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WEAKLY p-SUMMABLE SEQUENCES AND FIXED POINT THEORY IN BANACH LATTICES

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Abstract. Using weakly p-summable and Dunford-Pettis (resp. weakly p-summable and almost Dunford-Pettis) sequences, some geometric properties on Banach lattices are studied. Moreover, by the concept of relatively compact Dunford-Pettis property (briefly, $\mathrm{DP}_{rc}\mathrm{P}$) and strong $\mathrm{DP}_{rc}\mathrm{P}$, Banach lattices in which some of these properties coincide are characterized. As an application, Banach lattices with the Right fixed point property of order p are considered. In particular, it is established that for a Banach space X and a suitable Banach lattice F, a Banach lattice $\mathcal{M} \subset K(X, F)$ has the Right fixed point property of order p (resp. strong Right fixed point property of order p) if each evaluation operator ψ_{y^*} on $\mathcal{M}$ is Dunford-Pettis p-convergent (resp. almost Dunford-Pettis p-convergent), where $\psi_{y^*} : \mathcal{M} \to X^*$ is defined by $\psi_{y^*}(T) = T^*y^*$ for $y^* \in F^*$ and $T \in \mathcal{M}$.
Key Words and Phrases: Right topology, fixed point property, weak orthogonality, Dunford-Pettis set, almost Dunford-Pettis set.
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1. Introduction and preliminaries

Let D be a norm bounded subset of a Banach space X. A mapping $T : D \subset X \to X$ is *non-expansive* if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in D$. The fixed point set of T is $Fix(T) := \{x \in D : Tx = x\}$. A Banach space X has the *fixed point property (fpp)* if for every non-empty closed bounded convex subset D of X and every non-expansive map $T : D \to D$ we have $Fix(T) \neq \emptyset$. When the same holds for every nonempty weakly compact convex subset of X, we say that X has the *weak fixed point property (wfpp)*. Clearly we have fpp $\Rightarrow$ wfpp and if X is reflexive, then these two properties coincide. There are Banach spaces such as c_0, ℓ_1 with the wfpp and without the fpp. Also there are Banach spaces without wfpp such as $L^1[0,1], \ell_\infty$.

Only reflexive subspaces of $L^1[0,1]$ have the fpp [18, Remark 4.4]. A Banach space X has weak normal structure if for each weakly compact convex subset D there is an element $u \in D$ such that $\sup_{v \in D} \|u - v\|$ <diam (D). Every Banach space with

the Schur property and every Banach space with the weak normal structure have the wfpp [2, 16, 22].

For a closed subspace $\mathcal{M} \subset K(X,Y)$ (the Banach space of all compact operators between two Banach spaces X and Y), $\mathcal{M}^*$ has the Schur property if and only if all of the evaluation operators ϕ_x and ψ_{y^*} on $\mathcal{M}$ are compact, for $x \in X$, $y^* \in Y^*$ and $T \in \mathcal{M}$ [23] and these results are improved in the Banach lattice setting in [6].

A Banach lattice E is called *weak orthogonal* if for each weakly null sequence (x_n) in E, $\||x_n| \wedge |x|\| \to 0$ for all $x \in E$. A Banach lattice E has the *WORTH property* (resp. *non-strictly Opial condition*) if for each weakly null sequence (x_n) in E and $x \in E$, $\limsup_n \|x_n - x\| = \limsup_n \|x_n + x\|$ (resp. $\limsup_n \|x_n\| \leq \limsup_n \|x_n + x\|$) [13, 21].

If A is a norm bounded subset of X such that for each weakly null sequence (x_n^*) in X^*, $\lim_{n\to\infty} \sup_{a\in A} |\langle a, x_n^*\rangle| = 0$, then A is called a Dunford-Pettis set. Banach spaces in which Dunford-Pettis sets are relatively compact are said to have the Dunford-Pettis relatively compact property (DP$_{rc}$P). A Banach space X has the Dunford-Pettis property if each relatively weakly compact set in X is a Dunford-Pettis set [11].

Recently by the Right topology instead of weak topology, the concept of Right fixed point property for Banach spaces and Banach lattices was introduced [14]. Right topology on a Banach space X is a locally convex topology on X which is obtained by restriction of the Mackey topology $\tau(X^{**}, X^*)$ to X (which is the topology of uniform convergence on relatively $\sigma(X^*, X^{**})$ compact subsets of X^*) [17, 20]. It was proved that a sequence in a Banach space is Right null if and only if it is weakly null and Dunford-Pettis [15]. A Banach lattice E is called *Right orthogonal* if for each Right null sequence (x_n) in E, $\||x_n| \wedge |x|\| \to 0$ for all $x \in E$. A Banach lattice E has the *Right WORTH property* (resp. *non-strictly Right Opial condition*) if for each Right null sequence (x_n) in E and $x \in E$, $\limsup_n \|x_n - x\| = \limsup_n \|x_n + x\|$ (respectively, $\limsup_n \|x_n\| \leq \limsup_n \|x_n + x\|$) [14].

In order to study the Right fpp of order p, we need some definitions and notions. For each $1 \leq p < \infty$, a sequence (x_n) of a Banach space X is called *weakly p-summable* if for each $x^* \in X^*$, $(x^*(x_n)) \in \ell_p$ and (x_n) is said to be *weakly p-convergent* to an $x \in X$ if the sequence $(x_n - x) \in \ell_p^w(X)$, where $\ell_p^w(X)$ is the space of all weakly p-summable sequences in X. A bounded set A in a Banach space X is called *weakly p-compact*, if each sequence in A has a weakly p-convergent subsequence in A. Also a Banach space X is *weakly p-compact*, if the closed unit ball B_X is a weakly p-compact set. An operator T on a Banach space X is called weakly p-compact, if $T(B_X)$ is a weakly p-compact set. The space of all weakly p-compact operators is denoted by W_p. The reader can find some useful and additional properties about these concepts in [9].

An operator $T : E \to Y$ is called Dunford-Pettis p-convergent (resp. almost Dunford-Pettis p-convergent) if it carries weakly p-summable and Dunford-Pettis (resp. weakly p-summable and almost Dunford-Pettis) sequences in E into norm null ones in Y. A Banach space X has the Schur property of order p (or p-Schur property) if each weakly p-compact set in X is relatively compact. Alternatively X has the p-Schur property if and only if each weakly p-summable sequence in X is norm null. It is clear that Schur property implies the p-Schur property, but the converse is false. Banach spaces in which weakly p-compact and Dunford-Pettis sets are relatively compact are said to have the Dunford-Pettis relatively compact property of order p (p-DP$_{rc}$P). Also Banach lattices in which weakly p-compact and almost Dunford-Pettis sets are relatively compact are said to have the strong Dunford-Pettis relatively compact property of order p (strong p-DP$_{rc}$P).

Using weakly p-summable sequences, wfpp of order p was studied. A Banach lattice E is called *weak orthogonal of order p* if for each weakly p-summable sequence (x_n) in E, $\||x_n| \wedge |x|\| \to 0$ for all $x \in E$. A Banach lattice E has the *WORTH property of order p* (resp. *non-strictly Opial condition of order p*) if for each weakly p-summable sequence (x_n) in E and $x \in E$, $\limsup_n \|x_n - x\| = \limsup_n \|x_n + x\|$ (resp. $\limsup_n \|x_n\| \leq \limsup_n \|x_n + x\|$) [5].

