is bounded, so are $\{u_n\}, \{y_n\}, \{z_n\}, \{w_n\}, \{f(x_n)\}, \{At_n\}$, and $\{S^n w_n\}$.

Claim 2. We assert that

$$\begin{aligned}
&(1 - \beta_n)\left\{\left[1 - \frac{\alpha_n(\tau + \delta)}{2}\right][\|x_n - u_n\|^2 + \|u_n - z_n\|^2] + \|q_n - P_C(q_n)\|^2\right\} \\
&+ \beta_n(1 - \beta_n)\|x_n - P_C(q_n)\|^2 \leq \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 + \alpha_n M_1,
\end{aligned}$$

for some $M_1 > 0$. In fact, thanks to $u_n = \sigma_n x_n + (1 - \sigma_n)S_n u_n$, we ahve

$$\|u_n - x^*\|^2 \leq \sigma_n \langle x_n - x^*, u_n - x^* \rangle + (1 - \sigma_n)\|u_n - x^*\|^2,$$

which hence yields

$$2\|u_n - x^*\|^2 \le 2\langle x_n - x^*, u_n - x^*\rangle = \|x_n - x^*\|^2 + \|u_n - x^*\|^2 - \|x_n - u_n\|^2.$$

This immediately implies that

$$\|u_n - x^*\|^2 \le \|x_n - x^*\|^2 - \|x_n - u_n\|^2.$$

From $z_n = P_{C_n}(u_n)$, we obtain

$$\|z_n - x^*\|^2 \le \|u_n - x^*\|^2 - \|u_n - z_n\|^2 \le \|x_n - x^*\|^2 - \|x_n - u_n\|^2 - \|u_n - z_n\|^2.$$

Since $x_{n+1} = \beta_n x_n + (1-\beta_n)P_C(q_n)$, where $q_n = (I - \alpha_n\rho F)S^n w_n + \alpha_n f(x_n)$, by Lemma 2.9, we deduce that

$$\begin{aligned}
&\|x_{n+1} - x^*\|^2 \\
&\le (1-\beta_n)\{\|q_n - x^*\|^2 - \|q_n - P_C(q_n)\|^2\}2 \\
&\quad - \beta_n(1-\beta_n)\|x_n - P_C(q_n)\|^+ \beta_n\|x_n - x^*\|^2 \\
&\le (1-\beta_n)\{[\alpha_n\|f(x_n) - f(x^*)\| + \|(I-\alpha_n\rho F)S^n w_n - (I-\alpha_n\rho F)x^*\|]^2 \\
&\quad + 2\alpha_n\langle (f-\rho F)x^*, q_n - x^*\rangle - \|q_n - P_C(q_n)\|^2\} \\
&\quad - \beta_n(1-\beta_n)\|x_n - P_C(q_n)\|^2 + \beta_n\|x_n - x^*\|^2 \\
&\le (1-\beta_n)\{\alpha_n\delta\|x_n - x^*\|^2 + [(1-\alpha_n\tau) + \theta_n]\|w_n - x^*\|^2 \\
&\quad + 2\alpha_n\langle (f-\rho F)x^*, q_n - x^*\rangle - \|q_n - P_C(q_n)\|^2\} \\
&\quad - \beta_n(1-\beta_n)\|x_n - P_C(q_n)\|^2 + \beta_n\|x_n - x^*\|^2,
\end{aligned} \tag{3.16}$$

which together with $w_n = Gz_n$ ensures that

$$\begin{aligned}
&\|x_{n+1} - x^*\|^2 \\
&\le \left[1 - \frac{\alpha_n(1-\beta_n)(\tau-\delta)}{2}\right]\|x_n - x^*\|^2 \\
&\quad - (1-\beta_n)\left\{\left[1 - \frac{\alpha_n(\tau+\delta)}{2}\right][\|x_n - u_n\|^2 + \|u_n - z_n\|^2] + \|q_n - P_C(q_n)\|^2\right\} \\
&\quad - \beta_n(1-\beta_n)\|x_n - P_C(q_n)\|^2 + 2\alpha_n(1-\beta_n)\langle (f-\rho F)x^*, q_n - x^*\rangle \\
&\le \|x_n - x^*\|^2 - \beta_n(1-\beta_n)\|x_n - P_C(q_n)\|^2 + \alpha_n M_1 \\
&\quad - (1-\beta_n)\left\{\left[1 - \frac{\alpha_n(\tau+\delta)}{2}\right][\|x_n - u_n\|^2 + \|u_n - z_n\|^2] + \|q_n - P_C(q_n)\|^2\right\},
\end{aligned} \tag{3.17}$$

where $\sup_{n\ge 1} 2\|(f-\rho F)x^*\|\|q_n - x^*\| \le M_1$ for some $M_1 > 0$. This hence attains the claim.

Claim 3. We assert that

$$(1-\beta_n)\left[1 - \frac{\alpha_n(\tau+\delta)}{2}\right]\left[\frac{\tau_n}{2\lambda\tilde{L}}\|\varepsilon_\lambda(u_n)\|^2\right]^2 \le \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 + \alpha_n M_1.$$

In fact, we claim that for some $\tilde{L} > 0$,

$$\|z_n - x^*\|^2 \leq \|u_n - x^*\|^2 - \left[\frac{\tau_n}{2\lambda\tilde{L}}\|\varepsilon_\lambda(u_n)\|^2\right]^2. \tag{3.18}$$

From the boundedness of $\{At_n\}$, one knows that there exists $\tilde{L} > 0$ such that $\|At_n\| \leq \tilde{L}$ for all $n \geq 1$. This implies that $|\bar{h}_n(u) - \bar{h}_n(v)| = |\langle At_n, u - v\rangle| \leq \tilde{L}\|u - v\|$ for all $u, v \in C_n$, which hence guarantees that $\bar{h}_n(\cdot)$ is $\tilde{L}$-Lipschitz continuous on C_n. By Lemmas 2.6 and 3.2, we have

$$\operatorname{dist}(u_n, C_n) \geq \frac{1}{\tilde{L}}\bar{h}_n(u_n) = \frac{\tau_n}{2\lambda\tilde{L}}\|\varepsilon_\lambda(u_n)\|^2. \tag{3.19}$$

Combining (3.13) and (3.19) yields

$$\|z_n - x^*\|^2 \leq \|u_n - x^*\|^2 - \left[\frac{\tau_n}{2\lambda\tilde{L}}\|\varepsilon_\lambda(u_n)\|^2\right]^2.$$

From (3.16), (3.15) and (3.18) it follows that

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 &\leq \beta_n\|x_n - x^*\|^2 + (1-\beta_n)\{\alpha_n\delta\|x_n - x^*\|^2 + [(1-\alpha_n\tau) + \theta_n] \\
&\quad \times \left[\|u_n - x^*\|^2 - \left[\frac{\tau_n}{2\lambda\tilde{L}}\|\varepsilon_\lambda(u_n)\|^2\right]^2\right] + 2\alpha_n\langle(f-\rho F)x^*, q_n - x^*\rangle\} \\
&\leq \left[1 - \frac{\alpha_n(1-\beta_n)(\tau-\delta)}{2}\right]\|x_n - x^*\|^2 - (1-\beta_n)\left[1 - \frac{\alpha_n(\tau+\delta)}{2}\right]\left[\frac{\tau_n}{2\lambda\tilde{L}}\|\varepsilon_\lambda(u_n)\|^2\right]^2 \\
&\quad + 2\alpha_n(1-\beta_n)\langle(f-\rho F)x^*, q_n - x^*\rangle \\
&\leq \|x_n - x^*\|^2 - (1-\beta_n)\left[1 - \frac{\alpha_n(\tau+\delta)}{2}\right]\left[\frac{\tau_n}{2\lambda\tilde{L}}\|\varepsilon_\lambda(u_n)\|^2\right]^2 + \alpha_n M_1.
\end{aligned}$$

This hence attains the claim.

