Fixed Point Theory, 26(2025), No. 2, 409-440
DOI: 10.24193/fpt-ro.2025.2.06
http://www.math.ubbcluj.ro/~nodeacj/sfptcj.html

MODIFIED SUBGRADIENT EXTRAGRADIENT FOR BILEVEL PSEUDOMONOTONE VARIATIONAL INEQUALITY WITH SPLIT COMMON FIXED POINT PROBLEM CONSTRAINT

LU-CHUAN CENG[1], DEBDAS GHOSH[2], HABIB UR REHMAN[3] AND XIAOPENG ZHAO[4]

[1]Department of Mathematics, Shanghai Normal University, Shanghai 200234, China
Email: zenglc@shnu.edu.cn

[2]Department of Mathematical Sciences, Indian Institute of Technology (BHU),
Varanasi 221005, India
Email: debdas.mat@iitbhu.ac.in

[3]School of Mathematics, Zhejiang Normal University, Jinhua, 321004, People's Republic of China
Email: hrehman@zjnu.edu.cn

[4]School of Mathematical Sciences, Tiangong University, Tianjin, 300387, China
Email: zhaoxiaopeng.2007@163.com (Corresponding author)

Abstract. This research presents a novel algorithm for addressing a bilevel pseudomonotone variational inequality problem (BPVIP) constrained by a split common fixed point problem (SCFPP), employing demimetric mappings within real Hilbert spaces. The BPVIP is characterized by a strongly monotone operator at the upper-level variational inequality problem (VIP) and a pseudomonotone operator at the lower-level VIP. To address this issue, we propose a modified subgradient extragradient method that integrates an inertial term, a correction term, and a self-adaptive step size strategy. Under mild conditions, we establish the strong convergence of the proposed algorithm. Subsequently, we apply the primary result to resolve a bilevel split pseudomonotone variational inequality problem (BSPVIP). The efficacy and performance of the algorithm are demonstrated through a numerical example.

Keywords and Phrases: Modified subgradient extragradient rule, bilevel pseudomonotone variational inequality problem, pseudomonotone operator, demimetric mapping, split common fixed point problem.
2020 Mathematics Subject Classification: 65Y05, 65K15, 68W10, 47H05, 47H10.

1. Introduction

Let $\mathcal{H}$ be a real Hilbert space and $C \subset \mathcal{H}$ be a nonempty, convex, and closed set. The inner product and norm in $\mathcal{H}$ are represented by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$, respectively. The projection of $\mathcal{H}$ onto C is denoted as P_C. For an operator $S : C \to \mathcal{H}$, the set of fixed points is defined as $\mathrm{Fix}(S)$. Let $\mathbb{R}$ denote the real numbers. Consider an operator $A : \mathcal{H} \to \mathcal{H}$. The classical VIP seeks $x^* \in C$ such that

$$\langle Ax^*, x - x^* \rangle \geq 0 \quad \forall x \in C.$$

The solution set of the VIP is denoted by $\mathrm{VI}(C, A)$. If $C = \bigcap_{i=1}^{N} \mathrm{Fix}(T_i) \neq \emptyset$, where $\{T_i\}_{i=1}^{N}$ is a finite collection of nonlinear self-mappings on $\mathcal{H}$, the VIP for a common fixed point problem (CFPP) is expressed as:

$$\text{Find } x^* \in \bigcap_{i=1}^{N} \mathrm{Fix}(T_i) \text{ such that } \langle Ax^*, y - x^* \rangle \geq 0 \quad \forall y \in C,$$

termed a hierarchical VIP. This framework has been proven useful in various applications, including signal recovery, power control, bandwidth allocation, optimal control, network location, beamforming, and machine learning. Consequently, it has garnered considerable research interest recently.

For $i = 1, 2, \ldots, N$, let $\mathcal{H}_i$ be a real Hilbert space, and define $\mathcal{T}_i : \mathcal{H} \to \mathcal{H}_i$ as a bounded linear operator. Consider the nonlinear operators $T_i : \mathcal{H} \to \mathcal{H}$ and $S_i : \mathcal{H}_i \to \mathcal{H}_i$. The problem SCFPP is formulated as follows:

$$\text{Find } x^* \in \bigcap_{i=1}^{N} \mathrm{Fix}(T_i) \text{ such that } \mathcal{T}_i x^* \in \mathrm{Fix}(S_i), \quad \forall i \in \{1, 2, \ldots, N\}. \tag{1}$$

To the best of our knowledge, the SCFPP generalizes the split feasibility problem, which has attracted significant research interest due to its applications in image reconstruction, computed tomography, and radiation therapy treatment planning. In recent years, the SCFPP has been studied across various mapping classes, with considerable focus on iterative algorithms designed to solve this problem. Eslamian and Kamandi [17] recently proposed a novel iterative algorithm for addressing the strongly monotone VIP over the solution set of the SCFPP, utilizing an inertial method combined with a correction term and a self-adaptive step size strategy. We now assume that the following conditions hold:

(i) For $i = 1, 2, \ldots, N$, the mapping $S_i : \mathcal{H}_i \to \mathcal{H}_i$ is a ξ_i-demimetric mapping with $\xi_i \in (-\infty, 1)$, ensuring that $I - S_i$ is demiclosed at zero.
(ii) For $i = 1, 2, \ldots, N$, the operator $\mathcal{T}_i : \mathcal{H} \to \mathcal{H}_i$ is a nonzero bounded linear operator, with its adjoint denoted by $\mathcal{T}_i^* : \mathcal{H}_i \to \mathcal{H}$, and $\Omega = \bigcap_{i=1}^{N} \mathcal{T}_i^{-1}\mathrm{Fix}(S_i) \neq \emptyset$.
(iii) The operator $F : \mathcal{H} \to \mathcal{H}$ is κ-Lipschitz continuous and η-strongly monotone.
(iv) The sequences $\{\gamma_n\} \subset [0, 1)$, $\{\sigma_n\} \subset (0, 1)$, and $\{\varepsilon_n\} \subset (0, \infty)$ satisfy: $\limsup_{n\to\infty} \gamma_n < 1$, $\sum_{n=0}^{\infty} \sigma_n = \infty$, $\lim_{n\to\infty} \sigma_n = 0$, and $\lim_{n\to\infty} \frac{\varepsilon_n}{\sigma_n} = 0$.

Algorithm 1.1 (see [17], Algorithm 1). *Initialization:* Let $\alpha > 0$, $\beta > 0$, and $x_1, x_0, y_0 \in \mathcal{H}$ be chosen arbitrarily.
Iterative Steps: Given the iterates x_{n-1}, x_n for $n \geq 1$, compute x_{n+1} as follows:
Step 1. Calculate $y_n = x_n + \alpha_n(x_{n-1} - x_n) + \beta_n(y_{n-1} - x_{n-1})$ for $\alpha_n \in [0, \overline{\alpha}_n]$ and

$\beta_n \in [0, \overline{\beta}_n]$, where

$$\overline{\alpha}_n = \begin{cases} \min\left\{\alpha, \frac{\varepsilon_n}{\|x_n - x_{n-1}\|}\right\} & \text{if } x_n \neq x_{n-1}, \\ \alpha & \text{otherwise} \end{cases},$$

$$\overline{\beta}_n = \begin{cases} \min\left\{\beta, \frac{\varepsilon_n}{\|y_{n-1} - x_{n-1}\|}\right\} & \text{if } y_{n-1} \neq x_{n-1}, \\ \beta & \text{otherwise} \end{cases}.$$

Step 2. Select two indices $i_n, \imath_n \in \{1, 2, \ldots, N\}$ such that

$$\|(I - S_{i_n})\mathcal{T}_{i_n} y_n\| = \max_{i \in \{1,2,\ldots,N\}} \|(I - S_i)\mathcal{T}_i y_n\|$$

and

$$\|(I - S_{\imath_n})\mathcal{T}_{\imath_n} y_n\| = \min_{i \in \{1,2,\ldots,N\}} \|(I - S_i)\mathcal{T}_i y_n\|.$$

Step 3. Compute $u_n = y_n - \tau_{n,i_n}\mathcal{T}_{i_n}^*(I - S_{i_n})\mathcal{T}_{i_n} y_n$ and

$$v_n = y_n - \tau_{n,\imath_n}\mathcal{T}_{\imath_n}^*(I - S_{\imath_n})\mathcal{T}_{\imath_n} y_n,$$

where the step sizes are selected such that for sufficiently small $\epsilon > 0$,

$$\tau_{n,i} \in \left(\epsilon, \frac{(1 - \xi_{i_n})\|(I - S_i)\mathcal{T}_i y_n\|^2}{\|\mathcal{T}_i^*(I - S_i)\mathcal{T}_i y_n\|^2} - \epsilon\right) \text{ if } (I - S_i)\mathcal{T}_i y_n \neq 0,$$

otherwise, $\tau_{n,i} = \tau_i$ is any nonnegative real number.

Step 4. Calculate $w_n = (1 - \gamma_n)u_n + \gamma_n v_n$ and $x_{n+1} = (I - \sigma_n F)w_n$.

Step 5. Set $n := n + 1$ and return to Step 1.

In [17], it was shown that the sequence $\{x_n\}$ generated by Algorithm 1.1 converges strongly to the unique solution $x^* \in \Omega$ of the VIP, defined by the condition

$$\langle Fx^*, y - x^*\rangle \geq 0, \quad \forall y \in \Omega.$$

Thong and Hieu [27] were pioneers in introducing the inertial subgradient extragradient method for finding an element of $\mathrm{VI}(C, A)$, where A is a pseudomonotone self-mapping on $\mathcal{H}$. The iterative scheme for generating the sequence $\{x_n\}$ is as follows for initial points $x_0, x_1 \in \mathcal{H}$:

$$\begin{cases} w_n = x_n + \alpha_n(x_n - x_{n-1}), \\ y_n = P_C(w_n - \ell A w_n), \\ C_n = \{y \in \mathcal{H} : \langle w_n - \ell A w_n - y_n, y_n - y\rangle \geq 0\}, \\ x_{n+1} = P_{C_n}(w_n - \ell A y_n), \quad \forall n \geq 1, \end{cases}$$

where $\ell \in \left(0, \frac{1}{L}\right)$ is a constant and L is the Lipschitz constant of A. They proved that under appropriate conditions, $\{x_n\}$ weakly converges to an element of $\mathrm{VI}(C, A)$. The literature on VIP is vast, and the inertial subgradient extragradient method has attracted considerable attention, leading to various extensions and modifications in multiple studies, including [**?**, **?**, 21, 26, 27, 29, 31] and other in [10, 11, 20, **?**, 32].

Consider two real Hilbert spaces, $\mathcal{H}$ and $\mathcal{H}_1$, with nonempty, closed, and convex subsets $C \subseteq \mathcal{H}$ and $Q \subseteq \mathcal{H}_1$. Define $\mathcal{T} : \mathcal{H} \to \mathcal{H}_1$ as a bounded linear operator, and let $A, F : \mathcal{H} \to \mathcal{H}$ and $B : \mathcal{H}_1 \to \mathcal{H}_1$ be nonlinear mappings. The bilevel split variational inequality problem (BSVIP) is formulated as follows:

$$\text{Find } z^* \in \Omega \text{ such that } \langle Fz^*, z - z^* \rangle \geq 0 \quad \forall z \in \Omega, \tag{2}$$

where $\Omega := \{z \in \mathrm{VI}(C, A) : \mathcal{T}z \in \mathrm{VI}(Q, B)\}$ is the solution set of the split variational inequality problem (SVIP) introduced by Censor et al. The SVIP seeks a solution $x^* \in \mathrm{VI}(C, A)$ such that $\mathcal{T}x^* \in \mathrm{VI}(Q, B)$. To approximate a solution to the SVIP, the following iterative method is proposed, starting with an initial point $x_1 \in \mathcal{H}$:

$$x_{n+1} = P_C(I - \lambda A)\left(x_n + \gamma \mathcal{T}^*(P_Q(I - \lambda B) - I)\mathcal{T}x_n\right) \quad \forall n \geq 1, \tag{3}$$

where A and B are inverse-strongly monotone mappings, and $\mathcal{T}$ is a non-zero bounded linear operator. Under suitable conditions, it is shown that the sequence $\{x_n\}$ converges weakly to a solution of the SVIP. Moreover, the VIP can be reformulated as a fixed point problem (FPP): $Sz = P_Q(z - \mu Bz)$, $\quad \mu > 0$, where $\mathrm{VI}(Q, B) = \mathrm{Fix}(S)$, and $\mathrm{Fix}(S)$ denotes the set of fixed points of the operator S. Thus, the BSVIP can be expressed as:

$$\text{Find } z^* \in \Omega \text{ such that } \langle Fz^*, z - z^* \rangle \geq 0 \quad \forall z \in \Omega, \tag{4}$$

with $\Omega := \{z \in \mathrm{VI}(C, A) : \mathcal{T}z \in \mathrm{Fix}(S)\}$. Inspired by problems (1) and (4), we introduce and study the BPVIP with a SCFPP constraint, involving demimetric mappings in real Hilbert spaces.

To tackle this problem, we propose a modified subgradient extragradient method based on the subgradient extragradient algorithm from [28] for solving the BPVIP. Let $A : \mathcal{H} \to \mathcal{H}$ be an L-Lipschitz continuous and pseudomonotone operator, ensuring that $\Omega = \mathrm{VI}(C, A) \neq \emptyset$. Additionally, let $F : \mathcal{H} \to \mathcal{H}$ be a κ-Lipschitz continuous and η-strongly monotone operator. The algorithm outlined in [28] is structured as follows.

Algorithm 1.2 (refer to Algorithm 1 in [28]). *Initialization:* Select a sequence $\{\alpha_n\} \subset (0, 1)$ such that $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$. Set $\lambda > 0$, $\mu \in (0, 1)$, $\rho \in \left(0, \frac{2\eta}{\kappa^2}\right)$, and choose an arbitrary $x_0 \in \mathcal{H}$.
Iterative Steps: For each $n \geq 0$, compute x_{n+1} as follows:
Step 1. Compute $y_n = P_C(x_n - \lambda_n Ax_n)$ and $z_n = P_{C_n}(x_n - \lambda_n Ay_n)$, where

$$C_n := \{y \in \mathcal{H} : \langle x_n - \lambda_n Ax_n - y_n, y - y_n \rangle \leq 0\}.$$

Step 2. Update x_{n+1} as $x_{n+1} = (I - \alpha_n \rho F)z_n$, and update the step size λ_n according to the following rule:

$$\lambda_{n+1} = \begin{cases} \min\left\{\mu \frac{\|x_n - y_n\|^2 + \|z_n - y_n\|^2}{2\langle Ax_n - Ay_n, z_n - y_n\rangle}, \lambda_n\right\}, & \text{if } \langle Ax_n - Ay_n, z_n - y_n \rangle > 0, \\ \lambda_n, & \text{otherwise.} \end{cases}$$

Step 3. Set $n := n + 1$ and repeat from Step 1.

It was shown in [28] that the sequence $\{x_n\}$ converges strongly to the unique solution $z^* \in \mathit{\Omega}$ of the BPVIP, characterized by the condition:

$$\langle Fz^*, z - z^* \rangle \geq 0 \quad \forall z \in \mathit{\Omega}.$$

In this paper, we build on the concepts from [17, 28] to establish the strong convergence of our proposed algorithm to the unique solution of the BPVIP with the SCFPP constraint under certain mild conditions. Moreover, the principal result is applied to address a BSPVIP. An illustrative example is included to demonstrate the effectiveness and applicability of our method.

The structure of this paper is as follows: Section 2 introduces several key concepts and foundational tools needed for subsequent sections. Section 3 presents the convergence analysis of the proposed algorithm. Finally, Section 4 applies our main results to solve the BPVIP with the SCFPP constraint, complemented by an illustrative example. Our findings extend and enhance the results presented in [17, 28].