In this article, we use weakly p-summable and Dunford-Pettis sequences to study the Right orthogonality of order p, Right WORTH property of order p, non-strictly Right Opial condition of order p and then the Right fpp of order p. Also, by weakly p-summable and almost Dunford-Pettis sequences, the strong version of these properties is introduced to consider the strong Right fpp of order p. Moreover, some results of the Dunford-Pettis property, weak Dunford-Pettis property, DP$_{rc}$P and strong DP$_{rc}$P are obtained.

The following general result will be introduced in our study: Right orthogonality of order p, non-strictly Right Opial condition of order p, Right WORTH property of order p (and also the strong version of them) in Banach lattices. As an application the connection between three properties and Right fpp of order p (or, strong Right fpp of order p) is established. We will show that for each Banach space X and a suitable Banach lattice F, a Banach lattice $\mathcal{M} \subset K(X, F)$ has the Right fpp of order p (resp. strong Right fpp of order p) if and only if each evaluation operator ψ_{y^*} on $\mathcal{M}$ is Dunford-Pettis p-convergent (resp. almost Dunford-Pettis p-convergent).

Recall some definitions and notations from Banach lattice theory. Throughout this article, X denotes the arbitrary Banach space, E denotes a Banach lattice and E^* refers to the dual of E. A norm bounded subset A of E is solid if $|x| \leq |y|$ for some $y \in A$ implies that $x \in A$. Each solid vector subspace of E is called an ideal. Also an ideal B of E is called a band if $\sup(A) \in B$ for every subset $A \subset B$ which has a supremum in E. A band B in a Banach lattice E is called a projection band if $E = B + B^{\perp}$, where $B^{\perp} = \{x \in E : |x| \wedge |y| = 0, \text{ for some } y \in B\}$. A Banach lattice E has the *positive Schur* property if each positive weakly null sequence in E is norm null [24]. A Banach lattice E has the weakly sequentially continuous lattice operations if for every weakly null sequence (x_n) in E, $|x_n| \xrightarrow{w} 0$. A Banach lattice E is called a KB-space if every increasing norm bounded sequence of E^+ is norm convergent. We refer the reader for undefined terminologies to the classical references [3, 19].

2. Right fixed point property of order p

In this section, we replace weakly p-compact sets by weakly p-compact Dunford-Pettis sets and instead of "wfpp of order p", we have the phrase "Right fpp of order p".

For the convenience, we use the notion p-Right compact (resp. almost p-Right compact) sets for weakly p-compact and Dunford-Pettis (resp. weakly p-compact and almost Dunford-Pettis) sets. Also, we use p-Right null (resp. almost p-Right null) sequences for weakly p-summable and Dunford-Pettis (resp. weakly p-summable and almost Dunford-Pettis) sequences.

Definition 2.1 A Banach space X has the *Right fixed point property of order p or p-Rfpp* if every non-expansive self-map $T : K \to K$ of each nonempty, closed convex and p-Right compact subset K of X has a fixed point.

It is important to note that a Banach space X has the $\mathrm{DP}_{rc}\mathrm{P}$ of order p (p-$\mathrm{DP}_{rc}\mathrm{P}$) if each weakly p-compact Dunford-Pettis set in X is relatively compact. Equivalently, X has the p-$\mathrm{DP}_{rc}\mathrm{P}$ if and only if for each p-Right null sequence (x_n) in X, $\|x_n\| \to 0$. Each Banach space with the p-$\mathrm{DP}_{rc}\mathrm{P}$ such as a reflexive Banach space has the p-Rfpp. Clearly, every Banach space with the Rfpp has the p-Rfpp, while the converse is false. For instance, $L^1[0,1]$ has the 1-Rfpp (by the 1-Schur property), but it does not have the Rfpp. Note that $L^1[0,1]$ is a Banach space with the Dunford-Pettis property and without the wfpp and so it does not have the Rfpp.

A non-empty closed convex subset D of a Banach space X is called a minimal T-invariant set for a mapping $T : X \to X$ if $T(D) \subset D$ and D has no non-empty T-invariant closed convex proper subset.

Theorem 2.2 *Let K be a non-empty weakly p-compact convex subset of a Banach space X. Then for a mapping $T : K \to K$ there is a closed convex subset of K which is minimal T-invariant.*

Proof. First, we show that each weakly closed subset of a weakly p-compact set in a Banach space is weakly p-compact. For this, let K be a weakly p-compact subset of X and A be a weakly closed subset of K. Let (x_n) be a sequence in A. Since $(x_n) \subset K$ and K is weakly p-compact, (x_n) has a weakly p-convergent subsequene (x_{n_k}) with $x_{n_k} \overset{w}{\to} x \in K$. Hence $(x_{n_k} - x)$ is a weakly p-summable and so weakly null sequence in K. Then (x_{n_k}) is a weakly convergent sequence in A. Since A is weakly closed, $x_{n_k} \overset{w}{\to} x \in A$ and then A is weakly p-compact. Now, consider the family $\mathcal{M}$ of all non-empty, closed convex (thus weakly p-compact) T-invariant subsets of K ordered by inclusion. By the weakly p-compactness, any chain of sets in $\mathcal{M}$ has non-empty intersection. By the Zorn'Lemma there is a minimal set in D which is T-invariant.

For every non-empty bounded closed convex minimal T-invariant set, we have $K = \overline{conv}T(K)$ and $\sup_{y\in K} \|x - y\| = diam(K)$ for each $x \in K$, where $T : K \to K$ is a non-expansive map. Note that $\overline{conv}T(K)$ is a closed convex T-invariant set in K.

Theorem 2.3 *If a Banach space X does not have the p-Rfpp, then for each $x_0 \in X$ there is a p-Right compact closed convex subset C_0 of X with $x_0 \in C_0$ and $diam(C_0) = \sup_{x,y\in C_0} \|x-y\| = 1$ and there is a non-expansive map $T_0 : C_0 \to C_0$ with $Fix(T_0) = \emptyset$.*

Proof. If a Banach space X does not have the p-Rfpp, then there is a p-Right compact closed convex set C with $diam(C) > 0$ and also a non-expansive map $T : C \to C$ with $Fix(T) = \emptyset$. Choose $x_1 \in C$ and let $C_0 = \frac{1}{diam(C)}(C - x_1) + x_0$ and consider

$$T_0 = \frac{1}{diam(C)}(T(d(x - x_0) + x_1) - x_1)) + x_0.$$

An easy calculation now verifies the claim.

Proposition 2.4 The p-Rfpp is separably determined.

Proof. If a Banach space X does not have the p-Rfpp, then there is a p-Right compact closed convex set C and a non-expansive map $T : C \to C$ with $Fix(T) = \emptyset$. Choose any point $c \in C$ and $K_1 = \{c\}$. Inductively define K_n by $K_{n+1} = \overline{conv}(T(K_n) \cup K_n)$. Then $\hat{K} = \overline{\cup_{n=1}^{\infty} K_n}$ is a separable closed convex p-Right compact subset of C.

