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SOME FIXED POINT THEOREMS IN THE FRAMEWORK OF f-METRIC SPACES

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Abstract. This paper deals with the notion of f-metric space, a genuine generalization of the concept of b-metric space. We present Matkowski, Kannan and Chatterjea type fixed point theorems in the context of f-metric spaces.

Key Words and Phrases: Matkowski's fixed point theorem, Kannan's fixed point theorem, Chatterjea's fixed point theorem, b-metric space, f-metric space.
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1. Introduction

There are many generalizations of the concept of metric spaces (see [17]). One particular extension was introduced by Vulpe et al. [29] (see also [6]), I.A. Bakhtin [5] and S. Czerwik [11], known under the label of b-metric space. Their idea was to replace the triangle inequality by a weaker axiom. More precisely we have the following:

Definition 1.1. Let X be a nonempty set. A map $d : X \times X \to [0, \infty)$ is called a b-metric if there exists $s \geq 1$ such that for all x, y and $z \in X$ the following conditions hold

$$
\begin{aligned}
&(b_1) \quad d(x, y) = 0 \text{ if and only if } x = y,\\
&(b_2) \quad d(x, y) = d(y, x),\\
&(b_3) \quad d(x, y) \leq s[d(x, z) + d(z, y)].
\end{aligned}
$$

The triplet (X, d, s) is called a b-metric space.

Czerwik proved Matkowski's fixed point theorem [19], within the context of b-metric spaces [12]. Some corrections have been added in [14]. Many other results concerning generalizations of fixed point theorems, in this framework, were provided (see, for example, [8, 22, 18, 20, 10, 1, 4] and [6] for an excellent survey concerning early developments in fixed point theory on b-metric spaces). On the other hand, the topological properties of b-metric spaces were studied in [2] and [9].

Some authors introduced generalizations of b-metric spaces by weakening the triangle inequality even further (see [24, 26, 21, 27, 3, 16]). Of particular interest for our paper is the one introduced by A.V. Arutyunov in [3], where the focus is on the topological properties of f-quasimetric spaces. Along these lines of research, E.S. Zhukovski obtained a Browder type fixed point result in the above mentioned framework (see [30]).

In this paper we generalize some classical fixed point theorems to the framework of f-metric spaces. More precisely, Theorem 3.1 is a Matkowski type fixed point result, Theorem 3.2 is a Kannan type fixed point result and Theorem 3.3 is a Chatterjea type fixed point result.

2. Preliminaries

In this section, we introduce the concept of f-metric space (see [9, 3] for a more general notion) and some useful properties.

In line with the previously mentioned paper by Arutyunov we define

Definition 2.1. Let $X \neq \emptyset$ be a set and a function $f : [0,\infty) \times [0,\infty) \to [0,\infty)$ such that

$$(\alpha_1) \quad \lim_{(x,y)\to(0,0)} f(x,y) = 0,$$
$$(\alpha_2) \quad f(x,y) \geq f(x',y'),$$

for all $x, y, x', y' \in [0,\infty)$ having the property that $x \geq x'$ and $y \geq y'$.

A map $d : X \times X \to [0,\infty)$ is called an f-metric if for all x, y and $z \in X$ the following conditions hold

$$(\beta_1) \quad d(x,y) = 0 \text{ if and only if } x = y,$$
$$(\beta_2) \quad d(x,y) = d(y,x),$$
$$(\beta_3) \quad d(x,y) \leq f(d(x,z), d(z,y)).$$

The triplet (X, d, f) is called an f-metric space.

Convergent and Cauchy sequences are introduced in a similar manner to metric spaces (see [9]).

Definition 2.2. Let (X, d, f) be an f-metric space.

i) A sequence $(x_n)_{n\in\mathbb{N}} \subseteq X$ is called convergent to $x \in X$ if $\lim\limits_{n\to\infty} d(x_n, x) = 0$.

ii) A sequence $(x_n)_{n\in\mathbb{N}} \subseteq X$ is called Cauchy, if for any $\varepsilon > 0$ there exists $n_\varepsilon \in \mathbb{N}$ such that $d(x_n, x_m) < \varepsilon$, for all $n, m \in \mathbb{N}$, $n, m > n_\varepsilon$.

iii) (X, d, f) is called complete if every Cauchy sequence is convergent to some $x \in X$.

Proposition 2.1. *Let (X, d, f) be an f-metric space. Then each convergent sequence $(x_n)_{n\in\mathbb{N}} \subseteq X$ has a unique limit.*

Proof. Suppose that for a convergent sequence $(x_n)_{n\in\mathbb{N}}$ there are two limits $l_1, l_2 \in X$. Then

$$d(l_1, l_2) \leq f(d(l_1, x_n), d(x_n, l_2)),$$

for all $n \in \mathbb{N}$. Based on α_1, taking the limit as $n \to \infty$, we conclude that $d(l_1, l_2) = 0$, so $l_1 = l_2$. □

Remark 2.1. For a function f as described in Definition 2.1 we have

$$f(x, y) \leq f(x, x) + f(y, y),$$

for all $x, y \in [0, \infty)$.

