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A HYBRID STEEPEST-DESCENT APPROXIMANTS SCHEME FOR CONVEX MINIMIZATION OVER SPLIT EQUILIBRIUM AND FIXED POINT PROBLEMS

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Abstract. In this paper, we propose a framework for the investigation of the convex minimization problem over the split equilibrium problems (SEP) and fixed point set of a finite family of multivalued demicontractive mappings in Hilbert spaces. We employ a hybrid steepest-descent approximants scheme which converges strongly to a common solution associated with the fixed point problem (FPP) and the SEP. Theoretical results comprise strong convergence results under suitable sets of constraints, as well as numerical results which are established for the underlying algorithm.
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1. Introduction

Throughout this paper, we assume that $\mathcal{H}$ is a real Hilbert space, and $\mathcal{D} \subseteq \mathcal{H}$ is a nonempty, closed and convex. Let $S : \mathcal{D} \to \mathcal{D}$ be a nonexpansive mapping (i.e., $\|Su - Sv\| \leq \|u - v\|$) and let $\Phi : \mathcal{H} \to \mathbb{R}$ be a convex and bounded below function. The minimization problem over the fixed point set of a mapping is defined as:

$$\text{find } u \in Fix(S) \text{ such that } \Phi(u) = \inf \Phi(Fix(S)), \tag{1.1}$$

where $Fix(S) = \{u \in \mathcal{D};\ Su = u\}$ denotes the fixed point set of S.

It is remarked that the problem (1.1) is equivalent to the following variational inequality problem VIP$(\acute{\Phi}, Fix(S))$ ([23]):

$$\text{find } u \in Fix(S) \text{ such that } \langle v - u, \acute{\Phi}(u) \rangle \geq 0, \text{ for all } v \in Fix(S), \tag{1.2}$$

provided that Φ is Gateaux differentiable over an open set including $Fix(S)$ where $\acute{\Phi}$ denotes the derivative of Φ.

For a slow decreasing sequence $\tilde{\alpha}_n \subset (0,1)$, the following class of hybrid steepest-descent approximants (HSDA):

$$t_{n+1} = S(t_n) - \tilde{\alpha}_{n+1}\acute{\Phi}(S(t_n)) \text{ for all } n \in \mathbb{N}, \tag{1.3}$$

is prominent for solving (1.2). The algorithm (1.3) converges strongly to the set of solutions of (1.2), involving a (quasi-)nonexpansive mapping S, under suitable set of conditions on $\Phi, \acute{\Phi}$ and $(\tilde{\alpha}_n)$ [33, 34]. A robust variant of HSDA, involving (asymptotically) quasi-shrinking operators, was analyzed in [35](see also [8, 5, 7, 10, 2, 1, 6, 9]).

In 2008, Maingé [26] studied the problem (1.1) involving a more general class of demicontractive and demiclosed mapping via the following Mann-type variant of the HSDA:

$$\begin{cases} y_n := t_n - \tilde{\alpha}_n \acute{\Phi}(t_n); \\ t_{n+1} := (1-\beta)y_n + \beta S y_n. \end{cases} \tag{1.4}$$

The following compact form of (1.4) coincides with the HSDA:

$$y_{n+1} = S_\beta y_n - \tilde{\alpha}_{n+1}\acute{\Phi}(S_\beta(y_n)), \tag{1.5}$$

where $S_\beta := (1-\beta)I + \beta S$ and I denotes the identity mapping. Thus the following natural question arises in view of the architecture of the approximants (1.4).
The theory of equilibrium problems is a systematic approach to study a diverse range of problems arising in the field of physics, optimization, variational inequalities, transportation, economics, network and noncooperative games, see, for example [12, 21, 22] and the references cited therein.

In 1994, Blum and Oettli [12] proposed a systematic mathematical formulation of equilibrium problems to solve a diverse range of problems occurring in various branches of sciences. Note that an equilibrium problem with respect to a bifunction G defined on a nonempty subset $\mathcal{D}$ of a real Hilbert space $\mathcal{H}_1$ aims to find a point $\bar{u} \in \mathcal{D}$ such that

$$G(\bar{u}, \bar{v}) \geq 0, \text{ for all } \bar{v} \in \mathcal{D}. \tag{1.6}$$

The set of equilibrium points or the set of solutions of the problem (1.6) is denoted by $EP(G)$. The current literature provides various classical iterative algorithms to solve the equilibrium problem. The class of split feasibility problems (SFP) has an extraordinary utility and broad applicability in medical image reconstruction, signal processing and computerized tomography [13, 20, 16, 15]. Some interesting and crucial results regarding the SFP with areas of feasible applications are established in [14, 18, 17]. The first prototype strategies for computing the optimal solution of the split common null point problem (SCNPP) can be found in [14]. Since then, different variants of these strategies have been proposed and analyzed for SCNPP and other instances of SFP [4, 3, 18, 17]. Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two real Hilbert spaces and $\ell : \mathcal{H}_1 \to \mathcal{H}_2$ be a bounded linear operator. Let $\mathcal{D}$ and Q be nonempty, closed and convex subsets of $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively.

We now finally introduce the formalism of the proposed problem.
Let $G_1 : \mathcal{D} \times \mathcal{D} \to \mathbb{R}$ and $G_2 : Q \times Q \to \mathbb{R}$ be two bifunctions, then SEP is to find:

$$\bar{u} \in \mathcal{D} \quad \text{such that } G_1(\bar{u}, \bar{v}) \geq 0 \text{ for all } \bar{v} \in \mathcal{D}, \tag{1.7}$$

and

$$y^* = \ell\bar{u} \in Q \quad \text{such that } G_2(y^*, y) \geq 0 \text{ for all } y \in Q. \tag{1.8}$$

The solution set of the SEP (1.7) and (1.8) is denoted by

$$\Omega := \{x^* \in \mathcal{D} : \bar{u} \in EP(G_1) \quad \text{and} \quad \ell\bar{u} \in EP(G_2)\}. \tag{1.9}$$

Another useful formalism for modeling various nonlinear phenomenon is the fixed point problem (FPP) of the operator under consideration. Most of the problems in diverse areas such as mathematical economics, variational inequality theory, control theory and game theory can be reformulated in terms of FPP. It is remarked that various nonlinear fixed point operators play equivalent important role in convex optimization problems(COP). In 2015, Takahashi et al.[31] investigated a unified formalism of equilibrium problem and FPP in Hilbert spaces. Since then, FPP associated with different nonlinear operators are jointly investigated with split equilibrium problem(SEP) in this domain. It is therefore natural to investigate FPP associated with an infinite family of operators jointly with SEP in Hilbert spaces.

A variety of strategies combining iterative optimization algorithms and fixed point algorithms have been introduced and analyzed to construct an optimal solution of the SEP and FPP. This pioneering work drives the mathematical research community to propose and analyze combined iterative algorithms to address two or more abstract mathematical problems.
In this paper we are interested to solve the convex minimization problems over the set of fixed point and split equilibrium problems in the setting of Hilbert space. We can reformulate (1.2)-(1.4) in the following form:

$$\text{find } \bar{u} \in (Fix(T) \cap \Omega) \text{ such that } \Phi(\bar{u}) = \inf \Phi((Fix(T) \cap \Omega). \tag{1.10}$$

Thus the following natural question arises in view of the architecture of the approximants (1.4):

Can one modify the approximants (1.4) to solve the convex minimization problem (1.1) over SEP and the fixed point set of multivalued mappings? Answering this question in the affirmative, we propose a HSDA for the convex minimization problem over SEP and the fixed point set of a finite family of multivalued demicontractive mappings in Hilbert spaces. As far as we know, such results have not so far appeared in the literature.

The rest of the paper is organized as follows: Section 2 contains some relevant preliminary concepts and results for convex minimization problem, SEP and fixed point problem. Section 3 comprises strong convergence results of the proposed a HSDA whereas Section 4 provides numerical results concerning the viability of the proposed algorithm.

2. Preliminaries

Let $\mathcal{CB}(\mathcal{D})$ denote the family of nonempty bounded and closed subsets of $\mathcal{D}$. The Hausdorff metric on $\mathcal{CB}(\mathcal{D})$ is defined as:

$$\mathcal{H}(\tilde{A},\tilde{B}) := \max\Big\{\sup_{u\in\tilde{A}} d(u,\tilde{B}), \sup_{v\in\tilde{B}} d(v,\tilde{A})\Big\},$$

for all $\tilde{A},\tilde{B}\in\mathcal{CB}(\mathcal{D})$ where $d(u,\tilde{B}) = \inf_{a\in\tilde{B}}\|u-a\|$.