If D is a p-Right compact minimal invariant set, then D is separable. If this were not the case then Proposition 2.4 would give the existence of a smaller invariant subset of D.

A closed convex subset K of a Banach space X has normal structure if every bounded convex subset H of K with more than one point contains a non-diametral point; that is there is $x_0 \in H$ such that $\sup_{x \in H} \|x_0 - x\| < diam(K)$.

For every subset K of a Banach space X, we have $r_u(K) = \sup_{x \in K} \|x - u\|$, $r(K) = \inf_{u \in K} r_u(K)$ and $C(K) = \{u \in K : r_u(K) = r(K)\}$ is the Chebyshev center of K.

Theorem 2.5 *Let K be a weakly p-compact convex subset of a Banach space X with normal structure. Then every non-expansive map $T : K \to K$ has a fixed point.*

Proof. As noted in Theorem 2.2, we may assume K is minimal T-invariant and so $\overline{conv}T(K) = K$. Let $u \in C(K)$, where $C(K)$ is the Chebyshev center of K. Then $r_u(K) = r(K)$. Since for every $v \in K$, $\|Tu - Tv\| < r(K)$, we have $T(K) \subset B(Tu, r_K)$. Consequently, $K = \overline{conv}T(K) \subset B(Tu, r(K))$ which implies that $r_{Tu}(K) = r(K)$ and then $Tu \in C(K)$. This shows that $C(K)$ is T-invariant contradicting the minimality of K unless $diam(K) = 0$. Thus K has a single point which is fixed under T.

Recall that a sequence (x_n) is an approximate fixed point sequence for a non-expansive map T if $\|Tx_n - x_n\| \to 0$. A bounded sequence (x_n) in a Banach space X is said to be diametral if $\lim_{n\to\infty} dist(x_{n+1}, conv\{x_1, x_2, ..x_n\}) = diam\{x_1, x_2, ..x_n\}$. It follows from [16, Lemma 4.1] that a closed convex subset K of a Banach space X has normal structure if and only if it does not contain a diametral sequence.

Theorem 2.6 *Each Banach space with the p-Schur property has the p-weak normal structure.*

Proof. If X does not have the p-weak normal structure, then there is a weakly p-compact subset of X without the normal structure. Then there is a weakly p-summable sequence (x_n) in X such that for $C = \overline{conv}\{x_n\}$, $\lim_{n\to\infty} \|x - x_n\| = diam(C) = 1$, for all $x \in C$. That is, (x_n) is a non-constant weakly p-summable sequence which is diameterising for its closed convex hull. In particular, since $0 \in C$, we have $\lim_{n\to\infty} \|x_n\| = 1$. Therefore X does not have the p-Schur property which is a contradiction.

If K is a bounded closed convex subset of a Banach space and $T : K \to K$ is non-expansive, then there is an approximate fixed point sequence (x_n) in X for T. [18, Lemma 2.4]. This observation is crucial to the proof of the following theorem.

Theorem 2.7 *Let K be a minimal T-invariant weakly p-compact convex subset of a Banach space X with $T : K \to K$ be a non-expansive map. If (x_n) is an approximate fixed point sequence in K, then for each $x \in K$, $\lim_{n\to\infty} \|x - x_n\| = diam(K)$.*

Proof. Suppose that (x_n) is an approximate fixed point sequence in K, but $\lim_{n\to\infty} \|x - x_n\| < diam(K)$ for some $x \in K$.

Let $C = \{z \in K : \limsup_{n\to\infty} \|z - x_n\| \leq \limsup_{n\to\infty} \|x_n - x\|\}$. It is easy to see that C is a closed convex subset of a weakly p-compact set K. Hence C is a weakly p-compact and also for each $u \in K, \limsup_{n\to\infty} \|u - x_n\| \leq \limsup_{n\to\infty} \|x - x_n\|$ which implies that $T : C \to C$. By the minimality of K, we have $C = K$. However, if $diam(K) > 0$, then $r(K) < diam(K)$ contradicting the fact that K cannot have any diametral points. Therefore for each $x \in K$, $\lim_{n\to\infty} \|x - x_n\| = diam(K)$.

We say that a Banach space X has *Right normal structure of order p (p-Right normal structure)* if each nonempty convex and p-Right compact subset $K \subset X$ has normal structure. Similar to Theorem 2.5, we can prove that each Banach space with the p-Right normal structure has the p-Rfpp. The converse is false.

Example 2.8 c_0 does not have p-Right normal structure. In fact, the standard basis (e_n) is a weakly p-summable Dunford-Pettis sequence in c_0 which $\|e_n\| = 1$ for all n; that is, c_0 does not have the p-DP$_{rc}$P. Consider a weakly p-compact Dunford-Pettis convex subset $D = \overline{co}\{e_n : n \in \mathbb{N}\}$ of c_0. It is clear that $\lim_n \|x-e_n\| = diam(D) = 1$, for all $x \in D$. This implies that c_0 does not have p-Right normal structure. However, c_0 has the p-wfpp and so by the Dunford-Pettis property it has the p-Rfpp.

Theorem 2.10 *Each Banach space with the p-DP$_{rc}$P has the p-Right normal structure.*

Proof. If X does not have the p-Right normal structure, then there is a p-Right compact convex subset C with more than one point which is diametral in the sense that, for all $x \in C$, $\sup_{y\in C} \|y - x\| = diam(C)$. We can assume that C is minimal T-invariant. Also, there is a weakly p-summable Dunford-Pettis (p-Right null) sequence (x_n) in X such that for $C = \overline{conv}\{x_n\}$, $\lim_{n\to\infty} \|x - x_n\| = diam(C) = 1$, for all $x \in C$. Since $0 \in C$, we have a p-Right null sequence (x_n) in X with $\lim_{n\to\infty} \|x_n\| = 1$. This implies that X does not have the p-DP$_{rc}$P.

3. Right orthogonality of order p

Recently in [14] the authors introduced Right orthogonality, non-strictly Right Opial condition and Right WORTH property in Banach lattices and then considered Banach lattices in which these properties are equivalent. In this section replacing the weakly null and Dunford-Pettis sequences by weakly p-summable and Dunford-Pettis (resp. weakly p-summable and almost Dunford-Pettis) sequences, we will use the phrases "p-Right orthogonality, non-strictly p-Right Opial and p-Right WORTH property" and the strong version of them. For the convenience, we use the notion strong p-Right compact sets for weakly p-compact and almost Dunford-Pettis sets and almost p-Right null sequences for weakly p-summable and almost Dunford-Pettis sequences.

Definition 3.1 A Banach lattice E is called p-Right orthogonal (resp. strong p-Right orthogonal) if for each p-Right null (resp. almost p-Right null) sequence (x_n) in E, $\||x_n| \wedge |x|\| \to 0$ for all $x \in E$.

Recall that a Banach lattice E has the p-DP$_{rc}$P (resp. strong p-DP$_{rc}$P) if each weakly p-compact and Dunford-Pettis (resp. weakly p-compact and almost Dunford-Pettis) set in E is relatively compact, or equivalently, each p-Right null (resp. almost p-Right null) sequence in E is norm null. It is clear that each Banach lattice with the p-DP$_{rc}$P (resp. strong p-DP$_{rc}$P) is p-Right orthogonal (resp. strong p-Right orthogonal).

