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WEAK CONVERGENCE THEOREMS FOR A FINITE FAMILY OF MONOTONE NONEXPANSIVE MAPPINGS IN BANACH SPACES

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Abstract. In this paper, we prove weak convergence theorems for finite monotone nonexpansive mappings in uniformly convex Banach spaces endowed with a partial order.
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1. Introduction

Let E be a real Banach space, let C be a nonempty subset of E. A mapping $T : C \to C$ is called nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\|$$

for all $x, y \in C$. For a mapping $T : C \to C$, we denote by $F(T)$ the set of *fixed points* of T, i.e., $F(T) = \{z \in C : Tz = z\}$.

Ran and Reurings [15] proved an analogue of the classical Banach contraction principle [6] in metric spaces endowed with a partial order. In particular, they show how this extension is useful when dealing with some special matrix equations (see also [10, 14, 21, 22]).

Mann [12] introduced an iteration process for approximation of fixed points of a mapping T in a Hilbert space as follows:

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)Tx_n$$

for all $n \geq 1$, where $\{\alpha_n\}$ are sequences in $[0, 1]$. Later, Reich [13] discussed this iteration process in a uniformly convex Banach space whose norm is Frechet differentiable. Bin Dehaish and Khamsi [7] proved a weak convergence theorem of Mann's type [12] iteration for monotone nonexpansive mappings in Banach spaces endowed with a partial order (see also [1, 17]).

In this paper, we prove weak convergence theorems for finite monotone nonexpansive mappings in uniformly convex Banach spaces endowed with a partial order.

2. Preliminaries and notations

Throughout this paper, we assume that E is a real Banach space with norm $\|\cdot\|$ and endowed with a *partial order* $\preceq$ compatible with the linear structure of E, that is,

$$x \preceq y \text{ implies } x+z \preceq y+z,$$
$$x \preceq y \text{ implies } \lambda x \preceq \lambda y$$

for every $x, y, z \in E$ and $\lambda \geq 0$. As usual we adopt the convention $x \succeq y$ if and only if $y \preceq x$. It follows that all *order intervals* $[x, \rightarrow) = \{z \in E : x \preceq z\}$ and $(\leftarrow, y] = \{z \in E : z \in E : z \preceq y\}$ are convex. Moreover, we assume that each order intervals $[x, \rightarrow)$ and $(\leftarrow, y]$ are closed. Recall that an order interval is any of the subsets $[a, \rightarrow) = \{x \in X; a \preceq x\}$ or $(\leftarrow, a] = \{x \in X; x \preceq a\}$ for any $a \in E$. As a direct consequence of this, the subset

$$[a, b] = \{x \in X; a \preceq x \preceq b\} = [a, \rightarrow) \cap (\leftarrow, b]$$

is also closed and convex for each $a, b \in E$.

Let E be a real Banach space with norm $\|\cdot\|$ and endowed with a partial order $\preceq$ compatible with the linear structure of E. We will say that this Banach space $(E, \|\cdot\|, \preceq)$ is an *ordered Banach space*. Let C be a nonempty subset of E. A mapping $T : C \rightarrow C$ is called *nonexpansive* if

$$\|Tx - Ty\| \leq \|x - y\|$$

for all $x, y \in C$. A mapping $T : C \rightarrow C$ is called *monotone* if

$$Tx \preceq Ty$$

for each $x, y \in C$ such that $x \preceq y$ (see also [7]). For a mapping $T : C \rightarrow C$, we denote by $F(T)$ the set of *fixed points* of T, i.e., $F(T) = \{z \in C : Tz = z\}$. For a mapping $T : C \rightarrow C$ and $\varepsilon > 0$, we define the set $F_\varepsilon(T)$ to be

$$F_\varepsilon(T) = \{x \in C : \|Tx - x\| \leq \varepsilon\}.$$

We denote by E^* the topological dual space of E. We denote by $\mathbb{N}$ and $\mathbb{Z}^+$ the set of all positive integers and the set of all nonnegative integers, respectively. We also denote by $\mathbb{R}$ and $\mathbb{R}^+$ the set of all real numbers and the set of all nonnegative real numbers, respectively. We write $x_n \rightarrow x$ (or $\lim\limits_{n\to\infty} x_n = x$) to indicate that the sequence $\{x_n\}$ of vectors in E converges strongly to x. We also write $x_n \rightharpoonup x$ (or $\text{w-}\lim_{n\to\infty} x_n = x$) to indicate that the sequence $\{x_n\}$ of vectors in E converges weakly to x. We also denote by $\langle y, x^* \rangle$ the value of $x^* \in E^*$ at $y \in E$. For a subset A of E, $\text{co}A$ and $\overline{\text{co}}A$ mean the convex hull of A and the closure of the convex hull of A, respectively.