Claim 4. We assert that

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 &\leq [1 - \alpha_n(1-\beta_n)(\tau-\delta)]\|x_n - x^*\|^2 + \alpha_n(1-\beta_n)(\tau-\delta) \\
&\quad \times \left[\frac{2\langle(f-\rho F)x^*, q_n - x^*\rangle}{\tau-\delta} + \frac{\theta_n}{\alpha_n}\cdot\frac{M}{\tau-\delta}\right]
\end{aligned} \tag{3.20}$$

for some $M > 0$. In fact, from Lemma 2.9 and (3.15), one obtains

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 &\leq \beta_n\|x_n - x^*\|^2 + (1-\beta_n)\|\alpha_n(f(x_n) - f(x^*)) + (I - \alpha_n\rho F)S^n w_n \\
&\quad - (I - \alpha_n\rho F)x^* + \alpha_n(f-\rho F)x^*\|^2 \\
&\leq [1 - \alpha_n(1-\beta_n)(\tau-\delta)]\|x_n - x^*\|^2 \\
&\quad + \alpha_n(1-\beta_n)(\tau-\delta)\left[\frac{2\langle(f-\rho F)x^*, q_n - x^*\rangle}{\tau-\delta} + \frac{\theta_n}{\alpha_n}\cdot\frac{M}{\tau-\delta}\right],
\end{aligned}$$

where $\sup_{n\geq 1}\|x_n - x^*\|^2 \leq M$ for some $M > 0$.

Claim 5. We assert that $x_n \to x^* \in \Omega$, which is the solution to (3.12). In fact, putting $\Gamma_n = \|x_n - x^*\|^2$, we show the convergence of $\{\Gamma_n\}$ to zero by the two cases.

Case 1. $\exists n_0 \geq 1$ s.t. $\{\Gamma_n\}$ is nonincreasing. It is clear that the limit

$$\lim_{n\to\infty}\Gamma_n = d < +\infty \text{ and } \lim_{n\to\infty}(\Gamma_n - \Gamma_{n+1}) = 0.$$

From Claim 2 and $\{\beta_n\} \subset [a, b] \subset (0, 1)$, we obtain

$$\begin{aligned}
&(1-b)\left\{\left[1-\frac{\alpha_n(\tau+\delta)}{2}\right][\|x_n-u_n\|^2+\|u_n-z_n\|^2]+\|q_n-P_C(q_n)\|^2\right\}\\
&\quad+a(1-b)\|x_n-P_C(q_n)\|^2\\
&\le \|x_n-x^*\|^2-\|x_{n+1}-x^*\|^2+\alpha_n M_1=\Gamma_n-\Gamma_{n+1}+\alpha_n M_1.
\end{aligned}$$

Owing to the facts that $\alpha_n \to 0$ and $\Gamma_n - \Gamma_{n+1} \to 0$, one deduces from $\frac{\tau+\delta}{2} \in (0, 1)$ that

$$\begin{aligned}
\lim_{n\to\infty}\|x_n-u_n\| &= \lim_{n\to\infty}\|u_n-z_n\| = \lim_{n\to\infty}\|q_n-P_C(q_n)\|\\
&= \lim_{n\to\infty}\|x_n-P_C(q_n)\| = 0.
\end{aligned} \tag{3.21}$$

Hence it is readily known that

$$\|x_n-q_n\| \le \|x_n-P_C(q_n)\|+\|P_C(q_n)-q_n\| \to 0\ (n\to\infty),$$

$$\|x_{n+1}-x_n\| \le \|P_C(q_n)-x_n\| \to 0\ (n\to\infty),$$

and $\|S^n w_n - x_n\| \to 0$ $(n \to \infty)$.

Next, we show that $\|x_n - Gz_n\| \to 0$ as $n \to \infty$. Indeed, note that $y^* = J_{\mu_2}^{F_2}(x^* - \mu_2 B_2 x^*)$, $v_n = J_{\mu_2}^{F_2}(z_n - \mu_2 B_2 z_n)$, and $w_n = J_{\mu_1}^{F_1}(v_n - \mu_1 B_1 v_n)$. Then $w_n = Gz_n$. By Lemma 2.1, we have

$$\|v_n-y^*\|^2+\mu_2(2\beta-\mu_2)\|B_2z_n-B_2x^*\|^2 \le \|z_n-x^*\|^2$$

and

$$\|w_n-x^*\|^2+\mu_1(2\alpha-\mu_1)\|B_1v_n-B_1y^*\|^2 \le \|v_n-y^*\|^2.$$

Combining these two inequalities, we obtain from (3.15)

$$\|w_n-x^*\|^2 \le \|x_n-x^*\|^2-\mu_2(2\beta-\mu_2)\|B_2z_n-B_2x^*\|^2-\mu_1(2\alpha-\mu_1)\|B_1v_n-B_1y^*\|^2.$$

This together with (3.16) implies that

$$\begin{aligned}
&\|x_{n+1}-x^*\|^2\\
&\le (1-\beta_n)\Big\{\alpha_n\delta\|x_n-x^*\|^2+\left[(1-\alpha_n\tau)+\frac{\alpha_n(\tau-\delta)}{2}\right][\|x_n-x^*\|^2\\
&\quad-\mu_2(2\beta-\mu_2)\|B_2z_n-B_2x^*\|^2-\mu_1(2\alpha-\mu_1)\|B_1v_n-B_1y^*\|^2]+\alpha_n M_1\Big\}\\
&\quad+\beta_n\|x_n-x^*\|^2\\
&\le \|x_n-x^*\|^2-(1-\beta_n)\left[1-\frac{\alpha_n(\tau+\delta)}{2}\right]\{\mu_2(2\beta-\mu_2)\|B_2z_n-B_2x^*\|^2\\
&\quad+\mu_1(2\alpha-\mu_1)\|B_1v_n-B_1y^*\|^2\}+\alpha_n M_1.
\end{aligned}$$

Since $\beta_n \le b < 1$, $\mu_1 \in (0, 2\alpha)$, $\mu_2 \in (0, 2\beta)$, and $\lim_{n\to\infty} \alpha_n = 0$, we obtain from $\Gamma_n - \Gamma_{n+1} \to 0$ that

$$\lim_{n\to\infty}\|B_2z_n-B_2x^*\| = \lim_{n\to\infty}\|B_1v_n-B_1y^*\| = 0. \tag{3.22}$$