Proof. Indeed, for $x, y \in [0, \infty)$ with $x \leq y$, we have

$$f(x, y) \leq f(y, y) \leq f(x, x) + f(y, y).$$

The same conclusion holds for $x \geq y$. □

Example 1.

(1) Taking the function f given by $f(x, y) = x + y$, for all $x, y \in [0, \infty)$ we obtain the concept of metric space.
(2) Taking the function f given by $f(x, y) = s(x + y)$, for all $x, y \in [0, \infty)$ we obtain the concept of b-metric space with constant s due to Czerwik and Bakhtin.
(3) Taking the function f given by $f(x, y) = s \max\{x, y\}$, for all $x, y \in [0, \infty)$ we obtain the concept of quasi-metric space (see [25]).

We introduce the following method of constructing new f-metric spaces from metric spaces:

Proposition 2.2. *Let (X, d) be a metric space and $\theta : [0, \infty) \times [0, \infty) \to [0, \infty)$ such that*

(1) $\lim\limits_{(x,y)\to(0,0)} \theta(x, y) = 0$,
(2) $\theta(x, y) \geq \theta(x', y')$ *for all* $x, x', y, y' \in [0, \infty)$ *with* $x \geq x'$ *and* $y \geq y'$,
(3) $x \leq \theta(x, x)$ *for all* $x \in [0, \infty)$,
(4) $f(x, y) = \theta(x + y, x + y)$ *for all* $x, y \in [0, \infty)$,
(5) $\rho_d(x, y) = \theta(d(x, y), d(x, y))$ *for all* $x, y \in X$.

Then (X, ρ_d, f) is an f-metric space.

Proof. Note that from *2* and *3* we have

$$0 \leq \theta(0, 0) \leq \theta\left(\frac{1}{n}, \frac{1}{n}\right),$$

for all $n \in \mathbb{N}$. Taking the limit as $n \to \infty$ we find that $\theta(0, 0) = 0$.

Let $x, y \in X$ with $0 = \rho_d(x, y) = \theta(d(x, y), d(x, y)) \overset{3}{\geq} d(x, y)$. Consequently $d(x, y) = 0$, hence $x = y$.

Conversely, let $x, y \in X$ with $x = y$. We have

$$\rho_d(x, y) = \theta(d(x, y), d(x, y)) = \theta(0, 0) = 0.$$

Additionally

$$\rho_d(x,y) = \theta(d(x,y), d(x,y)) = \theta(d(y,x), d(y,x)) = \rho_d(y,x),$$

for all $x, y \in X$.

Conditions β_1 and β_2 are verified.

Moreover

$$\begin{aligned}\rho_d(x,y) &= \theta(d(x,y), d(x,y)) \\ &\overset{2}{\leq} \theta(d(x,z) + d(y,z), d(x,z) + d(y,z)) \\ &= f(d(x,z), d(y,z)) \\ &\overset{3}{\leq} f(\theta(d(x,z), d(x,z)), \theta(d(y,z), d(y,z))) \\ &\leq f(\rho_d(x,z), \rho_d(y,z)),\end{aligned}$$

for every x, y and $z \in X$, so condition β_3 is valid. □

A space constructed in this manner is given in the following example:

Example 2. For the metric space $(\mathbb{R}, |\cdot|)$, define

$$\theta(x,y) = e^{\frac{x+y}{2}} - 1,$$

for every $x, y \in [0, \infty)$.

Taking into account the previous result we get that $(\mathbb{R}, d, f)$ with

$$d(x,y) = \theta(|x-y|, |x-y|) = e^{|x-y|} - 1,$$

for every $x, y \in \mathbb{R}$ and

$$f(x,y) = \theta(x+y, x+y) = e^{x+y} - 1,$$

for every $x, y \in [0, \infty)$ is an f-metric space.

We claim that $(\mathbb{R}, d, f)$ is not a b-metric space, i.e. there is no s such that d verifies Definition 1.1 b_3.

Indeed, if this is not the case, there exists $s \geq 1$ such that

$$e^{|x-y|} - 1 \leq s[e^{|x-z|} - 1 + e^{|z-y|} - 1],$$

for all x, y and $z \in \mathbb{R}$. Taking $x > 0$, $y = -x$ and $z = 0$ we get $s \geq \frac{e^{2x}-1}{2(e^x-1)}$, for all $x > 0$. Clearly no s can satisfy this condition.

Remark 2.2. The previous example shows that the class of f-metric spaces is strictly larger than that of b-metric spaces.