2. Preliminaries

In this section, let C denote a nonempty closed convex subset of the real Hilbert space $\mathcal{H}$. A mapping $S : C \to \mathcal{H}$ is termed nonexpansive if it satisfies

$$\|Sx - Sy\| \leq \|x - y\| \quad \forall x, y \in C.$$

For a sequence $\{x_n\} \subset \mathcal{H}$, we write $x_n \to x$ (resp., $x_n \rightharpoonup x$) to indicate strong (resp., weak) convergence to the point x. For each $x \in \mathcal{H}$, there exists a unique nearest point in C, denoted $P_C x$, satisfying

$$\|x - P_C x\| \leq \|x - y\| \quad \forall y \in C.$$

Lemma 2.1. *(see* [18]*) The operator P_C is the metric projection from the Hilbert space $\mathcal{H}$ onto the convex set C. Then P_C possesses the following properties:*

(i) $\langle x - y, P_C x - P_C y\rangle \geq \|P_C x - P_C y\|^2 \quad \forall x, y \in \mathcal{H}$;
(ii) $z = P_C x \quad \Leftrightarrow \quad \langle x - z, y - z\rangle \leq 0 \quad \forall x \in \mathcal{H}, y \in C$;
(iii) $\|x - y\|^2 \geq \|x - P_C x\|^2 + \|y - P_C x\|^2 \quad \forall x \in \mathcal{H}, y \in C$;
(iv) $\|x - y\|^2 = \|x\|^2 - \|y\|^2 - 2\langle x - y, y\rangle \quad \forall x, y \in \mathcal{H}$;
(v) $\|sx + (1-s)y\|^2 = s\|x\|^2 + (1-s)\|y\|^2 - s(1-s)\|x - y\|^2 \quad \forall x, y \in \mathcal{H}, s \in [0, 1]$.

Definition 2.2. (see [19]) Let $S : C \to \mathcal{H}$ be an operator. The operator is defined as follows:

(i) *L-Lipschitz continuous* if there exists $L > 0$ such that
$$\|Sx - Sy\| \leq L\|x - y\| \quad \forall x, y \in C;$$
(ii) *α-strongly monotone* if there exists $\alpha > 0$ such that
$$\langle Sx - Sy, x - y\rangle \geq \alpha\|x - y\|^2 \quad \forall x, y \in C;$$
(iii) *monotone* if $\langle Sx - Sy, x - y\rangle \geq 0 \quad \forall x, y \in C$;
(iv) *pseudomonotone* if $\langle Sx, y - x\rangle \geq 0 \Rightarrow \langle Sy, y - x\rangle \geq 0 \quad \forall x, y \in C$;
(v) *quasimonotone* if $\langle Sx, y - x\rangle > 0 \Rightarrow \langle Sy, y - x\rangle \geq 0 \quad \forall x, y \in C$;
(vi) *ξ-demicontractive* if there exists $\xi \in (0, 1)$ such that
$$\|Sx - p\|^2 \leq \|x - p\|^2 + \xi\|x - Sx\|^2 \quad \forall x \in C,\ p \in \mathrm{Fix}(S) \neq \emptyset;$$
(vii) *ξ-demimetric* if there exists $\xi \in (-\infty, 1)$ such that
$$\langle x - Sx, x - p\rangle \geq \frac{1 - \xi}{2}\|x - Sx\|^2 \quad \forall x \in C,\ p \in \mathrm{Fix}(S) \neq \emptyset;$$

(viii) *sequentially weakly continuous* if for any sequence $\{x_n\} \subset C$ with $x_n \rightharpoonup x$, then $Sx_n \rightharpoonup Sx$.

Remark 2.3. It is straightforward to verify that (ii) $\Longrightarrow$ (iii), (iii) $\Longrightarrow$ (iv), and (iv) $\Longrightarrow$ (v). However, the converse implications do not generally hold.

Definition 2.4. A mapping $S : C \to \mathcal{H}$ is said to satisfy the demiclosedness principle if the operator $I - S$ is demiclosed at zero. For any sequence $\{x_n\} \subset C$ such that $x_n \rightharpoonup x$ and $(I - S)x_n \to 0$, it follows that $x \in \mathrm{Fix}(S)$.

Lemma 2.5. (see [25]) *If $S : C \to \mathcal{H}$ is a ξ-demimetric mapping, then the set* $\mathrm{Fix}(S)$ *is both closed and convex.*

Lemma 2.6. (see [30]) *Let $\lambda \in (0, 1]$ and let $S : C \to \mathcal{H}$ be a nonexpansive mapping. We define $S^\lambda : C \to \mathcal{H}$ as $S^\lambda x = Sx - \lambda\nu F(Sx) \quad \forall x \in C$, where $F : \mathcal{H} \to \mathcal{H}$ is κ-Lipschitzian and η-strongly monotone. Then, S^λ qualifies as a contraction if $0 < \nu < \frac{2\eta}{\kappa^2}$, specifically satisfying:*

$$\|S^\lambda x - S^\lambda y\| \le (1 - \lambda\zeta)\|x - y\| \quad \forall x, y \in C,$$

where $\zeta = 1 - \sqrt{1 - \nu(2\eta - \nu\kappa^2)} \in (0, 1]$.

Lemma 2.7. *Assuming that $A : C \to \mathcal{H}$ is pseudomonotone and continuous, an element $u \in C$ constitutes a solution to the variational inequality $\langle Au, v - u\rangle \ge 0 \quad \forall v \in C$ if and only if $\langle Av, v - u\rangle \ge 0 \quad \forall v \in C$.*

Proof. The assertion can be readily verified. □

Lemma 2.8. (see [30]) *Let $\{a_n\}$ denote a sequence of nonnegative numbers that satisfies the condition*

$$a_{n+1} \le (1 - \lambda_n)a_n + \lambda_n\gamma_n \quad \forall n \ge 1,$$

where $\{\lambda_n\}$ and $\{\gamma_n\}$ are sequences of real numbers fulfilling the criteria:
(i) $\{\lambda_n\} \subset [0, 1]$ *and* $\sum_{n=1}^{\infty} \lambda_n = \infty$, *and*
(ii) $\limsup_{n\to\infty} \gamma_n \le 0$ *or* $\sum_{n=1}^{\infty} |\lambda_n\gamma_n| < \infty$.
Then, it follows that $\lim_{n\to\infty} a_n = 0$.

Lemma 2.9. (see [22]) *Let $\{\mathbf{\Gamma}_m\}$ represent a sequence of real numbers that does not decrease at infinity, in the sense that there exists a subsequence $\{\mathbf{\Gamma}_{m_k}\} \subset \{\mathbf{\Gamma}_m\}$ such that*

$$\mathbf{\Gamma}_{m_k} < \mathbf{\Gamma}_{m_k+1} \quad \forall k \ge 1.$$

Define the sequence of integers $\{\phi(m)\}_{m\ge m_0}$ as follows:

$$\phi(m) = \max\{k \le m : \mathbf{\Gamma}_k < \mathbf{\Gamma}_{k+1}\},$$

where $m_0 \ge 1$ satisfies $\{k \le m_0 : \mathbf{\Gamma}_k < \mathbf{\Gamma}_{k+1}\} \ne \emptyset$. The following statements hold:
(i) $\phi(m_0) \le \phi(m_0 + 1) \le \cdots$ *and* $\phi(m) \to \infty$;
(ii) $\mathbf{\Gamma}_{\phi(m)} \le \mathbf{\Gamma}_{\phi(m)+1}$ *and* $\mathbf{\Gamma}_m \le \mathbf{\Gamma}_{\phi(m)+1} \quad \forall m \ge m_0$.

3. Main results and convergence criteria

In this section, let $\mathcal{H}$ and $\mathcal{H}_i$ be real Hilbert spaces for each $i = 1, 2, \ldots, N$. Assume that the feasible set C is nonempty, closed, and convex in $\mathcal{H}$. To conduct the convergence analysis of the proposed method for addressing the BPVIP with the SCFPP constraint, we impose the following assumptions:

(C1) The mapping $F : \mathcal{H} \to \mathcal{H}$ is η-strongly monotone and κ-Lipschitz continuous.

(C2) The operator $A : \mathcal{H} \to \mathcal{H}$ is pseudomonotone and L-Lipschitz continuous, and it satisfies $\|Au\| \leq \liminf_{n\to\infty} \|Au_n\|$ for any sequence $\{u_n\} \subset C$ such that $u_n \rightharpoonup u$.

(C3) For each $i = 1, 2, \ldots, N$, $S_i : \mathcal{H}_i \to \mathcal{H}_i$ is a ξ_i-demimetric operator with $\xi_i \in (-\infty, 1)$, and $I - S_i$ is demiclosed at zero.

(C4) For each $i = 1, 2, \ldots, N$, the operator $\mathcal{T}_i : \mathcal{H} \to \mathcal{H}_i$ is a nonzero bounded linear operator, and its adjoint $\mathcal{T}_i^* : \mathcal{H}_i \to \mathcal{H}$ satisfies

$$\Omega = \mathrm{VI}(C, A) \cap \left(\bigcap_{i=1}^{N} \mathcal{T}_i^{-1} \mathrm{Fix}(S_i) \right) \neq \emptyset.$$

(C5) The sequences $\{\gamma_n\} \subset [0, 1)$, $\{\sigma_n\} \subset (0, 1)$, and $\{\varepsilon_n\} \subset (0, \infty)$ satisfy $0 < \liminf_{n\to\infty} \gamma_n \leq \limsup_{n\to\infty} \gamma_n < 1$, $\sum_{n=0}^{\infty} \sigma_n = \infty$, $\lim_{n\to\infty} \sigma_n = 0$, and $\lim_{n\to\infty} \frac{\varepsilon_n}{\sigma_n} = 0$.

Under these assumptions, we now formulate the BPVIP with the SCFPP constraint as follows:

Find $x^* \in \Omega = \mathrm{VI}(C, A) \cap \left(\bigcap_{i=1}^{N} \mathcal{T}_i^{-1} \mathrm{Fix}(S_i) \right)$ such that $\langle Fx^*, y - x^* \rangle \geq 0 \; \forall y \in \Omega$.

Remark 3.1. First, observe that $\mathrm{VI}(C, A)$ is nonempty, closed, and convex in $\mathcal{H}$. We claim that Ω is also nonempty, closed, and convex in $\mathcal{H}$. Indeed, the conditions (C3)-(C4) guarantee that $\bigcap_{i=1}^{N} \mathcal{T}_i^{-1} \mathrm{Fix}(S_i)$ is a closed and convex set. To demonstrate this, let $\{x_n\}$ be a sequence in $\bigcap_{i=1}^{N} \mathcal{T}_i^{-1} \mathrm{Fix}(S_i)$ such that $x_n \to x^*$. By the continuity of $\mathcal{T}_i$, we have $\mathcal{T}_i x_n \to \mathcal{T}_i x^*$ for each $i = 1, 2, \ldots, N$. Using the fact that $I - S_i$ is demiclosed at zero, it follows that $\mathcal{T}_i x^* \in \mathrm{Fix}(S_i)$, implying that $x^* \in \mathcal{T}_i^{-1} \mathrm{Fix}(S_i)$. Hence, $\bigcap_{i=1}^{N} \mathcal{T}_i^{-1} \mathrm{Fix}(S_i)$ is closed.

Next, we establish the convexity of $\bigcap_{i=1}^{N} \mathcal{T}_i^{-1} \mathrm{Fix}(S_i)$. Let $x, y \in \bigcap_{i=1}^{N} \mathcal{T}_i^{-1} \mathrm{Fix}(S_i)$ and $\alpha \in [0, 1]$. This implies that $\mathcal{T}_i x, \mathcal{T}_i y \in \mathrm{Fix}(S_i)$ for each i. Since $\mathrm{Fix}(S_i)$ is convex (by Lemma 2.5), it follows that $\alpha \mathcal{T}_i x + (1 - \alpha) \mathcal{T}_i y \in \mathrm{Fix}(S_i)$. By the linearity of $\mathcal{T}_i$, we deduce that $\mathcal{T}_i(\alpha x + (1 - \alpha) y) = \alpha \mathcal{T}_i x + (1 - \alpha) \mathcal{T}_i y \in \mathrm{Fix}(S_i)$. Therefore, $\alpha x + (1 - \alpha) y \in \bigcap_{i=1}^{N} \mathcal{T}_i^{-1} \mathrm{Fix}(S_i)$, proving that $\bigcap_{i=1}^{N} \mathcal{T}_i^{-1} \mathrm{Fix}(S_i)$ is convex.

As a result, $\Omega = \mathrm{VI}(C, A) \cap \left(\bigcap_{i=1}^{N} \mathcal{T}_i^{-1} \mathrm{Fix}(S_i) \right)$ is nonempty, closed, and convex in $\mathcal{H}$. Given that Ω is a nonempty, closed, and convex set, and by condition (C1), we conclude that there exists a unique solution $x^* \in \Omega$ to the following variational

Algorithm 1

Initialization: Set $\lambda_1 > 0$, $\epsilon > 0$, $\{\tau_i\}_{i=1}^N \subset [0,\infty)$, $\mu \in (0,1)$, $\alpha, \beta \in [0,1]$, and choose $x_1, x_0, w_0 \in \mathcal{H}$ arbitrarily.

Step 1: Given current iterates x_{n-1} and x_n $(n \geq 1)$, compute

$$w_n = x_n + \alpha_n(x_n - x_{n-1}) + \beta_n(w_{n-1} - x_{n-1}),$$

where $\alpha_n \in [0, \overline{\alpha}_n]$ and $\beta_n \in [0, \overline{\beta}_n]$, with

$$\overline{\alpha}_n = \begin{cases} \min\left\{\alpha, \frac{\epsilon_n}{\|x_n - x_{n-1}\|}\right\} & \text{if } x_n \neq x_{n-1}, \\ \alpha & \text{otherwise,} \end{cases}$$

and

$$\overline{\beta}_n = \begin{cases} \min\left\{\beta, \frac{\epsilon_n}{\|w_{n-1} - x_{n-1}\|}\right\} & \text{if } w_{n-1} \neq x_{n-1}, \\ \beta & \text{otherwise.} \end{cases}$$

Step 2: Compute $y_n = P_C(w_n - \lambda_n A w_n)$ and $q_n = P_{C_n}(w_n - \lambda_n A y_n)$, where the half-space

$$C_n = \{y \in \mathcal{H} : \langle w_n - \lambda_n A w_n - y_n, y_n - y\rangle \geq 0\}.$$

Step 3: Choose two indices $i_n, \imath_n \in \{1, 2, \dots, N\}$ such that

$$\|(I - S_{i_n})\mathcal{T}_{i_n} q_n\| = \max_{i \in \{1,2,\dots,N\}} \|(I - S_i)\mathcal{T}_i q_n\|$$

and

$$\|(I - S_{\imath_n})\mathcal{T}_{\imath_n} q_n\| = \min_{i \in \{1,2,\dots,N\}} \|(I - S_i)\mathcal{T}_i q_n\|.$$

Step 4: Compute $u_n = q_n - \tau_{n,i_n} \mathcal{T}_{i_n}^*(I - S_{i_n})\mathcal{T}_{i_n} q_n$ and

$$v_n = q_n - \tau_{n,\imath_n} \mathcal{T}_{\imath_n}^*(I - S_{\imath_n})\mathcal{T}_{\imath_n} q_n,$$

where $\tau_{n,i}$ is chosen as a bounded sequence satisfying

$$0 < \epsilon \leq \tau_{n,i} \leq \frac{(1 - \xi_{i_n})\|(I - S_i)\mathcal{T}_i q_n\|^2}{\|\mathcal{T}_i^*(I - S_i)\mathcal{T}_i q_n\|^2} - \epsilon \quad \text{if } (I - S_i)\mathcal{T}_i q_n \neq 0, \tag{5}$$

otherwise, set $\tau_{n,i} = \tau_i$.

Step 5: Calculate $z_n = (1 - \gamma_n)u_n + \gamma_n v_n$ and $x_{n+1} = (I - \sigma_n F)z_n$, and update

$$\lambda_{n+1} = \begin{cases} \min\left\{\mu \frac{\|w_n - y_n\|^2 + \|q_n - y_n\|^2}{2\langle A w_n - A y_n, q_n - y_n\rangle}, \lambda_n\right\} & \text{if } \langle A w_n - A y_n, q_n - y_n\rangle > 0, \\ \lambda_n & \text{otherwise.} \end{cases} \tag{6}$$

Set $n := n + 1$ and return to Step 1.