A Banach space E is said to be strictly convex if

$$\frac{\|x + y\|}{2} < 1$$

for $x, y \in E$ with $\|x\| = \|y\| = 1$ and $x \neq y$. In a strictly convex Banach space, we have that if

$$\|x\| = \|y\| = \|(1 - \lambda)x + \lambda y\|$$

for $x, y \in E$ and $\lambda \in (0,1)$, then $x = y$. For every ε with $0 \le \varepsilon \le 2$, we define the modulus $\delta(\varepsilon)$ of convexity of E by

$$\delta(\varepsilon) = \inf \left\{ 1 - \frac{\|x+y\|}{2} \ : \ \|x\| \le 1, \|y\| \le 1, \|x-y\| \ge \varepsilon \right\}.$$

A Banach space E is said to be uniformly convex if $\delta(\varepsilon) > 0$ for every $\varepsilon > 0$. If E is uniformly convex, then for r, ε with $r \ge \varepsilon > 0$, we have $\delta\left(\frac{\varepsilon}{r}\right) > 0$ and

$$\left\| \frac{x+y}{2} \right\| \le r\left(1 - \delta\left(\frac{\varepsilon}{r}\right)\right)$$

for every $x, y \in E$ with $\|x\| \le r$, $\|y\| \le r$ and $\|x-y\| \ge \varepsilon$. It is well-known that a uniformly convex Banach space is reflexive and strictly convex.

The following lemmas were proved in [9].

Lemma 2.1 ([9]). *Let E be a uniformly convex Banach space and let C be a nonempty bounded closed convex subset of E. Then, for each $\varepsilon > 0$, there exists $\delta > 0$ such that*

$$\overline{\text{co}} F_\delta(T) \subset F_\varepsilon(T)$$

for every nonexpansive mapping T of C into itself.

Lemma 2.2 ([9]). *Let E be a uniformly convex Banach space and let C be a nonempty bounded closed convex subset of E. Then,*

$$\lim_{n\to\infty} \sup_{\substack{x\in C \\ T\in N(C)}} \left\| \frac{1}{n}\sum_{i=0}^{n-1} T^i x - T\left(\frac{1}{n}\sum_{i=0}^{n-1} T^i x\right) \right\| = 0$$

where $N(C)$ denotes the set of all nonexpansive mappings of C into itself.

The following theorem was proved in [8].

Theorem 2.3 ([8]). *Let C be a nonempty bounded closed convex subset of a uniformly convex Banach space E and let T be a nonexpansive mapping of C into itself. Let $\{x_n\}$ be a sequence in C such that it converges weakly to an element u in C and $\{x_n - Tx_n\}$ converges strongly to 0. Then, u is a fixed point of T.*

The following lemma was proved in [23].

Lemma 2.4 ([23]). *Let $p > 1$ and $r > 0$ be two fixed real numbers. Let E be a uniformly convex Banach space. Then, there is a continuous, strictly increasing and convex function $g : [0,\infty) \to [0,\infty)$ such that $g(0) = 0$ and*

$$\begin{aligned} &\|\lambda x + (1-\lambda)y\|^p \\ &\quad \le \lambda\|x\|^p + (1-\lambda)\|y\|^p - \{\lambda^p(1-\lambda) + \lambda(1-\lambda)^p\} g(\|x-y\|) \end{aligned}$$

for all $x, y \in B_r$ and λ with $0 < \lambda < 1$ where $B_r = \{x \in E : \|x\| \le r\}$

3. Lemmas

In this paper, we consider the following iteration process for approximation of common fixed points of finite nonexpansive mappings in a Banach space: Let C be a nonempty closed convex subset of an ordered Banach space E. Let $T_1, T_2, \dots, T_r$ be finite nonexpansive mappings of C into itself such that $T_iT_j = T_jT_i$ for every $i, j = 1, 2, \dots, r$ and $\bigcap_{i=1}^{r} F(T_i)$ is nonempty. For $x_1 \in C$, $\{x_n\}$ is defined by

$$x_{n+1} = \alpha_n x_n + (1-\alpha_n) \cdot \frac{1}{(n+1)^r} \sum_{i_1=0}^{n} \cdots \sum_{i_r=0}^{n} {T_1}^{i_1} \cdots {T_r}^{i_r} x_n$$

for every $n \in \mathbb{N}$, where $\{\alpha_n\}$ is a sequence in $[0,1]$ (see also [12]).

Let C be a nonempty subset of an ordered Banach space E and let T be a mapping of C into itself. A sequence $\{x_n\}$ in C is said to be an *approximate fixed point sequence* of a mapping T of C into itself if

$$\lim_{n\to\infty} \|x_n - Tx_n\| = 0$$

(see also [13, 18]). Let $T_1, T_2, \dots, T_r$ be mappings of C into itself. A sequence $\{x_n\}$ in C is said to be an *approximate common fixed point sequence* of mappings $T_1, T_2, \dots, T_r$ of C if for every $k = 1, 2, \dots, r$,

$$\lim_{n\to\infty} \|x_n - T_k x_n\| = 0.$$

A sequence $\{x_n\}$ in E is said to be *monotone increasing* if

$$x_1 \preceq x_2 \preceq x_3 \preceq \cdots$$

(see also [7]).