Let $S:\mathcal{D}\to\mathcal{CB}(\mathcal{D})$ be a multivalued mapping, then u is said to be: (i) a fixed point of S if $u\in S(u)$ and (ii) an endpoint of S if $S(u)=\{u\}$. If S satisfies the endpoint condition then $Fix(S)$ is convex. Recall that the multivalued mapping S is said to be (i) nonexpansive if $\mathcal{H}(Su,Sv)^2 \leq \|u-v\|^2$ for all $(u,v)\in\mathcal{D}\times\mathcal{D}$, (ii) quasi-nonexpansive if $Fix(S)\neq\emptyset$ and $d(Sv,u)^2\leq\|v-u\|^2$ for all $v\in\mathcal{D}$ and $u\in Fix(S)$ and (iii) demicontractive [19] if $Fix(S)\neq\emptyset$ and there exists $\eta\in[0,1)$ such that $d(Sv,u)^2\leq\|u-v\|^2+\eta d(v,Sv)^2$ for all $v\in\mathcal{D}$ and $u\in Fix(S)$. The class of multivalued demicontractive mappings contains properly the class of multivalued quasi-nonexpansive mappings [24].

Definition 2.1. Let $\mathcal{D}$ be a nonempty, closed and convex subset of a Hilbert space $\mathcal{H}$ and $S:\mathcal{D}\to\mathcal{CB}(\mathcal{D})$ be a multivalued mapping. Then $I-S$ is said to be demiclosed at 0 if for any sequence (t_k) in $\mathcal{D}$ which converges weakly to $u\in\mathcal{D}$ and the sequence $(\|t_n-y_n\|)$ converges strongly to 0, where $y_n\in St_n$, then $u\in Fix(S)$.

For every point $u\in\mathcal{H}$, there exists a unique nearest point in $\mathcal{D}$, denote by $\mathcal{P}_{\mathcal{D}}u$, such that $\|u-\mathcal{P}_{\mathcal{D}}u\|\leq\|u-v\|\ \forall\ u,v\in\mathcal{D}$. The mapping $\mathcal{P}_{\mathcal{D}}$ is called the metric projection of $\mathcal{H}$ onto $\mathcal{D}$. It is well known that $\mathcal{P}_{\mathcal{D}}$ is nonexpansive and satisfies $\langle u-\mathcal{P}_{\mathcal{D}}u, b-\mathcal{P}_{\mathcal{D}}u\rangle\leq 0\ \forall\ b\in\mathcal{D}$.

Assumption 2.2. Let $\mathcal{D}$ be a nonempty, closed and convex subset of a Hilbert space $\mathcal{H}_1$. Let $G_1:\mathcal{D}\times\mathcal{D}\to\mathbb{R}$ be a bifunction satisfying the following conditions:
$(A1)$: $G_1(x,x)=0$ for all $x\in\mathcal{D}$;
$(A2)$: G_1 is monotone, i.e., $G_1(x,y)+G_1(y,x)\leq 0$ for all $x,y\in\mathcal{D}$;
$(A3)$: for each $x,y,z\in\mathcal{D}$, $\limsup_{t\to 0} G_1(kz+(1-k)x,y)\leq G_1(x,y)$;
$(A4)$: for each $x\in\mathcal{D}$, $y\mapsto G_1(x,y)$ is convex and lower semi-continuous.

Now, we proposed an iterative scheme for solving (1.10) in the more general case when S are a finite family of η-demicontractive multivalued mapping and split equilibrium problems, respectively.

Now we consider the proposed iterative scheme:
Let $G_1:\mathcal{D}\times\mathcal{D}\to\mathbb{R}$ and $G_2:Q\times Q\to\mathbb{R}$ be two bifunction satisfying Assumption 2.2 such that G_2 is upper semicontinous. Let $\ell:\mathcal{H}_1\to\mathcal{H}_2$ be a bounded linear operator and let ℓ^* be the adjoint operator of ℓ and let $S_j:\mathcal{D}\to\mathcal{CB}(\mathcal{D})$, $j\in\{1,2,\cdots,N\}$ is a finite family of η-demicontractive multivalued mapping. Let $\Phi:\mathcal{D}\to\mathbb{R}\cup(-\infty,+\infty]$ is a proper, convex and bounded below function. Assume that $\Pi:=\Omega\cap\bigcap_{j=1}^{N} Fix(S_j)\neq\emptyset$. To this end, in a more general frame, we investigate the asymptotic convergence of the sequence (t_n) generated, with an arbitrary t_0 in $\mathcal{H}$, we examine the following

iteration method:

$$\begin{cases} x_n := t_n - \tilde{\alpha}_n \acute{\Phi}(t_n); \\ y_n := T^{G_1}_{m_n}(I - \hbar\ell^*(I - T^{G_2}_{m_n})\ell)x_n; \\ t_{n+1} := (1-\gamma_n)x_n + \gamma_n \sum_{j=1}^N \mu_j w_n^{(j)}, \ j = \{1, 2, \cdots, N\} \end{cases} \tag{2.1}$$

where $w_n^{(j)} \in S_j y_n$, for $j = \{1, 2, \cdots, N\}$, $\mu_j \in (0, 1)$ such that $\sum_{j=1}^N \mu_j = 1$, ($\tilde{\alpha}_n \subset [0, 1)$), $\gamma_n \in (0, 1)$ and and $\hbar \in (0, \frac{1}{\vartheta})$ such that ϑ is the spectral radius of $\ell^*\ell$. These are the following conditions needed in throughout paper:

(C1) $\gamma_n \in (0, \frac{1}{2})$ and $\gamma_n(1-\gamma_n) > 0$;
(C2) $\liminf_{n\to\infty} m_n > 0$;
(C3) $\lim_{n\to\infty} \tilde{\alpha}_n = 0$;
(C4) $\sum_{n\geq 0} \tilde{\alpha}_n = +\infty$;
(C5) $\acute{\Phi}$ is L-Lipschitz continuous on $\mathcal{H}$ (for some $L \geq 0$); i.e.

$$\|\acute{\Phi}(u) - \acute{\Phi}(v)\| \leq L\|u - v\|, \text{ for all } u, v \in \mathcal{H}.$$

(C6) $\acute{\Phi}$ is Ψ-strongly monotone on $\mathcal{H}$ (for some $\Psi > 0$); i.e.

$$\langle \acute{\Phi}(u) - \acute{\Phi}(v), u - v\rangle \geq \Psi\|u - v\|^2, \text{ for all } u, v \in \mathcal{H}.$$

It is noted that the unique existence of the solution of (1.10) is ensured by the conditions (C5) and (C6) (see for instance [33]).

The following lemmas are helpful to prove the strong convergence results in the next section.

Lemma 2.3. *Let $u, v, n \in \mathcal{H}$ and $a \in [0, 1] \subset \mathbb{R}$, then*

(1) $\|u + v\|^2 \leq \|u\|^2 + 2\langle v, u + v\rangle$;
(2) $\|u - v\|^2 = \|u\|^2 - \|v\|^2 - 2\langle u - v, v\rangle$;
(3) $\|au + (1-a)v - n\|^2 = a\|u - n\|^2 + (1-a)\|v - n\|^2 - a(1-a)\|u - v\|^2$.

Lemma 2.4 ([32])**.** *Let $\mathcal{D}$ be a nonempty, closed and convex subset of a real Hilbert space $\mathcal{H}$ and $S : \mathcal{D} \to \mathcal{CB}(\mathcal{D})$ be a η-demicontractive multivalued mapping. Then, we have, (i) $Fix(S)$ is closed; (ii) if S satisfies the endpoint condition, then $Fix(S)$ is convex.*

Lemma 2.5 ([28])**.** *Let $\mathcal{D}$ be a nonempty closed convex subset of a real Hilbert space $\mathcal{H}$. For every $c, d, e \in \mathcal{H}$ and $\rho \in \mathbb{R}$, the set is closed and convex, which is defined as:*

$$C = \{f \in \mathcal{D} : \|d - f\|^2 \leq \|c - f\|^2 + \langle e, f\rangle + \rho\},$$

Lemma 2.6 ([25])**.** *Let $\mathcal{D}$ be a nonempty, closed and convex subset of a real Hilbert space $\mathcal{H}_1$ and let $G_1 : \mathcal{D} \times \mathcal{D} \to \mathbb{R}$ be a bifunction satisfying Assumption 2.2. For $m > 0$ and $x \in \mathcal{H}_1$, there exists $z \in \mathcal{D}$ such that*

$$G_1(z, y) + \frac{1}{m}\langle y - z, z - x\rangle \geq 0, \text{ for all } y \in \mathcal{D}.$$