We say that lattice operations in a Banach lattice E are *p-Right weakly sequentially continuous (resp. almost p-Right weakly sequentially continuous)* if for every p-Right null (resp. almost p-Right null) sequence (x_n) in E, $|x_n| \xrightarrow{w} 0$.

We may then characterize p-Right orthogonal (resp. strong p-Right orthogonal) Banach lattices as follows:

Theorem 3.2 *Let E be a Banach lattice. Then the statements in each of the following two collections are equivalent:*

(1) (a) *E is p-Right orthogonal,*
(b) *E has order continuous norm and p-Right weakly sequentially continuous lattice operations.*

(2) (a) *E is strong p-Right orthogonal,*
(b) *E has order continuous norm and almost p-Right weakly sequentially continuous lattice operations.*

Proof. We will show that $(a) \Leftrightarrow (b)$ in case (1). The arguments for the remaining implications of the theorem follow the same pattern.

$(a) \Rightarrow (b)$ Let E be a p-Right orthogonal Banach lattice. First, we show that E has order continuous norm. For this, let (x_n) be an order-bounded disjoint sequence in E^+. It is enough to show $\|x_n\| \to 0$. Since, (x_n) is order bounded, there is an element $e \in E^+$ such that $(x_n) \subset [0, e]$. By [7], an order-bounded disjoint sequence (x_n) is weakly p-summable and by [4, Theorem 2.8] it is Dunford-Pettis; that is, (x_n) is p-Right null. By the p-Right orthogonality, $\||x_n| \wedge e\| = \||x_n|\| = \|x_n\| \to 0$ and so E has order continuous norm.

Next, we show that for each p-Right null sequence (x_n) in E, the sequence $|x_n| \xrightarrow{w} 0$. Since E is p-Right orthogonal, $\||x_n| \wedge |x|\| \to 0$ for all $x \in E$. From [3, Theorem 13.6], for each $f \in E^*_+$ there is an element $u \in E^+$ such that $f(|x_n| - u)^+ \to 0$ for all n. Hence

$$f(|x_n|) = f(|x_n| - u)^+ + f(|x_n| \wedge |x|) \to 0.$$

Thus E has the p-Right weakly sequentially continuous lattice operations.

$(b) \Rightarrow (a)$ Let (x_n) be a p-Right null sequence in E. We show that $\||x_n| \wedge |x|\| \to 0$ for all $x \in E$. Let $y_n := |x_n| \wedge |x|$. Since E has p-Right weakly sequentially continuous lattice operations, then positive sequences $(|x_n|)$ and so (y_n) are weakly null. If $\|y_n\| \nrightarrow 0$, then by [19, Corollary 2.3.5], there is a disjoint positive subsequence (y_{n_k}) with $\|y_{n_k}\| \nrightarrow 0$ which is a contradiction (since (y_{n_k}) is an order bounded disjoint sequence in E, by the order continuity of the norm on E it must be norm null).

Clearly each Banach lattice with almost p-Right weakly sequentially continuous lattice operations has p-Right weakly sequentially continuous lattice operations. Also each strong p-Right orthogonal Banach lattice is p-Right orthogonal. But the following example shows that the converse is false.

Example 3.3 $L^2[0,1]$ is reflexive and then it has the p-DP$_{rc}$P; that is each p-Right null in $L^2[0,1]$ is norm null. Hence $L^2[0,1]$ is p-Right orthogonal and so it has p-Right weakly sequentially continuous. But $L^2[0,1]$ does not have almost p-Right weakly sequentially continuous lattice operations and is not strong p-Right orthogonal. In fact, the *Rademacher sequence* (r_n) in $L^2[0,1]$ are weakly p-summable for each $p \geq 2$ and almost Dunford-Pettis (since $-\mathbf{1} \leq r_n \leq \mathbf{1}$, for all n and each order interval is almost Dunford-Pettis, in general), but $|r_n| = 1$ for all $n \in \mathbb{N}$. This implies that $L^2[0,1]$ does not have almost p-Right weakly sequentially continuous lattice operations and so by Theorem 3.2 it cannot be strong p-Right orthogonal for each $p \geq 2$. Note that $L^2[0,1]$ has the 1-Schur property and so it is strong strong 1-Right orthogonal.

By the fact that c_0 does not have the p-DP$_{rc}$P, each Banach lattice with the p-DP$_{rc}$P does not contain a copy of c_0 and so it is a KB-space. Hence we have the following result:

Proposition 3.4 Each Banach lattice E with the p-DP$_{rc}$P and without the almost p-Right weakly sequentially continuous lattice operations is p-Right orthogonal, but it is not strong p-Right orthogonal.

Proof. Since each weakly null and Dunford-Pettis sequence in a Banach lattice E with the p-DP$_{rc}$P is norm null, we have E is p-Right orthogonal. Also E is a KB-space and so it has order continuous norm. On the other hand, E does not have almost p-Right weakly sequentially continuous lattice operations and so by Theorem 3.2 it cannot be strong p-Right orthogonal.

A Banach lattice E is p-weak orthogonal if and only if E has order continuous norm and p-weak sequentially continuous lattice operations [14]. It is clear that each p-weak orthogonal Banach lattice is p-Right orthogonal. The converse is false.

Example 3.5 Each Dunford-Pettis set in $L^2[0,1]$ is relatively compact; that is $L^2[0,1]$ has the p-DP$_{rc}$P and so it is p-Right orthogonal. But $L^2[0,1]$ is not p-weak orthogonal. Since $Id_{L^2[0,1]} \in W_2$ (see, [9, Proposition 3.6]); the *Rademacher sequence* (r_n) in $L^2[0,1]$ is weakly 2-summable. So (r_n) is weakly p-summable for each $p \geq 2$, but $|r_n| = 1$ for all $n \in \mathbb{N}$. This implies that $L^2[0,1]$ does not have p-weak sequentially continuous lattice operations and so it cannot be p-weak orthogonal, for each $p \geq 2$. Note that $L^2[0,1]$ is 1-Right orthogonal.

Recall that a Banach lattice E has the *Dunford-Pettis property of order p (resp. weak Dunford-Pettis property of order p)* if each weakly p-compact set in E is a Dunford-Pettis (resp. an almost Dunford-Pettis) set. Each Banach lattice with the Dunford-Pettis property (resp. weak Dunford-Pettis property) has the Dunford-Pettis property of order p (resp. weak Dunford-Pettis property of order p), but the converse is false. See [8, 9] for more information.

It is clear that each p-weak orthogonal Banach lattice is strong p-Right orthogonal, but the converse can be hold in each discrete Banach lattice. Also, we have the following corollary:

Corollary 3.6 *For a Banach lattice E with the p-weak Dunford-Pettis property the following statements are equivalent.*

(a) E is p-weak orthogonal,

(b) E is strong p-Right orthogonal.