3. Main results

For a map $T : X \to X$ and $n \in \mathbb{N}$ by $T^{[n]}$ we mean $\underbrace{T \circ T \circ \cdots \circ T}_{n \text{ times}}$.

Theorem 3.1. *Let (X, d, f) be a complete f-metric space and $\varphi : [0, \infty) \to [0, \infty)$ satisfying the following conditions:*

$$\lim_{n\to\infty} \varphi^{[n]}(t) = 0 \quad \text{for all } t > 0, \tag{3.1}$$

$$\varphi \text{ is increasing .} \tag{3.2}$$

If $T : X \to X$ verifies

$$d(Tx, Ty) \leq \varphi(d(x, y)), \tag{3.3}$$

for all $x, y \in X$, then T is a Picard operator, i.e. it has a unique fixed point $x^* \in X$ and

$$\lim_{n\to\infty} d(T^{[n]}x, x^*) = 0,$$

for all $x \in X$.

Proof. It is known that

$$0 = \varphi(0) \leq \varphi(t) < t, \tag{3.4}$$

for all $t > 0$ (see, for example, Remark 3.1 in [9]).

From (3.1) and (3.4) we find that

$$\lim_{n\to\infty} \varphi^{[n]}(t) = 0 \quad \text{for all } t \geq 0. \tag{3.5}$$

Let $x_0 \in X$ and set $x_n = T^{[n]}x_0,\ n \in \mathbb{N}$.

Inequality (3.3) implies

$$d(T^{[n]}x, T^{[n]}y) \leq \varphi^{[n]}(d(x, y)), \tag{3.6}$$

for all $x, y \in X,\ n \in \mathbb{N}$.

Then

$$\begin{aligned} d(T^{[n]}x_{mn}, x_{mn}) &= d(T^{[nm]}x_n, T^{[nm]}x_0) \\ &\leq \varphi^{[nm]}(d(x_n, x_0)), \end{aligned} \tag{3.7}$$

for all $m, n \in \mathbb{N}$.

Taking the limit with $m \to \infty$ we obtain

$$\lim_{m\to\infty} d(T^{[n]}x_{mn}, x_{mn}) = 0, \tag{3.8}$$

for all $n \in \mathbb{N}$.

Let $\varepsilon > 0$ be arbitrarily chosen, but fixed.

Based on Definition 2.1 α_1 there exist $\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4$ with

$$0 < \varepsilon_1 \leq \varepsilon_2 \leq \varepsilon_3 \leq \varepsilon_4,$$

such that

$$f(\varepsilon_1, \varepsilon_1) \leq \varepsilon_2, f(\varepsilon_2, \varepsilon_2) \leq \varepsilon_3, f(\varepsilon_3, \varepsilon_3) \leq \varepsilon_4, f(\varepsilon_4, \varepsilon_4) \leq \varepsilon.$$

Due to (3.5) and (3.8) we can choose $N, M \in \mathbb{N}$ such that

$$\varphi^{[n]}(f(\varepsilon_1, \varepsilon_1)) \leq \varepsilon_1, \quad \text{for all } n \geq N. \tag{3.9}$$

$$d(T^{[N]}x_{mN}, x_{mN}) \leq \varepsilon_1, \quad \text{for all } m \geq M. \tag{3.10}$$

For any $z \in X$ and $m \geq M$ such that $d(z, x_{mN}) \leq f(\varepsilon_1, \varepsilon_1)$, we have

$$\begin{aligned} d(T^{[N]}z, T^{[N]}x_{mN}) &\overset{(3.6)}{\leq} \varphi^{[N]}(d(z, x_{mN})) \\ &\leq \varphi^{[N]}(f(\varepsilon_1, \varepsilon_1)) \overset{(3.9)}{\leq} \varepsilon_1. \end{aligned} \tag{3.11}$$

This implies

$$\begin{aligned} d(T^{[N]}z, x_{mN}) &\leq f(d(T^{[N]}z, T^{[N]}x_{mN}), d(T^{[N]}x_{mN}, x_{mN})) \\ &\overset{(3.10)\&(3.11)}{\leq} f(\varepsilon_1, \varepsilon_1), \end{aligned} \tag{3.12}$$

for all $z \in X$ and $m \geq M$ with $d(z, x_{mN}) \leq f(\varepsilon_1, \varepsilon_1)$.

We claim that

$$d(x_{(m+i)N}, x_{mN}) \leq f(\varepsilon_1, \varepsilon_1), \tag{3.13}$$

for all $i, m \in \mathbb{N}$, $m \geq M$.

Indeed, we have $d(x_{mN}, x_{mN}) = 0 \leq f(\varepsilon_1, \varepsilon_1)$ and $x_{(m+1)N} = T^{[N]}x_{mN}$, and choosing $z = x_{mN}$ in (3.12) we deduce that $d(x_{(m+1)N}, x_{mN}) \leq f(\varepsilon_1, \varepsilon_1)$, for all $m > M$. So (3.13) is valid for $i = 1$.