On the other hand, by Proposition 2.3 one has

$$2\|w_n - x^*\|^2 \le \|v_n - y^*\|^2 + \|w_n - x^*\|^2 - \|v_n - w_n + x^* - y^*\|^2 \\ + 2\mu_1\|B_1y^* - B_1v_n\|\|w_n - x^*\|,$$

which hence leads to

$$\|w_n - x^*\|^2 \le \|v_n - y^*\|^2 - \|v_n - w_n + x^* - y^*\|^2 + 2\mu_1\|B_1y^* - B_1v_n\|\|w_n - x^*\|.$$

Similarly,

$$\|v_n - y^*\|^2 \le \|z_n - x^*\|^2 - \|z_n - v_n + y^* - x^*\|^2 + 2\mu_2\|B_2x^* - B_2z_n\|\|v_n - y^*\|.$$

Combining these two inequalities, we get from (3.15) that

$$\|w_n - x^*\|^2 \le \|x_n - x^*\|^2 - \|z_n - v_n + y^* - x^*\|^2 - \|v_n - w_n + x^* - y^*\|^2 \\ +2\mu_1\|B_1y^* - B_1v_n\|\|w_n - x^*\| + 2\mu_2\|B_2x^* - B_2z_n\|\|v_n - y^*\|.$$

This together with (3.16) ensures that

$$\begin{aligned}&\|x_{n+1} - x^*\|^2 \\ &\le -(1-\beta_n)\left[1 - \frac{\alpha_n(\tau+\delta)}{2}\right]\{\|z_n - v_n + y^* - x^*\|^2 + \|v_n - w_n + x^* - y^*\|^2\} \\ &\quad + 2\mu_1\|B_1y^* - B_1v_n\|\|w_n - x^*\| + 2\mu_2\|B_2x^* - B_2z_n\|\|v_n - y^*\| \\ &\quad + \alpha_n M_1 + \|x_n - x^*\|^2,\end{aligned}$$

which immediately leads to

$$(1-\beta_n)\left[1 - \frac{\alpha_n(\tau+\delta)}{2}\right]\{\|z_n - v_n + y^* - x^*\|^2 + \|v_n - w_n + x^* - y^*\|^2\}$$
$$\le \Gamma_n - \Gamma_{n+1} + 2\mu_1\|B_1y^* - B_1v_n\|\|w_n - x^*\| + 2\mu_2\|B_2x^* - B_2z_n\|\|v_n - y^*\| + \alpha_n M_1.$$

Since $\beta_n \le b < 1$, $\lim_{n\to\infty}\alpha_n = 0$, and $\limsup_{n\to\infty}(\Gamma_n - \Gamma_{n+1}) = 0$, we deduce from (3.22) and the boundedness of $\{w_n\}$ and $\{v_n\}$ that

$$\lim_{n\to\infty}\|z_n - v_n + y^* - x^*\| = \lim_{n\to\infty}\|v_n - w_n + x^* - y^*\| = 0.$$

Therefore,

$$\|z_n - Gz_n\| = \|z_n - w_n\| \le \|z_n - v_n + y^* - x^*\| + \|v_n - w_n + x^* - y^*\| \to 0 \ (n\to\infty).$$

Also, we have by (3.21) that

$$\|x_n - z_n\| \le \|x_n - u_n\| + \|u_n - z_n\| \to 0 \quad (n\to\infty). \tag{3.23}$$

This immediately yields

$$\|x_n - Gz_n\| = \|x_n - w_n\| \le \|x_n - z_n\| + \|z_n - w_n\| \to 0 \quad (n\to\infty). \tag{3.24}$$

From $u_n = \sigma_n x_n + (1-\sigma_n)S_n u_n$ and $\liminf_{n\to\infty}(1-\sigma_n) > 0$, we have

$$\lim_{n\to\infty}\|S_n u_n - x_n\| = 0. \tag{3.25}$$

Meanwhile, from Claim 3, we obtain

$$(1-\beta_n)\left[1 - \frac{\alpha_n(\tau+\delta)}{2}\right]\left[\frac{\tau_n}{2\lambda\tilde{L}}\|\varepsilon_\lambda(u_n)\|^2\right]^2 \le \Gamma_n - \Gamma_{n+1} + \alpha_n M_1.$$

Since $\beta_n \le b < 1,\ \alpha_n \to 0$, and $\Gamma_n - \Gamma_{n+1} \to 0$, one gets

$$\lim_{n\to\infty} \left[\frac{\tau_n}{2\lambda\tilde{L}}\|\varepsilon_\lambda(u_n)\|^2\right]^2 = 0,$$

which together with Lemma 3.4, yields

$$\lim_{n\to\infty}\|u_n - y_n\| = 0. \tag{3.26}$$

By the boundedness of $\{x_n\}$, we know that $\exists$ subsequence $\{x_{n_i}\} \subset \{x_n\}$ s.t.

$$\limsup_{n\to\infty}\langle (f-\rho F)x^*, x_n - x^*\rangle = \lim_{i\to\infty}\langle (f-\rho F)x^*, x_{n_i} - x^*\rangle. \tag{3.27}$$

Since H is reflexive and $\{x_n\}$ is bounded, we might assume that $x_{n_i} \rightharpoonup \hat{x}$. Thus it follows from (3.27) that

$$\limsup_{n\to\infty}\langle (f-\rho F)x^*, x_n - x^*\rangle = \langle (f-\rho F)x^*, \hat{x} - x^*\rangle. \tag{3.28}$$

Since $x_n - x_{n+1} \to 0,\ x_n - Gz_n \to 0,\ u_n - y_n \to 0,\ x_n - z_n \to 0$, and $x_{n_i} \rightharpoonup \hat{x}$, by Lemma 3.3, we deduce that $\hat{x} \in \Omega$. Thus, using (3.12) and (3.28), one has

$$\limsup_{n\to\infty}\langle (f-\rho F)x^*, x_n - x^*\rangle = \langle (f-\rho F)x^*, \hat{x} - x^*\rangle \le 0, \tag{3.29}$$

which together with (3.21) indicates

$$\limsup_{n\to\infty}\langle (f-\rho F)x^*, q_n - x^*\rangle \le 0. \tag{3.30}$$

Note that

$$\limsup_{n\to\infty}\left[\frac{2\langle (f-\rho F)x^*, q_n - x^*\rangle}{\tau-\delta} + \frac{\theta_n}{\alpha_n}\cdot\frac{M}{\tau-\delta}\right] \le 0.$$

Consequently, applying Lemma 2.4 to (3.20), one has $\lim_{n\to\infty}\|x_n - x^*\|^2 = 0$.