Moreover, define a operator $T^G_m : \mathcal{H}_1 \to \mathcal{D}$ by

$$T^{G_1}_m(x) = \left\{ z \in C : G_1(z, y) + \frac{1}{m}\langle y - z, z - x\rangle \geq 0, \quad \text{for all } y \in C \right\},$$

for all $x \in \mathcal{H}_1$. *Then, we have the following observations:*

(1) : *for each* $x \in \mathcal{H}_1, T_m^{G_1} \neq \emptyset$*;*
(2) : $T_m^{G_1}$ *is single-valued;*
(3) : $T_m^{G_1}$ *is firmly nonexpansive;*
(4) : $Fix(T_m^{G_1}) = EP(G_1)$*;*
(5) : $EP(G_1)$ *is closed and convex.*

It is remarked that if $G_2 : Q \times Q \to \mathbb{R}$ is a bifunction satisfying Assumption 2.2, where Q is a nonempty, closed and convex subset of a Hilbert space $\mathcal{H}_2$. Then for each $r > 0$ and $w \in \mathcal{H}_2$, we define the operator:

$$T_r^{G_2}(w) = \left\{ d \in \mathcal{D} : G_2(d,e) + \frac{1}{r}\langle e-d, d-w\rangle \geq 0, \quad \text{for all } e \in Q \right\}.$$

Similarly, we have the following relations:

(1) : for each $w \in \mathcal{H}_2, T_r^{G_2} \neq \emptyset$;
(2) : $T_r^{G_2}$ is single-valued;
(3) : $T_r^{G_2}$ is firmly nonexpansive;
(4) : $Fix(T_r^{G_2}) = EP(G_2)$;
(5) : $EP(G_2)$ is closed and convex.

We need the following result to establish the strong convergence results of iterative scheme (2.1). First we show that $\ell^*(I - T_{m_k}^{G_2})\ell$ is a $\frac{1}{\vartheta}$-ism operator. For this, we utilize the firmly nonexpansiveness of $T_{m_k}^{G_2}$ which implies that $(I - T_{m_n}^{G_2})$ is a 1-ism operator. Now, observe that

$$\begin{aligned}
\|\ell^*(I - T_{m_k}^{G_2})\ell x - \ell^*(I - T_{m_k}^{G_2})\ell y\|^2 &= \langle \ell^*(I - T_{m_k}^{G_2})(\ell x - \ell y), \ell^*(I - T_{m_k}^{G_2})(\ell x - \ell y)\rangle \\
&= \langle (I - T_{m_k}^{G_2})(\ell x - \ell y), \ell^*\ell(I - T_{m_k}^{G_2})(\hbar x - \ell y)\rangle \\
&\leq \vartheta\langle (I - T_{m_k}^{G_2})(\ell x - \ell y), (I - T_{m_k}^{G_2})(\ell x - \ell y)\rangle \\
&= \vartheta\|(I - T_{m_k}^{G_2})(\ell x - \ell y)\|^2 \\
&\leq \vartheta\langle x - y, \ell^*(I - T_{m_k}^{G_2})(\ell x - \ell y)\rangle,
\end{aligned}$$

for all $x, y \in \mathcal{H}_1$. So, we observe that, $\ell^*(I - T_{m_k}^{G_2})\ell$ is a $\frac{1}{\vartheta}$-ism.
Moreover, $I - \hbar\ell^*(I - T_{m_k}^{G_2})\ell$ is nonexpansive provided $\hbar \in (0, \frac{1}{\vartheta})$.

Lemma 2.7. *Let* $G_1 : \mathcal{D} \times \mathcal{D} \to \mathbb{R}$ *and* $G_2 : Q \times Q \to \mathbb{R}$ *be two bifunction satisfying Assumption 2.2 such that* G_2 *is upper semicontinous. Let* $\ell : \mathcal{H}_1 \to \mathcal{H}_2$ *be a bounded linear operator and let* ℓ^* *be the adjoint operator of* ℓ. *Let* S_j, $j \in \{1, 2, \cdots, N\}$ *be a finite family of* η*-demicontractive multivalued mappings on* $\mathcal{H}$ *and* $\varPhi$ *be a convex, bounded below and Gateaux differentiable function on* $\mathcal{H}$ *with derivative* $\acute{\varPhi}$. *Assume furthermore the conditions* $(C1)-(C3)$ *and* $(C6)$ *hold. Then the sequence* (t_k) *given by (2.1) satisfies for all* $k \geq 0$,

$$U_{n+1} - U_n + \frac{1}{2}(1 - 2L\tilde{\alpha}_n)\|t_{n+1} - t_n\|^2 \leq -\tilde{\alpha}_n\langle t_n - \bar{u}, \acute{\varPhi}(t_n)\rangle, \tag{2.2}$$

where $\bar{u} \in \Pi$ *and*

$$U_n := \frac{1}{2}\|t_n - \bar{u}\|^2 + \tilde{\alpha}_n(\varPhi(t_n) - \inf\varPhi). \tag{2.3}$$

Proof. Let $\bar{u} \in \Pi$. Now, it follows from iterative scheme (2.1) that utilizing Lemma 2.3-2.6, we obtain

$$\begin{aligned}
\|t_{n+1} - \bar{u}\|^2 &= \|(1-\gamma_n)(x_n - \bar{u}) - \gamma_n(\sum_{j=1}^{M} \mu_j w_n^{(j)} - \bar{u})\|^2 \\
&= \| \sum_{j=1}^{M} \mu_j \big[(1-\gamma_n)(x_n - \bar{u}) - \gamma_n(w_n^{(j)} - \bar{u})\big] \|^2 \\
&\leq \sum_{j=1}^{M} \mu_j \big[\|(1-\gamma_n)(x_n - \bar{u}) - \gamma_n(w_n^{(j)} - \bar{u})\|^2\big] \\
&= \sum_{j=1}^{M} \mu_j \big[(1-\gamma_n)\|x_n - \bar{u}\|^2 + \gamma_n\|w_n^{(j)} - \bar{u}\|^2 - (1-\gamma_n)\gamma_n\|x_n - w_k^{(j)}\|^2\big] \\
&= \sum_{j=1}^{M} \mu_j \big[(1-\gamma_n)\|x_n - \bar{u}\|^2 + \gamma_n d(w_k^{(j)}, S_j\bar{u})^2 - (1-\gamma_n)\gamma_n\|x_n - w_n^{(j)}\|^2\big] \\
&\leq \sum_{j=1}^{M} \mu_j \big[(1-\gamma_n)\|x_n - \bar{u}\|^2 + \gamma_n \mathcal{H}(S_j y_n, S_j\bar{u})^2 - (1-\gamma_n)\gamma_n\|x_n - w_n^{(j)}\|^2\big] \\
&\leq \sum_{j=1}^{M} \mu_j \big[(1-\gamma_n)\|x_n - \bar{u}\|^2 + \gamma_n\|y_n - \bar{u}\|^2 - (1-\gamma_n)\gamma_n\|x_n - w_n^{(j)}\|^2\big].
\end{aligned} \tag{2.4}$$

Since $T^{G_1}_{m_n}\bar{u} = \bar{u}$ and we obtain

$$\begin{aligned}
\|y_n - \bar{u}\|^2 &= \|T^{G_1}_{m_n}(I - \hbar\ell^*(I - T^{G_2}_{m_n})\ell)x_n - \bar{u}\|^2 \\
&\leq \|x_n - \hbar\ell^*(I - T^{G_2}_{m_n})\ell x_n - \bar{u}\|^2 \\
&\leq \|x_n - \bar{u}\|^2 + \hbar^2\|\ell^*(I - T^{G_2}_{m_n})\ell x_n\|^2 \\
&\quad +2\hbar\langle \bar{u} - x_n, \ell^*(I - T^{G_2}_{m_n})\ell x_n\rangle.
\end{aligned} \tag{2.5}$$

Thus, we have

$$\begin{aligned}
\|y_n - \bar{u}\|^2 &\leq \|x_n - \bar{u}\|^2 + \hbar^2\langle \ell x_n - T^{G_2}_{m_n}\ell x_n, \ell^*\ell(I - T^{G_2}_{m_n})\ell x_n\rangle \\
&\quad +2\hbar\langle \bar{u} - x_n, \ell^*(I - T^{G_2}_{m_n})\ell x_n\rangle.
\end{aligned} \tag{2.6}$$