Suppose that, for a fixed $i \in \mathbb{N}$, $d(x_{(m+i)N}, x_{mN}) \leq f(\varepsilon_1, \varepsilon_1)$. By considering $z = x_{(m+i)N}$ in (3.12) since $x_{(m+i+1)N} = T^{[N]}x_{(m+i)N}$ we conclude that $d(x_{(m+i+1)N}, x_{mN}) \leq f(\varepsilon_1, \varepsilon_1)$, for all $m > M$. According to the principle of mathematical induction, inequality (3.13) is true.

For $m_1, m_2 \in \mathbb{N}$ such that $m_1, m_2 \geq M$, we have

$$\begin{aligned} d(x_{m_1N}, x_{m_2N}) &\leq f(d(x_{m_1N}, x_{mN}), d(x_{mN}, x_{m_2N})) \\ &\overset{(3.13)}{\leq} f(f(\varepsilon_1, \varepsilon_1), f(\varepsilon_1, \varepsilon_1)) \leq f(\varepsilon_2, \varepsilon_2) \leq \varepsilon_3, \end{aligned} \tag{3.14}$$

where $m \in \mathbb{N}$ was chosen such that $m > M$.

Fix $p \in \{0, 1, \ldots, N-1\}$. Then

$$d(x_{mN+p}, x_{mN}) \overset{(3.6)}{\leq} \varphi^{[mN]}(d(x_p, x_0)),$$

for all $m \in \mathbb{N}$.

Taking the limit as $m \to \infty$ in the above inequality, via (3.5), we get

$$\lim_{m \to \infty} d(x_{mN+p}, x_{mN}) = 0,$$

for all $p \in \{0, 1, \ldots, N-1\}$.

Consequently there exists $M_1 \in \mathbb{N}$ such that

$$d(x_{mN+p}, x_{mN}) \leq \varepsilon_3, \tag{3.15}$$

for all $m \geq M_1$ and $p \in \{0, 1, \ldots, N-1\}$.

Let us consider $q_1, q_2 \in \mathbb{N}$, $q_1, q_2 \geq \max\{M, M_1\}N$. Then, there exist $p_1, p_2 \in \{0, 1, \ldots, N-1\}$ and $m_1, m_2 \geq \max\{M, M_1\}$ such that $q_1 = m_1N + p_1$ and $q_2 = m_2N + p_2$.

We have

$$\begin{aligned} d(x_{q_1}, x_{q_2}) &\leq f(d(x_{q_1}, x_{m_1N}), d(x_{m_1N}, x_{q_2})) \\ &= f(d(x_{m_1N+p_1}, x_{m_1N}), d(x_{m_1N}, x_{q_2})) \\ &\leq f(d(x_{m_1N+p_1}, x_{m_1N}), f(d(x_{m_1N}, x_{m_2N}), d(x_{m_2N}, x_{m_2N+p_2}))) \\ &\overset{(3.14)\&(3.15)}{\leq} f(\varepsilon_3, f(\varepsilon_3, \varepsilon_3)) \leq f(\varepsilon_3, \varepsilon_4) \leq f(\varepsilon_4, \varepsilon_4) \leq \varepsilon. \end{aligned}$$

Thus we conclude that $(x_n)_{n\in\mathbb{N}}$ is Cauchy. Since (X, d, f) is complete there exists $x^* \in X$ so that

$$\lim_{n\to\infty} d(x_n, x^*) = 0.$$

We claim that x^* is a fixed point for T. Indeed, we have

$$\begin{aligned} 0 \leq d(x^*, Tx^*) &\leq f(d(x^*, x_{n+1}), d(x_{n+1}, Tx^*)) \\ &\leq f(d(x^*, x_{n+1}), \varphi(d(x_n, x^*))) \\ &\overset{(4)}{\leq} f(d(x^*, x_{n+1}), d(x_n, x^*)), \end{aligned}$$

for all $n \in \mathbb{N}$ and taking the limit as $n \to \infty$ we obtain $x^* = Tx^*$.

Suppose x^* and y^* are distinct fixed points for T. Then

$$d(x^*, y^*) = d(Tx^*, Ty^*) \leq \varphi(d(x^*, y^*)) \overset{(3.4)}{<} d(x^*, y^*).$$

The above contradiction shows that T has a unique fixed point. $\square$

We mention that even though a Theorem similar to Theorem 3.1 was obtained in [30], our proof is different from the one presented by Zhukovski.

Remark 3.1. By choosing $\varphi(x) = cx, c \in [0, 1)$ in Theorem 3.1 we find the analogue of Banach's contraction principle in the framework of f-metric spaces.