Case 2. $\exists\{\Gamma_{n_i}\} \subset \{\Gamma_n\}$ s.t. $\Gamma_{n_i} < \Gamma_{n_i+1}, \forall i \in \mathcal{N}$, with $\mathcal{N}$ being the set of all natural numbers. Let $\phi : \mathcal{N} \to \mathcal{N}$ be defined as $\phi(n) := \max\{i \le n : \Gamma_i < \Gamma_{i+1}\}$. Using Lemma 2.8, we have $\Gamma_{\phi(n)} \le \Gamma_{\phi(n)+1}$ and $\Gamma_n \le \Gamma_{\phi(n)+1}$. By Claim 2, one has

$$\begin{aligned}(1-b)\left\{\left[1-\frac{\alpha_{\phi(n)}(\tau+\delta)}{2}\right][\|x_{\phi(n)}-u_{\phi(n)}\|^2+\|u_{\phi(n)}-z_{\phi(n)}\|^2] + \|q_{\phi(n)}-P_C(q_{\phi(n)})\|^2\right\}\\ +a(1-b)\|x_{\phi(n)} - P_C(q_{\phi(n)})\|^2 \le \Gamma_{\phi(n)} - \Gamma_{\phi(n)+1} + \alpha_{\phi(n)}M_1,\end{aligned} \tag{3.31}$$

which immediately guarantees that

$$\begin{aligned}\lim_{n\to\infty}\|x_{\phi(n)} - u_{\phi(n)}\| &= \lim_{n\to\infty}\|u_{\phi(n)} - z_{\phi(n)}\| = \lim_{n\to\infty}\|q_{\phi(n)} - P_C(q_{\phi(n)})\|\\ &= \lim_{n\to\infty}\|x_{\phi(n)} - P_C(q_{\phi(n)})\| = 0.\end{aligned}$$

By Claim 3, we have

$$(1-\beta_{\phi(n)})\left[1-\frac{\alpha_{\phi(n)}(\tau+\delta)}{2}\right]\left[\frac{\tau_{\phi(n)}}{2\lambda\tilde{L}}\|\varepsilon_\lambda(u_{\phi(n)})\|^2\right]^2 \le \Gamma_{\phi(n)} - \Gamma_{\phi(n)+1} + \alpha_{\phi(n)}M_1,$$

which yields

$$\lim_{n\to\infty}\left[\frac{\tau_{\phi(n)}}{2\lambda\tilde{L}}\|\varepsilon_\lambda(u_{\phi(n)})\|^2\right]^2 = 0.$$

Using the similar arguments to those of Case 1, we obtain

$$\lim_{n\to\infty}\|x_{\phi(n)+1}-x_{\phi(n)}\|=0,$$
$$\lim_{n\to\infty}\|x_{\phi(n)}-S_{\phi(n)}u_{\phi(n)}\|=\lim_{n\to\infty}\|x_{\phi(n)}-z_{\phi(n)}\|=0,$$
$$\lim_{n\to\infty}\|x_{\phi(n)}-Gz_{\phi(n)}\|=\lim_{n\to\infty}\|u_{\phi(n)}-y_{\phi(n)}\|=0,$$

and

$$\limsup_{n\to\infty}\langle(f-\rho F)x^*,q_{\phi(n)}-x^*\rangle\le 0. \tag{3.32}$$

On the other hand, from (3.20) we obtain

$$\begin{aligned}&\alpha_{\phi(n)}(1-\beta_{\phi(n)})(\tau-\delta)\Gamma_{\phi(n)}\\ &\le\alpha_{\phi(n)}(1-\beta_{\phi(n)})(\tau-\delta)\left[\frac{2\langle(f-\rho F)x^*,q_{\phi(n)}-x^*\rangle}{\tau-\delta}+\frac{\theta_{\phi(n)}}{\alpha_{\phi(n)}}\cdot\frac{M}{\tau-\delta}\right],\end{aligned}$$

which immediately attains

$$\limsup_{n\to\infty}\Gamma_{\phi(n)}\le\limsup_{n\to\infty}\left[\frac{2\langle(f-\rho F)x^*,q_{\phi(n)}-x^*\rangle}{\tau-\delta}+\frac{\theta_{\phi(n)}}{\alpha_{\phi(n)}}\cdot\frac{M}{\tau-\delta}\right]\le 0.$$

Thus $\lim_{n\to\infty}\|x_{\phi(n)}-x^*\|^2=0$. In addition, one has

$$\|x_{\phi(n)+1}-x^*\|^2-\|x_{\phi(n)}-x^*\|^2\le 2\|x_{\phi(n)+1}-x_{\phi(n)}\|\|x_{\phi(n)}-x^*\|+\|x_{\phi(n)+1}-x_{\phi(n)}\|^2.$$

Thanks to $\Gamma_n\le\Gamma_{\phi(n)+1}$, one sees that

$$\begin{aligned}\|x_n-x^*\|^2\le\|x_{\phi(n)+1}-x^*\|^2&\le\|x_{\phi(n)}-x^*\|^2+2\|x_{\phi(n)+1}-x_{\phi(n)}\|\|x_{\phi(n)}-x^*\|\\ &+\|x_{\phi(n)+1}-x_{\phi(n)}\|^2\to 0\quad(n\to\infty).\end{aligned}$$

That is, $x_n\to x^*$ as $n\to\infty$. □

Theorem 3.2. *If $S:C\to C$ is nonexpansive and $\{x_n\}$ is the sequence constructed in the modified version of Algorithm 3.1, that is, for any starting $x_1\in C$,*

$$\begin{cases}u_n=\sigma_nx_n+(1-\sigma_n)S_nu_n,\\ y_n=P_C(u_n-\lambda Au_n),\\ t_n=(1-\tau_n)u_n+\tau_ny_n,\\ z_n=P_{C_n}(u_n),\\ v_n=J_{\mu_2}^{F_2}(z_n-\mu_2B_2z_n),\\ w_n=J_{\mu_1}^{F_1}(v_n-\mu_1B_1v_n),\\ x_{n+1}=\beta_nx_n+(1-\beta_n)P_C[(I-\alpha_n\rho F)Sw_n+\alpha_nf(x_n)]\quad\forall n\ge 1,\end{cases}\tag{3.33}$$

where, for each $n\ge 1$, C_n and τ_n are chosen as in Algorithm 3.1, then $x_n\to x^*\in\Omega$, where $x^*\in\Omega$ is the solution to the HVI, $\langle(\rho F-f)x^*,y-x^*\rangle\ge 0\ \forall y\in\Omega$.

Proof. We divide the proof of the theorem into several claims.

Claim 1. The boundedness of $\{x_n\}$ follows the Claim 1 of Theorem 3.1 immediately.

Claim 2. We assert that

$$\begin{aligned}&(1-\beta_n)\{(1-\alpha_n\tau)[\|x_n-u_n\|^2+\|u_n-z_n\|^2]+\|q_n-P_C(q_n)\|^2\}\\ &+\beta_n(1-\beta_n)\|x_n-P_C(q_n)\|^2\le\|x_n-x^*\|^2-\|x_{n+1}-x^*\|^2+\alpha_nM_1,\end{aligned}$$

for some $M_1 > 0$. In fact, putting $\theta_n = 0$, we get from (3.16) that

$$\begin{aligned}
&\|x_{n+1} - x^*\|^2 \\
&\le [1 - \alpha_n(1 - \beta_n)(\tau - \delta)]\|x_n - x^*\|^2 - (1 - \beta_n)\{(1 - \alpha_n\tau)[\|x_n - u_n\|^2 + \|u_n - z_n\|^2] \\
&\quad + \|q_n - P_C(q_n)\|^2\} + \alpha_n M_1 - \beta_n(1 - \beta_n)\|x_n - P_C(q_n)\|^2 \\
&\le \|x_n - x^*\|^2 - (1 - \beta_n)\{(1 - \alpha_n\tau)[\|x_n - u_n\|^2 + \|u_n - z_n\|^2] + \|q_n - P_C(q_n)\|^2\} \\
&\quad + \alpha_n M_1 - \beta_n(1 - \beta_n)\|x_n - P_C(q_n)\|^2,
\end{aligned}$$

where $\sup_{n\ge1} 2\|(f - \rho F)x^*\|\|q_n - x^*\| \le M_1$ for some $M_1 > 0$. This attains the desired assertion.