Moreover, we have

$$\begin{aligned}
\hbar^2\langle \ell x_n - T^{G_2}_{m_n}\ell x_n, \ell^*\ell(I - T^{G_2}_{m_n})\ell x_n\rangle &\leq \vartheta\hbar^2\langle \ell x_n - T^{G_2}_{m_n}\ell x_n, \ell x_n - T^{G_2}_{m_n}\ell x_n\rangle \\
&= \vartheta\hbar^2\|\ell x_n - T^{G_2}_{m_n}\ell x_n\|^2.
\end{aligned} \tag{2.7}$$

Note that

$$\begin{aligned}
&2\hbar\langle \bar{u}-x_n, \ell^*(I-T^{G_2}_{m_n})\ell x_n\rangle \\
&\quad = 2\hbar\langle \ell(\bar{u}-x_n), \ell x_n - T^{G_2}_{m_n}\ell x_n\rangle \\
&\quad = 2\hbar[\langle \ell\bar{u}-T^{G_2}_{m_n}\ell x_n, \ell x_n - T^{G_2}_{m_n}\ell x_n\rangle - \|\ell x_n - T^{G_2}_{m_n}\ell x_n\|^2] \\
&\quad \leq 2\hbar[\frac{1}{2}\|\ell x_n - T^{G_2}_{m_n}\ell x_n\|^2 - \|\ell x_n - T^{G_2}_{m_n}\ell x_n\|^2] \\
&\quad = -\hbar\|\ell x_n - T^{G_2}_{m_n}\ell x_n\|^2. \qquad (2.8)
\end{aligned}$$

Utilizing (2.6)-(2.8), we have

$$\begin{aligned}
\|y_n-\bar{u}\|^2 &\leq \|x_n-\bar{u}\|^2 + \vartheta\hbar^2\|\ell x_n - T^{G_2}_{m_n}\ell x_n\|^2 - \hbar\|\ell x_n - T^{G_2}_{m_n}\ell x_n\|^2 \\
&\leq \|x_n-\bar{u}\|^2 + \hbar(\vartheta\hbar-1)\|\ell x_n - T^{G_2}_{m_n}\ell x_n\|^2. \qquad (2.9)
\end{aligned}$$

Since $\hbar \in (0, \frac{1}{L})$, the estimate (2.9) implies that

$$\|y_n-\bar{u}\|^2 = \|x_n-\bar{u}\|^2. \qquad (2.10)$$

Utilizing (2.10), the estimate (2.4) implies that

$$\|t_{n+1}-\bar{u}\|^2 = \|x_n-\bar{u}\|^2 - (1-\gamma_n)\gamma_n\|x_n - w^{(j)}_k\|^2. \qquad (2.11)$$

From (2.1), we have

$$\sum_{j=1}^{M}\mu_j\|w^{(j)}_n - x_n\| = \frac{1}{\gamma_n}(t_{n+1}-x_n).$$

Setting $\Xi := \frac{1}{\gamma_n}(1-\gamma_n)$, we get from

$$\|t_{n+1}-\bar{u}\|^2 \leq \|x_n-\bar{u}\|^2 - \Xi\|t_{n+1}-x_n\|^2, \qquad (2.12)$$

so therefore, if $\gamma_k \in (0, \frac{1}{2})$ (so that $\Xi \geq 1$), we obtain

$$\|t_{n+1}-\bar{u}\|^2 \leq \|x_n-\bar{u}\|^2 - \|t_{n+1}-x_n\|^2, \qquad (2.13)$$

From (2.1) and (C3), we have

$$\begin{aligned}
\|x_n-\bar{u}\|^2 &= \|(t_n-\bar{u}) - \tilde{\alpha}_n\acute{\Phi}(t_n)\|^2 \\
&= \|t_n-\bar{u}\|^2 - 2\tilde{\alpha}_n\langle t_n-\bar{u}, \acute{\Phi}(t_n)\rangle + \alpha_n^2\|\acute{\Phi}(t_n)\|^2. \qquad (2.14)
\end{aligned}$$

Moreover

$$\begin{aligned}
\|x_n-t_{n+1}\|^2 &= \|(t_{n+1}-t_n) + \tilde{\alpha}_n\acute{\Phi}(t_n)\|^2 \\
&= \|t_{n+1}-t_n\|^2 + 2\tilde{\alpha}_n\langle t_{n+1}-t_n, \acute{\Phi}(t_n)\rangle + \tilde{\alpha}_k^2\|\acute{\Phi}(t_n)\|^2 \\
&= \|t_{n+1}-t_n\|^2 + 2\tilde{\alpha}_k\langle t_{n+1}-t_n, \acute{\Phi}(t_n) - \acute{\Phi}(t_{n+1})\rangle \\
&\quad +2\tilde{\alpha}_n\langle t_{n+1}-t_n, \acute{\Phi}(t_{n+1})\rangle + \tilde{\alpha}_n^2\|\acute{\Phi}(t_n)\|^2. \qquad (2.15)
\end{aligned}$$

Using the L-Lipschitz continuity of $\acute{\Phi}$ and the convexity of Φ, we obtain

$$\langle t_{n+1}-t_n, \acute{\Phi}(t_n) - \acute{\Phi}(t_{n+1})\rangle \geq -L\|t_{n+1}-t_n\|^2$$

and

$$\langle t_{n+1}-t_n, \acute{\Phi}(t_{n+1})\rangle \geq \Phi(t_{n+1}) - \Phi(t_n),$$

utilizing the above estimate in (2.15), we get

$$\begin{aligned}\|t_{n+1}-x_n\|^2 &\geq \|t_{n+1}-t_n\|^2-2\tilde{\alpha}_n L\|t_n-t_{n+1}\|^2+2\tilde{\alpha}_n(\Phi(t_{n+1})-\Phi(t_n))+\tilde{\alpha}_n^2\|\acute{\Phi}(t_n)\|^2\\ &=(1-2L\tilde{\alpha}_n)\|t_{n+1}-t_n\|^2+2\tilde{\alpha}_n(\Phi(t_{n+1})-\Phi(t_n))+\tilde{\alpha}_n^2\|\acute{\Phi}(t_n)\|^2. \qquad (2.16)\end{aligned}$$

So therefore, from (2.14), (2.16) in (2.13), we get

$$\begin{aligned}\|t_{n+1}-\bar{u}\|^2 \quad \leq \quad & \|t_n-\bar{u}\|^2-2\tilde{\alpha}_n\langle t_n-\bar{u},\acute{\Phi}(t_n)\rangle\\ & -(1-2L\tilde{\alpha}_n)\|t_{n+1}-t_n\|^2-2\tilde{\alpha}_n(\Phi(t_{n+1})-\Phi(t_n)),\end{aligned}$$

Rearranging the above statement, we have

$$\begin{aligned}&\|t_{n+1}-\bar{u}\|^2+2\alpha_{n+1}(\Phi(t_{n+1})-\inf\Phi)\\ &\quad\leq \|t_n-\bar{u}\|^2+2\tilde{\alpha}_n(\Phi(t_n)-\inf\Phi)-2\tilde{\alpha}_n\langle t_n-\bar{u},\acute{\Phi}(t_n)\rangle\\ &\qquad-(1-2L\tilde{\alpha}_n)\|t_{n+1}-t_n\|^2-2(\tilde{\alpha}_n-\alpha_{n+1})(\Phi(t_{n+1})-\inf\Phi).\end{aligned}$$

Note that, if $\tilde{\alpha}_n$ is non-increasing, we have $(\tilde{\alpha}_n-\alpha_{n+1})(\Phi(t_{n+1})-\inf\Phi)\geq 0$, that is

$$\begin{aligned}&\frac{1}{2}\|t_{n+1}-\bar{u}\|^2+\alpha_{n+1}(\Phi(t_{n+1})-\inf\Phi)\\ &\quad\leq\frac{1}{2}\|t_n-\bar{u}\|^2+\tilde{\alpha}_n(\Phi(t_n)-\inf\Phi)-\tilde{\alpha}_n\langle t_n-\bar{u},\acute{\Phi}(t_n)\rangle\\ &\qquad-\frac{1}{2}(1-2L\tilde{\alpha}_n)\|t_{n+1}-t_n\|^2.\end{aligned}$$

The following results can easily be derived from [26, Lemma 2.2 & 2.3].