Example 3. Consider X the f-metric space stated in Example 2.2. and the function $\varphi : [0, \infty) \to [0, \infty)$ given by $\varphi(x) = \frac{1}{2}x$, for all $x \in [0, \infty)$.

Let $T : X \to X$ defined by

$$Tx = \ln\left(\frac{e^{-|x|} + 1}{2}\right),$$

for all $x \in \mathbb{R}$.

Then T is a Picard operator.

Proof. For $x, y \in \mathbb{R}$ we have

$$d(Tx, Ty) = e^{\left|\ln\left(\frac{e^{-|x|}+1}{2}\right) - \ln\left(\frac{e^{-|y|}+1}{2}\right)\right|} - 1 = e^{\left|\ln\left(\frac{e^{-|x|}+1}{e^{-|y|}+1}\right)\right|} - 1.$$

Assume $|x| \leq |y|$, we deduce

$$d(Tx,Ty) = e^{\ln\left(\frac{e^{-|x|}+1}{e^{-|y|}+1}\right)} - 1 = \frac{e^{-|x|} - e^{-|y|}}{e^{-|y|}+1} = \frac{\frac{e^{|y|}}{e^{|x|}} - 1}{e^{|y|}+1} = \frac{e^{|y|-|x|} - 1}{e^{|y|}+1}$$
$$\leq \frac{1}{e^{|y|}+1}(e^{|y-x|} - 1) \leq \frac{1}{2}(e^{|y-x|} - 1) = \varphi(d(x,y)).$$

Since the case $|y| \leq |x|$ is similar we find that

$$d(Tx,Ty) \leq \varphi(d(x,y)),$$

for all $x, y \in X$.

Consequently T satisfies the conditions imposed by Theorem 3.1, therefore it is a Picard operator. □

Definition 3.1. Let (X,d,f) be a complete f-metric space. We say that $T : X \to X$ is a Kannan map if there exists $a \in [0, \frac{1}{2})$ such that

$$d(Tx,Ty) \leq a(d(x,Tx) + d(y,Ty)),$$

for all $x, y \in X$.

The method used in Theorem 3.1 can be adapted to show the following

Theorem 3.2. *Let (X,d,f) be a complete f-metric space and let $T : X \to X$ be a Kannan map. Then there is $x^* \in X$ such that $\lim_{n\to\infty} d(T^{[n]}x, x^*) = 0$, for all $x \in X$.*

Proof. Let us consider a fixed $x \in X$ and set $x_n = T^{[n]}x, n \in \mathbb{N}$. Since T is a Kannan map we have

$$d(T^{[n+1]}x, T^{[n]}x) \leq a[d(T^{[n-1]}x, T^{[n]}x) + d(T^{[n+1]}x, T^{[n]}x)],$$

so

$$d(T^{[n+1]}x, T^{[n]}x) \leq \frac{a}{1-a} d(T^{[n-1]}x, T^{[n]}x),$$

for all $n \in \mathbb{N}$.

Inductively, we obtain

$$d(T^{[n+1]}x, T^{[n]}x) \leq \left(\frac{a}{1-a}\right)^n d(Tx,x), \tag{3.1}$$

for all $n \in \mathbb{N}$.

Taking in (3.1) the limit as $n \to \infty$ we get that

$$\lim_{n\to\infty} d(x_{n+1}, x_n) = \lim_{n\to\infty} d(T^{[n+1]}x, T^{[n]}x) = 0. \tag{3.2}$$

We also have

$$\begin{aligned}
d(T^{[n]}x_{mn}, x_{mn}) &= d(T^{[mn+n]}x_0, T^{[mn]}x_0) \\
&\overset{\text{Definition 3.1}}{\leq} a[d(T^{[mn+n-1]}x_0, T^{[mn+n]}x_0) + d(T^{[mn-1]}x_0, T^{[mn]}x_0)] \\
&\overset{(3.1)}{\leq} a\left[\left(\frac{a}{1-a}\right)^{mn+n-1} d(Tx_0, x_0) + \left(\frac{a}{1-a}\right)^{mn-1} d(Tx_0, x_0)\right] \\
&\leq ad(Tx_0, x_0)\left(\frac{a}{1-a}\right)^{mn-1}\left[1 + \left(\frac{a}{1-a}\right)^{n}\right],
\end{aligned}$$

for all $m, n \in \mathbb{N}$.