Note that p-Dunford-Pettis property implies the p-weak Dunford-Pettis property. Also each p-Right orthogonal Banach lattice has order continuous norm. Thus we have the following corollary:

Corollary 3.7 *If E is discrete or E has the p-Dunford-Pettis property, then the following statements are equivalent.*

(a) E is p-weak orthogonal,

(a) E is p-Right orthogonal,

(b) E is strong p-Right orthogonal.

A Banach lattice E is Right orthogonal if and only if E has order continuous norm and Right sequentially continuous lattice operations [14]. Each Right orthogonal Banach lattice is p-Right orthogonal, but the converse is false. Consider, each non-discrete AL-space with the p-Schur property. Note that these spaces have order continuous norm and Dunford-Pettis property but none of them have Right sequentially continuous lattice operations, and so they cannot be Right orthogonal. But all of Banach lattices with the p-Schur property are p-Right orthogonal. For instance, $L^1[0,1]$ is a 1-Right orthogonal Banach lattice which is not Right orthogonal.

Example 3.8 $L^1[0,1]$ has the 1-Schur property and so it is a 1-Right orthogonal Banach lattice. Also $L^1[0,1]$ has order continuous norm, but it is not Right orthogonal. Indeed, the *Rademacher sequences* (r_n) in $L^1[0,1]$ are weakly null and Dunford-Pettis, but $|r_n| = 1$ for all $n \in \mathbb{N}$. This implies that $L^1[0,1]$ does not have the Right weakly sequentially continuous lattice operations and so it is not Right orthogonal.

Proposition 3.9 Each Banach lattice E with the p-DP$_{rc}$P and without the Right weakly sequentially continuous lattice operations is p-Right orthogonal, but it is not Right orthogonal.

Proof. Since c_0 does not have the p-DP$_{rc}$P, each Banach lattice with the p-DP$_{rc}$P does not contain a copy of c_0; that is a KB-space and so it has order continuous norm. On the other hand, E does not have the Right weakly sequentially continuous lattice operations and so it cannot be Right orthogonal.

Using p-Right null (resp. almost p-Right null) sequences we have the following definition.

Definition 3.10 Let E be a Banach lattice. Then:

(a) E has the p-Right WORTH (resp. strong p-Right WORTH) property if for each p-Right null (resp. almost p-Right null) sequence (x_n) in E and $x \in E$, $\limsup_n \|x_n - x\| = \limsup_n \|x_n + x\|$.

(b) E has the non-strictly p-Right Opial (resp. non-strictly strong p-Right Opial) condition if for each p-Right null (resp. almost p-Right null) sequence (x_n) in E and $x \in E$, $\limsup_n \|x_n\| \leq \limsup_n \|x_n + x\|$.

The following examples consider the relation between these concepts which will be used:

- Each Banach lattice with the Right WORTH property (resp. non-strictly Right Opial condition) has the p-Right WORTH property (resp. non-strictly p-Right Opial condition), but the converse is not true. In fact, $L^1[0,1]$ has the 1-Right WORTH property (resp. non-strictly 1-Right Opial condition), but it does not have the Right WORTH property (resp. non-strictly Right Opial condition).
- Each Banach lattice with the p-WORTH property (resp. non-strictly p-Opial condition) has the p-Right WORTH property (resp. non-strictly p-Right

Opial condition), but the converse is not true. All reflexive spaces $L^p[0,1]$ $(1 < p < \infty)$ have the p-Right WORTH property (resp. non-strictly p-Right Opial condition) but only $L^2[0,1]$ has the p-WORTH property (resp. non-strictly p-Opial condition).

- Each Banach lattice with the strong p-Right WORTH property (resp. non-strictly strong p-Right Opial condition) has the p-Right WORTH property (resp. non-strictly p-Right Opial condition) too, but the converse is not true. In fact, $L^2[0,1]$ has the p-Right WORTH property (resp. non-strictly p-Right Opial condition), but it does not have the strong p-Right WORTH property (resp. non-strictly strong p-Right Opial condition).

The following conditions on the underlying Banach lattices ensure that these three concepts are equivalent.

Theorem 3.11 *If a σ-Dedekind complete Banach lattice E has p-Right (resp. almost p-Right) weakly sequentially continuous lattice operations, then the following are equivalent:*

(a) *E is p-Right orthogonal (resp. strong p-Right orthogonal),*
(b) *E has the p-Right WORTH (resp. strong p-Right WORTH) property,*
(c) *E has the non-strictly p-Right Opial (resp. non-strictly strong p-Right Opial) condition.*

Proof. $(a) \Rightarrow (b)$ Let (x_n) be a p-Right null (resp. almost p-Right null) sequence in E. Then, $||x_n + x| - |x_n - x|| = 2(|x_n| \wedge |x|)$ for all $x \in E$ and so by the p-Right orthogonality (resp. strong p-Right orthogonality), we have

$$\||x_n + x| - |x_n - x|\| = 2(\||x_n| \wedge |x|\|) \to 0.$$

Thus E has the p-Right WORTH (resp. strong p-Right WORTH) property.

$(b) \Rightarrow (c)$ Let (x_n) be a p-Right null (resp. almost p-Right null) sequence in E. Then, by the p-Right WORTH (resp. strong p-Right WORTH) property, we have $\limsup_n \|x_n - x\| = \limsup_n \|x_n + x\|$ for all $x \in E$. Hence,

$$\limsup_n \|x_n\| \leq \frac{1}{2}(\limsup_n \|x_n - x\| + \limsup_n \|x_n + x\|) = \limsup_n \|x_n + x\|$$

and so E has the non-strictly p-Right Opial (resp. non-strictly strong p-Right Opial) condition.

$(c) \Rightarrow (a)$ If E has the non-strictly p-Right Opial (resp. non-strictly strong p-Right Opial) condition, then E does not contain a copy of ℓ_∞. Indeed, the standard unit basis (e_n) in ℓ_∞ is a weakly p-summable and Dunford-Pettis sequence and $\|e_n\| = 1$ for all n. Also, one can find an element $x \in \ell_\infty$ with $1 = \limsup_n \|e_n\| > \limsup_n \|e_n + x\|$. On the other hand E is σ-Dedekind complete and so by [19, Corollary 2.4.3] it has order continuous norm. Since E has the p-Right (resp. almost p-Right) weakly sequentially continuous lattice operations, then it is p-Right orthogonal (resp. strong p-Right orthogonal).

Note that both conditions mentioned in Theorem 3.11 are needed. Indeed, $L^1[0,1]$ is a σ-Dedekind complete Banach lattice with the non-strictly p-Right Opial condition which is not p-Right orthogonal. Also, c is a non-σ-Dedekind complete Banach lattice with the non-strictly p-Right Opial condition which is not p-Right orthogonal. In fact, c has the p-Right weakly sequentially continuous lattice operations, but $L^1[0,1]$ does not have the p-Right weakly sequentially continuous lattice operations.

We have the following corollary as summary:

Corollary 3.12 *If a σ-Dedekind complete Banach lattice E is discrete, the following are equivalent:*

(a) *E is p-weak orthogonal,*
(b) *E is p-Right orthogonal (resp. strong p-Right orthogonal),*
(c) *E has the p-Right WORTH (resp. strong p-Right WORTH) property,*
(d) *E has the non-strictly p-Right Opial (resp. non-strictly strong p-Right Opial) condition.*

The norm of a Banach lattice is said to be uniformly monotone if for given $\epsilon > 0$ there is a $\delta > 0$ such that if $x, y \geq 0$ with $\|y\| = 1$ and $\|x + y\| \leq 1 + \delta$ then $\|x\| \leq \epsilon$. Each Banach lattice with uniformly monotone norm has order continuous norm (since it does not contain any copy of c_0).