Lemma 2.8. *Let the conditions (C1)-(C4) and (C6) hold, given for any $\bar{u}\in\Pi$ and any $\varepsilon\in(0,2)$, the sequence (t_n)(for $n\in\mathbb{N}$) given by (2.1) satisfies,*

$$\langle t_n-\bar{u},\acute{\Phi}(t_n)\rangle \geq \frac{1}{1+\Psi\varepsilon\tilde{\alpha}_n}(\Psi\varepsilon U_n-(D_\varepsilon+d\Psi\varepsilon\tilde{\alpha}_n)), \qquad (2.17)$$

where

$$\begin{aligned}U_n &:= \frac{1}{2}\|t_n-\bar{u}\|^2+\tilde{\alpha}_n(\Phi(t_n)-\inf\Phi),\\ d &:= \Phi(\bar{u})-\inf\Phi,\\ D_\varepsilon &:= \frac{\|\acute{\Phi}(\bar{u})\|^2}{2(2-\varepsilon)\Psi}.\end{aligned}$$

Further, assume that the conditions (C1)-(C3) and (C6) hold and suppose $(\tilde{\alpha}_k)\subset(0,\frac{1}{2L}]$ (when $L\neq 0$). Then we have

$$U_n \leq U_0 e^{-\frac{\Psi\varepsilon}{1+\Psi\varepsilon\alpha_0}(\sum_{r=0}^{k}\alpha_r-\alpha_0)}+(D_\varepsilon+d\Psi\varepsilon\alpha_0)\frac{1+2\Psi\varepsilon\alpha_0}{\Psi\varepsilon}e^{\frac{2\Psi\varepsilon}{1+\Psi\varepsilon\alpha_0}}. \qquad (2.18)$$

Proof. See proof in [26].

Lemma 2.9. *Let the conditions (C1)-(C4) and (C6) hold, given for any $\bar{u}\in\Pi$ and any $\varepsilon\in(0,2)$, the sequence (t_n)(for $n\in\mathbb{N}$) generated by (2.1) is bounded.*

Proof. This result is easily can see in consequence of Lemma 2.8.

3. Strong Convergence Analysis

In this section, we first introduce the necessary results required in the sequel to formulate the proposed scheme (2.1) for a convex minimization problems over the set of SEP and FPP associated with a finite family of η_j-demicontractive multivalued mapping in Hilbert spaces. Consequently, we formulate our hybrid stepeast descend iterative scheme along with the required control conditions for further investigation in such settings.

We need the following result to establish the strong convergence results of scheme (2.1).

Lemma 3.1. *Let the demiclosed principal and the conditions (C5) and (C6) hold and assume the sequence (t_n) generated by (2.1) is bounded and satisfies $\|t_{n+1} - x_n\| \to 0$ and $\|y_n - x_n\| \to 0$. Then $t_n \rightharpoonup \bar{u}$, $\bar{u} \in \Pi$ and we have*

$$\liminf_{n\to\infty} \langle t_n - \bar{u}, \acute{\Phi}(\bar{u})\rangle \geq 0,$$

where $\bar{u}$ is the solution of (1.10).

Proof. Let (t_{n_s}) be a subsequence of (t_n) which converges weakly to an element $\bar{u}$ in $\mathcal{H}$. Assume that $\|t_{n+1} - t_n\| \to 0$, $\tilde{\alpha}_n \to 0$ and (t_n) is bounded, consequently, t_{n_s} is weakly converges to $\bar{u}$ and $y_{n_s} := t_{n_s} - \tilde{\alpha}_{n_s}\acute{\Phi}(t_{n_s})$ converges weakly to $\bar{u}$, utilizing (C5) and boundedness of $\acute{\Phi}(t_{n_s})$, we have $\tilde{\alpha}_{n_s}\|\acute{\Phi}(t_{n_s})\| \to 0$. From (2.1), we get

$$\sum_{j=1}^{N} \mu_j \|w_{n_s}^{(j)} - x_{n_s}\| = \frac{1}{\gamma_n}\|t_{n_s+1} - x_{n_s}\| \to 0,\ j = 1, 2, \cdots, N.$$

It now from the following triangle inequality, we have

$$\|w_{n_s}^{(j)} - y_{n_s}\| \leq \|w_{n_s}^{(j)} - x_{n_s}\| + \|x_{n_s} - y_{n_s}\|,$$

that

$$\lim_{n\to\infty} \|w_{n_s}^{(j)} - y_{n_s}\| = 0. \tag{3.1}$$

Notably, S_j is demiclosed, then we obtain $x^* = S_j x^*$, $j \in \{1, 2, \cdots, N\}$. Next, we show that $x^* \in \Omega$.

Rearranging the the estimates (2.9) and Lemma 2.4, we have,

$$-\hbar(\vartheta\hbar - 1)\|\ell x_n - T_{m_n}^{G_2}\ell x_n\|^2 \quad \leq \quad \big(\|x_n - \bar{u}\| + \|y_n - \bar{u}\|\big)\|x_n - y_n\|. \tag{3.2}$$

Since $\hbar(\vartheta\hbar - 1) < 0$ and (3.2) that

$$\lim_{n\to\infty} \|\ell x_n - T_{m_n}^{G_2}\ell x_n\| = 0. \tag{3.3}$$

Note that $T^{G_1}_{m_n}$ is firmly nonexpansive and $I-\hbar\ell^*(I-T^{G_2}_{m_n})\ell$ is nonexpansive, therefore we have

$$\begin{aligned}
\|y_n-\bar{u}\|^2 &= \|T^{G_1}_{m_n}(x_n-\hbar\ell^*(I-T^{G_2}_{m_n})\ell x_n)-T^{G_1}_{m_n}\bar{u}\|^2\\
&\leq \langle T^{G_1}_{m_n}(x_n-\hbar\ell^*(I-T^{G_2}_{m_n})\ell x_n)-T^{G_1}_{m_n}\bar{u}, x_n-\hbar\ell^*(I-T^{G_2}_{m_n})\ell x_n-\bar{u}\rangle\\
&= \langle y_n-\bar{u}\ ,\ x_n-\hbar\ell^*(I-T^{G_2}_{m_n})\ell x_n-\bar{u}\rangle\\
&= \frac{1}{2}\{\|y_n-\bar{u}\|^2+\|x_n-\hbar\ell^*(I-T^{G_2}_{m_n})\ell x_n-\bar{u}\|^2\\
&\quad -\|y_n-x_n+\hbar\ell^*(I-T^{G_2}_{m_n})\ell x_n\|^2\}\\
&\leq \frac{1}{2}\{\|y_n-\bar{u}\|^2+\|x_n-\bar{u}\|^2-\|y_n-x_n+\hbar\ell^*(I-T^{G_2}_{m_n})\ell x_n\|^2\}\\
&= \frac{1}{2}\{\|y_n-\bar{u}\|^2+\|x_n-\bar{u}\|^2-(\|y_n-x_n\|^2+\hbar^2\|\ell^*(I-T^{G_2}_{x_n})\ell x_n\|^2\\
&\quad +2\hbar\langle y_n-x_n\ ,\ \ell^*(I-T^{G_2}_{m_n})\ell x_n\rangle)\}.
\end{aligned}$$

So, we have

$$\|y_n-\bar{u}\|^2 \quad\leq\quad \|x_n-\bar{u}\|^2-\|y_n-x_n\|^2+2\hbar\langle y_n-x_n,\ell^*(I-T^{G_2}_{m_n})\ell x_n\rangle. \tag{3.4}$$

Therefore, we have

$$\begin{aligned}
\|y_n-x_n\|^2 &\leq \|x_n-\bar{u}\|^2-\|y_n-\bar{u}\|^2+2\hbar\|y_n-x_n\|\|\ell^*(I-T^{G_2}_{m_n})\ell x_n\|\\
&\leq \big(\|x_n-\bar{u}\|+\|y_n-\bar{u}\|\big)\|x_n-y_n\|+2\hbar\|y_n-x_n\|\|\ell^*(I-T^{G_2}_{m_n})\ell x_n\|\\
&.
\end{aligned} \tag{3.5}$$

Utilizing (3.3) and (A1), we have

$$\lim_{n\to\infty}\|y_n-x_n\|=0. \tag{3.6}$$

From the estimate (3.6) and the triangle inequality $\|y_n-t_n\|\leq\|y_n-x_n\|+\|x_n-t_n\|$, we have

$$\lim_{n\to\infty}\|y_n-t_n\|=0. \tag{3.7}$$

Note that the existence of a convergent subsequence (x_{n_j}) of (x_n) such that $x_{n_j}\rightharpoonup\bar{u}\in\mathcal{H}_1$ as $j\to\infty$ follows from the boundedness of (x_n). This also infers that $y_{n_j}\rightharpoonup\bar{u}$, as $j\to\infty$. To establish the claim, we first prove that $\bar{u}\in\Omega$.
Show that $\bar{u}\in\Omega$.
Let $\bar{u}\in EP(G_1)$. For any $p\in\mathcal{H}_1$, we have

$$G_1(y_n,p)+\frac{1}{m_n}\langle p-y_n,y_n-x_n-\hbar\ell^*(I-T^{G_2}_{m_n}\ell x_n)\rangle\geq 0.$$