Taking the limit as $m \to \infty$ in the above relation we get

$$\lim_{m\to\infty} d(T^{[n]}x_{mn}, x_{mn}) = 0,$$

i.e.

$$\lim_{m\to\infty} d(x_{(m+1)n}, x_{mn}) = 0,$$

for all $n \in \mathbb{N}$

Let $\varepsilon > 0$, arbitrarily chosen, but fixed. Based on Definition 2.1 α_1 there exist $\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4$ with

$$0 < \varepsilon_1 \leq \varepsilon_2 \leq \varepsilon_3 \leq \varepsilon_4,$$

such that

$$f(\varepsilon_1, \varepsilon_1) \leq \varepsilon_2, f(\varepsilon_2, \varepsilon_2) \leq \varepsilon_3, f(\varepsilon_3, \varepsilon_3) \leq \varepsilon_4, f(\varepsilon_4, \varepsilon_4) \leq \varepsilon.$$

Choose N and $M \in \mathbb{N}$ such that

$$\left(\frac{a}{1-a}\right)^{n-1} \leq \frac{\sqrt{\varepsilon_1}}{2a}, \quad \text{for all } n \geq N. \tag{3.3}$$

$$d(x_{(m+1)N}, x_{mN}) \leq \varepsilon_1, \quad \text{for all } m \geq M. \tag{3.4}$$

$$d(x_{mN+1}, x_{mN}) \leq \sqrt{\varepsilon_1}, \quad \text{for all } m \geq M. \tag{3.5}$$

For any $z \in X$ that satisfies $d(z, x_{mN}) \leq f(\varepsilon_1, \varepsilon_1)$ and $d(Tz, z) \leq \sqrt{\varepsilon_1}$, we have

$$\begin{aligned}
d(T^{[N]}z, T^{[N]}x_{mN}) &\overset{\text{Definition 3.1}}{\leq} a[d(T^{[N-1]}z, T^{[N]}z) + d(T^{[N-1]}x_{mN}, T^{[N]}x_{mN})] \\
&\overset{(3.1)}{\leq} a\left(\frac{a}{1-a}\right)^{N-1}[d(Tz, z) + d(x_{mN+1}, x_{mN})] \\
&\overset{(3.3)\&(3.5)}{\leq} a\frac{\sqrt{\varepsilon_1}}{2a}[\sqrt{\varepsilon_1} + \sqrt{\varepsilon_1}] = \varepsilon_1.
\end{aligned} \tag{3.6}$$

Therefore

$$\begin{aligned}
d(T^{[N]}z, x_{mN}) &\leq f(d(T^{[N]}z, x_{(m+1)N}), d(x_{(m+1)N}, x_{mN})) \\
&\overset{(3.4)\&(3.6)}{\leq} f(\varepsilon_1, \varepsilon_1),
\end{aligned} \tag{3.7}$$

for all $m \geq M$.

Claim.

$$d(x_{(m+i)N}, x_{mN}) \leq f(\varepsilon_1, \varepsilon_1),$$

for all $i, m \in \mathbb{N}$, $m > M$.

Justification of the claim. Since $d(x_{mN}, x_{mN}) = 0 \le f(\varepsilon_1, \varepsilon_1)$, via (3.5), choosing $z = x_{mN}$ in (3.7) we deduce that $d(x_{(m+1)N}, x_{mN}) \le f(\varepsilon_1, \varepsilon_1)$.

Suppose that for a fixed $i \in \mathbb{N}$, $d(x_{(m+i)N}, x_{mN}) \le f(\varepsilon_1, \varepsilon_1)$.

Set $z = x_{(m+i)N}$. Since $x_{(m+i+1)N} = T^{[N]}x_{(m+i)N}$ and $m + i \ge m \ge M$ we get from (3.5) that $d(Tx_{(m+i)N}, x_{(m+i)N}) \le \sqrt{\varepsilon_1}$.

We deduce from (3.7) that

$$d(x_{(m+i+1)N}, x_{mN}) \le f(\varepsilon_1, \varepsilon_1),$$

for all $m \in \mathbb{N}$.

Via the principle of mathematical induction, the justification of the claim is completed.

For $m_1, m_2 \in \mathbb{N}$ such that $m_1, m_2 \ge M$, we have

$$\begin{aligned} d(x_{m_1 n}, x_{m_2 n}) &\le f(d(x_{m_1 n}, x_{mn}), d(x_{mn}, x_{m_2 n})) \\ &\overset{Claim}{\le} f(f(\varepsilon_1, \varepsilon_1), f(\varepsilon_1, \varepsilon_1)) \le f(\varepsilon_2, \varepsilon_2) \le \varepsilon_3. \end{aligned} \tag{3.8}$$

Fix $p \in \{0, 1, \dots, N-1\}$. Then

$$d(x_{mN+p}, x_{mN}) \overset{\text{Definition 3.1}}{\le} a[d(x_{mN+p-1}, x_{mN+p}) + d(x_{mN-1}, x_{mN})],$$

for all $m \ge M$.

Taking the limit with $m \to \infty$, based on (3.2), we get

$$\lim_{m \to \infty} d(x_{mN+p}, x_{mN}) = 0,$$

for all $p \in \{0, 1, \dots, N-1\}$.