Claim 3. We prove that

$$(1 - \beta_n)(1 - \alpha_n\tau)\left[\frac{\tau_n}{2\lambda\tilde{L}}\|\varepsilon_\lambda(u_n)\|^2\right]^2 \le \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 + \alpha_n M_1.$$

Indeed, From the Claim 3 of Theorem 3.1, one can derive the desired assertion immediately.

Claim 4. We prove that

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 &\le [1 - \alpha_n(1 - \beta_n)(\tau - \delta)]\|x_n - x^*\|^2 \\
&\quad + \alpha_n(1 - \beta_n)(\tau - \delta)\cdot\frac{2\langle(f - \rho F)x^*, q_n - x^*\rangle}{\tau - \delta}.
\end{aligned}$$

Indeed, this directly follows from the Claim 4 of Theorem 3.1.

Claim 5. We assert that $\{x_n\}$ converges strongly to the unique solution $x^* \in \Omega$ of HVI (3.16). Indeed, using the Claim 5 of Theorem 3.1, one gets the desired assertion immediately. □

Next we present another modified subgradient extragradient algorithm with line-search process.

Algorithm 3.2.

Initial step: Give $\nu > 0$, $\ell \in (0, 1)$, and $\lambda \in (0, \frac{1}{\nu})$ and choose an initial $x_1 \in C$ arbitrarily.

Iterative steps: Given the current iterate x_n, compute x_{n+1} below:

Step 1. Calculate $u_n = \sigma_n x_n + (1 - \sigma_n)S_n u_n$, $y_n = P_C(u_n - \lambda A u_n)$ and

$$\varepsilon_\lambda(u_n) := u_n - y_n.$$

Step 2. Calculate $t_n = u_n - \tau_n\varepsilon_\lambda(u_n)$, where $\tau_n := \ell^{j_n}$ and j_n is the smallest nonnegative integer j satisfying

$$2\langle Au_n - A(u_n - \ell^j\varepsilon_\lambda(u_n)), u_n - y_n\rangle \le \nu\|\varepsilon_\lambda(u_n)\|^2.$$

Step 3. Calculate $z_n = P_{C_n}(u_n)$, where $C_n := \{u \in C : \bar{h}_n(u) \le 0\}$ and

$$\bar{h}_n(u) = \frac{2\lambda\langle At_n, u - u_n\rangle + \tau_n\|\varepsilon_\lambda(u_n)\|^2}{2\lambda}.$$

Step 4. Calculate $v_n = J_{\mu_2}^{F_2}(z_n - \mu_2 B_2 z_n)$, $w_n = J_{\mu_1}^{F_1}(v_n - \mu_1 B_1 v_n)$ and

$$x_{n+1} = \beta_n u_n + (1 - \beta_n)P_C[(I - \alpha_n\rho F)S^n w_n + \alpha_n f(z_n)].$$

Set $n := n + 1$ and return to Step 1.

It is worth noting that (3.13)-(3.16) and Lemmas 3.1-3.4 remain true for Algorithm 3.2.

Theorem 3.3. *Suppose that $\{x_n\}$ is the sequence constructed in Algorithm 3.2. Then $x_n \to x^* \in \Omega$ provided $S^n x_n - S^{n+1}x_n \to 0$, where $x^* \in \Omega$ is the solution to the HVI: $\langle(\rho F - f)x^*, y - x^*\rangle \geq 0$ for all y in Ω.*

Proof. In what follows, under the assumption $S^n x_n - S^{n+1}x_n \to 0$, one can divide the proof into the following several claims.

Claim 1. We assert that $\{x_n\}$ is bounded. Indeed, for

$$x^* \in \Omega = \bigcap_{n=0}^{\infty} \mathrm{Fix}(S_n) \cap \mathrm{Fix}(G) \cap \mathrm{VI}(C, A)$$

we have $S_n x^* = x^*$, $Gx^* = x^*$, and $P_C(x^* - \lambda A x^*) = x^*$. Using (3.15), we obtain from Lemma 2.9 that

$$\begin{aligned}
\|x_{n+1} - x^*\| &\leq \beta_n\|u_n - x^*\| + (1-\beta_n)\|\alpha_n(f(z_n) - f(x^*)) + (I - \alpha_n\rho F)S^n w_n \\
&\quad -(I - \alpha_n\rho F)x^* + \alpha_n(f - \rho F)x^*\| \\
&\leq \beta_n\|x_n - x^*\| + (1-\beta_n)\{[\alpha_n\delta + (1-\alpha_n\tau) + \theta_n]\|x_n - x^*\| + \alpha_n\|(f-\rho F)x^*\|\} \\
&\leq \left[1 - \frac{\alpha_n(1-\beta_n)(\tau-\delta)}{2}\right]\|x_n - x^*\| + \frac{\alpha_n(1-\beta_n)(\tau-\delta)}{2} \cdot \frac{2\|(f-\rho F)x^*\|}{\tau - \delta} \\
&\leq \max\left\{\|x_n - x^*\|, \frac{2\|(f-\rho F)x^*\|}{\tau-\delta}\right\}.
\end{aligned}$$

By induction, $\|x_n - x^*\| \leq \max\left\{\|x_1 - x^*\|, \frac{2\|(f-\rho F)x^*\|}{\tau-\delta}\right\}$. Thus $\{x_n\}$ is bounded, so $\{u_n\}, \{y_n\}, \{z_n\}, \{w_n\}, \{f(z_n)\}, \{At_n\}$, and $\{S^n w_n\}$.