This implies that

$$G_1(y_n,p)+\frac{1}{m_n}\langle p-\ell_n,y_n-x_n\rangle-\frac{1}{m_n}\langle p-\ell_n,\hbar\ell^*(I-T^{G_2}_{m_n}\ell x_n)\rangle\geq 0.$$

From the Assumption 2.2(A2), we have

$$\frac{1}{m_n}\langle p-y_n,y_n-t_n\rangle-\frac{1}{m_n}\langle p-y_n,\hbar\ell^*(I-T^{G_2}_{m_n}\hbar x_n)\rangle\geq -G_1(y_n,p)\geq G_1(p,y_n).$$

So, we have

$$\frac{1}{m_{n_s}}\langle p-y_{n_s},y_{n_s}-t_{n_s}\rangle-\frac{1}{m_{n_s}}\langle p-y_{n_s},\hbar\ell^*(I-T^{G_2}_{m_n}\ell x_{n_s})\rangle\geq G(p,y_{n_s}). \tag{3.8}$$

Utilizing (3.3) and (C2), we get that $y_{n_s}\rightharpoonup\bar{u}$. Moreover, from (3.3) and the Assumption 2.2(A4), we get

$$G(p,\bar{u})\leq 0,\ \text{for all}\ p\ \in\ \mathcal{H}_1.$$

Let $q_k=kp+(1-k)\bar{u}$ for some $1\geq t>0$ and $p\in\mathcal{H}_1$. Since $\bar{u}\in\mathcal{H}$, consequently, $q_k\in\mathcal{H}$ and hence $G_1(q_k,p)\leq 0$. Using Assumption 2.2((A1) and (A4)), it follows that

$$\begin{aligned}0&=G_1(q_k,q_k)\\&\leq kG_1(q_k,p)+(1-k)G_1(q_k,p)\\&\leq k(G_1(q_k,p)).\end{aligned}$$

This implies that

$$G_1(q_k,p)\geq 0,\ \text{for all}\ p\in C.$$

Let $k\to 0$, we have

$$G_1(\bar{u},p)\geq 0,\ \text{for all}\ p\in C.$$

Thus, $\bar{u}\in EP(G_1)$. Similarly, we can show that $\bar{u}\in EP(G_2)$. Since ℓ is a bounded linear operator, we have $\ell t_{n_s}\rightharpoonup\ell\bar{u}$. It follows from (3.8) that

$$T^{G_2}_{m_{n_s}}\ell y_{n_s}\rightharpoonup\hbar\bar{u}\ as\ s\ \to\ \infty. \tag{3.9}$$

Now, from Lemma 2.7 we have

$$G_2(T^{G_2}_{m_{n_s}}\ell x_{n_s},p)+\frac{1}{m_{n_s}}\langle p-T^{G_2}_{m_{n_s}}\ell x_{n_s},T^{G_2}_{m_{n_s}}\hbar x_{n_s}-\ell x_{n_s}\rangle\geq 0,$$

for all $p\in\mathcal{H}_1$. Since G_2 is upper semicontinuous in the first argument and from (3.9), we have

$$G_2(\ell\bar{u},p)\geq 0,$$

for all $p\in\mathcal{H}_1$. This implies that, $\ell\bar{u}\in EP(G_2)$. Therefore, $\bar{u}\in\Omega$.
Since $t_{n_s}\rightharpoonup\bar{u}\in\mathcal{H}$ as $s\to\infty$, therefore we have $t_{n_s+1}\rightharpoonup\bar{u}$ and $t_{n_s}\rightharpoonup\bar{u}$ as $s\to\infty$. Moreover, from $x_n\rightharpoonup\bar{u}$ and $\|x_n-y_n\|\to 0$ as $n\to\infty$ imply that $y_n\rightharpoonup\bar{u}$.
The term $\langle t_n-\bar{u},\acute{\Phi}(\bar{u})\rangle$ is bounded, as (t_n) is bounded. So, there exists a subsequence (t_{n_s}) weakly converges to a point $x^*\in\mathcal{H}$, so therefore $x^*\in\Pi$ and such that

$$\liminf_{n\to\infty}\langle t_n-\bar{u},\acute{\Phi}(\bar{u})\rangle=\lim_{s\to\infty}\langle t_{n_s}-\bar{u},\acute{\Phi}(\bar{u})\rangle,$$

hence $\liminf_{n\to\infty}\langle t_n-\bar{u},\acute{\Phi}(\bar{u})\rangle=\langle x^*-\bar{u},\acute{\Phi}(\bar{u})\rangle$. As $\bar{u}$ is a solution of (1.10), we have $\langle x^*-\bar{u},\acute{\Phi}(\bar{u})\rangle\geq 0$. This is required result.

Lemma 3.2. *Let the demiclosed principal and the conditions (C4),(C5) and (C6) hold and let the sequence (t_n) generated by (2.1) has a subsequence (t_{n_s}) such that:*

(I) $(t_{n_s})\subset\Gamma:=\{x^*\in\mathcal{H}:\langle x^*-\bar{u},\acute{\Phi}(\bar{u})\rangle\leq 0\}$, *where $\bar{u}$ is the solution of (1.10).*

(II) $\|t_{n_s+1}-t_{n_s}\|\to 0$ *as $n\to\infty$.*

Then, (t_{n_s}) converges strongly to $\bar{u}$.

Proof. It is observed that using (C6), we have $\Psi\|t_{n_s}-\bar{u}\|^2 \leq \langle t_{n_s}-\bar{u}, \acute{\Phi}(t_{n_s})-\acute{\Phi}(\bar{u})\rangle$, then (I), it yields

$$\Psi\|t_{n_s}-\bar{u}\|^2 \leq -\langle t_{n_s}-\bar{u}, \acute{\Phi}(\bar{u})\rangle. \tag{3.10}$$

From (3.10), we obtain $\|t_{n_s}-\bar{u}\| \leq \frac{\acute{\Phi}(\bar{u})}{\Psi}$ so therefore, (t_{n_s}) and as well Γ are bounded. Consequently, a subsequence $(t_{n_s}) \in \mathcal{H}$ converges weakly to a point $x^* \in \mathcal{H}$ and utilizing (II), we obtain $\|t_{n_s}-t_{n_s+1}\| \to 0$ as $n \to \infty$. Moreover, from (2.1), we have

$$\begin{aligned} \gamma_n\|x_{n_s}-\sum_{j=1}^{N}\mu_j w_{n_s}^{(j)}\| &\leq \gamma_n\sum_{j=1}^{N}\mu_j\|x_{n_s}-w_{n_s}^{(j)}\| \\ &= \frac{1}{\gamma_s}\|t_{n_s+1}-x_{n_s}\| \to 0,\ as\ n\to\infty. \end{aligned} \tag{3.11}$$

By using (C6) and since $(\tilde{\alpha}_n) \to 0$, y_{n_s} converges weakly to $\bar{u}$. Note that, $\bar{u} \in \Pi$, (as proved in Lemma 3.1) and utilizing (3.10) and (1.10) entails

$$\limsup_{n\to+\infty}\|t_{n_s}-\bar{u}\|^2 \leq -(\frac{1}{\Psi})\langle x^*-\bar{u}, \acute{\Phi}(\bar{u})\rangle \leq 0,$$

hence $\lim_{s\to+\infty}\|t_{n_s}-\bar{u}\| = 0$. This is the required result.

Lemma 3.3. *Let the demiclosed principal and conditions(C1)-(C6) hold then the sequence (t_n) given by (2.1) satisfies:*

(I) $\|t_{n+1}-t_n\| \to 0$.
(II) $\lim_{n\to\infty}\|t_n-\bar{u}\|$ *exists,*

where $\bar{u}$ is the solution of (1.10). Then (t_n) converges strongly to $\bar{u}$.