Theorem 3.13 *Let E be a Banach lattice. The following assertions hold:*

(a) *If E is discrete with the p-Right normal structure, then E is weak orthogonal.*
(b) *If E is p-Right orthogonal with uniformly monotone norm, then E has the p-Right normal structure.*

Proof.

(a) Since c_0 does not have p-Right normal structure, E does not contain any copy of c_0 and so it has order continuous norm. Each discrete Banach lattice with order continuous norm, is weak orthogonal.

(b) If E is p-Right orthogonal, then it has order continuous norm and the p-Right weakly sequentially continuous lattice operations. On the other hand E has uniformly monotone norm and similar to the result in [10] it has the p-Right normal structure.

Corollary 3.14 *Let E be a discrete Banach lattice with uniformly monotone norm. The following assertions hold:*

(a) *E has the p-Right normal structure,*
(b) *E is weak orthogonal.*
(c) *E is Right orthogonal,*
(d) *E is p-Right orthogonal.*

Definition 3.15 A Banach lattice E has the *strong Right fixed point property of order p or strong p-Rfpp* if every non-expansive self-map $T : K \to K$ of each nonempty, closed convex and weakly p-compact almost Dunford-Pettis (almost p-Right compact) subset K of E has a fixed point.

Clearly, every Banach lattice with the strong p-DP$_{rc}$P has the strong p-Rfpp. Also, every Banach lattice with the strong p-Rfpp has the p-Rfpp while the converse is false. For instance, $L^1[0,1]$ has the 1-Rfpp (by the 1-Schur property), but it does not have the Rfpp. Note that $L^1[0,1]$ is a Banach space with the Dunford-Pettis property and without the wfpp and so it does not have the Rfpp.

We say that a Banach lattice E has the *p-Right Opial (resp. strong p-Right Opial) condition* if for each p-Right null (resp. almost p-Right null) sequence (x_n) in E and $x \in E$ we have $\limsup_n \|x_n\| < \limsup_n \|x_n + x\|$. We know that each Banach lattice with the Opial condition has the wfpp and similarly it can be proved that each Banach lattice with the p-Right Opial (resp. strong p-Right Opial)condition has the p-Rfpp (resp. strong p-Rfpp).

There is another version of p-Right Opial condition, the so-called uniformly p-Right Opial (resp. strong p-Right Opial) condition. All p-Right Opials condition, non-strict p-Right Opial condition, uniformly p-Right Opial condition (and also the strong versions of them) have an important role for the p-Rfpp (also the strong p-Rfpp) in Banach lattices. A Banach lattice E has uniformly p-Right Opial's (resp. strong p-Right Opial's) condition if for each $c > 0$ there is an $r > 0$ such that $1 + r \leq \liminf_n \|x_n + x\|$ for each $x \in E$ with $\|x\| \geq c$ and each p-Right null (resp. almost p-Right null) sequence (x_n) in E and $\liminf_n \|x_n\| \geq 1$. In general, uniformly p-Right Opial's condition (resp. strong p-Right Opial's) implies the p-Right Opial's (resp. strong p-Right Opial's) condition and so the non-strictly p-Right Opial's (resp. strong p-Right Opial's) condition.

We can prove that each Banach lattice with uniformly monotone norm and p-Right weakly sequentially continuous lattice operations has the uniformly p-Right Opial's condition.

Definition 3.16 A Banach lattice E has the:

(a) p-Right (resp. strong p-Right) (m_q) property $(1 \leq q < \infty)$ if for all p-Right null (resp. almost p-Right null) sequence (x_n) in E and $x \in E$, $\limsup_n \|x_n + x\|^q = \limsup_n \|x_n\|^q + \|x\|^q$.

(b) p-Right (resp. strong p-Right) (m_∞) property if for all p-Right null (resp. almost p-Right null) sequence (x_n) in E and $x \in E$, $\limsup_n \|x_n + x\| = \limsup_n \|x_n\| \vee \|x\|$.

The following theorem characterizes p-DP$_{rc}$P (resp. strong p-DP$_{rc}$P) with two properties.

Theorem 3.17 *A Banach lattice E has the p-DP$_{rc}$P (resp. strong p-DP$_{rc}$P) if and only if E has the p-Right (resp. strong p-Right) (m_∞) and p-Right (resp. strong p-Right) (m_1) property.*

Proof. If E has the p-Right (resp. strong p-Right) (m_∞) and p-Right (resp. strong p-Right) (m_1) property, then for all p-Right null (resp. almost p-Right null) sequence (x_n) in E and $x \in E$,

$$\limsup_n \|x_n + x\| = \limsup_n \|x_n\| + \|x\| = \limsup_n \|x_n\| \vee \|x\|.$$

Hence $\limsup_n \|x_n\| = 0$ or $\|x\| = 0$, for all $x \in E$. This implies that $\lim_n \|x_n\| = 0$ and so E has the p-DP$_{rc}$P (resp. strong p-DP$_{rc}$P).

In the other direction assume that E has the p-DP$_{rc}$P (resp. strong p-DP$_{rc}$P) and (x_n) be a p-Right (resp. almost p-Right) null sequence (x_n) in E. Then $\lim_n \|x_n\| = 0$ and so

$$\limsup_n \|x_n + x\| = \|x\| = \|x\| + \limsup_n \|x_n\| = \limsup_n \|x_n\| \vee \|x\|.$$

Therefore E has the p-Right (resp. strong p-Right) (m_∞) and p-Right (resp. strong p-Right) (m_1) property.

4. positive p-Rfpp and some operator spaces

Using the positive or disjoint weakly p-summable and Dunford-Pettis sequences instead of weakly p-summable and Dunford-Pettis ones, the positive version of p-Right orthogonality, p-Right WORTH property and non-strictly p-Right Opial condition is studied in this section. We also discuss their applications to study the positive p-Rfpp.

We say that a Banach lattice E has the *positive p-DP$_{rc}$P* if each p-Right null sequence in E with the positive terms is norm null. By the same technique in [12, Theorem 3.15], we have a Banach lattice E has the positive p-DP$_{rc}$P if and only if each positive almost p-Right null sequence in E is norm null. It is easily verified that each Banach lattice with the positive p-DP$_{rc}$P is positive p-Right orthogonal.