---

inequality problem (VIP):

$$\langle F x^*, y - x^*\rangle \geq 0 \quad \forall y \in \Omega. \tag{7}$$

Remark 3.2. From the definitions of $\overline{\alpha}_n$ and $\overline{\beta}_n$, we deduce that

$$\lim_{n\to\infty} \frac{\alpha_n}{\sigma_n}\|x_n - x_{n-1}\| = 0 \quad \text{and} \quad \lim_{n\to\infty} \frac{\beta_n}{\sigma_n}\|w_{n-1} - x_{n-1}\| = 0.$$

Indeed, since $\alpha_n\|x_n - x_{n-1}\| \leq \varepsilon_n$ for all $n \geq 1$, and using the fact that

$$\lim_{n\to\infty} \frac{\varepsilon_n}{\sigma_n} = 0,$$

it follows that

$$\frac{\alpha_n}{\sigma_n}\|x_n - x_{n-1}\| \leq \frac{\varepsilon_n}{\sigma_n} \to 0 \quad \text{as} \quad n \to \infty.$$

Similarly, we conclude that

$$\lim_{n\to\infty} \frac{\beta_n}{\sigma_n}\|w_{n-1} - x_{n-1}\| = 0.$$

Lemma 3.3. *Let $\{\lambda_n\}$ be the sequence generated by* (6)*. Then, $\{\lambda_n\}$ is nonincreasing, with $\lambda_n \geq \lambda := \min\{\lambda_1, \frac{\mu}{L}\}$ for all $n \geq 1$, and $\lim_{n\to\infty} \lambda_n \geq \lambda := \min\{\lambda_1, \frac{\mu}{L}\}$.*

Proof. From (6), it is clear that $\lambda_n \geq \lambda_{n+1}$ for all $n \geq 1$. Furthermore, note that

$$\left.\begin{array}{l}\frac{1}{2}(\|w_n - y_n\|^2 + \|q_n - y_n\|^2) \geq \|w_n - y_n\|\|q_n - y_n\|, \\ \langle Aw_n - Ay_n, q_n - y_n\rangle \leq L\|w_n - y_n\|\|q_n - y_n\|\end{array}\right\} \Rightarrow \lambda_{n+1} \geq \min\{\lambda_n, \frac{\mu}{L}\}.$$

□

The following lemmas are instrumental in the convergence analysis of our algorithm.

Lemma 3.4. *Let $\{x_n\}$ be the sequence generated by Algorithm 1. Then, the stepsize $\tau_{n,i}$ defined in* (5) *is well-defined.*

Proof. It suffices to show that $\|\mathcal{T}_i^*(I - S_i)\mathcal{T}_i q_n\|^2 \neq 0$. Let $p \in \Omega$ be arbitrary. Since S_i is a ξ_i-demimetric mapping, we have

$$\begin{aligned}\|q_n - p\|\|\mathcal{T}_i^*(I - S_i)\mathcal{T}_i q_n\| &\geq \langle q_n - p, \mathcal{T}_i^*(I - S_i)\mathcal{T}_i q_n\rangle \\ &= \langle \mathcal{T}_i q_n - \mathcal{T}_i p, (I - S_i)\mathcal{T}_i q_n\rangle \\ &\geq \tfrac{1-\xi_i}{2}\|(I - S_i)\mathcal{T}_i q_n\|^2.\end{aligned}$$

If $(I - S_i)\mathcal{T}_i q_n \neq 0$, then $\|(I - S_i)\mathcal{T}_i q_n\|^2 > 0$. Therefore, $\|\mathcal{T}_i^*(I - S_i)\mathcal{T}_i q_n\|^2 > 0$. □

Lemma 3.5. *Let $\{w_n\}$, $\{y_n\}$, and $\{q_n\}$ be the sequences generated by Algorithm 1. Then, for all $p \in \Omega$, we have:*

$$\|q_n - p\|^2 \leq \|w_n - p\|^2 - \left(1 - \mu\frac{\lambda_n}{\lambda_{n+1}}\right)\|w_n - y_n\|^2 - \left(1 - \mu\frac{\lambda_n}{\lambda_{n+1}}\right)\|q_n - y_n\|^2.$$

Proof. First, from the definition of the sequence $\{\lambda_n\}$, we assert that

$$2\langle Aw_n - Ay_n, q_n - y_n\rangle \leq \frac{\mu}{\lambda_{n+1}}\|w_n - y_n\|^2 + \frac{\mu}{\lambda_{n+1}}\|q_n - y_n\|^2 \quad \forall n \geq 1. \tag{8}$$

Specifically, if $\langle Aw_n - Ay_n, q_n - y_n\rangle \leq 0$, then inequality (8) holds. Otherwise, from (6), we can derive (8). Furthermore, for any $p \in \Omega \subset C \subset C_n$, we can observe that

$$\begin{aligned}\|q_n - p\|^2 &= \|P_{C_n}(w_n - \lambda_n Ay_n) - P_{C_n}p\|^2 \\ &\leq \langle q_n - p, w_n - \lambda_n Ay_n - p\rangle \\ &= \tfrac{1}{2}\|q_n - p\|^2 + \tfrac{1}{2}\|w_n - p\|^2 - \tfrac{1}{2}\|q_n - w_n\|^2 - \langle q_n - p, \lambda_n Ay_n\rangle,\end{aligned}$$

which leads to

$$\|q_n - p\|^2 \leq \|w_n - p\|^2 - \|q_n - w_n\|^2 - 2\langle q_n - p, \lambda_n Ay_n\rangle. \tag{9}$$

Since $p \in \mathrm{VI}(C, A)$, we know that $\langle Ap, x - p\rangle \geq 0$ for all $x \in C$. By the pseudomonotonicity of A on C, it follows that $\langle Ax, x - p\rangle \geq 0$ for any $x \in C$. Choosing $x := y_n \in C$, we derive $\langle Ay_n, p - y_n\rangle \leq 0$. Thus,

$$\langle Ay_n, p - q_n\rangle = \langle Ay_n, p - y_n\rangle + \langle Ay_n, y_n - q_n\rangle \leq \langle Ay_n, y_n - q_n\rangle. \tag{10}$$

Substituting (10) into (9), we obtain

$$\|q_n - p\|^2 \leq \|w_n - p\|^2 - \|q_n - y_n\|^2 - \|y_n - w_n\|^2 + 2\langle w_n - \lambda_n Ay_n - y_n, q_n - y_n\rangle. \tag{11}$$

Since $q_n = P_{C_n}(w_n - \lambda_n Ay_n)$, it follows that $q_n \in C_n$, and thus,

$$\begin{aligned} 2\langle w_n - \lambda_n Ay_n - y_n, q_n - y_n\rangle &= 2\langle w_n - \lambda_n Aw_n - y_n, q_n - y_n\rangle \\ &\quad + 2\lambda_n\langle Aw_n - Ay_n, q_n - y_n\rangle \\ &\leq 2\lambda_n\langle Aw_n - Ay_n, q_n - y_n\rangle. \end{aligned}$$

Combining this with (8) yields

$$2\langle w_n - \lambda_n Ay_n - y_n, q_n - y_n\rangle \leq \mu\frac{\lambda_n}{\lambda_{n+1}}\|w_n - y_n\|^2 + \mu\frac{\lambda_n}{\lambda_{n+1}}\|q_n - y_n\|^2. \tag{12}$$

By substituting (12) into (11), we arrive at the desired result. □

Next, we demonstrate that the sequence $\{x_n\}$ generated by Algorithm 1 is bounded.

Lemma 3.6. *Let $\{x_n\}$ denote the sequence produced by Algorithm 1. Then, the sequence $\{x_n\}$ is bounded.*

Proof. Consider $x^* \in \Omega$ as the unique solution to the VIP defined in (7). This implies that there exists a unique solution $x^* \in \Omega$ for the BPVIP with the SCFPP constraint. By the definition of w_n and applying the triangle inequality, we obtain:

$$\begin{aligned} \|w_n - x^*\| &= \|x_n + \alpha_n(x_n - x_{n-1}) + \beta_n(w_{n-1} - x_{n-1}) - x^*\| \\ &\leq \|x_n - x^*\| + \alpha_n\|x_n - x_{n-1}\| + \beta_n\|w_{n-1} - x_{n-1}\| \\ &= \|x_n - x^*\| + \sigma_n \cdot \tfrac{\alpha_n}{\sigma_n}\|x_n - x_{n-1}\| + \sigma_n \cdot \tfrac{\beta_n}{\sigma_n}\|w_{n-1} - x_{n-1}\| \\ &\leq \|x_n - x^*\| + \sigma_n M_1, \end{aligned} \tag{13}$$

where

$$\sup_{n\geq 1}\left\{\frac{\alpha_n}{\sigma_n}\|x_n - x_{n-1}\| + \frac{\beta_n}{\sigma_n}\|w_{n-1} - x_{n-1}\|\right\} \leq M_1$$

for some constant $M_1 > 0$.

Utilizing the fact that S_{i_n} is a ξ_{i_n}-demimetric mapping, we have:

$$\begin{aligned} \|u_n - x^*\|^2 &= \|q_n - \tau_{n,i_n}\mathcal{T}_{i_n}^*(I - S_{i_n})\mathcal{T}_{i_n}q_n - x^*\|^2 \\ &= \|q_n - x^*\|^2 - 2\langle q_n - x^*, \tau_{n,i_n}\mathcal{T}_{i_n}^*(I - S_{i_n})\mathcal{T}_{i_n}q_n\rangle \\ &\quad + \|\tau_{n,i_n}\mathcal{T}_{i_n}^*(I - S_{i_n})\mathcal{T}_{i_n}q_n\|^2 \\ &= \|q_n - x^*\|^2 - 2\tau_{n,i_n}\langle \mathcal{T}_{i_n}q_n - \mathcal{T}_{i_n}x^*, (I - S_{i_n})\mathcal{T}_{i_n}q_n\rangle \\ &\quad + \tau_{n,i_n}^2\|\mathcal{T}_{i_n}^*(I - S_{i_n})\mathcal{T}_{i_n}q_n\|^2 \\ &\leq \|q_n - x^*\|^2 - 2\tau_{n,i_n}\tfrac{1-\xi_{i_n}}{2}\|(I - S_{i_n})\mathcal{T}_{i_n}q_n\|^2 \\ &\quad + \tau_{n,i_n}^2\|\mathcal{T}_{i_n}^*(I - S_{i_n})\mathcal{T}_{i_n}q_n\|^2 \\ &= \|q_n - x^*\|^2 \\ &\quad + \tau_{n,i_n}\left(\tau_{n,i_n}\|\mathcal{T}_{i_n}^*(I - S_{i_n})\mathcal{T}_{i_n}q_n\|^2 - (1 - \xi_{i_n})\|(I - S_{i_n})\mathcal{T}_{i_n}q_n\|^2\right). \end{aligned} \tag{14}$$

For each $n \geq 1$, based on the definition of τ_{n,i_n} in (5), it follows that:

$$\|\mathcal{T}_{i_n}^*(I - S_{i_n})\mathcal{T}_{i_n} q_n\|^2(\epsilon + \tau_{n,i_n}) \leq (1 - \xi_{i_n})\|(I - S_{i_n})\mathcal{T}_{i_n} q_n\|^2.$$

Consequently, we can express this as:

$$\epsilon\|\mathcal{T}_{i_n}^*(I - S_{i_n})\mathcal{T}_{i_n} q_n\|^2 \leq (1 - \xi_{i_n})\|(I - S_{i_n})\mathcal{T}_{i_n} q_n\|^2 - \tau_{n,i_n}\|\mathcal{T}_{i_n}^*(I - S_{i_n})\mathcal{T}_{i_n} q_n\|^2.$$

This leads to the inequality:

$$\begin{aligned}\tau_{n,i_n}\epsilon\|\mathcal{T}_{i_n}^*(I - S_{i_n})\mathcal{T}_{i_n} q_n\|^2 \leq \tau_{n,i_n}[&(1 - \xi_{i_n})\|(I - S_{i_n})\mathcal{T}_{i_n} q_n\|^2 \\ &- \tau_{n,i_n}\|\mathcal{T}_{i_n}^*(I - S_{i_n})\mathcal{T}_{i_n} q_n\|^2].\end{aligned} \tag{15}$$

Thus, combining inequalities (14) and (15), we derive:

$$\|u_n - x^*\|^2 \leq \|q_n - x^*\|^2 - \tau_{n,i_n}\epsilon\|\mathcal{T}_{i_n}^*(I - S_{i_n})\mathcal{T}_{i_n} q_n\|^2. \tag{16}$$

Similarly, we obtain:

$$\|v_n - x^*\|^2 \leq \|q_n - x^*\|^2 - \tau_{n,\imath_n}\epsilon\|\mathcal{T}_{\imath_n}^*(I - S_{\imath_n})\mathcal{T}_{\imath_n} q_n\|^2. \tag{17}$$

From the convexity of the function $\|\cdot\|^2$ with the inequalities (16) and (17), we have

$$\begin{aligned}\|z_n - x^*\|^2 \quad &\leq (1 - \gamma_n)\|u_n - x^*\|^2 + \gamma_n\|v_n - x^*\|^2 \\ &\leq \|q_n - x^*\|^2 - (1 - \gamma_n)\tau_{n,i_n}\epsilon\|\mathcal{T}_{i_n}^*(I - S_{i_n})\mathcal{T}_{i_n} q_n\|^2 \\ &\quad -\gamma_n\tau_{n,\imath_n}\epsilon\|\mathcal{T}_{\imath_n}^*(I - S_{\imath_n})\mathcal{T}_{\imath_n} q_n\|^2.\end{aligned} \tag{18}$$

Additionally, by Lemma 3.3, we have $\lim_{n\to\infty} \lambda_n \geq \lambda := \min\left\{\lambda_1, \frac{\mu}{L}\right\}$, which leads to $\lim_{n\to\infty}\left(1 - \mu\frac{\lambda_n}{\lambda_{n+1}}\right) = 1 - \mu > 0$. Without loss of generality, we can assume that $1 - \mu\frac{\lambda_n}{\lambda_{n+1}} > 0$ for all $n \geq 1$. Thus, by Lemma 3.5, we have

$$\begin{aligned}\|q_n - x^*\|^2 \quad &\leq \|w_n - x^*\|^2 - \left(1 - \mu\frac{\lambda_n}{\lambda_{n+1}}\right)\|w_n - y_n\|^2 \\ &\quad - \left(1 - \mu\frac{\lambda_n}{\lambda_{n+1}}\right)\|q_n - y_n\|^2 \\ &\leq \|w_n - x^*\|^2.\end{aligned} \tag{19}$$

By combining (13), (18), and (19), we obtain

$$\|z_n - x^*\| \leq \|q_n - x^*\| \leq \|w_n - x^*\| \leq \|x_n - x^*\| + \sigma_n M_1 \quad \forall n \geq 1. \tag{20}$$

Now, take $\nu \in \left(0, \frac{2\eta}{\kappa^2}\right)$. Since $\lim_{n\to\infty} \sigma_n = 0$, there exists $n_0 \geq 1$ such that for all $n \geq n_0$, $\sigma_n \leq \nu$. Hence, $\frac{\sigma_n}{\nu} \in (0, 1]$. From Lemma 2.6, it follows that for all $n \geq n_0$,

$$\begin{aligned}\|(I - \sigma_n F)z_n - (I - \sigma_n F)x^*\| \quad &= \left\|(I - \tfrac{\sigma_n}{\nu} \cdot \nu F)z_n - (I - \tfrac{\sigma_n}{\nu} \cdot \nu F)x^*\right\| \\ &\leq \left(1 - \tfrac{\sigma_n}{\nu} \cdot \zeta\right)\|z_n - x^*\|,\end{aligned} \tag{21}$$

where $\zeta = 1 - \sqrt{1 - \nu(2\eta - \nu\kappa^2)} \in (0, 1]$. Using inequalities (20) and (21), we deduce that

$$\begin{aligned}
\|x_{n+1} - x^*\| &= \|z_n - \sigma_n F z_n - x^*\| \\
&= \|(I - \sigma_n F)z_n - (I - \sigma_n F)x^* - \sigma_n F x^*\| \\
&\le \|(I - \sigma_n F)z_n - (I - \sigma_n F)x^*\| + \sigma_n \|F x^*\| \\
&\le \left(1 - \tfrac{\sigma_n}{\nu}\zeta\right) \|z_n - x^*\| + \sigma_n \|F x^*\| \\
&\le \left(1 - \tfrac{\sigma_n}{\nu}\zeta\right) \|x_n - x^*\| + \sigma_n M_1 + \sigma_n \|F x^*\| \\
&= \left(1 - \tfrac{\sigma_n}{\nu}\zeta\right) \|x_n - x^*\| + \tfrac{\sigma_n}{\nu}\zeta \cdot \tfrac{\nu(M_1 + \|F x^*\|)}{\zeta} \\
&\le \max\left\{\|x_n - x^*\|, \tfrac{\nu(M_1 + \|F x^*\|)}{\zeta}\right\}.
\end{aligned}$$