The following lemma plays an important role in this paper (see also [2]).

Lemma 3.1. *Let E be a uniformly convex Banach space, let C be a nonempty bounded closed convex subset of E. Let $T_1, T_2, \dots, T_r$ be nonexpansive mappings of C into itself such that $T_iT_j = T_jT_i$ for every $i, j = 1, 2, \dots, r$. For any $n \in \mathbb{N}$, put*

$$S_n^{l_1,\dots,l_r} x = \frac{1}{n^r} \sum_{i_1=0}^{n-1} \cdots \sum_{i_r=0}^{n-1} {T_1}^{i_1+l_1} \cdots {T_r}^{i_r+l_r} x \quad \text{for every} \quad x \in C.$$

Then, for every $k = 1, 2, \dots, r$,

$$\lim_{n\to\infty} \sup_{\substack{x\in C \\ l_1,\dots,l_r \in \mathbb{Z}^+}} \left\| S_n^{l_1,\dots,l_r} x - T_k \left(S_n^{l_1,\dots,l_r} x \right) \right\| = 0. \tag{3.1}$$

Proof. Let $\varepsilon > 0$. From Lemma 2.1, we know that there exists $\delta > 0$ such that

$$\overline{\text{co}} F_\delta(U) \subset F_\varepsilon(U) \tag{3.2}$$

for every nonexpansive mapping U of C into itself. Let $k = 1, 2, \dots, r$. By Lemma 2.2, we also have

$$\lim_{n\to\infty} \sup_{y\in C} \left\| \frac{1}{n} \sum_{i=0}^{n-1} {T_k}^i y - T_k \left(\frac{1}{n} \sum_{i=0}^{n-1} {T_k}^i y \right) \right\| = 0.$$

Then, there exists $n_1 \in \mathbb{N}$ such that

$$\sup_{y \in C} \left\| \frac{1}{n} \sum_{i=0}^{n-1} {T_k}^i y - T_k \left(\frac{1}{n} \sum_{i=0}^{n-1} {T_k}^i y \right) \right\| < \delta$$

for every $n > n_1$. Then, we obtain that

$$\frac{1}{n} \sum_{i=0}^{n-1} {T_k}^i y \subset F_\delta(T_k) \subset \overline{\mathrm{co}} F_\delta(T_k) \tag{3.3}$$

for every $y \in C$ and $n \geq n_1$. We prove

$$\frac{1}{n^r} \sum_{i_1=0}^{n-1} \cdots \sum_{i_r=0}^{n-1} {T_1}^{i_1+l_1} \cdots {T_r}^{i_r+l_r} x \in \overline{\mathrm{co}} F_\delta(T_k) \tag{3.4}$$

for every $x \in C$, $l_1, l_2, \ldots, l_r \in \mathbb{Z}^+$ and $n \in \mathbb{N}$ with $n \geq n_1$. If not, for some $x_0 \in C$, $m_1, m_2, \ldots, m_r \in \mathbb{Z}^+$ and $n \geq n_1$,

$$\frac{1}{n^r} \sum_{i_1=0}^{n-1} \cdots \sum_{i_r=0}^{n-1} {T_1}^{i_1+m_1} \cdots {T_r}^{i_r+m_r} x_0 \notin \overline{\mathrm{co}} F_\delta(T_k)\,.$$

From the separation theorem, there exists $y_1^* \in E^*$ such that

$$\left\langle \frac{1}{n^r} \sum_{i_1=0}^{n-1} \cdots \sum_{i_r=0}^{n-1} {T_1}^{i_1+m_1} \cdots {T_r}^{i_r+m_r} x_0, y_1^* \right\rangle < \inf \left\{ \langle z, y_1^* \rangle : z \in \overline{\mathrm{co}} F_\delta(T_k) \right\}. \tag{3.5}$$