Proof. It is observed that from Lemma 2.9, (t_n) is a bounded sequence. Suppose that $\lim_{n\to\infty}\|t_n-\bar{u}\| = \mu > 0$ and utilizing Lemma 3.1, we have $\liminf_{n\to\infty}\langle t_n-\bar{u}, \acute{\Phi}(\bar{u})\rangle \geq 0$ and also from (C6), we get

$$\langle t_n-\bar{u}, \acute{\Phi}(t_n)\rangle \geq \Psi\|t_n-\bar{u}\|^2 + \langle t_n-\bar{u}, \acute{\Phi}(\bar{u})\rangle.$$

After simplification, We obtain

$$\liminf_{n\to\infty}\langle t_n-\bar{u}, \acute{\Phi}(t_n)\rangle \geq \Psi\mu^2.$$

It deduced from Lemma 2.7 that there exists $n_0 \geq 0$ such that for $n \geq n_0$,

$$V_{n+1}-V_n \leq -\tilde{\alpha}_n(\frac{1}{2}\Psi\mu^2),$$

where $V_n := \frac{1}{2}\|t_n-\bar{u}\|^2 + \tilde{\alpha}_n(\Phi(t_n)-\inf\Phi)$, it yields

$$(\frac{1}{2}\Psi\mu^2)\sum_{s=n_0}^{n}\tilde{\alpha}_n \leq V_{n_0}-V_{n+1}.$$

It is observe from the above estimate, if $\sum\tilde{\alpha}_n = \infty$, then the last inequality is inappropriate as $n \to \infty$, because (t_n) is bounded, so its right hand side is supposed to be bounded, while the left hand side approaches to $+\infty$. Hence, as consequence $\mu = 0$. This is the required result.

Theorem 3.4. *Let $G_1 : \mathcal{D} \times \mathcal{D} \to \mathbb{R}$ and $G_2 : Q \times Q \to \mathbb{R}$ be two bifunction satisfying Assumption 2.2 such that G_2 is upper semicontinous. Let $\ell : \mathcal{H}_1 \to \mathcal{H}_2$ be a bounded linear operator and let ℓ^* be the adjoint operator of ℓ. Let $S_j : \mathcal{D} \to \mathcal{CB}(\mathcal{D})$ is a finite family of η_j-demicontractive multivalued mapping and Φ be a convex, bounded below and Gateaux differentiable function on $\mathcal{H}$ with derivative $\acute{\Phi}$. Assume that $\Pi := \Omega \cap \bigcap_{j=1}^{N} Fix(S_j) \neq \emptyset$. Suppose that (C1)-(C6) hold then the sequence (t_n) given by (2.1) converges strongly to $\bar{u}$, where $\bar{u}$ is the unique solution of (1.10).*

Proof. It follows from Lemma 2.8 that if $V_n = \frac{1}{2}\|t_n - \bar{u}\|^2 + \tilde{\alpha}_n(\Phi(t_n) - \inf \Phi)$, then both (V_n) and (t_n) are bounded. Hence, there exists a constant $M \geq 0$ such that $\|\langle t_n - \bar{u}, \acute{\Phi}(t_n)\rangle\| \leq M$ for all $n \geq 0$, utilizing Lemma 2.7, it yields

$$V_{n+1} - V_n + \frac{1}{2}(1 - 2L\tilde{\alpha}_n)\|t_{n+1} - t_n\|^2 \leq M\tilde{\alpha}_n. \tag{3.12}$$

For simplification, we consider the following two cases:
Case A. In the first instance, we assume that (V_n) is monotone, i.e., for large enough n_0, $(V_n)_{n \geq n_0}$ is either non-increasing or non-decreasing. In addition, (n_k) is bounded and hence it is convergent. Using (C1), that $\lim_{n\to+\infty}\|t_n - \bar{u}\|$ exists, that is, $\lim_{n\to\infty}\|t_n - \bar{u}\| = d$ (*say*). Utilizing (3.12) and $\lim_{n\to\infty}\|V_{n+1} - V_n\| = 0$, we have

$$\lim_{n\to\infty}\|t_{n+1} - t_n\| = 0. \tag{3.13}$$

Now, consider the re-arranging estimate (2.4) and using (C1), we have

$$\begin{aligned} \gamma_n(1 - \gamma_n)\sum_{j=1}^{M}\mu_j\|x_n - w_n^{(j)}\|^2 &\leq \|t_n - \bar{u}\|^2 - \|t_{n+1} - \bar{u}\|^2 \\ &\leq (\|t_n - \bar{u}\| + \|t_{n+1} - \bar{u}\|)\|t_n - t_{n+1}\|. \end{aligned}$$

Letting $n \to \infty$ and utilizing (3.13), we have

$$\gamma_n(1 - \gamma_n)\sum_{j=1}^{M}\mu_j\|x_n - w_n^{(j)}\|^2 = 0. \tag{3.14}$$

It is observed that

$$\sum_{j=1}^{N}\mu_j\|w_n^{(j)} - x_n\| = \frac{1}{\gamma_n}\|t_{n+1} - x_n\| \to 0, \ j = 1, 2, \cdots, N.$$

The above estimate implies that

$$\lim_{n\to\infty}\|x_n - t_{n+1}\| = 0. \tag{3.15}$$

From (3.13), (3.15) and the following triangular inequality:

$$\|x_n - t_n\| \leq \|x_n - t_{n+1}\| + \|t_{n+1} - t_n\|,$$

we get,

$$\lim_{n\to\infty}\|x_n - t_n\| = 0. \tag{3.16}$$

Note that $\limsup_{n\to\infty}\|y_n-\bar{u}\|\leq\limsup_{n\to\infty}\|t_n-\bar{u}\|\leq d$. Moreover, from the estimate (2.5), we get $d\leq\liminf_{n\to\infty}\|y_n-\bar{u}\|$. In total, we have $\lim_{n\to\infty}\|y_n-\bar{u}\|=d$. That is,

$$\lim_{n\to\infty}\|y_n-\bar{u}\|=\lim_{n\to\infty}\|T^{G_1}_{m_n}(I-\hbar\ell^*(I-T^{G_2}_{m_n})\ell)x_n-\bar{u}\|^2\|=d. \tag{3.17}$$

Since $\limsup_{n\to\infty}\|x_n-\bar{u}\|\leq d$ and by the virtue of the variant of estimate (2.9) $\limsup_{n\to\infty}\|T^{G_1}_{m_n}(I-\hbar\ell^*(I-T^{G_2}_{m_n})\ell)x_n-T^{G_1}_{m_n}(\bar{u})\|\leq d$. It now follows from Lemma 2.9, that

$$\lim_{n\to\infty}\|x_n-T^{G_1}_{m_n}(I-\hbar\ell^*(I-T^{G_2}_{m_n})\ell)x_n\|=0. \tag{3.18}$$

Also observe that, $\|x_n-y_n\|=\|x_n-T^{G_1}_{m_n}(I-\hbar\ell^*(I-T^{G_2}_{m_n})\ell)x_n\|$. It now follows from the estimate (3.22) that

$$\lim_{n\to\infty}\|x_n-y_n\|=0. \tag{3.19}$$

Consider this alternative version of the estimate (2.11) through utilizing the estimate (2.9):

$$\|t_{n+1}-\bar{u}\|^2 \quad\leq\quad \|x_n-\bar{u}\|^2-\hbar(1-\vartheta\hbar)\|\ell x_n-T^{G_2}_{m_n}\ell x_n\|^2.$$

The above estimate implies that

$$\begin{aligned}\hbar(1-\vartheta\hbar)\|\ell x_n-T^{G_2}_{m_n}\ell x_n\|^2 &\leq \|x_n-\bar{u}\|^2-\|t_{n+1}-\bar{u}\|^2\\ &\leq (\|x_n-\bar{u}\|-\|t_{n+1}-\bar{u}\|^2)\|x_n-t_{n+1}\|.\end{aligned}$$