By induction, we conclude that

$$\|x_n - x^*\| \le \max\left\{\|x_{n_0} - x^*\|, \frac{\nu(M_1 + \|F x^*\|)}{\zeta}\right\} \text{ for all } n \ge n_0.$$

Thus, the sequence $\{x_n\}$ is bounded, and so are the sequences $\{q_n\}$, $\{u_n\}$, $\{v_n\}$, $\{w_n\}$, $\{y_n\}$, $\{z_n\}$, and $\{F z_n\}$. □

Lemma 3.7. *Let $\{q_n\}, \{w_n\}, \{x_n\}, \{y_n\}, \{z_n\}$ be the sequences generated by Algorithm 1. Suppose that $w_n - y_n \to 0$, $q_n - y_n \to 0$, and $q_n - u_n \to 0$. Then, $\omega_w(\{x_n\}) \subset \Omega$, where $\omega_w(\{x_n\}) = \{z \in \mathcal{H} : x_{n_k} \rightharpoonup z$ for some $\{x_{n_k}\} \subset \{x_n\}\}$.*

Proof. Let z be an arbitrary element of $\omega_w(\{x_n\})$. Then, there exists a subsequence $\{x_{n_k}\} \subset \{x_n\}$ such that $x_{n_k} \rightharpoonup z \in \mathcal{H}$. From Algorithm 1, we have

$$w_n - x_n = \alpha_n(x_n - x_{n-1}) + \beta_n(w_{n-1} - x_{n-1}) \quad \forall n \ge 1.$$

Thus, we obtain

$$\begin{aligned}
\|w_n - x_n\| &= \|\alpha_n(x_n - x_{n-1}) + \beta_n(w_{n-1} - x_{n-1})\| \\
&\le \alpha_n \|x_n - x_{n-1}\| + \beta_n \|w_{n-1} - x_{n-1}\| \\
&= \sigma_n \tfrac{\alpha_n}{\sigma_n} \|x_n - x_{n-1}\| + \sigma_n \tfrac{\beta_n}{\sigma_n} \|w_{n-1} - x_{n-1}\|.
\end{aligned}$$

Using Remark 3.2, we conclude that

$$\lim_{n\to\infty} \|w_n - x_n\| = 0.$$

Since $w_n - x_n \to 0$, we can infer that there exists a subsequence $\{w_{n_k}\} \subset \{w_n\}$ such that $w_{n_k} \rightharpoonup z \in \mathcal{H}$. Next, we will show that $z \in \Omega$. Indeed, since $y_n = P_C(w_n - \lambda_n A w_n)$, we have $\langle w_n - \lambda_n A w_n - y_n, y_n - y\rangle \ge 0$ $\forall y \in C$. This leads to

$$\frac{1}{\lambda_n}\langle w_n - y_n, y - y_n\rangle + \langle A w_n, y_n - w_n\rangle \le \langle A w_n, y - w_n\rangle \quad \forall y \in C. \tag{22}$$

By the Lipschitz continuity of A, the sequence $\{A w_{n_k}\}$ is bounded. Note that $\lambda_n \ge \min\{\lambda_1, \frac{\mu}{L}\}$. Therefore, from (22), we obtain $\liminf_{k\to\infty}\langle A w_{n_k}, y - w_{n_k}\rangle \ge 0$ $\forall y \in C$. Moreover, observe that

$$\langle A y_n, y - y_n\rangle = \langle A y_n - A w_n, y - w_n\rangle + \langle A w_n, y - w_n\rangle + \langle A y_n, w_n - y_n\rangle.$$

Since $w_n - y_n \to 0$, and given the L-Lipschitz continuity of A, we find that $A w_n - A y_n \to 0$, which, in conjunction with (22), yields

$$\liminf_{k\to\infty}\langle A y_{n_k}, y - y_{n_k}\rangle \ge 0 \quad \forall y \in C. \tag{23}$$

Now, consider a sequence $\{\delta_k\} \subset (0,1)$ such that $\delta_k \downarrow 0$ as $k \to \infty$. For all $k \geq 1$, let m_k denote the smallest positive integer satisfying

$$\langle Ay_{n_j}, y - y_{n_j}\rangle + \delta_k \geq 0 \quad \forall j \geq m_k. \tag{24}$$

Since $\{\delta_k\}$ is decreasing, it follows that $\{m_k\}$ is increasing. Again, based on the assumption regarding A, we have $\liminf_{k\to\infty} \|Ay_{n_k}\| \geq \|Az\|$. If $Az = 0$, then z is a solution, i.e., $z \in \mathrm{VI}(C, A)$. Now, suppose $Az \neq 0$. Then, we have

$$0 < \|Az\| \leq \liminf_{k\to\infty} \|Ay_{n_k}\|.$$

Without loss of generality, we may assume that $Ay_{n_k} \neq 0$ for all $k \geq 1$. Noticing that $\{y_{m_k}\} \subset \{y_{n_k}\}$ and $Ay_{n_k} \neq 0$ for all $k \geq 1$, we define

$$\upsilon_{m_k} = \frac{Ay_{m_k}}{\|Ay_{m_k}\|^2},$$

which gives us $\langle Ay_{m_k}, \upsilon_{m_k}\rangle = 1$ for all $k \geq 1$. From (24), we have

$$\langle Ay_{m_k}, y + \delta_k \upsilon_{m_k} - y_{m_k}\rangle \geq 0 \quad \forall k \geq 1.$$

Furthermore, due to the pseudomonotonicity of A, we have

$$\langle A(y + \delta_k \upsilon_{m_k}), y + \delta_k \upsilon_{m_k} - y_{m_k}\rangle \geq 0 \quad \forall k \geq 1.$$

This leads to the conclusion:

$$\langle Ay, y - y_{m_k}\rangle \geq \langle Ay - A(y + \delta_k \upsilon_{m_k}), y + \delta_k \upsilon_{m_k} - y_{m_k}\rangle - \delta_k \langle Ay, \upsilon_{m_k}\rangle \quad \forall k \geq 1. \tag{25}$$

We now claim that $\lim_{k\to\infty} \delta_k \upsilon_{m_k} = 0$. Indeed, since $x_{n_k} \rightharpoonup z$ and $x_n - y_n \to 0$ (due to $w_n - x_n \to 0$ and $w_n - y_n \to 0$), it follows that $y_{n_k} \rightharpoonup z$. As the sequence $\{y_n\} \subset C$, we can conclude that $z \in C$. Note that $\{y_{m_k}\} \subset \{y_{n_k}\}$ and $\delta_k \downarrow 0$ as $k \to \infty$. Therefore, we obtain

$$0 \leq \limsup_{k\to\infty} \|\delta_k \upsilon_{m_k}\| = \limsup_{k\to\infty} \frac{\delta_k}{\|Ay_{m_k}\|} \leq \frac{\limsup_{k\to\infty} \delta_k}{\liminf_{k\to\infty} \|Ay_{n_k}\|} = 0.$$

Hence, we conclude that $\delta_k \upsilon_{m_k} \to 0$.

Next, we claim that $z \in \Omega$. Indeed, letting $k \to \infty$, we deduce that the right-hand side of (25) tends to zero due to the uniform continuity of A, the boundedness of $\{w_{m_k}\}$ and $\{\upsilon_{m_k}\}$, and the limit $\lim_{k\to\infty} \delta_k \upsilon_{m_k} = 0$. Thus, we have

$$\langle Ay, y - z\rangle = \liminf_{k\to\infty} \langle Ay, y - y_{m_k}\rangle \geq 0 \quad \forall y \in C.$$

By Lemma 2.7, it follows that $z \in \mathrm{VI}(C, A)$. Furthermore, we assert that $\mathcal{T}_i z \in \mathrm{Fix}(S_i)$ for $i = 1, 2, \ldots, N$. In fact, noticing that $u_n = q_n - \tau_{n,i_n} \mathcal{T}_{i_n}^* (I - S_{i_n}) \mathcal{T}_{i_n} q_n$, and given that $0 < \epsilon \leq \tau_{n,i_n}$ and $q_n - u_n \to 0$, we obtain

$$\epsilon \|\mathcal{T}_{i_n}^* (I - S_{i_n}) \mathcal{T}_{i_n} q_n\| \leq \tau_{n,i_n} \|\mathcal{T}_{i_n}^* (I - S_{i_n}) \mathcal{T}_{i_n} q_n\| = \|q_n - u_n\| \to 0 \quad (n \to \infty),$$

which, together with the ξ_{i_n}-demimetricness of S_{i_n}, leads to

$$\begin{aligned} \tfrac{1-\xi_{i_n}}{2} \max_{i\in\{1,2,\ldots,N\}} \|(I - S_i)\mathcal{T}_i q_n\|^2 \quad &= \tfrac{1-\xi_{i_n}}{2} \|(I - S_{i_n})\mathcal{T}_{i_n} q_n\|^2 \\ &\leq \langle (I - S_{i_n})\mathcal{T}_{i_n} q_n, \mathcal{T}_{i_n}(q_n - x^*)\rangle \\ &\leq \|\mathcal{T}_{i_n}^* (I - S_{i_n})\mathcal{T}_{i_n} q_n\| \|q_n - x^*\| \to 0 \\ &\quad (n \to \infty). \end{aligned} \tag{26}$$

This ensures that $(I-S_i)\mathcal{T}_i q_n \to 0$ for $i=1,2,\dots,N$. Furthermore, from $w_n - y_n \to 0$, $q_n - y_n \to 0$ (due to the assumptions), and $w_n - x_n \to 0$, it follows that

$$\|q_n - x_n\| \le \|q_n - y_n\| + \|y_n - w_n\| + \|w_n - x_n\| \to 0 \quad (n \to \infty),$$

which, together with $x_{n_k} \rightharpoonup z$, leads to $q_{n_k} \rightharpoonup z$. Since each $\mathcal{T}_i : \mathcal{H} \to \mathcal{H}_i$ is a bounded linear operator, we know that $\mathcal{T}_i$ is weakly continuous from $\mathcal{H}$ to $\mathcal{H}_i$ for $i=1,2,\dots,N$. Hence, we obtain that $\mathcal{T}_i q_{n_k} \rightharpoonup \mathcal{T}_i z$ for $i=1,2,\dots,N$. By using the demiclosedness assumption of each $(I-S_i)$ at zero, we conclude from $(I-S_i)\mathcal{T}_i q_{n_k} \to 0$ that $\mathcal{T}_i z \in \mathrm{Fix}(S_i)$ for $i=1,2,\dots,N$. As a result, we have $z \in \bigcap_{i=1}^N \mathcal{T}_i^{-1}\mathrm{Fix}(S_i)$. Therefore, it follows that

$$z \in \mathrm{VI}(C,A) \cap \left(\bigcap_{i=1}^N \mathcal{T}_i^{-1}\mathrm{Fix}(S_i) \right) = \Omega.$$

This completes the proof. □

Theorem 3.8. *Let $\{x_n\}$ be the sequence generated by Algorithm 1. Then $\{x_n\}$ converges strongly to the unique solution $z^* \in \Omega$ of the BPVIP with the SCFPP constraint.*

Proof. First, by Lemma 3.6, we establish that $\{x_n\}$ is bounded. It is known that there exists a unique solution $x^* \in \Omega$ to the BPVIP with the SCFPP constraint, which means that the VIP (7) has the unique solution $x^* \in \Omega$. By the definition of w_n and applying the Cauchy-Schwarz inequality, we have

$$\begin{aligned}
\|w_n - x^*\|^2 &= \|x_n - x^*\|^2 + \|\alpha_n(x_n - x_{n-1}) + \beta_n(w_{n-1} - x_{n-1})\|^2 \\
&\quad + 2\langle x_n - x^*, \alpha_n(x_n - x_{n-1}) + \beta_n(w_{n-1} - x_{n-1})\rangle \\
&= \|x_n - x^*\|^2 + \alpha_n^2\|x_n - x_{n-1}\|^2 + \beta_n^2\|w_{n-1} - x_{n-1}\|^2 \\
&\quad + 2\alpha_n\beta_n\langle x_n - x_{n-1}, w_{n-1} - x_{n-1}\rangle \\
&\quad + 2\alpha_n\langle x_n - x^*, x_n - x_{n-1}\rangle \\
&\quad + 2\beta_n\langle x_n - x^*, w_{n-1} - x_{n-1}\rangle \\
&\le \|x_n - x^*\|^2 + \alpha_n^2\|x_n - x_{n-1}\|^2 + \beta_n^2\|w_{n-1} - x_{n-1}\|^2 \\
&\quad + 2\alpha_n\beta_n\|x_n - x_{n-1}\|\|w_{n-1} - x_{n-1}\| + 2\alpha_n\|x_n - x^*\|\|x_n - x_{n-1}\| \\
&\quad + 2\beta_n\|x_n - x^*\|\|w_{n-1} - x_{n-1}\| \\
&= \|x_n - x^*\|^2 + \beta_n\|w_{n-1} - x_{n-1}\|\left(2\|x_n - x^*\| + \beta_n\|w_{n-1} - x_{n-1}\|\right) \\
&\quad + \alpha_n\|x_n - x_{n-1}\|\left(\alpha_n\|x_n - x_{n-1}\| + 2\beta_n\|w_{n-1} - x_{n-1}\| + 2\|x_n - x^*\|\right).
\end{aligned}$$

Since $\{x_n\}, \{\alpha_n\}, \{\beta_n\}$, and $\{w_n\}$ are bounded, it follows that

$$\|w_n - x^*\|^2 \le \|x_n - x^*\|^2 + M_2\alpha_n\|x_n - x_{n-1}\| + M_3\beta_n\|w_{n-1} - x_{n-1}\|, \tag{27}$$

where $\sup_{n\ge1}\{\alpha_n\|x_n - x_{n-1}\| + 2\beta_n\|w_{n-1} - x_{n-1}\| + 2\|x_n - x^*\|\} \le M_2$ and $\sup_{n\ge1}\{2\|x_n - x^*\| + \beta_n\|w_{n-1} - x_{n-1}\|\} \le M_3$ for some constants $M_2 > 0$ and $M_3 > 0$.

To demonstrate the conclusion of the theorem, we will divide the remainder of the proof into several steps.