Then, from (3.3), we obtain

$$\begin{aligned} \inf \left\{ \langle z, y_1^* \rangle : z \in \overline{\mathrm{co}} F_\delta(T_k) \right\} &\leq \inf \left\{ \left\langle \frac{1}{n} \sum_{i=0}^{n-1} {T_k}^i x, y_1^* \right\rangle : x \in C, n \geq n_1 \right\} \\ &\leq \left\langle \frac{1}{n} \sum_{i=0}^{n-1} {T_k}^i y, y_1^* \right\rangle \end{aligned}$$

for all $y \in C$ and $n \in \mathbb{N}$ with $n \geq n_1$. Then, we have that for any $i_1, i_2, \ldots, i_r \in \mathbb{Z}^+$,

$$\begin{aligned} &\inf \left\{ \langle z, y_1^* \rangle : z \in \overline{\mathrm{co}} F_\delta(T_k) \right\} \\ &\leq \left\langle \frac{1}{n} \sum_{i=0}^{n-1} {T_k}^i ({T_1}^{i_1+m_1} \cdots T_{k-1}^{i_{k-1}+m_{k-1}} {T_k}^{m_k} T_{k+1}^{i_{k+1}+m_{k+1}} \cdots {T_r}^{i_r+m_r} x_0), y_1^* \right\rangle. \end{aligned}$$

Put

$$U_{i_1,\ldots,i_r}^{m_1,\ldots,m_r} x_0 = {T_1}^{i_1+m_1} \cdots T_{k-1}^{i_{k-1}+m_{k-1}} {T_k}^{m_k} T_{k+1}^{i_{k+1}+m_{k+1}} \cdots {T_r}^{i_r+m_r} x_0.$$

Then, it follows that

$$\begin{aligned}
&\inf\left\{\langle z, y_1^*\rangle : z\in \overline{\mathrm{co}}F_\delta(T_k)\right\}\\
&\leq \frac{1}{n^{r-1}}\sum_{i_1=0}^{n-1}\cdots\sum_{i_{k-1}=0}^{n-1}\sum_{i_{k+1}=0}^{n-1}\cdots\sum_{i_r=0}^{n-1}\left\langle \frac{1}{n}\sum_{i=0}^{n-1} {T_k}^i(U_{i_1,\ldots,i_r}^{m_1,\ldots,m_r}x_0), y_1^*\right\rangle\\
&= \frac{1}{n^r}\sum_{i_1=0}^{n-1}\cdots\sum_{i_r=0}^{n-1}\left\langle {T_1}^{i_1+m_1}\cdots {T_r}^{i_r+m_r}x_0, y_1^*\right\rangle .
\end{aligned}$$

Therefore, by (3.5), we have

$$\begin{aligned}
&\inf\left\{\langle z, y_1^*\rangle : z\in \overline{\mathrm{co}}F_\delta(T_k)\right\}\\
&\leq \frac{1}{n^r}\sum_{i_1=0}^{n-1}\cdots\sum_{i_r=0}^{n-1}\left\langle {T_1}^{i_1+m_1}\cdots {T_r}^{i_r+m_r}x_0, y_1^*\right\rangle\\
&= \left\langle \frac{1}{n^r}\sum_{i_1=0}^{n-1}\cdots\sum_{i_r=0}^{n-1} {T_1}^{i_1+m_1}\cdots {T_r}^{i_r+m_r}x_0, y_1^*\right\rangle\\
&< \inf\left\{\langle z, y_1^*\rangle : z\in \overline{\mathrm{co}}F_\delta(T_k)\right\}.
\end{aligned}$$

This is a contradiction and hence (3.4) hold for every $l_1, l_2, \ldots, l_r \in \mathbb{Z}^+, x \in C$ and $n \geq n_1$. Then, from (3.2), we have

$$S_n^{l_1,\ldots,l_r}x = \frac{1}{n^r}\sum_{i_1=0}^{n-1}\cdots\sum_{i_r=0}^{n-1} {T_1}^{i_1+l_1}\cdots {T_r}^{i_r+l_r}x \in \overline{\mathrm{co}}F_\delta(T_k) \subset F_\varepsilon(T_k)$$

for every $l_1, l_2, \ldots, l_r \in \mathbb{Z}^+, x \in C$ and $n \geq n_1$. We ramark that k is arbitrary. Then, it follows that

$$\sup_{\substack{x\in C\\ l_1,\cdots,l_r\in\mathbb{Z}^+}} \left\| S_n^{l_1,\ldots,l_r}x - T_k\left(S_n^{l_1,\ldots,l_r}x\right)\right\| < \varepsilon$$

for every $n \geq n_1$ and hence we obtain (3.1). □

Lemma 3.2. *Let E be a Banach space and let C be a nonempty closed convex subset of E. Let $T_1, T_2, \ldots, T_r$ be nonexpansive mappings of C into itself such that $T_iT_j = T_jT_i$ for every $i, j = 1, 2, \ldots, r$ and $\bigcap_{i=1}^r F(T_i)$ is nonempty. Suppose $x_1 \in C$ and $\{x_n\}$ is given by*

$$x_{n+1} = \alpha_n x_n + (1-\alpha_n)\frac{1}{(n+1)^r}\sum_{i_1=0}^{n}\cdots\sum_{i_r=0}^{n} {T_1}^{i_1}\cdots {T_r}^{i_r}x_n \quad \textit{for every} \quad n\in\mathbb{N},$$

where $\{\alpha_n\}$ is a sequence in $[0,1]$. Let w be a common fixed point of $T_1, T_2, \ldots, T_r$. Then, $\lim_{n\to\infty}\|x_n - w\|$ exists.