Consequently, there exists $M_1 \in \mathbb{N}$ such that

$$d(x_{mN+p}, x_{mN}) \le \varepsilon_3, \tag{3.9}$$

for all $m \ge M_1$ and $p \in \{0, 1, \dots, N-1\}$.

Let us consider $q_1, q_2 \in \mathbb{N}$, $q_1, q_2 \ge \max\{M, M_1\}N$. Then, there exist $p_1, p_2 \in \{0, 1, \dots, N-1\}$ and $m_1, m_2 \ge \max\{M, M_1\}$ such that $q_1 = m_1 N + p_1$, $q_2 = m_2 N + p_2$ and we have

$$\begin{aligned} d(x_{q_1}, x_{q_2}) &\le f(d(x_{q_1}, x_{m_1 N}), d(x_{m_1 N}, x_{q_2})) \\ &= f(d(x_{m_1 N + p_1}, x_{m_1 N}), d(x_{m_1 N}, x_{q_2})) \\ &\le f(d(x_{m_1 N + p_1}, x_{m_1 N}), f(d(x_{m_1 N}, x_{m_2 N}), d(x_{m_2 N}, x_{m_2 N + p_2}))) \\ &\overset{(3.8)\&(3.9)}{\le} f(\varepsilon_3, f(\varepsilon_3, \varepsilon_3)) \le f(\varepsilon_4, \varepsilon_4) \le \varepsilon. \end{aligned}$$

Therefore $(x_n)_{n \in \mathbb{N}}$ is Cauchy, and since (X, d, f) is complete there exists $x^* \in X$ such that

$$\lim_{n \to \infty} d(x_n, x^*) = 0.$$

As

$$\begin{aligned} d(T^{[n]}y, x^*) &\leq f(d(T^{[n]}y, T^{[n]}x), d(T^{[n]}x, x^*)) \\ &\overset{\text{Definition 3.1}}{\leq} f(ad(T^{[n-1]}y, T^{[n]}y) + ad(T^{[n-1]}x, T^{[n]}x), d(T^{[n]}x, x^*)), \end{aligned}$$

for all $n \in \mathbb{N}$ and $y \in X$, taking the limit as $n \to \infty$ in the above inequality, we conclude that

$$\lim_{n\to\infty} d(T^{[n]}y, x^*) = 0,$$

for all $y \in X$. □

Proposition 3.1. *Let (X, d, f) be an f-metric space and $T : X \to X$ a Kannan map. Then T has at most one fixed point.*

Proof. Suppose there are $x^*, y^* \in X$ such that $Tx^* = x^*$ and $Ty^* = y^*$. Then

$$d(x^*, y^*) = d(Tx^*, Ty^*) \leq a[d(x^*, Tx^*) + d(y^*, Ty^*)] = 0,$$

and consequently $x^* = y^*$. □

A sufficient condition such that a Kannan map has a fixed point is the following

Theorem 3.3. *Let (X, d, f) be an f-metric space with f continuous and $T : X \to X$ a Kannan map. If the constant a from Definition 3.1 satisfies*

$$f(ax, 0) < x,$$

for all $x \in (0, diam(X)]$, then T has a unique fixed point.

Proof. From Theorem 3.2 there exists $x^* \in X$ such that $\lim_{n\to\infty} T^{[n]}x = x^*$ for all $x \in X$. Suppose $d(x^*, Tx^*) > 0$. Then

$$\begin{aligned} d(Tx^*, x^*) &\leq f(d(Tx^*, T^{[n]}x), d(T^{[n]}x, x^*)) \\ &\leq f(a[d(Tx^*, x^*) + d(T^{[n]}x, T^{[n-1]}x)], d(T^{[n]}x, x^*)). \end{aligned}$$

Taking the limit as $n \to \infty$ in the above inequality we obtain the following contradiction

$$d(Tx^*, x^*) \leq f(ad(Tx^*, x^*), 0) < d(Tx^*, x^*).$$

Therefore $d(Tx^*, x^*) = 0$, that is $Tx^* = x^*$. □

Remark 3.2. Let us note that for f as in Example 2.1, 1) Theorem 3.3 yields the well known Kannan fixed point theorem.

Definition 3.2. Let (X, d, f) be an f-metric space and let $T : X \to X$. We say that T is a Chatterjea map if there exists $b \in [0, \frac{1}{2})$ such that

$$d(Tx, Ty) \leq b[d(x, Ty) + d(y, Tx)],$$

for all $x, y \in X$.

The proof of the following theorem is based on an idea similar to the one used in [18].

Theorem 3.4. *Let (X, d, f) be a complete f-metric space and let $T : X \to X$ be a Chatterjea map such that the sequence $(d(T^{[n]}x, x))_{n\in\mathbb{N}}$ is bounded for any $x \in X$.*

Then T is a Picard operator, i.e. there exists a unique fixed point x^* of T and $\lim_{n\to\infty} d(T^{[n]}x, x^*) = 0$ for all $x \in X$.