Step 1. We claim that

$$\begin{aligned}
&(1-\tfrac{\sigma_n}{\nu}\zeta)\Big[(1-\mu\tfrac{\lambda_n}{\lambda_{n+1}})\left(\|w_n-y_n\|^2+\|q_n-y_n\|^2\right)\\
&\quad+(1-\gamma_n)\epsilon^2\|\mathcal{T}_{i_n}^*(I-S_{i_n})\mathcal{T}_{i_n}q_n\|^2+\gamma_n\epsilon^2\|\mathcal{T}_{\imath_n}^*(I-S_{\imath_n})\mathcal{T}_{\imath_n}q_n\|^2\Big]\\
&\le\|x_n-x^*\|^2-\|x_{n+1}-x^*\|^2+\sigma_nM_5,\quad\forall n\ge n_0,
\end{aligned}$$

for some $M_5>0$. Indeed, noticing the inequality $\|x+y\|^2\le\|x\|^2+2\langle y,x+y\rangle$ for all $x,y\in\mathcal{H}$, we get

$$\begin{aligned}
\|x_{n+1}-x^*\|^2&=\|(I-\sigma_nF)z_n-x^*\|^2\\
&=\|(I-\sigma_nF)z_n-(I-\sigma_nF)x^*-\sigma_nFx^*\|^2\\
&\le\|(I-\sigma_nF)z_n-(I-\sigma_nF)x^*\|^2-2\sigma_n\langle Fx^*,x_{n+1}-x^*\rangle\\
&\le(1-\tfrac{\sigma_n}{\nu}\zeta)^2\|z_n-x^*\|^2+2\sigma_n\langle Fx^*,x^*-x_{n+1}\rangle\\
&\le(1-\tfrac{\sigma_n}{\nu}\zeta)\|z_n-x^*\|^2+2\sigma_n\|Fx^*\|\|x^*-x_{n+1}\|\\
&\le(1-\tfrac{\sigma_n}{\nu}\zeta)\|z_n-x^*\|^2+\sigma_nM_4\quad\forall n\ge n_0,
\end{aligned}\tag{28}$$

where $\sup_{n\ge1}\{2\|Fx^*\|\|x^*-x_{n+1}\|\}\le M_4$ for some $M_4>0$. Using Lemma 3.5, we deduce from (18), (27), and (28) that for all $n\ge n_0$,

$$\begin{aligned}
&\|x_{n+1}-x^*\|^2\le(1-\frac{\sigma_n}{\nu}\zeta)\|z_n-x^*\|^2+\sigma_nM_4\\
&\le(1-\frac{\sigma_n}{\nu}\zeta)[\|q_n-x^*\|^2-(1-\gamma_n)\tau_{n,i_n}\epsilon\|\mathcal{T}_{i_n}^*(I-S_{i_n})\mathcal{T}_{i_n}q_n\|^2\\
&\quad-\gamma_n\tau_{n,\imath_n}\epsilon\|\mathcal{T}_{\imath_n}^*(I-S_{\imath_n})\mathcal{T}_{\imath_n}q_n\|^2]+\sigma_nM_4\\
&\le(1-\frac{\sigma_n}{\nu}\zeta)\Big\{\|w_n-x^*\|^2-(1-\mu\frac{\lambda_n}{\lambda_{n+1}})\|w_n-y_n\|^2-(1-\mu\frac{\lambda_n}{\lambda_{n+1}})\|q_n-y_n\|^2\\
&\quad-(1-\gamma_n)\tau_{n,i_n}\epsilon\|\mathcal{T}_{i_n}^*(I-S_{i_n})\mathcal{T}_{i_n}q_n\|^2-\gamma_n\tau_{n,\imath_n}\epsilon\|\mathcal{T}_{\imath_n}^*(I-S_{\imath_n})\mathcal{T}_{\imath_n}q_n\|^2\Big\}+\sigma_nM_4\\
&\le(1-\frac{\sigma_n}{\nu}\zeta)\big\{\|x_n-x^*\|^2+M_2\alpha_n\|x_n-x_{n-1}\|+M_3\beta_n\|w_{n-1}-x_{n-1}\|\\
&\quad-(1-\mu\frac{\lambda_n}{\lambda_{n+1}})\|w_n-y_n\|^2-(1-\mu\frac{\lambda_n}{\lambda_{n+1}})\|q_n-y_n\|^2\\
&\quad-(1-\gamma_n)\tau_{n,i_n}\epsilon\|\mathcal{T}_{i_n}^*(I-S_{i_n})\mathcal{T}_{i_n}q_n\|^2-\gamma_n\tau_{n,\imath_n}\epsilon\|\mathcal{T}_{\imath_n}^*(I-S_{\imath_n})\mathcal{T}_{\imath_n}q_n\|^2\big\}+\sigma_nM_4\\
&\le\|x_n-x^*\|^2+M_2\alpha_n\|x_n-x_{n-1}\|+M_3\beta_n\|w_{n-1}-x_{n-1}\|\\
&\quad-(1-\frac{\sigma_n}{\nu}\zeta)\Big[(1-\mu\frac{\lambda_n}{\lambda_{n+1}})\|w_n-y_n\|^2+(1-\mu\frac{\lambda_n}{\lambda_{n+1}})\|q_n-y_n\|^2\\
&\quad+(1-\gamma_n)\epsilon^2\|\mathcal{T}_{i_n}^*(I-S_{i_n})\mathcal{T}_{i_n}q_n\|^2+\gamma_n\epsilon^2\|\mathcal{T}_{\imath_n}^*(I-S_{\imath_n})\mathcal{T}_{\imath_n}q_n\|^2\Big]+\sigma_nM_4\\
&=\|x_n-x^*\|^2-(1-\frac{\sigma_n}{\nu}\zeta)\Big[(1-\mu\frac{\lambda_n}{\lambda_{n+1}})\|w_n-y_n\|^2+(1-\mu\frac{\lambda_n}{\lambda_{n+1}})\|q_n-y_n\|^2\\
&\quad+(1-\gamma_n)\epsilon^2\|\mathcal{T}_{i_n}^*(I-S_{i_n})\mathcal{T}_{i_n}q_n\|^2+\gamma_n\epsilon^2\|\mathcal{T}_{\imath_n}^*(I-S_{\imath_n})\mathcal{T}_{\imath_n}q_n\|^2\Big]+\sigma_nM_5,
\end{aligned}\tag{29}$$

where $\sup_{n\ge1}\{M_2\frac{\alpha_n}{\sigma_n}\|x_n-x_{n-1}\|+M_3\frac{\beta_n}{\sigma_n}\|w_{n-1}-x_{n-1}\|+M_4\}\le M_5$ for some $M_5>0$.

Step 2. We claim that

$$\begin{aligned}\|x_{n+1}-x^*\|^2 \quad &\le (1-\tfrac{\sigma_n}{\nu}\zeta)\|x_n-x^*\|^2+\tfrac{\sigma_n}{\nu}\zeta\Big\{\tfrac{M_2\nu}{\zeta}\cdot\tfrac{\alpha_n}{\sigma_n}\|x_n-x_{n-1}\| \\ &\quad+\tfrac{M_3\nu}{\zeta}\cdot\tfrac{\beta_n}{\sigma_n}\|w_{n-1}-x_{n-1}\|+\tfrac{2\nu}{\zeta}\langle Fx^*,x^*-x_{n+1}\rangle\Big\} \quad \forall n\ge n_0.\end{aligned}$$

Indeed, from (20), (27), and (28), it follows that

$$\begin{aligned}\|x_{n+1}-x^*\|^2 &\le (1-\frac{\sigma_n}{\nu}\zeta)\|z_n-x^*\|^2+2\sigma_n\langle Fx^*,x^*-x_{n+1}\rangle \\ &\le (1-\frac{\sigma_n}{\nu}\zeta)\|w_n-x^*\|^2+2\sigma_n\langle Fx^*,x^*-x_{n+1}\rangle \\ &\le (1-\frac{\sigma_n}{\nu}\zeta)\left[\|x_n-x^*\|^2+M_2\alpha_n\|x_n-x_{n-1}\|+M_3\beta_n\|w_{n-1}-x_{n-1}\|\right] \\ &\quad+2\sigma_n\langle Fx^*,x^*-x_{n+1}\rangle \\ &\le (1-\frac{\sigma_n}{\nu}\zeta)\|x_n-x^*\|^2+M_2\alpha_n\|x_n-x_{n-1}\|+M_3\beta_n\|w_{n-1}-x_{n-1}\| \\ &\quad+2\sigma_n\langle Fx^*,x^*-x_{n+1}\rangle \\ &= (1-\frac{\sigma_n}{\nu}\zeta)\|x_n-x^*\|^2+\frac{\sigma_n}{\nu}\zeta\left\{\frac{M_2\nu}{\zeta}\cdot\frac{\alpha_n}{\sigma_n}\|x_n-x_{n-1}\|\right. \\ &\quad\left.+\frac{M_3\nu}{\zeta}\cdot\frac{\beta_n}{\sigma_n}\|w_{n-1}-x_{n-1}\|+\frac{2\nu}{\zeta}\langle Fx^*,x^*-x_{n+1}\rangle\right\} \quad \forall n\ge n_0.\end{aligned} \tag{30}$$

Step 3. We claim that $\{x_n\}$ converges strongly to the unique solution $x^*\in\Omega$ to the VIP (7). Indeed, putting $\mathbf{\Gamma}_n=\|x_n-x^*\|^2$, we show the convergence of $\{\mathbf{\Gamma}_n\}$ to zero by the following two cases. For the sake of simplicity, we replace all integers $n\ge n_0$ with $n\ge 1$ in the following proof.

Case 1. Suppose there exists an integer $m_0\ge 1$ such that $\{\mathbf{\Gamma}_n\}$ is nonincreasing. Then $\lim_{n\to\infty}\mathbf{\Gamma}_n=d<+\infty$ and $\lim_{n\to\infty}(\mathbf{\Gamma}_n-\mathbf{\Gamma}_{n+1})=0$. From (29), we obtain

$$\begin{aligned}&(1-\tfrac{\sigma_n}{\nu}\zeta)\Big[(1-\mu\tfrac{\lambda_n}{\lambda_{n+1}})(\|w_n-y_n\|^2+\|q_n-y_n\|^2) \\ &\quad+(1-\gamma_n)\epsilon^2\|\mathcal{T}_{i_n}^*(I-S_{i_n})\mathcal{T}_{i_n}q_n\|^2+\gamma_n\epsilon^2\|\mathcal{T}_{\imath_n}^*(I-S_{\imath_n})\mathcal{T}_{\imath_n}q_n\|^2\Big] \\ &\le\|x_n-x^*\|^2-\|x_{n+1}-x^*\|^2+\sigma_nM_5=\mathbf{\Gamma}_n-\mathbf{\Gamma}_{n+1}+\sigma_nM_5.\end{aligned}$$

Since $\lim_{n\to\infty}(1-\mu\frac{\lambda_n}{\lambda_{n+1}})=1-\mu>0$, $0<\liminf_{n\to\infty}\gamma_n\le\limsup_{n\to\infty}\gamma_n<1$, $\sigma_n\to 0$, and $\mathbf{\Gamma}_n-\mathbf{\Gamma}_{n+1}\to 0$, we conclude that

$$\lim_{n\to\infty}\|w_n-y_n\|=\lim_{n\to\infty}\|q_n-y_n\|=0,$$

and

$$\lim_{n\to\infty}\|\mathcal{T}_{i_n}^*(I-S_{i_n})\mathcal{T}_{i_n}q_n\|=\lim_{n\to\infty}\|\mathcal{T}_{\imath_n}^*(I-S_{\imath_n})\mathcal{T}_{\imath_n}q_n\|=0. \tag{31}$$

Noticing $u_n=q_n-\tau_{n,i_n}\mathcal{T}_{i_n}^*(I-S_{i_n})\mathcal{T}_{i_n}q_n$, $v_n=q_n-\tau_{n,\imath_n}\mathcal{T}_{\imath_n}^*(I-S_{\imath_n})\mathcal{T}_{\imath_n}q_n$, and the boundedness of $\{\tau_{n,i}\}$, from (31) we obtain that

$$\|q_n-u_n\|=\tau_{n,i_n}\|\mathcal{T}_{i_n}^*(I-S_{i_n})\mathcal{T}_{i_n}q_n\|\to 0 \quad (n\to\infty),$$

and

$$\|q_n - v_n\| = \tau_{n,\imath_n} \|\mathcal{T}_{\imath_n}^*(I - S_{\imath_n})\mathcal{T}_{\imath_n} q_n\| \to 0 \quad (n \to \infty). \tag{32}$$

So it follows that

$$\begin{aligned} \|q_n - z_n\| &\leq (1 - \gamma_n)\|q_n - u_n\| + \gamma_n \|q_n - v_n\| \\ &\leq \|q_n - u_n\| + \|q_n - v_n\| \to 0 \quad (n \to \infty), \end{aligned}$$

and

$$\|w_n - z_n\| \leq \|w_n - y_n\| + \|y_n - q_n\| + \|q_n - z_n\| \to 0 \quad (n \to \infty). \tag{33}$$

Since $w_n - x_n \to 0$, $\sigma_n \to 0$, and $\{Fz_n\}$ is bounded, from Algorithm 1 we obtain that

$$\begin{aligned} \|x_{n+1} - x_n\| &\leq \|x_{n+1} - z_n\| + \|z_n - w_n\| + \|w_n - x_n\| \\ &= \sigma_n \|Fz_n\| + \|z_n - w_n\| + \|w_n - x_n\| \to 0 \quad (n \to \infty). \end{aligned} \tag{34}$$

In addition, from the boundedness of $\{x_n\}$, it follows that there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty} \langle Fx^*, x^* - x_n \rangle = \lim_{k\to\infty} \langle Fx^*, x^* - x_{n_k} \rangle. \tag{35}$$

Since $\mathcal{H}$ is reflexive and $\{x_n\}$ is bounded, we may assume, without loss of generality, that $x_{n_k} \rightharpoonup \widetilde{x}$. Thus, from (35), one gets

$$\begin{aligned} \limsup_{n\to\infty} \langle Fx^*, x^* - x_n \rangle &= \lim_{k\to\infty} \langle Fx^*, x^* - x_{n_k} \rangle \\ &= \langle Fx^*, x^* - \widetilde{x} \rangle. \end{aligned} \tag{36}$$

Since $w_n - y_n \to 0$, $q_n - y_n \to 0$, and $q_n - u_n \to 0$, by Lemma 3.7, we deduce that $\widetilde{x} \in \omega_w(\{x_n\}) \subset \Omega$. Hence, from (7) and (36), one gets

$$\limsup_{n\to\infty} \langle Fx^*, x_n - x^* \rangle = \langle Fx^*, x^* - \widetilde{x} \rangle \leq 0. \tag{37}$$

This, together with (34), leads to

$$\begin{aligned} &\limsup_{n\to\infty} \langle Fx^*, x^* - x_{n+1} \rangle \\ &= \limsup_{n\to\infty} [\langle Fx^*, x_n - x_{n+1} \rangle + \langle Fx^*, x^* - x_n \rangle] \\ &\leq \limsup_{n\to\infty} [\|Fx^*\| \|x_n - x_{n+1}\| + \langle Fx^*, x^* - x_n \rangle] \leq 0. \end{aligned} \tag{38}$$

Note that $\left\{\frac{\sigma_n}{\nu}\zeta\right\} \subset [0,1]$, $\sum_{n=1}^{\infty} \frac{\sigma_n}{\nu}\zeta = \infty$, and

$$\limsup_{n\to\infty} \left\{ \frac{M_2 \nu}{\zeta} \cdot \frac{\alpha_n}{\sigma_n} \|x_n - x_{n-1}\| + \frac{M_3 \nu}{\zeta} \cdot \frac{\beta_n}{\sigma_n} \|w_{n-1} - x_{n-1}\| + \frac{2\nu}{\zeta} \langle Fx^*, x^* - x_{n+1} \rangle \right\} \leq 0.$$

Consequently, applying Lemma 2.8 to (30), one has $\lim_{n\to\infty} \|x_n - x^*\|^2 = 0$.