Proof. Let w be a common fixed point of $T_1, T_2, \dots, T_r$. Then, we have

$$\begin{aligned}
\|x_{n+1} - w\| &= \left\| \alpha_n(x_n - w) + (1-\alpha_n)\left(\frac{1}{(n+1)^r}\sum_{i_1=0}^{n}\cdots\sum_{i_r=0}^{n} T_{i_1}^{i_1}\cdots {T_r}^{i_r}x_n - w\right)\right\| \\
&\le \alpha_n\|x_n - w\| + (1-\alpha_n)\left\|\frac{1}{(n+1)^r}\sum_{i_1=0}^{n}\cdots\sum_{i_r=0}^{n} {T_1}^{i_1}\cdots {T_r}^{i_r}x_n - w\right\| \\
&\le \alpha_n\|x_n - w\| + (1-\alpha_n)\cdot\frac{1}{(n+1)^r}\sum_{i_1=0}^{n}\cdots\sum_{i_r=0}^{n} \|{T_1}^{i_1}\cdots {T_r}^{i_r}x_n - w\| \\
&\le \alpha_n\|x_n - w\| + (1-\alpha_n)\|x_n - w\| = \|x_n - w\|
\end{aligned}$$

and hence $\lim_{n\to\infty}\|x_n - w\|$ exists. □

Lemma 3.3. *Let C be a nonempty closed convex subset of an ordered uniformly convex Banach space E. Let $T_1, T_2, \dots, T_r$ be monotone nonexpansive mappings of C into itself such that $T_iT_j = T_jT_i$ for every $i, j = 1, 2, \dots, r$, and $\bigcap_{i=1}^{r} F(T_i)$ is nonempty. Suppose $x_1 \in C$ and $\{x_n\}$ is given by*

$$x_{n+1} = \alpha_n x_n + (1-\alpha_n)\frac{1}{(n+1)^r}\sum_{i_1=0}^{n}\cdots\sum_{i_r=0}^{n} {T_1}^{i_1}\cdots {T_r}^{i_r}x_n \quad \textit{for every} \quad n \in \mathbb{N},$$

where $\{\alpha_n\}$ is a sequence in $[0, a]$ for some $0 < a < 1$. Then, $\{x_n\}$ is an approximate common fixed point sequence of mappings $T_1, T_2, \dots, T_r$ of C, i.e., for every $k = 1, 2, \dots, r$,

$$\lim_{n\to\infty}\|x_n - T_k x_n\| = 0.$$

Proof. For $x_1 \in C$ and $f \in \bigcap_{i=1}^{r} F(T_i)$, put $L = \|x_1 - f\|$ and set

$$X = \{u \in E : \|u - f\| \le L\} \cap C.$$

Then, X is a nonempty bounded closed convex subset of C which is T_k-invariant for every $k = 1, 2, \dots, r$, and contains $x_1 \in C$. So, without loss of generality, we may assume that C is bounded. Let w be a common fixed point of $T_1, T_2, \dots, T_r$. Then, from Lemma 2.4, there exists a continuous, strictly increasing and convex function

$g : [0, \infty) \to [0, \infty)$ such that $g(0) = 0$ and

$$
\begin{aligned}
\|x_{n+1} - w\|^2 &= \left\| \alpha_n (x_n - w) + (1 - \alpha_n) \left(\frac{1}{(n+1)^r} \sum_{i_1=0}^{n} \cdots \sum_{i_r=0}^{n} {T_1}^{i_1} \cdots {T_r}^{i_r} x_n - w \right) \right\|^2 \\
&\le \alpha_n \|x_n - w\|^2 + (1 - \alpha_n) \left\| \frac{1}{(n+1)^r} \sum_{i_1=0}^{n} \cdots \sum_{i_r=0}^{n} {T_1}^{i_1} \cdots {T_r}^{i_r} x_n - w \right\|^2 \\
&\quad - \alpha_n (1 - \alpha_n) g \left(\left\| x_n - \frac{1}{(n+1)^r} \sum_{i_1=0}^{n} \cdots \sum_{i_r=0}^{n} {T_1}^{i_1} \cdots {T_r}^{i_r} x_n \right\| \right) \\
&\le \alpha_n \|x_n - w\|^2 + (1 - \alpha_n) \|x_n - w\|^2 \\
&\quad - \alpha_n (1 - \alpha_n) g \left(\left\| x_n - \frac{1}{(n+1)^r} \sum_{i_1=0}^{n} \cdots \sum_{i_r=0}^{n} {T_1}^{i_1} \cdots {T_r}^{i_r} x_n \right\| \right) \\
&= \|x_n - w\|^2 \\
&\quad - \alpha_n (1 - \alpha_n) g \left(\left\| x_n - \frac{1}{(n+1)^r} \sum_{i_1=0}^{n} \cdots \sum_{i_r=0}^{n} {T_1}^{i_1} \cdots {T_r}^{i_r} x_n \right\| \right)
\end{aligned}
$$