The above estimate then implies from the estimate (3.15) that

$$\lim_{k\to\infty}\|\ell x_n-T^{G_2}_{m_n}\ell x_n\|^2=0. \tag{3.20}$$

Hence from Lemma 3.3, we deduce that $\bar{u}\in\Pi$.
Case B. Conversely, suppose (V_n) is not monotone sequence and for all $n\geq n_0$ (for some n_0 large enough) let a mapping $\omega:\mathbb{N}\to\mathbb{N}$ defined by

$$\omega(n):=\max\{s\in\mathbb{N};s\leq n,V_n\leq V_{n+1}\}. \tag{3.21}$$

Note that, ω is a non-decreasing sequence imply that $\omega(n)\to+\infty$ as $n\to+\infty$ and $V_{\omega_n}\leq V_{\omega(n)+1}$ for $n\geq n_0$, so therefor by using (3.12), it yields

$$\frac{1}{2}(1-2L\alpha_{\omega(n)})\|t_{\omega(n)+1}-t_{\omega(n)}\|^2\leq M\alpha_{\omega_n}\to 0, \tag{3.22}$$

hence, $\|t_{\omega(n)+1}-t_{\omega(n)}\|\to 0$. Utilizing Lemma 2.7, for any $k\geq 0$, the inequality $V_{k+1}<V_k$ holds provided that $t_k\notin\Gamma:=\{t\in\mathcal{H};\langle t-\bar{u},\acute{\Phi}(t)\rangle\leq 0\}$. Consequently, we have $t_{\omega(n)}\in\Gamma$ for all $n\geq n_0$ (since $V_{\omega(n)}\leq V_{\omega(n)+1}$). By Lemma 3.2, we conclude that $\|t_{\omega(n)}-\bar{u}\|\to 0$ and it follows that $\lim_{n\to\infty}V_{\omega(n)}=\lim_{n\to\infty}V_{\omega(n)+1}=0$. Moreover, for $n\geq n_0$, it is mention that $V_n\leq V_{\omega(n)+1}$ if $n\neq\omega(n)$ that is , if $\omega(n)<n$, because we have $V_k>V_{k+1}$ for $\omega(n)+1\leq k\leq n-1$. It follows that for all $n\geq n_0$, $0\leq V_n\leq\max\{V_{\omega(n)},V_{\omega(n)+1}\}\to 0$, hence $\lim_{n\to\infty}V_n=0$. This completes the proof.

4. Numerical experiment and Results

This section provides effective viability of our algorithm supported by a suitable example.

Example 4.1. Let $\mathcal{H}_1 = \mathcal{H}_2 = (\mathbb{R}, \langle \cdot, \cdot \rangle, |\cdot|)$ where $\langle t, u \rangle = tu$ and $\mathcal{D} \subset \mathcal{H}_1$. Let $G_1 : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ be a bifunction defined as $G_1(x, y) = 2x(y-x)$ and let $G_2 : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ be a bifunction defined as $G_2(p, q) = p(q-p)$. Consider the operator $\ell : \mathbb{R} \to \mathbb{R}$ are defined as $\ell(t) = 3t$. Suppose $\Phi : \mathbb{R} \to (-\infty, \infty]$ is defined as $\Phi(x) = \frac{1}{2}\|\tilde{A}x - \epsilon\|^2$, with $\tilde{A}x = 0 = \epsilon$. Then Φ is a proper, convex and lower semicontinuouos mapping, since $\tilde{A}$ is a continuous linear mapping (see[27]). Let a multivalued mapping $S : \mathbb{R} \to \mathcal{CB}(\mathbb{R})$ be defined as follows: if $t \in (-\infty, 0]$, take $S(t) = [-(\frac{7t}{3}), -4t]$. Then there exists unique sequences (t_n), (x_n), (y_n) and (t_{n+1}) generated by the iterative method in (2.1) then t_n converge strongly to a point $\bar{u}$.

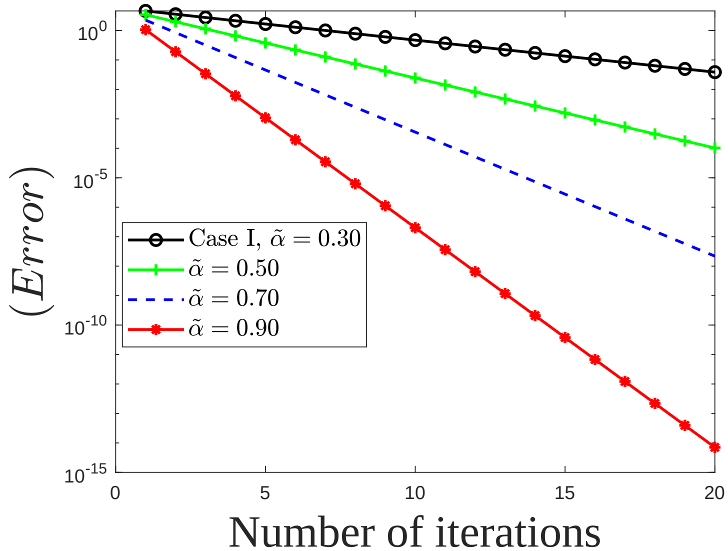
It is noted that the bifunctions G_1 and G_2 satisfy the Assumptions (A_1)-(A_4) and G_2 is upper semicontinuous with $\Omega = 0$. Moreover, ℓ is bounded linear operator on $\mathbb{R}$ with adjoint opertator ℓ^* and $\|\ell\| = \|\ell^*\| = 3$ and $\bigcap_{j=1}^{N} Fix(S_j) = 0$. Hence $\Pi := \Omega \cap \bigcap_{j=1}^{N} Fix(S_j) = 0$. In order to compute t_{n+1}, for each $j \in \{1, 2, \cdots, N\}$, take $S_j = S$. By Example 2.2 in Ref. [24], we know that S is a multivalued demicontractive operator with a constant $\eta = \frac{96}{121}$. Choose $w_k^{(j)} = -5t_n, j \in \{1, 2, \cdots, N\}$, $\gamma_n = \frac{1}{100*k+1}$, $N = 3 \times 10^4$. The numerical experiment of a hybrid steepest descent approximants defined in iterative scheme 2.1. The stopping criteria is defined as Error$=E_n = \|t_{n+1} - t_n\| < 10^{-5}$. The different cases of x_0 are giving as following:
Case I: $x_0 = 3.7$
Case II: $x_0 = 5$

The error plotting $\|t_{n+1} - t_n\|$ against the iterative scheme 2.1 for each case in Table 1 has shown in Figure 1.

Table 1. Computations of iterative scheme 2.1 with different values of $\tilde{\alpha}_k$.

	No. of Iterations		CPU Time	
	Case I	Case II	Case I	Case II
Scheme. 2.1 ($\tilde{\alpha}$=0.90)	20	34	0.022654	0.0264547
Scheme. 2.1 ($\tilde{\alpha}$=0.70)	91	63	0.037654	0.046367
Scheme. 2.1 ($\tilde{\alpha}$=0.50)	171	84	0.044364	0.069274
Scheme. 2.1 ($\tilde{\alpha}$=0.30)	233	102	0.068654	0.0743659

Case I



Case II

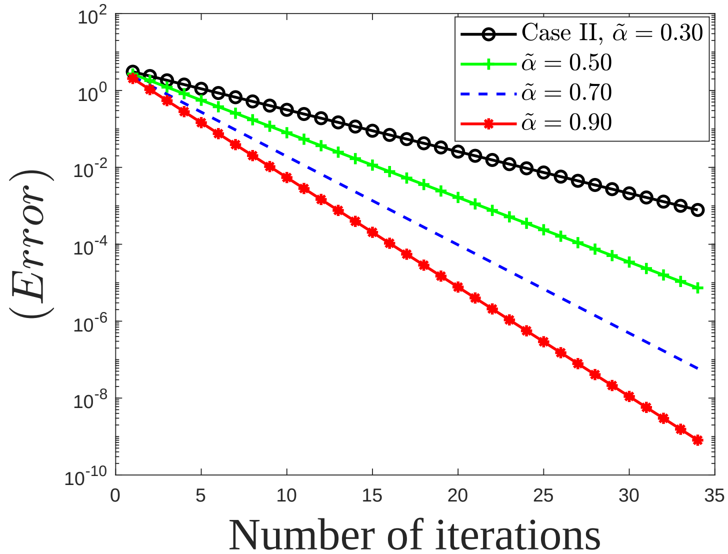


FIGURE 1. Computations of a number of iterations and error for iterative scheme 2.1

Remark 4.2.

(1) The example presented above is serving a purpose to show the impact of different values of $\tilde{\alpha}_n$ on our proposed iterative scheme.
(2) The numerical results presented in Table 1 and Figure **??** indicate that our proposed iterative scheme is efficient, easy to implement and does well for any values of $\tilde{\alpha} \neq 0$ in both number of iterations and CPU time required.
(3) We observe that the CPU time of iterative scheme 2.1 increases, but the number of iterations decreases when the parameter $\tilde{\alpha}$ approaches 1.
(4) We observe from the numerical implementation above as well that our proposed iterative scheme outperformed in the number of iterations and CPU time required to reach the stopping criterion.

5. Conclusions

In this paper, we have devised a hybrid steepest descent approximants method for computing the convex minimization problems over SEP and the set of FPP problems in Hilbert space. The theoretical framework of the algorithm has been strengthened with an appropriate numerical example. We would like to emphasize that the above mentioned problems occur naturally in many applications, therefore, iterative algorithms are inevitable in this field of investigation. As a consequence, our theoretical framework constitutes an important topic of future research.

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