Case 2. Suppose that $\exists \{\mathbf{\Gamma}_{n_k}\} \subset \{\mathbf{\Gamma}_n\}$ such that $\mathbf{\Gamma}_{n_k} < \mathbf{\Gamma}_{n_k+1}$ for all $k \in \mathcal{N}$, where $\mathcal{N}$ is the set of all positive integers. Define the mapping $\phi : \mathcal{N} \to \mathcal{N}$ by

$$\phi(n) := \max\{k \leq n : \mathbf{\Gamma}_k < \mathbf{\Gamma}_{k+1}\}.$$

By Lemma 2.9, we have

$$\mathbf{\Gamma}_{\phi(n)} \leq \mathbf{\Gamma}_{\phi(n)+1} \quad \text{and} \quad \mathbf{\Gamma}_n \leq \mathbf{\Gamma}_{\phi(n)+1}.$$

From (29), we obtain

$$\begin{aligned}
&(1-\tfrac{\sigma_{\phi(n)}}{\nu}\zeta)\Big[(1-\mu\tfrac{\lambda_{\phi(n)}}{\lambda_{\phi(n)+1}})\left(\|w_{\phi(n)}-y_{\phi(n)}\|^2+\|q_{\phi(n)}-y_{\phi(n)}\|^2\right)\\
&\quad+(1-\gamma_{\phi(n)})\epsilon^2\|\mathcal{T}^*_{i_{\phi(n)}}(I-S_{i_{\phi(n)}})\mathcal{T}_{i_{\phi(n)}}q_{\phi(n)}\|^2\\
&\quad+\gamma_{\phi(n)}\epsilon^2\|\mathcal{T}^*_{\imath_{\phi(n)}}(I-S_{\imath_{\phi(n)}})\mathcal{T}_{\imath_{\phi(n)}}q_{\phi(n)}\|^2\Big]\\
&\le\|x_{\phi(n)}-x^*\|^2-\|x_{\phi(n)+1}-x^*\|^2+\sigma_{\phi(n)}M_5\\
&=\mathbf{\Gamma}_{\phi(n)}-\mathbf{\Gamma}_{\phi(n)+1}+\sigma_{\phi(n)}M_5,
\end{aligned}\tag{39}$$

which immediately implies that

$$\lim_{n\to\infty}\|w_{\phi(n)}-y_{\phi(n)}\|=\lim_{n\to\infty}\|q_{\phi(n)}-y_{\phi(n)}\|=0,$$

and

$$\lim_{n\to\infty}\|\mathcal{T}^*_{i_{\phi(n)}}(I-S_{i_{\phi(n)}})\mathcal{T}_{i_{\phi(n)}}q_{\phi(n)}\|=\lim_{n\to\infty}\|\mathcal{T}^*_{\imath_{\phi(n)}}(I-S_{\imath_{\phi(n)}})\mathcal{T}_{\imath_{\phi(n)}}q_{\phi(n)}\|=0.$$

Using the same inferences as in the proof of Case 1, we deduce that

$$\lim_{n\to\infty}\|q_{\phi(n)}-u_{\phi(n)}\|=\lim_{n\to\infty}\|q_{\phi(n)}-v_{\phi(n)}\|=0,$$

$$\lim_{n\to\infty}\|w_{\phi(n)}-z_{\phi(n)}\|=\lim_{n\to\infty}\|x_{\phi(n)+1}-x_{\phi(n)}\|=0,$$

and

$$\limsup_{n\to\infty}\langle Fx^*,x^*-x_{\phi(n)+1}\rangle\le 0.\tag{40}$$

On the other hand, from (30), we obtain

$$\begin{aligned}
\tfrac{\sigma_{\phi(n)}}{\nu}\zeta\mathbf{\Gamma}_{\phi(n)}\quad&\le\mathbf{\Gamma}_{\phi(n)}-\mathbf{\Gamma}_{\phi(n)+1}+\tfrac{\sigma_{\phi(n)}}{\nu}\zeta\Big[\tfrac{M_2\nu}{\zeta}\cdot\tfrac{\alpha_{\phi(n)}}{\sigma_{\phi(n)}}\cdot\|x_{\phi(n)}-x_{\phi(n)-1}\|\\
&\quad+\tfrac{M_3\nu}{\zeta}\cdot\tfrac{\beta_{\phi(n)}}{\sigma_{\phi(n)}}\cdot\|w_{\phi(n)-1}-x_{\phi(n)-1}\|+\tfrac{2\nu}{\zeta}\langle Fx^*,x^*-x_{\phi(n)+1}\rangle\Big].
\end{aligned}$$

This leads to

$$\begin{aligned}
\limsup_{n\to\infty}\mathbf{\Gamma}_{\phi(n)}\le\limsup_{n\to\infty}&\Big[\frac{M_2\nu}{\zeta}\cdot\frac{\alpha_{\phi(n)}}{\sigma_{\phi(n)}}\cdot\|x_{\phi(n)}-x_{\phi(n)-1}\|\\
&+\frac{M_3\nu}{\zeta}\cdot\frac{\beta_{\phi(n)}}{\sigma_{\phi(n)}}\cdot\|w_{\phi(n)-1}-x_{\phi(n)-1}\|+\frac{2\nu}{\zeta}\langle Fx^*,x^*-x_{\phi(n)+1}\rangle\Big]\le 0.
\end{aligned}$$

Thus, we have $\lim_{n\to\infty}\|x_{\phi(n)}-x^*\|^2=0$. Additionally, observe that

$$\begin{aligned}
&\|x_{\phi(n)+1}-x^*\|^2-\|x_{\phi(n)}-x^*\|^2\\
&=2\langle x_{\phi(n)+1}-x_{\phi(n)},x_{\phi(n)}-x^*\rangle+\|x_{\phi(n)+1}-x_{\phi(n)}\|^2\\
&\le 2\|x_{\phi(n)+1}-x_{\phi(n)}\|\|x_{\phi(n)}-x^*\|+\|x_{\phi(n)+1}-x_{\phi(n)}\|^2.
\end{aligned}\tag{41}$$

Given that $\mathbf{\Gamma}_n\le\mathbf{\Gamma}_{\phi(n)+1}$, we obtain

$$\begin{aligned}
&\|x_n-x^*\|^2\\
&\le\|x_{\phi(n)+1}-x^*\|^2\\
&\le\|x_{\phi(n)}-x^*\|^2+2\|x_{\phi(n)+1}-x_{\phi(n)}\|\|x_{\phi(n)}-x^*\|+\|x_{\phi(n)+1}-x_{\phi(n)}\|^2\to 0\\
&\quad(n\to\infty).
\end{aligned}$$

That is, $x_n\to x^*$ as $n\to\infty$. This completes the proof. □

Remark 3.9. In comparison to the corresponding results presented by Eslamian and Kamandi [17] and Thong et al. [28], our findings enhance and extend these results in the following ways:

(i) The problem of locating an element in $\bigcap_{i=1}^{N} \mathcal{T}_i^{-1}\mathrm{Fix}(S_i)$, as discussed in [17], is broadened to develop the BPVIP with the SCFPP constraint. Specifically, we consider the problem of finding

$$x^* \in \Omega = \mathrm{VI}(C, A) \cap \left(\bigcap_{i=1}^{N} \mathcal{T}_i^{-1}\mathrm{Fix}(S_i) \right)$$

such that

$$\langle Fx^*, p - x^* \rangle \geq 0 \ \forall p \in \Omega,$$

where A is a pseudomonotone and Lipschitzian mapping. The inertial method with a correction term and a self-adaptive step size strategy introduced in [17] is extended to develop our modified subgradient extragradient rule for addressing the BPVIP with the SCFPP constraint. This development is grounded in the subgradient extragradient method, which incorporates adaptive step sizes, an adaptive inertial technique, and a hybrid deepest-descent method.

(ii) The problem of finding an element of $\mathrm{VI}(F, \mathrm{VI}(C, A))$, as described in [28], is also extended to develop the BPVIP with the SCFPP constraint. Specifically, we aim to find

$$x^* \in \Omega = \mathrm{VI}(C, A) \cap \left(\bigcap_{i=1}^{N} \mathcal{T}_i^{-1}\mathrm{Fix}(S_i) \right)$$

such that

$$\langle Fx^*, p - x^* \rangle \geq 0 \ \forall p \in \Omega.$$

The subgradient extragradient algorithm for solving the BPVIP, as presented in [28], is extended to formulate our modified subgradient extragradient rule for resolving the BPVIP with the SCFPP constraint, which builds upon the subgradient extragradient method with adaptive step sizes, an adaptive inertial technique, and a hybrid deepest-descent method.

Remark 3.10. In particular, when $N = 1$, the above BPVIP (7) with the SCFPP constraint reduces to the BSPVIP:

$$\text{Find } x^* \in \Omega \text{ such that } \langle Fx^*, y - x^* \rangle \geq 0 \quad \forall y \in \Omega, \tag{42}$$

where

$$\Omega = \mathrm{VI}(C, A) \cap \mathcal{T}_1^{-1}(\mathrm{Fix}(S_1)) = \{z \in \mathrm{VI}(C, A) : \mathcal{T}_1 z \in \mathrm{Fix}(S_1)\}.$$

In this case, Algorithm 1 is rewritten as follows.

Theorem 3.11. *Let $\{x_n\}$ be the sequence generated by Algorithm 2. Then, the sequence $\{x_n\}$ converges strongly to the unique solution $z^* \in \Omega$ of the BSPVIP* (42).

Algorithm 2

1: **Initialize:** Set $\lambda_1 > 0$, $\epsilon > 0$, $\tau_1 \in [0, \infty)$, $\mu \in (0, 1)$, $\alpha, \beta \in [0, 1]$, and $x_1, x_0, w_0 \in \mathcal{H}$ arbitrarily.

2: **Step 1:** Given the current iterates x_{n-1}, x_n $(n \geq 1)$, compute

$$w_n = x_n + \alpha_n(x_n - x_{n-1}) + \beta_n(w_{n-1} - x_{n-1}),$$

where $\alpha_n \in [0, \overline{\alpha}_n]$ and $\beta_n \in [0, \overline{\beta}_n]$, with

$$\overline{\alpha}_n = \begin{cases} \min\left\{\alpha, \frac{\epsilon_n}{\|x_n - x_{n-1}\|}\right\} & \text{if } x_n \neq x_{n-1}, \\ \alpha & \text{otherwise,} \end{cases}$$

and

$$\overline{\beta}_n = \begin{cases} \min\left\{\beta, \frac{\epsilon_n}{\|w_{n-1} - x_{n-1}\|}\right\} & \text{if } w_{n-1} \neq x_{n-1}, \\ \beta & \text{otherwise.} \end{cases}$$

3: **Step 2:** Calculate $y_n = P_C(w_n - \lambda_n A w_n)$ and $q_n = P_{C_n}(w_n - \lambda_n A y_n)$, where the half-space

$$C_n = \{y \in \mathcal{H} : \langle w_n - \lambda_n A w_n - y_n, y_n - y\rangle \geq 0\}.$$

4: **Step 3:** Compute $z_n = q_n - \tau_{n,1}\mathcal{T}_1^*(I - S_1)\mathcal{T}_1 q_n$, where $\tau_{n,1}$ is chosen as the bounded sequence satisfying

$$0 < \epsilon \leq \tau_{n,1} \leq \frac{(1-\xi_1)\|(I - S_1)\mathcal{T}_1 q_n\|^2}{\|\mathcal{T}_1^*(I - S_1)\mathcal{T}_1 q_n\|^2} - \epsilon \quad \text{if } (I - S_1)\mathcal{T}_1 q_n \neq 0,$$

otherwise set $\tau_{n,1} = \tau_1$.

5: **Step 4:** Calculate $x_{n+1} = (I - \sigma_n F)z_n$, and update

$$\lambda_{n+1} = \begin{cases} \min\left\{\mu \frac{\|w_n - y_n\|^2 + \|q_n - y_n\|^2}{2\langle A w_n - A y_n, q_n - y_n\rangle}, \lambda_n\right\} & \text{if } \langle A w_n - A y_n, q_n - y_n\rangle > 0, \\ \lambda_n & \text{otherwise.} \end{cases}$$

6: Set $n := n + 1$ and return to Step 1.

4. Numerical illustrations

This section presents a series of numerical experiments aimed at demonstrating the effectiveness of the proposed methods. The primary objective of these experiments is to offer insights into the selection of various parameters within the proposed algorithms and to investigate the influence of these parameters on the algorithm's performance.

Throughout these numerical experiments, the projection onto a convex set is computed using the built-in MATLAB function `fmincon`. In the graphs provided, the error term (E_n) denotes the norm value $||x_{n+1} - x_n||$. The total number of iterations and CPU time (measured in seconds) required to satisfy the termination criterion $||x_{n+1} - x_n|| < 10^{-6}$ are represented by n and t, respectively. All MATLAB code was executed using MATLAB R2012b on a Lenovo Core(TM) i9-13900H 2.60 GHz laptop with 32.0 GB (31.7 GB usable) RAM.

Example 4.1. In the following, we present an illustrative example to demonstrate the utility and effectiveness of the proposed rule. Let us set the parameters as follows: $\lambda_1 = \epsilon = \tau_i = \frac{1}{5}$, $\mu = \alpha = \beta = \frac{1}{3}$, $\gamma_n = \frac{2}{3}$, $\varepsilon_n = \frac{1}{3(n+1)^2}$, and $\sigma_n = \frac{1}{3(n+1)}$.

We construct an example where the set

$$\Omega = \mathrm{VI}(C, A) \cap \left(\bigcap_{i=1}^{N} \mathcal{T}_i^{-1} \mathrm{Fix}(S_i) \right) \neq \emptyset,$$

with the following conditions: $A : \mathcal{H} \to \mathcal{H}$ is a pseudomonotone and Lipschitz continuous mapping, $\mathcal{T}_i : \mathcal{H} \to \mathcal{H}_i$ is a bounded linear operator, and $S_i : \mathcal{H}_i \to \mathcal{H}_i$ is a ξ_i-demimetric mapping for $i = 1, 2, \ldots, N$. We set $\mathcal{H}_1 = \mathcal{H} = \mathbb{R}$ and use the inner product $\langle a, b \rangle = ab$ with the induced norm $\| \cdot \| = | \cdot |$. Furthermore, let the set $C = [-2, 2]$. The initial points x_1, x_0, and w_0 are chosen arbitrarily in C. Define the mapping $F(x) = \frac{1}{2}x - \frac{1}{4}\sin(x)$ for all $x \in \mathcal{H}$.

We define the mapping $S_1 : \mathcal{H}_1 \to \mathcal{H}_1$ as: $S_1(x) = \frac{3}{5}x + \frac{1}{5}\sin(x), \quad \forall x \in \mathcal{H}_1$. Assume that $\mathcal{T}_1(x) = x$ for all $x \in \mathcal{H}$. Then, $\mathcal{T}_1$ is a bounded linear operator from $\mathcal{H}$ into $\mathcal{H}_1$. It is evident that S_1 is a ξ_1-demicontractive mapping with $\xi_1 = \frac{1}{5}$, and that: $\mathrm{Fix}(S_1) = \{0\}$. In fact, S_1 is a ξ_1-strictly pseudocontractive mapping with $\xi_1 = \frac{1}{5}$ because:

$$\begin{aligned} \|S_1(x) - S_1(y)\|^2 &= \left\| \frac{3}{5}(x - y) + \frac{1}{5}(\sin(x) - \sin(y)) \right\|^2 \\ &\leq \|x - y\|^2 + \frac{1}{5}\|(I - S)x - (I - S)y\|^2. \end{aligned}$$

Let $A : \mathcal{H} \to \mathcal{H}$ be defined as:

$$A(x) := \frac{1}{1 + |\sin(x)|} - \frac{1}{1 + |x|}.$$

We now demonstrate that A is pseudomonotone and Lipschitz continuous. For any $v, w \in \mathcal{H}$, we have:

$$\begin{aligned} \|A(v) - A(w)\| &\leq \left| \frac{\|w\| - \|v\|}{(1 + \|w\|)(1 + \|v\|)} \right| + \left| \frac{\|\sin(w)\| - \|\sin(v)\|}{(1 + \|\sin(w)\|)(1 + \|\sin(v)\|)} \right| \\ &\leq \frac{\|v - w\|}{(1 + \|v\|)(1 + \|w\|)} + \frac{\|\sin(v) - \sin(w)\|}{(1 + \|\sin(v)\|)(1 + \|\sin(w)\|)} \\ &\leq \|v - w\| + \|\sin(v) - \sin(w)\| \leq 2\|v - w\|. \end{aligned}$$

This establishes that A is L-Lipschitz continuous with $L = 2$. Next, we claim that A is pseudomonotone. For any $v, w \in \mathcal{H}$, it is readily established that:

$$\begin{aligned} \langle A(v), w - v \rangle &= \left(\frac{1}{1 + |\sin(v)|} - \frac{1}{1 + |v|} \right)(w - v) \geq 0, \\ &\Rightarrow \langle A(w), w - v \rangle = \left(\frac{1}{1 + |\sin(w)|} - \frac{1}{1 + |w|} \right)(w - v) \geq 0. \end{aligned}$$

We assert that $\mathrm{VI}(C, A) = \{0\}$. In fact, it is evident that $0 \in \mathrm{VI}(C, A)$. Suppose there exists a $v \in \mathrm{VI}(C, A)$ such that $v \neq 0$. Then, we have $\langle Av, w - v \rangle \geq 0 \; \forall w \in C$, and

$$Av = \frac{1}{1+|\sin v|} - \frac{1}{1+|v|} > 0.$$

This implies that $w - v \geq 0 \ \forall w \in C$, leading to a contradiction. Consequently, we have $\Omega := \mathrm{VI}(C, A) \cap \mathcal{T}_1^{-1}\mathrm{Fix}(S_1) = \{0\} \neq \emptyset$. Furthermore, we observe that

$$0 < \frac{1}{5} = \epsilon \leq \tau_{n,1} \leq \frac{(1-\xi_1)\|(I-S_1)\mathcal{T}_1 q_n\|^2}{\|\mathcal{T}_1^*(I-S_1)\mathcal{T}_1 q_n\|^2} - \epsilon = \frac{3}{5}$$

if $(I - S_1)\mathcal{T}_1 q_n \neq 0$, and $\tau_{n,1} = \tau_1 = \frac{1}{5}$ otherwise. Thus, we set $\tau_{n,1} = \frac{1}{5} \ \forall n \geq 1$. It is also clear that $\mathcal{T}_1^*(I - S_1)\mathcal{T}_1 q_n = (I - S_1)q_n$ and $(I - \sigma_n F)z_n = \left(I - \frac{1}{3(n+1)}F\right) z_n$. In this case, we define

$$\alpha_n = \begin{cases} \min\left\{\frac{\frac{1}{3(n+1)^2}}{\|x_n - x_{n-1}\|}, \frac{1}{3}\right\} & \text{if } x_n \neq x_{n-1}, \\ \frac{1}{3} & \text{otherwise,} \end{cases}$$

and

$$\beta_n = \begin{cases} \min\left\{\frac{\frac{1}{3(n+1)^2}}{\|w_{n-1} - x_{n-1}\|}, \frac{1}{3}\right\} & \text{if } w_{n-1} \neq x_{n-1}, \\ \frac{1}{3} & \text{otherwise.} \end{cases}$$

We can then rewrite Algorithm 2 as follows:

$$\begin{cases} w_n = x_n + \alpha_n(x_n - x_{n-1}) + \beta_n(w_{n-1} - x_{n-1}), \\ y_n = P_C(w_n - \lambda_n A w_n), \\ q_n = P_{C_n}(w_n - \lambda_n A y_n), \\ z_n = q_n - \frac{1}{5}(q_n - S_1 q_n), \\ x_{n+1} = \left(I - \frac{1}{3(n+1)}F\right) z_n \quad \forall n \geq 1, \end{cases}$$

where for each $n \geq 1$, C_n and λ_n are chosen as in Algorithm 2. Therefore, by using Theorem 3.11, we conclude that $\{x_n\}$ converges to $0 \in \Omega$.