for all $n \in \mathbb{N}$. Then, since $\alpha_n \le a$, we have

$$
\begin{aligned}
\alpha_n (1 - a) g & \left(\left\| x_n - \frac{1}{(n+1)^r} \sum_{i_1=0}^{n} \cdots \sum_{i_r=0}^{n} {T_1}^{i_1} \cdots {T_r}^{i_r} x_n \| \right\| \right) \\
&\le \alpha_n (1 - \alpha_n) g \left(\left\| x_n - \frac{1}{(n+1)^r} \sum_{i_1=0}^{n} \cdots \sum_{i_r=0}^{n} {T_1}^{i_1} \cdots {T_r}^{i_r} x_n \| \right\| \right) \\
&\le \|x_n - w\|^2 - \|x_{n+1} - w\|^2.
\end{aligned}
$$

So, from Lemma 3.2, we obtain

$$
\lim_{n \to \infty} \alpha_n g \left(\left\| x_n - \frac{1}{(n+1)^r} \sum_{i_1=0}^{n} \cdots \sum_{i_r=0}^{n} {T_1}^{i_1} \cdots {T_r}^{i_r} x_n \right\| \right) = 0.
$$

Since g satisfies the suitable condition, we have

$$
\lim_{n \to \infty} \alpha_n \left\| x_n - \frac{1}{(n+1)^r} \sum_{i_1=0}^{n} \cdots \sum_{i_r=0}^{n} {T_1}^{i_1} \cdots {T_r}^{i_r} x_n \right\| = 0. \tag{3.6}
$$

It follows from the definition of $\{x_n\}$ that

$$
\begin{aligned}
x_{n+1} &- \frac{1}{(n+1)^r} \sum_{i_1=0}^{n} \cdots \sum_{i_r=0}^{n} {T_1}^{i_1} \cdots {T_r}^{i_r} x_n \\
&= \alpha_n \left(x_n - \frac{1}{(n+1)^r} \sum_{i_1=0}^{n} \cdots \sum_{i_r=0}^{n} {T_1}^{i_1} \cdots {T_r}^{i_r} x_n \right).
\end{aligned}
$$

Since

$$\begin{aligned}
&\|T_k x_{n+1} - x_{n+1}\| \\
&\leq \left\| T_k x_{n+1} - T_k \left(\frac{1}{(n+1)^r} \sum_{i_1=0}^{n} \cdots \sum_{i_r=0}^{n} {T_1}^{i_1} \cdots {T_r}^{i_r} x_n \right) \right\| \\
&+ \left\| T_k \left(\frac{1}{(n+1)^r} \sum_{i_1=0}^{n} \cdots \sum_{i_r=0}^{n} {T_1}^{i_1} \cdots {T_r}^{i_r} x_n \right) - \frac{1}{(n+1)^r} \sum_{i_1=0}^{n} \cdots \sum_{i_r=0}^{n} {T_1}^{i_1} \cdots {T_r}^{i_r} x_n \right\| \\
&+ \left\| \frac{1}{(n+1)^r} \sum_{i_1=0}^{n} \cdots \sum_{i_r=0}^{n} {T_1}^{i_1} \cdots {T_r}^{i_r} x_n - x_{n+1} \right\| \\
&\leq \left\| T_k \left(\frac{1}{(n+1)^r} \sum_{i_1=0}^{n} \cdots \sum_{i_r=0}^{n} {T_1}^{i_1} \cdots {T_r}^{i_r} x_n \right) - \frac{1}{(n+1)^r} \sum_{i_1=0}^{n} \cdots \sum_{i_r=0}^{n} {T_1}^{i_1} \cdots {T_r}^{i_r} x_n \right\| \\
&+ 2 \left\| \frac{1}{(n+1)^r} \sum_{i_1=0}^{n} \cdots \sum_{i_r=0}^{n} {T_1}^{i_1} \cdots {T_r}^{i_r} x_n - x_{n+1} \right\| \\
&= \left\| T_k \left(\frac{1}{(n+1)^r} \sum_{i_1=0}^{n} \cdots \sum_{i_r=0}^{n} {T_1}^{i_1} \cdots {T_r}^{i_r} x_n \right) - \frac{1}{(n+1)^r} \sum_{i_1=0}^{n} \cdots \sum_{i_r=0}^{n} {T_1}^{i_1} \cdots {T_r}^{i_r} x_n \right\| \\
&+ 2\alpha_n \left\| \frac{1}{(n+1)^r} \sum_{i_1=0}^{n} \cdots \sum_{i_r=0}^{n} {T_1}^{i_1} \cdots {T_r}^{i_r} x_n - x_n \right\|,
\end{aligned}$$