Theorem 4.1 *For a Banach lattice E, the following are equivalent:*

(a) *E has order continuous norm,*
(b) *E is positive p-Right orthogonal,*
(c) *E is disjoint p-Right orthogonal.*

Proof. $(a) \Rightarrow (b)$ Let (x_n) be a positive p-Right null sequence in E. We show that $\||x_n| \wedge |x|\| \to 0$ for all $x \in E$. Let $y_n := |x_n| \wedge |x|$, then (y_n) is a positive p-Right null sequence in E. If $\|y_n\| \nrightarrow 0$ then by [19, Corollary 2.3.5], there is a disjoint positive subsequence (y_{n_k}) with $\|y_{n_k}\| \nrightarrow 0$ which is a contradiction (since (y_{n_k}) is an order bounded disjoint sequence in E and it must be norm null, by the order continuity of the norm on E).
$(b) \Rightarrow (c)$ First, note that for each disjoint weakly p-summable sequence (x_n) in a Banach lattice E, the sequence $(|x_n|)$ is also weakly p-summable [26, Corollary 3.1.6]. Also it can be proved that each disjoint weakly null Dunford-Pettis sequence $(|x_n|)$ is Dunford-Pettis, [12, Lemma 3.7]. Second, let (x_n) be a disjoint p-Right null sequence in E. Then $(|x_n|)$ is weakly p-summable and Dunford-Pettis. Let $y_n := |x_n| \wedge |x|$, for all $x \in E$. Then (y_n) is a positive weakly p-summable and Dunford-Pettis sequence in E and so $\|y_n\| \to 0$.
$(c) \Rightarrow (a)$ Let (x_n) be an order bounded disjoint sequence in E. From [3, Theorem 4.14], it is enough to show $\|x_n\| \to 0$. Since (x_n) is an order bounded sequence, then there is an element $e \in E^+$ such that $(x_n) \subset [0, e]$. Also the sequence (x_n) is weakly p-summable [7, Theorem 2.8] and Dunford-Pettis [4, Theorem 2.8]. By the disjoint p-Right orthogonality, $\||x_n| \wedge |x|\| \to 0$ for all $x \in E$. Hence, $\||x_n| \wedge e\| = \||x_n|\| = \|x_n\| \to 0$ and so E has order continuous norm.

Since ℓ_∞ does not have the positive non-strictly p-Right Opial condition, a Banach lattice with positive non-strictly p-Right Opial condition does not contain a copy of ℓ_∞. We know that a σ-Dedekind complete Banach lattice has order continuous norm if and only if it does not contain a copy of ℓ_∞. Hence a σ-Dedekind complete Banach lattice is positive p-Right orthogonal if and only if it has the positive p-Right WORTH property if and only if it has the positive non-strictly p-Right Opial condition.

It is clear that p-Right orthogonality (resp. p-Right WORTH property or non-strictly p-Right Opial condition) implies the positive p-Right orthogonality (resp. positive p-Right WORTH property or non-strictly positive p-Right Opial condition). The converse is false. Consider $L^1[0,1]$.

Corollary 4.2 *For a Banach lattice E with the p-Right weakly sequentially continuous lattice operations, the following are valid:*

(a) *E is positive p-Right orthogonal if and only if E is p-Right orthogonal.*
(b) *E has the positive p-Right WORTH property if and only if E has the p-Right WORTH property.*
(c) *E has the positive non-strictly p-Right Opial condition if and only if E has the non-strictly p-Right Opial condition.*

Following the discussion in section 3 in connection with so called p-Rfpp, we now introduce the notion of positive p-Rfpp:

Definition 4.3 A Banach lattice E has the *positive p-Rfpp* if every non-expansive self-map $T : K \to K$ of each nonempty, convex and weakly p-compact Dunford-Pettis with the positive terms subset K of E has a fixed point.

Each p-Right orthogonal Banach lattice has the p-Rfpp and by the same techniques we can show that each positive p-Right orthogonal Banach lattice has the positive p-Rfpp. On the other hand, each Banach lattice with the p-Rfpp has the positive p-Rfpp, but the converse is false. In fact, $L^1[0,1]$ has the positive Schur property and so it is a positive p-Right orthogonal Banach lattice. Then $L^1[0,1]$ has the positive p-Rfpp, but only reflexive subspaces of $L^1[0,1]$ have the p-Rfpp. Note that all AL-spaces have the positive Schur property and so they have the positive Rfpp.

Based on our discussion and by the same arguments in [6, 25, 13], we can assume that X is an arbitrary Banach space and then improve [25, Theorem 2.6] for R-fpp (or the positive R-fpp) for a suitable Banach lattice of some compact operator spaces from a Banach space into a Banach lattice as follows. First, we define two classes of operators which will be needed in Lemma 4.5.

Definition 4.4 An operator $T : E \to X$ is called:

(a) Dunford-Pettis p-convergent (briefly, DPpc) if for every p-Right null sequence (x_n) in E, $\|Tx_n\| \to 0$.
(b) almost Dunford-Pettis p-convergent (briefly, almost DPpc) if for every almost p-Right null sequence (x_n) in E, $\|Tx_n\| \to 0$.

It is clear that each DPpc operator is almost DPpc, but the converse is not true. In fact, the identity operator on each Banach lattice with the p-DP_{rc}P and without the strong p-DP_{rc}P such as $L^2[0,1]$ is almost DPpc, but it is not DPpc. To continue, we need the following two lemmas.

For the first lemma, we may use Remark 2.3 in [1] in a similar fashion to prove that:

Lemma 4.5 *Let X and Y be two Banach spaces and $\mathcal{M} \subset L(X,Y)$ be a Banach lattice.*

(a) *If all evaluation operators ϕ_x are DPpc (resp. almost DPpc), then the operator $T \to TS$ from $\mathcal{M}$ into $K(X,Y)$ is DPpc (resp. almost DPpc) for all compact operators $S \in K(X)$.*
(b) *If all evaluation operators ψ_{y^*} are DPpc (resp. almost DPpc), then the operator $T \to T^*K$ from $\mathcal{M}$ into $K(Y^*, X^*)$ is DPpc (resp. almost DPpc), for all compact operator $K \in K(Y^*)$.*

Proof. (a) Let (T_n) be a p-Right null (resp. an almost p-Right null) sequence in $\mathcal{M}$. We show that $\|T_nS\| \to 0$ for all $S \in K(X)$. Since for every $x \in X$, the evaluation operator ϕ_x is DPpc (resp. almost DPpc), $\|T_nx\| \to 0$, that is, (T_n) converges to 0 strongly. Since the sequence (T_n) is bounded, T_n converges uniformly to 0 on compact subsets. Hence, T_nS converges in norm to 0, for every $S \in K(X)$.

Similarly, T_n^*K converges uniformly to 0, for every $K \in K(Y^*)$. Therefore, for every $S \in K(X)$ and $K \in K(Y^*)$, the mappings $\mathcal{M} \to K(X,Y)$ and $\mathcal{M} \to K(Y^*, X^*)$ defined by $T \to TS$ and $T \to T^*K$ are DPpc (resp. almost DPpc) operators.

For the second lemma (which can be proved by the same arguments in [6, 25]) we recall some notations. Let F be a discrete Banach lattice with complete disjoint systems consisting of discrete elements $\{u_i\}_{i\in I}$. Then $W = \sum_{i\in I} I_{u_i}$ is a projection band and $F = W + W^{\perp}$. The projection $P_W : E \to E$ defined by $P_W(x) = x_1$ is the band projection onto W and $\|P_W\| = 1$.