In this section, we present a series of numerical experiments based on Example 4.1 to evaluate the performance of Algorithm 2. The evaluation is carried out in terms of the *execution time* (t, measured in seconds) and the *number of iterations* (n) required for the algorithm to converge. Specifically, we aim to investigate how the performance of Algorithm 2 is affected by varying the following parameters:

(i) *Experiment 1:* Different initial values of x_0, x_1, and w_0, where these values belong to the space $\mathcal{H}$.
(ii) *Experiment 2:* Different values of the parameter $\alpha \in [0, 1]$.
(iii) *Experiment 3:* Different values of the parameter $\beta \in [0, 1]$.
(iv) *Experiment 4:* Different values of the parameter $\lambda_1 > 0$.
(v) *Experiment 5:* Different values of the parameter $\tau \in [0, +\infty)$.
(vi) *Experiment 6:* Different values of the parameter $\mu \in (0, 1)$.
(vii) *Experiment 7:* Different values of the parameter $\sigma_n \in (0, 1)$.

In all the numerical experiments, the stopping criterion is set as

$$E_n = \|x_{n+1} - x_n\| \leq 10^{-6}.$$

Unless otherwise specified in individual experiments, the default values for the parameters and sequences are as follows:

$$w_0 = 0,\ x_0 = 1,\ x_1 = 2,\ \alpha = \frac{1}{3}, \beta = \frac{1}{3},\ \epsilon_n = \frac{1}{3(n+1)^2},$$

$$\lambda_1 = \frac{1}{5},\ \tau = \frac{1}{5},\ \sigma_n = \frac{1}{3(n+1)},\ \mu = \frac{1}{3}.$$

Experiment 1. (Impact of initial points w_0, x_0, and x_1): This experiment investigates how the initial points w_0, x_0, and x_1 influence the performance of Algorithm 2. The results for ten different triplets are presented in Figure 1 and Table 1, analyzing their effects on the number of iterations (n) and CPU time (t).

(i) The triplet $(-0.79, -1.92, -0.54)$ yields the lowest iteration count ($n = 180$) and the shortest CPU time ($t = 1.445890$), while the triplet $(-0.54, 0.79, -1.92)$ results in the highest iteration count ($n = 241$) and CPU time ($t = 1.858170$), indicating slower convergence.

(ii) Triplets comprising all negative values tend to converge more rapidly, whereas those with mixed-sign or varied values exhibit slower convergence.

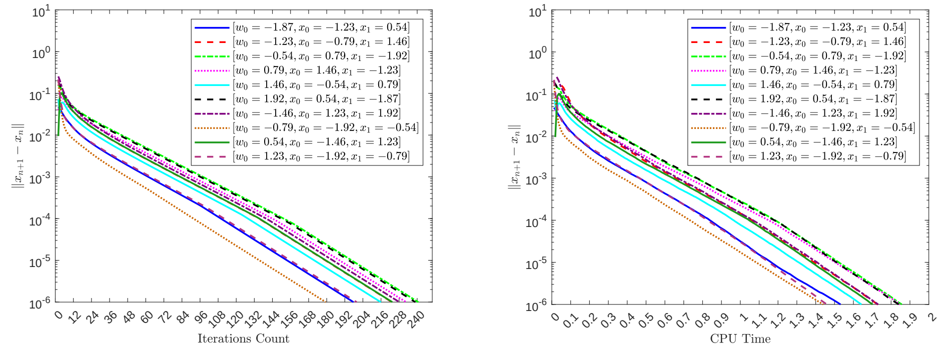
(iii) Triplets with similar or closely spaced values (e.g., $(-1.87, -1.23, 0.54)$) achieve quicker convergence compared to those with dispersed values.

(iv) The most efficient triplets are $(-0.79, -1.92, -0.54)$, $(1.23, -1.92, -0.79)$, and $(-1.87, -1.23, 0.54)$, which demonstrate lower iteration counts and CPU times.

(v) Overall, selecting initial values that are closer together or all negative significantly enhances the convergence speed.

TABLE 1. The numerical date link with Figure 1.

w_0	x_0	x_1	*Algorithm 2.*	
			(n)	(t)
-1.87	-1.23	0.54	198	1.533936
-1.23	-0.79	1.46	224	1.735602
-0.54	0.79	-1.92	241	1.858170
0.79	1.46	-1.23	233	1.849935
1.46	-0.54	0.79	216	1.643528
1.92	0.54	-1.87	239	1.842925
-1.46	1.23	1.92	229	1.738488
-0.79	-1.92	-0.54	180	1.445890
0.54	-1.46	1.23	224	1.704911
1.23	-1.92	-0.79	200	1.460150



(A) $\|x_{n+1} - x_n\|$ vs. Iterations Count (B) $\|x_{n+1} - x_n\|$ vs. CPU Time

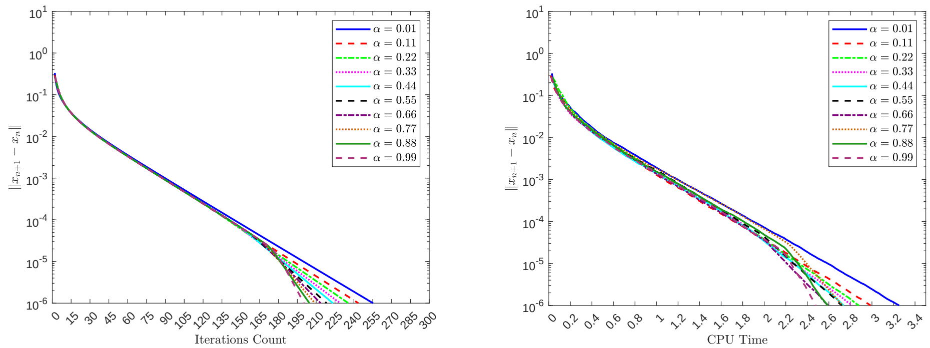
FIGURE 1. The error term graph for Algorithm 2, plotted against the iteration number (n) and elapsed time (t) for ten different x_1-values.

Experiment 2. (Impact of parameter α). This experiment assesses the effect of varying α on the performance of Algorithm 2, focusing on the number of iterations (n) and CPU time (t). The results are summarized in Figure 2 and Table 2.

(i) As α increases from 0.01 to 0.99, both the iteration count n and CPU time t decrease, indicating faster convergence. The number of iterations reduces from 256 at $\alpha = 0.01$ to 201 at $\alpha = 0.99$.
(ii) Similarly, CPU time decreases from 3.26 seconds at $\alpha = 0.01$ to 2.49 seconds at $\alpha = 0.99$, demonstrating improved efficiency.
(iii) Values of α in the range of 0.66 to 0.99 offer the best performance, with $\alpha = 0.99$ achieving the lowest iteration count and CPU time.
(iv) Increasing α enhances the algorithm's performance, especially within the range 0.66 to 0.99, with $\alpha = 0.99$ proving to be the most efficient.

TABLE 2. The numerical data linked with Figure 2.

Algorithm 2	α									
	0.01	0.11	0.22	0.33	0.44	0.55	0.66	0.77	0.88	0.99
(n)	256	244	236	230	224	219	214	209	205	201
(t)	3.258826	2.992441	2.883373	2.822021	2.744974	2.727800	2.599790	2.729572	2.591730	2.485404



(A) $\|x_{n+1} - x_n\|$ vs. Iterations Count (B) $\|x_{n+1} - x_n\|$ vs. CPU Time

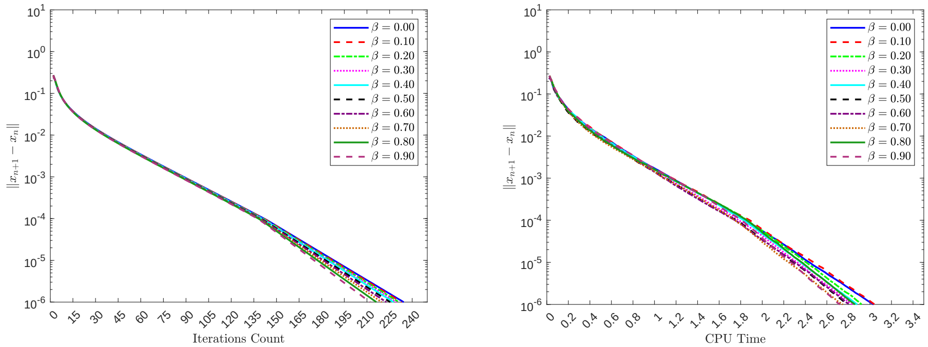
FIGURE 2. The error term graph for Algorithm 2, plotted against the iteration number (n) and elapsed time (t) for ten different α-values.

Experiment 3. (Impact of parameter β). This experiment investigates the influence of varying β on the performance of Algorithm 2, with particular attention to the number of iterations (n) and CPU time (t). The results are presented in Figure 3 and Table 3.

(i) As β increases from 0.00 to 0.90, the number of iterations decreases, with the maximum iteration count at $\beta = 0.00$ (235 iterations) and the minimum at $\beta = 0.90$ (213 iterations). This suggests faster convergence for higher values of β.
(ii) Although CPU time exhibits slight fluctuations, it generally declines as β increases. The lowest CPU time occurs at $\beta = 0.90$ (2.73 seconds), while the highest is observed at $\beta = 0.00$ (3.04 seconds), indicating improved efficiency for larger values of β.
(iii) The values $\beta = 0.70$ and $\beta = 0.80$ provide a favorable balance, producing relatively low iteration counts and reasonable CPU times.
(iv) In conclusion, increasing β enhances the algorithm's convergence, with $\beta = 0.70$ and $\beta = 0.80$ emerging as particularly effective choices for subsequent experiments.

TABLE 3. The numerical data linked with Figure 3.

Algorithm 2	β									
	0.00	0.10	0.20	0.30	0.40	0.50	0.60	0.70	0.80	0.90
(n)	235	234	232	230	228	226	223	220	217	213
(t)	3.042121	3.054295	2.929429	2.836859	2.888161	2.801054	2.808738	2.751537	2.869400	2.734657



(A) $\|x_{n+1} - x_n\|$ vs. Iterations Count (B) $\|x_{n+1} - x_n\|$ vs. CPU Time

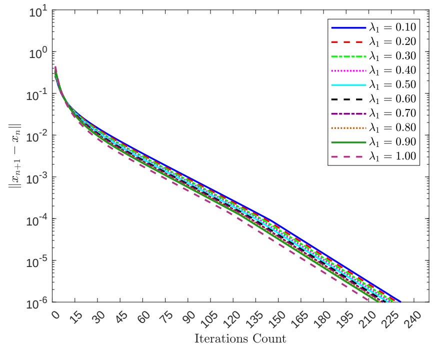
FIGURE 3. The error term graph for Algorithm 2, plotted against the iteration number (n) and elapsed time (t) for ten different β-values.

Experiment 4. (Impact of parameter λ_1). This experiment investigates the influence of varying λ_1 on the performance of Algorithm 2, with a focus on the number of iterations (n) and CPU time (t). The results are presented in Figure 4 and Table 4. Key observations include:

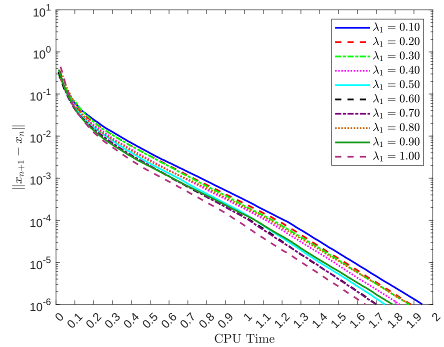
(i) As λ_1 increases, the number of iterations decreases from 232 at $\lambda_1 = 0.10$ to 212 at $\lambda_1 = 1.00$, indicating faster convergence with higher values of λ_1.
(ii) CPU time also decreases with increasing λ_1, ranging from approximately 1.949 seconds at $\lambda_1 = 0.10$ to 1.646 seconds at $\lambda_1 = 1.00$, reflecting a reduction in computational effort.
(iii) Among the tested values, $\lambda_1 = 1.00$ yields the lowest number of iterations and CPU time, suggesting that it is the most efficient choice. Values around $\lambda_1 = 0.50$ and $\lambda_1 = 0.60$ also perform well, providing a balance between speed and stability.
(iv) In summary, increasing λ_1 significantly enhances both the number of iterations and CPU time required for convergence. Higher values of λ_1 are recommended for achieving faster results, and testing values above 1.00 may provide additional insights into the convergence behavior.

TABLE 4. The numerical data linked with Figure 4.

Algorithm 2	λ_1									
	0.10	0.20	0.30	0.40	0.50	0.60	0.70	0.80	0.90	1.00
(n)	232	229	227	225	223	222	220	218	217	212
(t)	1.949502	1.883964	1.876402	1.826792	1.747356	1.709095	1.702436	1.879257	1.789440	1.646142



(A) $\|x_{n+1} - x_n\|$ vs. Iterations Count



(B) $\|x_{n+1} - x_n\|$ vs. CPU Time

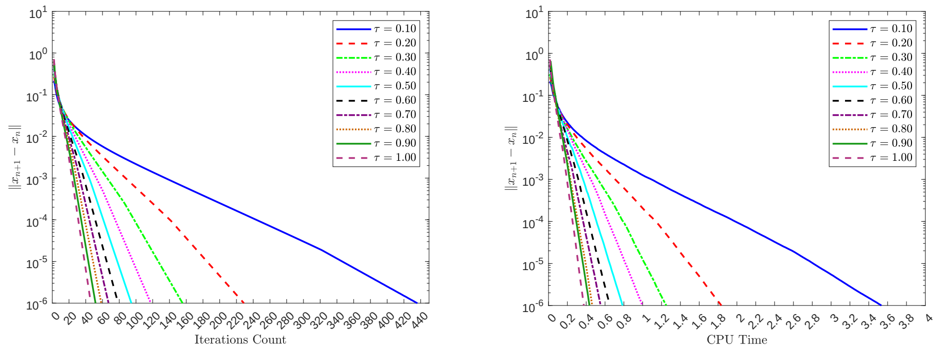
FIGURE 4. The error term graph for Algorithm 2, plotted against the iteration number (n) and elapsed time (t) for ten different λ_1-values.