from (3.6) and Lemma 3.1, we have, for every $k = 1, 2, \ldots, r$,

$$\lim_{n \to \infty} \|x_n - T_k x_n\| = 0. \qquad \square$$

Lemma 3.4. *Let C be a nonempty closed convex subset of an ordered Banach space E. Let $T_1, T_2, \ldots, T_r$ be monotone nonexpansive mappings of C into itself such that $T_i T_j = T_j T_i$ for every $i, j = 1, 2, \ldots, r$, and $\bigcap_{i=1}^{r} F(T_i)$ is nonempty. Assume that $x \preceq T_k x$ for every $k = 1, 2, \ldots, r$ and $x \in C$. Suppose $x_1 \in C$ and $\{x_n\}$ is given by*

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) \frac{1}{(n+1)^r} \sum_{i_1=0}^{n} \cdots \sum_{i_r=0}^{n} {T_1}^{i_1} \cdots {T_r}^{i_r} x_n \quad \text{for every} \quad n \in \mathbb{N},$$

where $\{\alpha_n\}$ is a sequence in $[0, 1]$. Then, $\{x_n\}$ is monotone increasing.

Proof. It follows from the assumption that $x \preceq T_k x \preceq T_m T_k x$ and $x \preceq T_k x \preceq {T_k}^2 x$. for every $m, k = 1, 2, \ldots, r$, We remark that all order intervals are convex. Then, we obtain that

$$x_n \preceq \frac{1}{(n+1)^r} \sum_{i_1=0}^{n} \cdots \sum_{i_r=0}^{n} {T_1}^{i_1} \cdots {T_r}^{i_r} x_n.$$

So, we also have that

$$\begin{aligned} x_n &= \alpha_n x_n + (1-\alpha_n)x_n \\ &\preceq \alpha_n x_n + (1-\alpha_n)\frac{1}{(n+1)^r}\sum_{i_1=0}^{n}\cdots\sum_{i_r=0}^{n} {T_1}^{i_1}\cdots {T_r}^{i_r} x_n = x_{n+1} \end{aligned}$$

for each $n \in \mathbb{N}$. Hence, we have that $\{x_n\}$ is monotone increasing. □

4. Weak convergence theorems of Mann's type iteration

In this section, we provide the approximation of common fixed points of families of monotone nonexpansive mappings in ordered Banach spaces. It is connected with the theory of differential equations (see [11]). We assume that E is an ordered Banach space in the sense that

$$x \preceq y \text{ implies } x+z \preceq y+z,$$

$$x \preceq y \text{ implies } \lambda x \preceq \lambda y$$

for every $x, y, z \in E$ and $\lambda \geq 0$. Note that this is a very typical situation. Let C be a nonempty, bounded, closed and convex subset of E. Let us fix $x \in C$. We consider the following initial value problem (IVP) for an unknown function $u(x,\cdot) : [0,\infty) \to C$.

$$\begin{cases} u(x,0) = x \\ u'(x,t) + (I - H_t)(u(x,t)) = 0, \end{cases} \tag{4.1}$$

where $H_t : C \to C$ are monotone nonexpansive mappings with respect to the order $\preceq$. The notation $u'(x,t)$ denotes the derivative of the function $t \mapsto u(x,t)$. We assume that $x \preceq H_t(x)$ for every $t \in \mathbb{R}^+$. Our aim is to construct a solution $u(x,\cdot)$ for the IVP (4.1), such that $x \preceq u(x,t)$ for every $t \in \mathbb{R}^+$. To achieve this we need to construct $u(x,\cdot)$ such that

$$u(x,t) = e^{-t}x + \int_0^t e^{s-t} H_s(u(s))ds.$$

It can be proved, using the standard methods of the Bochner integration, that the above formula gives us the required solution. Define then $T_t : C \to C$ by $T_t(x) = u(x,t)$ for all $t \in \mathbb{R}^+$. It can be proved that T_t is a monotone nonexpansive mapping for each $t \in \mathbb{R}^+$. Hence, it follows that $\mathcal{S} = \{T_t : t \in \mathbb{R}^+\}$ is a family of monotone nonexpansive mappings (see [11, 3, 4]).

We prove a weak convergence theorem for a family of monotone nonexpansive mappings in an ordered uniformly convex Banach space.