Claim 2. We prove that

$$\begin{aligned}
&(1-\beta_n)\left\{\left[1 - \frac{\alpha_n(\tau+\delta)}{2}\right][\|x_n - u_n\|^2 + \|u_n - z_n\|^2] + \|\tilde{q}_n - P_C(\tilde{q}_n)\|^2\right\} \\
&\quad +\beta_n(1-\beta_n)\|u_n - P_C(\tilde{q}_n)\|^2 \leq \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 + \alpha_n M_1,
\end{aligned}$$

for some $M_1 > 0$. In fact, $\|u_n - x^*\|^2 \leq \|x_n - x^*\|^2 - \|x_n - u_n\|^2$. Thus

$$\|z_n - x^*\|^2 \leq \|u_n - x^*\|^2 - \|u_n - z_n\|^2 \leq \|x_n - x^*\|^2 - \|x_n - u_n\|^2 - \|u_n - z_n\|^2.$$

Since $x_{n+1} = \beta_n u_n + (1-\beta_n)P_C(\tilde{q}_n)$, where $\tilde{q}_n = \alpha_n f(z_n) + (I - \alpha_n\rho F)S^n w_n$, we obtain from (3.15) that

$$\begin{aligned}
&\|x_{n+1} - x^*\|^2 \\
&\leq \beta_n\|u_n - x^*\|^2 + (1-\beta_n)\|P_C(\tilde{q}_n) - x^*\|^2 - \beta_n(1-\beta_n)\|u_n - P_C(\tilde{q}_n)\|^2 \\
&\leq \beta_n\|x_n - x^*\|^2 + (1-\beta_n)\{\alpha_n\delta\|x_n - x^*\|^2 + [(1-\alpha_n\tau) + \theta_n]\|w_n - x^*\|^2 \\
&\quad + 2\alpha_n\langle(f-\rho F)x^*, \tilde{q}_n - x^*\rangle - \|\tilde{q}_n - P_C(\tilde{q}_n)\|^2\} - \beta_n(1-\beta_n)\|u_n - P_C(\tilde{q}_n)\|^2
\end{aligned} \tag{3.34}$$

which together with $w_n = Gz_n$ guarantees that

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 &\leq \beta_n\|x_n - x^*\|^2 + (1-\beta_n)\{\alpha_n\delta\|x_n - x^*\|^2 + [(1-\alpha_n\tau) + \theta_n][\|x_n - x^*\|^2 \\
&\quad -\|x_n - u_n\|^2 - \|u_n - z_n\|^2] + 2\alpha_n\langle(f-\rho F)x^*, \tilde{q}_n - x^*\rangle - \|\tilde{q}_n - P_C(\tilde{q}_n)\|^2\}
\end{aligned}$$

$$
\begin{aligned}
&-\beta_n(1-\beta_n)\|u_n - P_C(\tilde{q}_n)\|^2 \\
&\leq \left[1 - \frac{\alpha_n(1-\beta_n)(\tau-\delta)}{2}\right]\|x_n - x^*\|^2 \\
&-(1-\beta_n)\left\{\left[1 - \frac{\alpha_n(\tau+\delta)}{2}\right][\|x_n - u_n\|^2 + \|u_n - z_n\|^2] + \|\tilde{q}_n - P_C(\tilde{q}_n)\|^2\right\} \\
&-\beta_n(1-\beta_n)\|u_n - P_C(\tilde{q}_n)\|^2 + 2\alpha_n(1-\beta_n)\langle (f-\rho F)x^*, \tilde{q}_n - x^*\rangle \\
&\leq \|x_n - x^*\|^2 - (1-\beta_n)\left\{\left[1 - \frac{\alpha_n(\tau+\delta)}{2}\right][\|x_n - u_n\|^2 + \|u_n - z_n\|^2] + \|\tilde{q}_n - P_C(\tilde{q}_n)\|^2\right\} \\
&-\beta_n(1-\beta_n)\|u_n - P_C(\tilde{q}_n)\|^2 + \alpha_n M_1, \qquad (3.35)
\end{aligned}
$$

where $\sup_{n\geq 1} 2\|(f-\rho F)x^*\|\|\tilde{q}_n - x^*\| \leq M_1$ for some $M_1 > 0$.

Claim 3. We assert that

$$
(1-\beta_n)\left[1 - \frac{\alpha_n(\tau+\delta)}{2}\right]\left[\frac{\tau_n}{2\lambda\tilde{L}}\|\varepsilon_\lambda(u_n)\|^2\right]^2 \leq \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 + \alpha_n M_1.
$$

In fact, using the similar arguments to those of (3.18) in the proof of Theorem 3.1, we can deduce that for some $\tilde{L} > 0$,

$$
\|z_n - x^*\|^2 \leq \|u_n - x^*\|^2 - \left[\frac{\tau_n}{2\lambda\tilde{L}}\|\varepsilon_\lambda(u_n)\|^2\right]^2. \qquad (3.36)
$$

From (3.34), (3.15), and (3.36) it follows that

$$
\begin{aligned}
&\|x_{n+1} - x^*\|^2 \leq \beta_n\|x_n - x^*\|^2 + (1-\beta_n)\{\alpha_n\delta\|x_n - x^*\|^2 + [(1-\alpha_n\tau) + \theta_n]\|w_n - x^*\|^2 \\
&+2\alpha_n\langle (f-\rho F)x^*, \tilde{q}_n - x^*\rangle\} \leq \beta_n\|x_n - x^*\|^2 + (1-\beta_n)\{\alpha_n\delta\|x_n - x^*\|^2 \\
&+[(1-\alpha_n\tau) + \theta_n]\left[\|u_n - x^*\|^2 - \left[\frac{\tau_n}{2\lambda\tilde{L}}\|\varepsilon_\lambda(u_n)\|^2\right]^2\right] + 2\alpha_n\langle (f-\rho F)x^*, \tilde{q}_n - x^*\rangle\} \\
&\leq \left[1 - \frac{\alpha_n(1-\beta_n)(\tau-\delta)}{2}\right]\|x_n - x^*\|^2 - (1-\beta_n)\left[1 - \frac{\alpha_n(\tau+\delta)}{2}\right]\left[\frac{\tau_n}{2\lambda\tilde{L}}\|\varepsilon_\lambda(u_n)\|^2\right]^2 \\
&+2\alpha_n(1-\beta_n)\langle (f-\rho F)x^*, \tilde{q}_n - x^*\rangle \\
&\leq \|x_n - x^*\|^2 - (1-\beta_n)\left[1 - \frac{\alpha_n(\tau+\delta)}{2}\right]\left[\frac{\tau_n}{2\lambda\tilde{L}}\|\varepsilon_\lambda(u_n)\|^2\right]^2 + \alpha_n M_1,
\end{aligned}
$$

which in turn yields the desired assertion.

Claim 4. We assert that

$$
\begin{aligned}
\|x_{n+1} - x^*\|^2 &\leq [1 - \alpha_n(1-\beta_n)(\tau-\delta)]\|x_n - x^*\|^2 + \alpha_n(1-\beta_n)(\tau-\delta) \\
&\times\left[\frac{2\langle (f-\rho F)x^*, \tilde{q}_n - x^*\rangle}{\tau-\delta} + \frac{\theta_n}{\alpha_n}\cdot\frac{M}{\tau-\delta}\right] \qquad (3.37)
\end{aligned}
$$

for some $M > 0$. In fact, from (3.34) and (3.15), one obtains

$$
\begin{aligned}
&\|x_{n+1} - x^*\|^2 \leq \beta_n\|x_n - x^*\|^2 + (1-\beta_n)\{\alpha_n\delta\|x_n - x^*\|^2 + (1-\alpha_n\tau + \theta_n)\|w_n - x^*\|^2 \\
&+2\alpha_n\langle (f-\rho F)x^*, \tilde{q}_n - x^*\rangle\} \\
&\leq [1 - \alpha_n(1-\beta_n)(\tau-\delta)]\|x_n - x^*\|^2
\end{aligned}
$$

$$+\alpha_n(1-\beta_n)(\tau-\delta)\left[\frac{2\langle (f-\rho F)x^*, \tilde{q}_n - x^*\rangle}{\tau-\delta} + \frac{\theta_n}{\alpha_n}\cdot\frac{M}{\tau-\delta}\right],$$

where $\sup_{n\geq 1}\|x_n - x^*\|^2 \leq M$ for some $M>0$.