Experiment 5. (Impact of parameter τ). This experiment evaluates the impact of varying τ on the performance of Algorithm 2, specifically concerning the number of iterations (n) and CPU time (t). We selected ten values of τ evenly distributed over the interval $[0, +\infty)$, with an initial focus on the practical range of $[0, 1]$. The results, as presented in Figure 5 and Table 5, indicate the following:

(i) As τ increases, the number of iterations significantly decreases. For $\tau = 0.10$, 437 iterations are required, whereas $\tau = 1.00$ reduces this count to 47, demonstrating faster convergence with higher values of τ.
(ii) CPU time also decreases with increasing τ, dropping from 3.540912 seconds at $\tau = 0.10$ to 0.382580 seconds at $\tau = 1.00$, which reflects improved computational efficiency.
(iii) Larger values of τ (specifically between 0.70 and 1.00) are more effective in minimizing both iterations and CPU time, with $\tau = 0.90$ and $\tau = 1.00$ providing optimal performance.
(iv) Overall, higher values of τ enhance both convergence speed and efficiency, with values approaching 1.00 proving to be the most effective.

TABLE 5. The numerical data linked with Figure 5.

Algorithm 2	τ									
	0.10	0.20	0.30	0.40	0.50	0.60	0.70	0.80	0.90	1.00
(n)	437	229	156	118	95	79	68	59	52	47
(t)	3.540912	1.835303	1.253118	1.000242	0.790663	0.654590	0.558803	0.467551	0.437773	0.382580



(A) $\|x_{n+1} - x_n\|$ vs. Iterations Count (B) $\|x_{n+1} - x_n\|$ vs. CPU Time

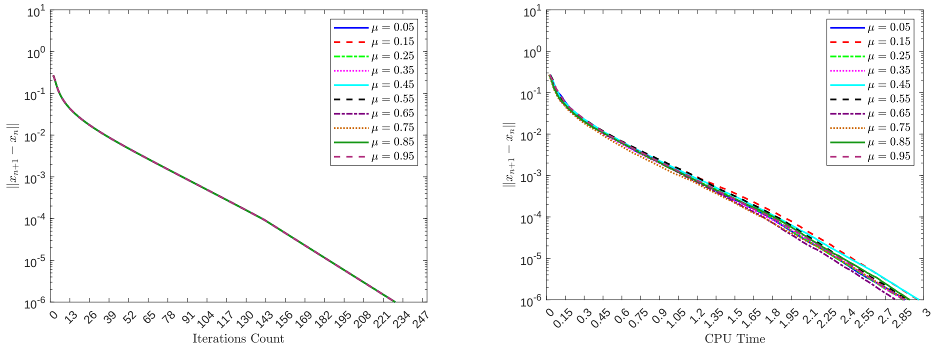
FIGURE 5. The error term graph for Algorithm 2, plotted against the iteration number (n) and elapsed time (t) for ten different τ-values.

Experiment 6. (Impact of parameter μ). This experiment investigates the effect of varying μ on the performance of Algorithm 2 in terms of the number of iterations (n) and CPU time (t). Ten values of μ were selected from the interval $(0, 1)$. As illustrated in Figure 6 and summarized in Table 6:

(i) The number of iterations remains constant at 229 for μ values between 0.05 and 0.95, suggesting that μ has no significant impact on the convergence behavior.
(ii) The CPU time exhibits slight variations, with the minimum at $\mu = 0.65$ (2.786849) and the maximum at $\mu = 0.15$ and $\mu = 0.45$ (2.962838, 2.967505).

TABLE 6. The numerical data linked with Figure 6.

Algorithm 2	μ									
	0.05	0.15	0.25	0.35	0.45	0.55	0.65	0.75	0.85	0.95
(n)	229	229	229	229	229	229	229	229	229	229
(t)	2.855605	2.962838	2.839915	2.849222	2.967505	2.874852	2.786849	2.862010	2.893535	2.846614



(A) $\|x_{n+1} - x_n\|$ vs. Iterations Count (B) $\|x_{n+1} - x_n\|$ vs. CPU Time

FIGURE 6. The error term graph for Algorithm 2, plotted against the iteration number (n) and elapsed time (t) for ten different μ-values.

Experiment 7. (Impact of parameter σ_n). This experiment examines the effect of varying σ_n on the performance of Algorithm 2, specifically in terms of the number of iterations (n) and CPU time (t). Ten values of σ_n were selected from the interval $(0, 1)$. Based on the numerical data presented in Figure 7 and Table 7, the following observations can be made regarding the influence of different σ_n values on the iteration count (n) and CPU time (t):

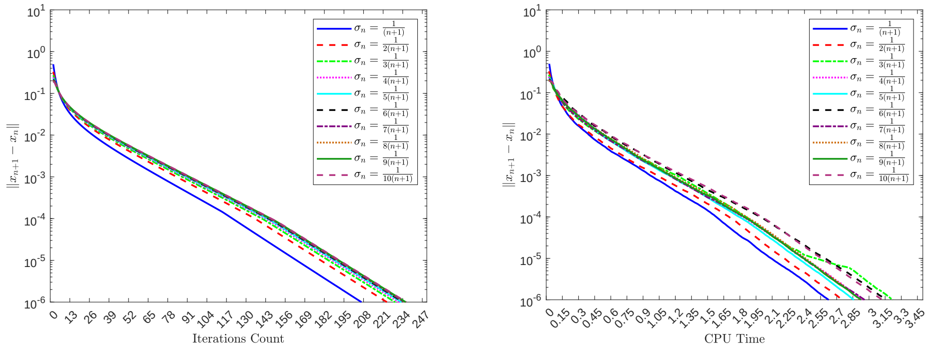
(i) The number of iterations increases as σ_n decreases (e.g., from $\sigma_n = \frac{1}{(n+1)}$ to $\sigma_n = \frac{1}{10(n+1)}$). Specifically, the iteration count ranges from 207 for $\sigma_n = \frac{1}{(n+1)}$ to 237 for $\sigma_n = \frac{1}{10(n+1)}$, indicating that smaller values of σ_n require more iterations for convergence.

(ii) CPU time also varies with different σ_n values. The shortest CPU time, 2.629895 seconds, is observed for $\sigma_n = \frac{1}{(n+1)}$, while the highest, 3.221875 seconds, occurs for $\sigma_n = \frac{1}{3(n+1)}$. Although there are fluctuations, the overall trend suggests that smaller values of σ_n generally lead to higher CPU times.

(iii) The selection of σ_n significantly affects the algorithm's performance. Larger values of σ_n, such as $\frac{1}{(n+1)}$, seem to be more favorable, resulting in faster convergence (fewer iterations) and reduced CPU time. Further reducing σ_n does not enhance performance and may slightly diminish the algorithm's efficiency.

TABLE 7. The numerical data linked with Figure 7.

Algorithm 2	σ_n									
	$\frac{1}{(n+1)}$	$\frac{1}{2(n+1)}$	$\frac{1}{3(n+1)}$	$\frac{1}{4(n+1)}$	$\frac{1}{5(n+1)}$	$\frac{1}{6(n+1)}$	$\frac{1}{7(n+1)}$	$\frac{1}{8(n+1)}$	$\frac{1}{9(n+1)}$	$\frac{1}{10(n+1)}$
(n)	207	224	229	232	234	235	236	236	237	237
(t)	2.629895	2.755611	3.221875	2.945376	2.868887	3.165253	2.968663	2.922523	2.934973	3.132077



(A) $\|x_{n+1} - x_n\|$ vs. Iterations Count (B) $\|x_{n+1} - x_n\|$ vs. CPU Time

FIGURE 7. The error term graph for Algorithm 2, plotted against the iteration number (n) and elapsed time (t) for ten different σ_n-values.

5. Conclusions

By incorporating an inertial method with a correction term and a self-adaptive stepsize strategy from [17], we developed a modified subgradient extragradient scheme to solve the BPVIP constrained by a SCFPP involving demimetric mappings in real Hilbert spaces. The BPVIP includes a strongly monotone operator in the upper-level VIP and a pseudomonotone operator in the lower-level VIP. We also established a strong convergence theorem under mild conditions. Our main result was applied to the BSPVIP, which has potential applications in fields such as image recognition, signal processing, and machine learning. To illustrate the effectiveness and practical utility of the proposed scheme, we provide an example. The method employs an adaptive stepsize that does not require prior knowledge of the operator norm and uses an inertial approach with a correction term to accelerate the algorithm's convergence.

6. List of abbreviations

The following abbreviations are used throughout this paper:

(a) Bilevel Pseudomonotone Variational Inequality Problem (BPVIP)
(b) Split Common Fixed Point Problem (SCFPP)
(c) Variational Inequality Problem (VIP)
(d) Bilevel Split Pseudomonotone Variational Inequality Problem (BSPVIP)
(e) Common Fixed Point Problem (CFPP)
(f) Bilevel Split Variational Inequality Problem (BSVIP)
(g) Split Variational Inequality Problem (SVIP)
(h) Fixed Point Problem (FPP)

Acknowledgment. The research of Lu-Cuan Ceng was supported by the 2020 Shanghai Leading Talents Program of the Shanghai Municipal Human Resources and Social Security Bureau (20LJ2006100), the Innovation Program of Shanghai Municipal Education Commission (15ZZ068) and the Program for Outstanding Academic

Leaders in Shanghai City (15XD1503100). Debdas Ghosh acknowledges the financial support of the research grants MATRICS (MTR/2021/000696) and Core Research Grant (CRG/2022/001347) by the Science and Engineering Research Board, India. Habib ur Rehman acknowledges financial support from Zhejiang Normal University, Jinhua, China (Grant YS304223974).

References

[1] L.C. Ceng, Q.H. Ansari, S. Schaible, *Hybrid extragradient-like methods for generalized mixed equilibrium problems, systems of generalized equilibrium problems and optimization problems*, J. Global Optim., **53**(2012), 69-96.

[2] L.C. Ceng, N. Hadjisavvas, N.C. Wong, *Strong convergence theorem by a hybrid extragradient-like approximation method for variational inequalities and fixed point problems*, J. Global Optim., **46**(2010), 635-646.

[3] L.C. Ceng, A. Latif, Q.H. Ansari, J.C. Yao, *Hybrid extragradient method for hierarchical variational inequalities*, Fixed Point Theory Appl., 2014, 2014:222, 35 pp.

[4] L.C. Ceng, Y.C. Liou, C.F. Wen, *Extragradient method for convex minimization problem*, J. Inequal. Appl., 2014, 2014:444, 40 pp.

[5] L.C. Ceng, Y.C. Liou, C.F. Wen, *A hybrid extragradient method for bilevel pseudomonotone variational inequalities with multiple solutions*, J. Nonlinear Sci. Appl., **9**(2016), 4052-4069.

[6] L.C. Ceng, Y.C. Liou, C.F. Wen, *Composite relaxed extragradient method for triple hierarchical variational inequalities with constraints of systems of variational inequalities*, J. Nonlinear Sci. Appl., **10**(2017), 2018-2039.

[7] L.C. Ceng, Y.C. Liou, C.F. Wen, Y.J. Wu, *Hybrid extragradient viscosity method for general system of variational inequalities*, J. Inequal. Appl., 2015, 2015:150, 43 pp.

[8] L.C. Ceng, C.T. Pang, C.F. Wen, *Multi-step extragradient method with regularization for triple hierarchical variational inequalities with variational inclusion and split feasibility constraints*, J. Inequal. Appl., 2014, 2014:492, 40 pp.

[9] L.C. Ceng, A. Petruşel, X. Qin, J.C. Yao, *A modified inertial subgradient extragradient method for solving pseudomonotone variational inequalities and common fixed point problems*, Fixed Point Theory, **21**(2020), 93-108.

[10] L.C. Ceng, A. Petruşel, X. Qin, J.C. Yao, *Two inertial subgradient extragradient algorithms for variational inequalities with fixed-point constraints*, Optimization, **70**(2021), 1337-1358.

[11] L.C. Ceng, A. Petruşel, J.C. Yao, Y. Yao, *Hybrid viscosity extragradient method for systems of variational inequalities, fixed points of nonexpansive mappings, zero points of accretive operators in Banach spaces*, Fixed Point Theory, **19**(2018), no. 2, 487-501.

[12] L.C. Ceng, D.R. Sahu, J.C. Yao, *A unified extragradient method for systems of hierarchical variational inequalities in a Hilbert space*, J. Inequal. Appl., 2014, 2014:460, 32 pp.

[13] L.C. Ceng, C.F. Wen, *Relaxed extragradient methods for systems of variational inequalities*, J. Inequal. Appl., 2015, 2015:140, 41 pp.

[14] J.Z. Chen, L.C. Ceng, Y.Q. Qiu, Z.R. Kong, *Extra-gradient methods for solving split feasibility and fixed point problems*, Fixed Point Theory Appl., 2015, 2015:192, 21 pp.

[15] S.V. Denisov, V.V. Semenov, L.M. Chabak, *Convergence of the modified extragradient method for variational inequalities with non-Lipschitz operators*, Cybern. Syst. Anal., **51**(2015), 757-765.

[16] Q.L. Dong, Y.Y. Lu, J.F. Yang, *The extragradient algorithm with inertial effects for solving the variational inequality*, Optimization, **65**(2016), 2217-2226.

[17] M. Eslamian, A. Kamandi, *A novel method for hierarchical variational inequality with split common fixed point constraint*, J. Appl. Math. Comput., 70(2024), 1837-1857.

[18] K. Goebel, S. Reich, *Uniform Convexity, Hyperbolic Geometry, and Nonexpansive Mappings*, Marcel Dekker, New York, 1984.

[19] P.V. Huy, L.H.M. Van, N.D. Hien, T.V. Anh, *Modified Tseng's extragradient methods with self-adaptive step size for solving bilevel split variational inequality problems*, Optimization, **71**(6)(2022), 1721-1748.

[20] L.O. Jolaoso, Y. Shehu, J.C. Yao, *Inertial extragradient type method for mixed variational inequalities without monotonicity*, Math. Comput. Simulation, **192**(2022), 353-369.

[21] R. Kraikaew, S. Saejung, *Strong convergence of the Halpern subgradient extragradient method for solving variational inequalities in Hilbert spaces*, J. Optim. Theory Appl., **163**(2014), 399-412.

[22] P.E. Maingé, *Strong convergence of projected subgradient methods for nonsmooth and non-strictly convex minimization*, Set-Valued Anal., **16**(2008), 899-912.

[23] Y. Shehu, Q.L. Dong, D. Jiang, *Single projection method for pseudo-monotone variational inequality in Hilbert spaces*, Optimization, **68**(2019), 385-409.

[24] Y. Shehu, O.S. Iyiola, *Strong convergence result for monotone variational inequalities*, Numer. Algorithms, **76**(2017), 259-282.

[25] W. Takahashi, *The split common fixed point problem and the shrinking projection method in Banach spaces*, J. Convex. Anal., **24**(2017), 1015-1028.

[26] D.V. Thong, D.V. Hieu, *Modified Tseng's extragradient algorithms for variational inequality problems*, J. Fixed Point Theory Appl., **20**(2018), no. 4, Paper No. 152, 18 pp.

[27] D.V. Thong, D.V. Hieu, *Modified subgradient extragradient method for variational inequality problems*, Numer. Algorithms, **79**(2018), 597-610.

[28] D.V. Thong, N.A. Triet, X.H. Li, Q.L. Dong, *Strong convergence of extragradient methods for solving bilevel pseudo-monotone variational inequality problems*, Numer. Algorithms, **83**(2020), 1123-1143.

[29] P.T. Vuong, *On the weak convergence of the extragradient method for solving pseudo-monotone variational inequalities*, J. Optim. Theory Appl., **176**(2018), 399-409.

[30] H.K. Xu, T.H. Kim, *Convergence of hybrid steepest-descent methods for variational inequalities*, J. Optim. Theory Appl., **119**(2003), 185-201.

[31] J. Yang, H. Liu, Z. Liu, *Modified subgradient extragradient algorithms for solving monotone variational inequalities*, Optimization, **67**(2018), 2247-2258.

[32] Y. Yao, Y.C. Liou, S.M. Kang, *Approach to common elements of variational inequality problems and fixed point problems via a relaxed extragradient method*, Comput. Math. Appl., **59**(2010), 3472-3480.

Received: June 4, 2023; Accepted: April 20, 2024.