Theorem 4.1. *Let C be a nonempty closed convex subset of an ordered uniformly convex Banach space E. Let $T_1, T_2, \ldots, T_r$ be monotone nonexpansive mappings of C into itself such that $T_iT_j = T_jT_i$ for every $i, j = 1, 2, \ldots, r$ and $\bigcap_{i=1}^r F(T_i)$ is nonempty. Assume that $x \preceq T_k x$ for every $k = 1, 2, \ldots, r$ and $x \in C$. Suppose $x_1 \in C$ and $\{x_n\}$ is given by*

$$x_{n+1} = \alpha_n x_n + (1-\alpha_n)\frac{1}{(n+1)^r}\sum_{i_1=0}^{n}\cdots\sum_{i_r=0}^{n} {T_1}^{i_1}\cdots {T_r}^{i_r} x_n \quad \textit{for every} \quad n \in \mathbb{N},$$

where $\{\alpha_n\}$ is a sequence in $[0,1]$. If $\{\alpha_n\}$ is chosen so that $\alpha_n \in [0,a]$ for some a with $0 < a < 1$, then the sequence $\{x_n\}$ converges weakly to a point of $\bigcap_{i=1}^r F(T_i)$.

Proof. As in the proof of Lemma 3.3, we may assume that C is bounded. It follows from Lemma 3.3 that the sequence $\{x_n\}$ is an approximate common fixed point sequence of $T_1, T_2, \dots, T_r$, i.e., for every $k = 1, 2, \dots, r$,

$$\lim_{n\to\infty} \|x_n - T_k x_n\| = 0.$$

Since E is reflexive, $\{x_n\}$ must contain a subsequence which converges weakly to a point in C. So, let $\{x_{n_i}\}$ and $\{x_{n_j}\}$ be two subsequences of $\{x_n\}$ such that $x_{n_i} \rightharpoonup z_1$ and $x_{n_j} \rightharpoonup z_2$, respectively. We remark that for every $k = 1, 2, \dots, r$,

$$\lim_{i\to\infty} \|x_{n_i} - T_k x_{n_i}\| = 0 \text{ and } \lim_{j\to\infty} \|x_{n_j} - T_k x_{n_j}\| = 0.$$

It follows from Theorem 2.3 that z_1 and z_2 are common fixed points of $T_1, T_2, \dots, T_r$. Next, we show $z_1 = z_2$ (see also [7]). Fix $m \geq 1$. By Lemma 3.4, $\{x_n\}$ is monotone increasing. Since $\{x_n\}$ is monotone increasing and the order interval $[x_m, \rightarrow)$ is weakly closed, we conclude that $z_i \in [x_m, \rightarrow)$ for $i = 1, 2$. Then, we also obtain that $\{x_n\} \subset (\leftarrow, z_i]$ for $i = 1, 2$. It follows from the same reason that $z_j \in (\leftarrow, z_i]$ for $i, j = 1, 2$. So, we have $z_1 = z_2$. Therefore, we obtain that $\{x_n\}$ converges weakly to a point of $\bigcap_{i=1}^r F(T_i)$. □

Using Theorem 4.1, we get some convergence theorems for monotone nonexpansive mappings in ordered uniformly convex Banach spaces (see also [17]).

Theorem 4.2. *Let C be a nonempty closed convex subset of an ordered uniformly convex Banach space E. Let S and T be monotone nonexpansive mappings of C into itself such that $ST = TS$ and $F(S) \cap F(T) \neq \emptyset$. Assume that $x \preceq Tx$ and $x \preceq Sx$ for each $x \in C$. Suppose $x_1 \in C$ and $\{x_n\}$ is given by*

$$x_{n+1} = \alpha_n x_n + (1-\alpha_n)\frac{1}{(n+1)^2}\sum_{i=0}^{n}\sum_{j=0}^{n} S^i T^j x_n \quad \text{for every} \quad n \in \mathbb{N},$$

where $\{\alpha_n\}$ is a sequence in $[0,1]$. If $\{\alpha_n\}$ is chosen so that $\alpha_n \in [0,a]$ for some a with $0 < a < 1$, $\{x_n\}$ converges weakly to a point of $F(T) \cap F(S)$.

Theorem 4.3. *Let C be a nonempty closed convex subset of an ordered uniformly convex Banach space E and let T be a monotone nonexpansive mapping of C into itself such that $F(T) \neq \emptyset$. Assume that $x \preceq Tx$ for each $x \in C$. Suppose $x_1 \in C$ and $\{x_n\}$ is given by*

$$x_{n+1} = \alpha_n x_n + (1-\alpha_n)\frac{1}{n+1}\sum_{i=0}^{n} T^i x_n \quad \text{for every} \quad n \in \mathbb{N},$$

where $\{\alpha_n\}$ is a sequence in $[0,1]$. If $\{\alpha_n\}$ is chosen so that $\alpha_n \in [0,a]$ for some a with $0 < a < 1$, $\{x_n\}$ converges weakly to a point of $F(T)$.

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