Lemma 4.6 *Let X be a Banach space, F be a discrete Banach lattice with order continuous norm and $\mathcal{M} \subset K(X,F)$ be a Banach lattice. Then the following assertions hold:*

(a) *If $K_1, K_2, ...K_n \in K(X,F)$ and $\epsilon > 0$, then there is a finite dimensional projection band $W \subset F$ such that $\|P_{W^\perp} K_i\| \leq \epsilon$ for all $i = 1, 2, ...n$.*
(b) *If all of the evaluation operators ψ_{y^*} are DPpc (resp. positive DPpc), (K_n) is a p-Right null (resp. positive p-Right null) sequence in $\mathcal{M}$, then there is a subsequence (K_{n_i}) of (K_n) and a sequence (U_i) of $K(X,F)$ such that* $\lim_{i\to\infty} \|U_i - K_{n_i}\| = 0$

Proof. (a) It follows from [6, Lemma 2.4].

(b) Let (ϵ_i) be a sequence of positive numbers that tend to 0 and $n_0 = 1$. By (a), we can find $m_0 \in \mathbb{N}$ such that $\|P_{W_{m_0}} K_{n_0} - K_{n_0}\| \leq \epsilon_0$, where $W_{m_0} = \sum_{i=1}^{i=m_0} Y_i$. Using Lemma 4.5, $\|P_{W_{m_0}} K_n\| \to 0$ as $n \to \infty$ and so we can find $n_1 > n_0$ such that $\|P_{W_{m_0}} K_n\| \leq \epsilon_1$ for all $n \geq n_1$. Similarly, choose $m_1 \geq m_0$ such that $\|P_{W_{m_1}} K_{n_1} - K_{n_1}\| \leq \epsilon_1$. Since $\|P_{W_{m_1}} K_n\| \to 0$, we can find $n_2 > n_1$ such that $\|P_{W_{m_1}} K_n\| \leq \epsilon_2$. Thus, we are constructing a sequence of pairs (n_i, m_i) with $n_1 \leq n_2 < ...$ and $m_1 \leq m_2 < ...$ with $\|P_{W_{m_{i-1}}} K_{n_i}\| \leq \epsilon_i$ and $\|P_{W_{m_i}} K_{n_i} - K_{n_i}\| \leq \epsilon_i$, for all integer i. If we consider $U_i = (P_{W_{m_i}} - P_{W_{m_{i-1}}})K_{n_i}$ for $i = 1, 2, ..,$ then we have $\|U_i - K_{n_i}\| \leq 2\epsilon$ and the proof is completed.

Now, we give some sufficient conditions for the p-Rfpp (resp. strong p-Rfpp) of a Banach lattice $\mathcal{M}$ of some compact operators from a Banach space X into a Banach lattice F with respect to DPpc-ness (resp. almost DPpc-ness) of all evaluation operators. The reader should note that following the same arguments as in the proof of Theorem 3.5 of [13] we can conclude the following theorem.

Theorem 4.7 *Let X be a Banach space, F be an AM-space with order continuous norm and $\mathcal{M} \subset K(X,F)$ be a Banach lattice. If all of the evaluation operators ψ_{y^*} are DPpc (resp. almost DPpc), then $\mathcal{M}$ has the p-Rfpp (resp. strong p-Rfpp).*

Proof. Assume by way of contradiction that $\mathcal{M}$ does not have the the p-Rfpp (resp. strong p-Rfpp). So there is a nonempty p-Right compact (resp. almost p-Right compact) convex and minimal subset C of $\mathcal{M}$ and a non-expansive map $T : C \to C$ with $Fix(T) = \emptyset$. Without loss of generality, we can assume that $diam(C) = 1$. Let (K_n) be an approximate fixed point sequence in C. Since C is p-Right compact (resp. almost p-Right compact), we can assume that (K_n) be is p-Right null (resp. almost p-Right null). Also, by Theorem 2.7 we have $\lim_n \|K - K_n\| = 1$, for each $K \in C$. On the other hand, for each $y^* \in F^*$ the evaluation operator ψ_{y^*} is DPpc (resp. almost DPpc) and so by Lemma 4.5, the operator $K \to TK$ from $\mathcal{M}$ into $K(X,F)$ is DPpc (resp. almost DPpc) for all compact operator $T \in K(F)$. Similar to [13, Theorem 3.5], $\mathcal{M}$ has the p-Right WORTH (resp. strong p-Right WORTH) property and so $\lim_n \|K_n + K\| = \lim_n \|K_n - K\| = 1$. Hence using the same arguments as in [25, Theorem 2.6], we conclude that $1 = diam(C) < \frac{3}{4}$. This leads to a contradiction and hence $\mathcal{M}$ has the p-Rfpp (resp. strong p-Rfpp).

The reader should note that if $T : X \to Y$ is an operator between two Banach spaces such that one of them has the p-DP$_{rc}$P such as a reflexive space (resp. strong p-DP$_{rc}$P such as a discrete KB-space), then T is DPpc (reps. almost DPpc). Similarly, if $T : E \to F$ is an operator between two Banach lattices such that one of them has the positive DP$_{rc}$P (such as $L^1[0,1]$), then each the operator T is almost DPpc.

We conclude our paper review with the following two examples. At first, using the fact that a Banach lattice E is an AM-space with order continuous norm if and only if E is lattice isometric to $c_0(\Omega)$, where Ω is a nonempty set, it follows that:

Example 4.8 Let X be a weakly p-compact Banach space and F be an AM-space with order continuous norm. Then each Banach lattice $\mathcal{M} \subset K(X,F)$ has the p-Rfpp. For instance, each Banach lattice $\mathcal{M} \subset K(\ell_2, c_0)$ has the p-Rfpp. In fact, B_{ℓ_2} is weakly p-compact , for all $p \geq 2$ and also ℓ_2 has the p-DP$_{rc}$P (also the DP$_{rc}$P). Then for the Banach lattice $\mathcal{M} \subset K(\ell_2, c_0)$ all evaluation operators $\psi_{y^*} : \mathcal{M} \to \ell_2$ are DPpc. Also, c_0 is an AM-space with order continuous norm and then by Theorem 4.7, $\mathcal{M}$ has the p-Rfpp.

The following example shows that order continuity of the norm on an AM-space F in Theorem 4.7 cannot be removed.

Example 4.9 Banach lattice ℓ_∞ is an AM-space without order continuous norm and ℓ_∞ can be embedded isometrically into $K(\ell_2, \ell_\infty)$. It is easy to see that all evaluation operators $\psi_{y^*} : \mathcal{M} \to \ell_2$ are DPpc, but ℓ_∞ (and so $K(\ell_2, \ell_\infty)$) does not have the p-Rfpp. Note that, since ℓ_∞ does not have the wfpp and it has the Dunford-Pettis property, it does not have the p-Rfpp.

Similar to [13, Example 3.9] if X is a Banach space and H is a Hilbert space such that all of the evaluation operators ψ_y with $y \in H$ are DPpc, then each Banach lattice $\mathcal{M} \subset K(X,H)$ has the p-Rfpp.

It should be noted that AL-spaces do not have the p-Rfpp, in general. For instance, only reflexive subspaces of $L^1[0,1]$ have the p-Rfpp. Also, ℓ_∞ is a σ-Dedekind complete AM-space without p-Rfpp.

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