Claim 5. We assert that $x_n \to x^* \in \Omega$, which is the solution of the HVI (3.12).

In fact, putting $\Gamma_n = \|x_n - x^*\|^2$, we demonstrate the convergence of $\{\Gamma_n\}$ to zero by the following two cases.

Case 1. $\exists n_0 \geq 1$ s.t. $\{\Gamma_n\}$ is nonincreasing. It is clear that

$$\lim_{n\to\infty}\Gamma_n = d < +\infty \text{ and } \lim_{n\to\infty}(\Gamma_n - \Gamma_{n+1}) = 0.$$

From Claim 2 and $\{\beta_n\}\subset[a,b]\subset(0,1)$, we obtain

$$(1-b)\left\{\left[1-\frac{\alpha_n(\tau+\delta)}{2}\right][\|x_n-u_n\|^2+\|u_n-z_n\|^2]+\|\tilde{q}_n - P_C(\tilde{q}_n)\|^2\right\}$$
$$+a(1-b)\|u_n - P_C(\tilde{q}_n)\|^2 \leq \Gamma_n - \Gamma_{n+1} + \alpha_n M_1.$$

Owing to the facts that $\alpha_n \to 0$ and $\Gamma_n - \Gamma_{n+1}\to 0$ and using $\frac{\tau+\delta}{2}\in(0,1)$, one deduces that

$$\begin{aligned}\lim_{n\to\infty}\|x_n-u_n\| &= \lim_{n\to\infty}\|u_n-z_n\| = \lim_{n\to\infty}\|\tilde{q}_n - P_C(\tilde{q}_n)\| \\ &= \lim_{n\to\infty}\|u_n - P_C(\tilde{q}_n)\| = 0. \end{aligned} \tag{3.38}$$

Hence it is readily known that

$$\|x_n-\tilde{q}_n\| \leq \|x_n-u_n\| + \|u_n-P_C(\tilde{q}_n)\| + \|P_C(\tilde{q}_n)-\tilde{q}_n\| \to 0\ (n\to\infty),$$
$$\|x_{n+1}-x_n\| \leq (1-\beta_n)\|P_C(\tilde{q}_n)-u_n\| + \|u_n-x_n\| \to 0\ (n\to\infty),$$

and

$$\begin{aligned}\|S^n w_n - x_n\| &\leq \|S^n w_n - u_n\| + \|u_n - x_n\| \\ &= \|\tilde{q}_n - u_n - \alpha_n(f(z_n) - \rho F S^n w_n)\| + \|u_n - x_n\| \to 0 \quad (n\to\infty).\end{aligned}$$

Next, we show that $\|x_n - Gz_n\|\to 0$ and $\|x_n - x_{n+1}\|\to 0$ as $n\to\infty$. Indeed, note that $y^* = J_{\mu_2}^{F_2}(x^* - \mu_2 B_2 x^*)$, $v_n = J_{\mu_2}^{F_2}(z_n - \mu_2 B_2 z_n)$ and $w_n = J_{\mu_1}^{F_1}(v_n - \mu_1 B_1 v_n)$. Then $w_n = Gz_n$. Using Lemma 2.1, from (3.15) we have

$$\|w_n - x^*\|^2 \leq \|x_n-x^*\|^2 - \mu_2(2\beta-\mu_2)\|B_2 z_n - B_2 x^*\|^2 - \mu_1(2\alpha-\mu_1)\|B_1 v_n - B_1 y^*\|^2.$$

This together with (3.34) implies that

$$(1-\beta_n)\left[1-\frac{\alpha_n(\tau+\delta)}{2}\right]\{\mu_2(2\beta-\mu_2)\|B_2 z_n - B_2 x^*\|^2 + \mu_1(2\alpha-\mu_1)\|B_1 v_n - B_1 y^*\|^2\}$$
$$\leq \Gamma_n - \Gamma_{n+1} + \alpha_n M_1.$$

This hence ensures that $\lim_{n\to\infty}\|B_2 z_n - B_2 x^*\| = \lim_{n\to\infty}\|B_1 v_n - B_1 y^*\| = 0$.

On the other hand, we get from (3.15) that

$$\|w_n - x^*\|^2 \leq \|x_n - x^*\|^2 - \|z_n - v_n + y^* - x^*\|^2 - \|v_n - w_n + x^* - y^*\|^2$$
$$+2\mu_1\|B_1 y^* - B_1 v_n\|\|w_n - x^*\| + 2\mu_2\|B_2 x^* - B_2 z_n\|\|v_n - y^*\|.$$

This together with (3.34), implies that

$$\|x_{n+1}-x^*\|^2 \leq \beta_n\|x_n - x^*\|^2 + (1-\beta_n)\{\alpha_n\delta\|x_n-x^*\|^2$$

$$
\begin{aligned}
&+[(1-\alpha_n\tau)+\theta_n]\|w_n-x^*\|^2+2\alpha_n\langle(f-\rho F)x^*,\tilde{q}_n-x^*\rangle\} \\
&\le \|x_n-x^*\|^2-(1-\beta_n)\left[1-\frac{\alpha_n(\tau+\delta)}{2}\right]\{\|z_n-v_n+y^*-x^*\|^2 \\
&+\|v_n-w_n+x^*-y^*\|^2\}+2\mu_1\|B_1y^*-B_1v_n\|\|w_n-x^*\| \\
&+2\mu_2\|B_2x^*-B_2z_n\|\|v_n-y^*\|+\alpha_nM_1,
\end{aligned}
$$

which immediately leads to

$$
\begin{aligned}
&(1-\beta_n)\left[1-\frac{\alpha_n(\tau+\delta)}{2}\right]\{\|z_n-v_n+y^*-x^*\|^2+\|v_n-w_n+x^*-y^*\|^2\} \\
&\le \Gamma_n-\Gamma_{n+1}+2\mu_1\|B_1y^*-B_1v_n\|\|w_n-x^*\|+2\mu_2\|B_2x^*-B_2z_n\|\|v_n-y^*\|+\alpha_nM_1.
\end{aligned}
$$

This hence ensures that $\lim_{n\to\infty}\|z_n-v_n+y^*-x^*\|=\lim_{n\to\infty}\|v_n-w_n+x^*-y^*\|=0$. Therefore,

$$
\|z_n-Gz_n\|=\|z_n-w_n\|\le\|z_n-v_n+y^*-x^*\|+\|v_n-w_n+x^*-y^*\|\to 0\ (n\to\infty). \tag{3.39}
$$

It follows from (3.38) and (3.39) that

$$
\|x_n-z_n\|\le\|x_n-u_n\|+\|u_n-z_n\|\to 0\ (n\to\infty),
$$

and

$$
\|x_n-Gz_n\|\le\|x_n-z_n\|+\|z_n-Gz_n\|\to 0\quad(n\to\infty).
$$

Also, using the similar arguments to those of (3.25) and (3.26) in the proof of Theorem 3.1, we can obtain that $\lim_{n\to\infty}\|S_nu_n-x_n\|=\lim_{n\to\infty}\|u_n-y_n\|=0$. By the boundedness of $\{x_n\}$, we know that $\exists\{x_{n_i}\}\subset\{x_n\}$ s.t.

$$
\limsup_{n\to\infty}\langle(f-\rho F)x^*,x_n-x^*\rangle=\lim_{i\to\infty}\langle(f-\rho F)x^*,x_{n_i}-x^*\rangle.
$$

Since H is reflexive and $\{x_n\}$ is bounded, we may assume that $x_{n_i}\rightharpoonup\hat{x}$. Thus it follows that $\limsup_{n\to\infty}\langle(f-\rho F)x^*,x_n-x^*\rangle=\langle(f-\rho F)x^*,\hat{x}-x^*\rangle$. Since $x_n-x_{n+1}\to 0$, $x_n-Gz_n\to 0$, $x_n-S_nu_n\to 0$, $u_n-y_n\to 0$, and $x_n-z_n\to 0$, and $x_{n_i}\rightharpoonup\hat{x}$, we infer by Lemma 3.3 that $\hat{x}\in\Omega$. Thus, using (3.12), one has $\limsup_{n\to\infty}\langle(f-\rho F)x^*,x_n-x^*\rangle\le 0$, which together with (3.38) yields

$$
\limsup_{n\to\infty}\langle(f-\rho F)x^*,\tilde{q}_n-x^*\rangle\le 0.
$$

Note that

$$
\limsup_{n\to\infty}\left[\frac{2\langle(f-\rho F)x^*,\tilde{q}_n-x^*\rangle}{\tau-\delta}+\frac{\theta_n}{\alpha_n}\cdot\frac{M}{\tau-\delta}\right]\le 0.
$$

Consequently, one has $\lim_{n\to\infty}\|x_n-x^*\|^2=0$.

Case 2. $\exists\{\Gamma_{n_i}\}\subset\{\Gamma_n\}$ s.t. $\Gamma_{n_i}<\Gamma_{n_i+1}\ \forall i\in\mathcal{N}$, with $\mathcal{N}$ being the set of all natural numbers. Let $\phi:\mathcal{N}\to\mathcal{N}$ be defined as $\phi(n):=\max\{i\le n:\Gamma_i<\Gamma_{i+1}\}$. Using Lemma 2.8, we get $\Gamma_{\phi(n)}\le\Gamma_{\phi(n)+1}$ and $\Gamma_n\le\Gamma_{\phi(n)+1}$. In the remainder of the proof, using the same references as in Case 2 of the proof of Theorem 3.1, we obtain the desired assertion. □

Theorem 3.4. *If $S : C \to C$ is nonexpansive and $\{x_n\}$ is the sequence constructed in the modified version of Algorithm 3.2, that is, for any starting $x_1 \in C$,*

$$\begin{cases} u_n = \sigma_n x_n + (1-\sigma_n) S_n u_n, \\ y_n = P_C(u_n - \lambda A u_n), \\ t_n = (1-\tau_n) u_n + \tau_n y_n, \\ z_n = P_{C_n}(u_n), \\ v_n = J_{\mu_2}^{F_2}(z_n - \mu_2 B_2 z_n), \\ w_n = J_{\mu_1}^{F_1}(v_n - \mu_1 B_1 v_n), \\ x_{n+1} = \beta_n u_n + (1-\beta_n) P_C[(I - \alpha_n \rho F) S w_n + \alpha_n f(z_n)] \quad \forall n \geq 1, \end{cases}$$

where, for each $n \geq 1$, C_n and τ_n are chosen as in Algorithm 3.2, then $x_n \to x^* \in \Omega$ where $x^* \in \Omega$ is the solution to the HVI: $\langle (\rho F - f)x^*, y - x^* \rangle \geq 0$ for all y in Ω.

Proof. We divide the proof of the theorem into the following several claims.

Claim 1. We assert that $\{x_n\}$ is bounded. Indeed, using the Claim 1 of Theorem 3.3, one derives the desired assertion immediately.

Claim 2. We assert that

$$\begin{aligned} &(1-\beta_n)\{(1-\alpha_n\tau)[\|x_n - u_n\|^2 + \|u_n - z_n\|^2] + \|\tilde{q}_n - P_C(\tilde{q}_n)\|^2\} \\ &+\beta_n(1-\beta_n)\|u_n - P_C(\tilde{q}_n)\|^2 \leq \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 + \alpha_n M_1, \end{aligned}$$

for some $M_1 > 0$. In fact, putting $\theta_n = 0$, we get from (3.34) that

$$\begin{aligned} &\|x_{n+1} - x^*\|^2 \\ &\leq [1 - \alpha_n(1-\beta_n)(\tau - \delta)]\|x_n - x^*\|^2 - (1-\beta_n)\{(1-\alpha_n\tau)[\|x_n - u_n\|^2 \\ &\quad + \|u_n - z_n\|^2] + \|\tilde{q}_n - P_C(\tilde{q}_n)\|^2\} + \alpha_n M_1 - \beta_n(1-\beta_n)\|u_n - P_C(\tilde{q}_n)\|^2 \\ &\leq \|x_n - x^*\|^2 - (1-\beta_n)\{(1-\alpha_n\tau)[\|x_n - u_n\|^2 + \|u_n - z_n\|^2] + \|\tilde{q}_n - P_C(\tilde{q}_n)\|^2\} \\ &\quad + \alpha_n M_1 - \beta_n(1-\beta_n)\|u_n - P_C(\tilde{q}_n)\|^2, \end{aligned}$$

where $\sup_{n\geq 1} 2\|(f - \rho F)x^*\|\|\tilde{q}_n - x^*\| \leq M_1$ for some $M_1 > 0$. This attains the desired assertion.

Claim 3. We assert that

$$(1-\beta_n)(1-\alpha_n\tau)\left[\frac{\tau_n}{2\lambda\tilde{L}}\|\varepsilon_\lambda(u_n)\|^2\right]^2 \leq \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 + \alpha_n M_1.$$

Indeed, using the Claim 3 of Theorem 3.3, one deduces the desired assertion directly.

Claim 4. We assert that

$$\begin{aligned} \|x_{n+1} - x^*\|^2 &\leq [1 - \alpha_n(1-\beta_n)(\tau - \delta)]\|x_n - x^*\|^2 \\ &\quad + \alpha_n(1-\beta_n)(\tau - \delta) \cdot \frac{2\langle (f - \rho F)x^*, \tilde{q}_n - x^* \rangle}{\tau - \delta}. \end{aligned}$$

Indeed, using the Claim 4 of Theorem 3.3, one obtains the desired assertion immediately.

Claim 5. We assert that $\{x_n\}$ converges strongly to the unique solution $x^* \in \Omega$ of the HVI (3.12). Indeed, using the Claim 5 of Theorem 3.3, one obtains the desired assertion immediately. $\square$

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