Proof. Let us consider $x \in X$ arbitrarily chosen, but fixed. For the sake of simplicity let us denote $d(T^{[n]}x, x)$ by a_n and $T^{[n]}x$ by x_n, for every $n \in \mathbb{N}$. Let M be such that $a_n < M$ for all $n \in \mathbb{N}$.

We will prove by double induction (see pp. 45 in [13]) that

$$d(x_n, x_m) \le \left(\frac{b}{1-b}\right)^{\min\{n,m\}} M, \tag{3.1}$$

for all $n, m \in \mathbb{N}$.

First we claim that

$$d(x_n, x_1) \le \frac{b}{1-b} M, \tag{3.2}$$

for all $n \in \mathbb{N}$.

Indeed, we have

$$d(x_2, x_1) \le b[d(x_1, x_1) + d(x, x_2)] \le bM \le \frac{b}{1-b} M.$$

Suppose that (3.2) is true for an arbitrary $n \in \mathbb{N}$. Then

$$\begin{aligned} d(x_{n+1}, x_1) &\le b[d(x_n, x_1) + d(x, x_{n+1})] \\ &\le b\left[\frac{b}{1-b} M + M\right] \le \frac{b}{1-b} M. \end{aligned}$$

Therefore (3.2) is true.

Similarly it can be shown that

$$d(x_1, x_m) \le \frac{b}{1-b} M, \tag{3.3}$$

for all $m \in \mathbb{N}$.

Assume

$$d(x_{n+1}, x_m) \le \left(\frac{b}{1-b}\right)^{\min\{n+1,m\}} M,$$

and

$$d(x_n, x_{m+1}) \le \left(\frac{b}{1-b}\right)^{\min\{n,m+1\}} M,$$

are true for some fixed $m, n \in \mathbb{N}$.

We have

$$\begin{aligned} d(x_{n+1}, x_{m+1}) &\le b[d(x_n, x_{m+1}) + d(x_m, x_{n+1})] \\ &\le b\left[\left(\frac{b}{1-b}\right)^{\min\{n+1,m\}} + \left(\frac{b}{1-b}\right)^{\min\{n,m+1\}}\right] M. \end{aligned}$$

The following cases occur:

(1) If $n < m$, we get

$$d(x_{n+1}, x_{m+1}) \leq b\left[\left(\frac{b}{1-b}\right)^{n+1} + \left(\frac{b}{1-b}\right)^{n}\right] M$$
$$\leq \left(\frac{b}{1-b}\right)^{n+1} M = \left(\frac{b}{1-b}\right)^{\min\{n+1,m+1\}} M,$$

i.e. (1) is valid.

(2) If $n = m$, (1) is obviously true.

(3) If $n > m$, we have

$$d(x_{n+1}, x_{m+1}) \leq b\left[\left(\frac{b}{1-b}\right)^{m} + \left(\frac{b}{1-b}\right)^{m+1}\right] M$$
$$\leq \left(\frac{b}{1-b}\right)^{m+1} M = \left(\frac{b}{1-b}\right)^{\min\{n+1,m+1\}} M,$$

so (1) is verified.

This concludes the proof of (1) by mathematical induction.

Since $\frac{b}{1-b} < 1$, via (3.1), we infer that the sequence$(x_n)_{n\in\mathbb{N}}$ is Cauchy and, since the space (X, d, f) is complete, there is $x^* \in X$ such that

$$\lim_{n\to\infty} d(x_n, x^*) = 0.$$

Note that

$$d(x_{n+1}, Tx^*) \leq f(d(x_{n+1}, x^*), d(Tx^*, x^*))$$
$$\overset{\text{Remark 2.1}}{\leq} f(d(x_{n+1}, x^*), d(x_{n+1}, x^*)) + f(d(Tx^*, x^*), d(Tx^*, x^*)),$$

for every $n \in \mathbb{N}$. In consequence

$$0 \leq \limsup_{n\to\infty} d(x_{n+1}, Tx^*) < \infty.$$

Since

$$d(x_{n+1}, Tx^*) \leq b[d(x_n, Tx^*) + d(x^*, x_{n+1})],$$

for every $n \in \mathbb{N}$, we deduce

$$\limsup_{n\to\infty} d(x_{n+1}, Tx^*) \leq b[\limsup_{n\to\infty} d(x_n, Tx^*) + \limsup_{n\to\infty} d(x^*, x_{n+1})]$$
$$\leq b \limsup_{n\to\infty} d(x_{n+1}, Tx^*),$$

so

$$\limsup_{n\to\infty} d(x_{n+1}, Tx^*) = 0,$$

and consequently

$$\lim_{n\to\infty} d(x_n, Tx^*) = 0.$$

According to Proposition 2.1 the limit of $(x_n)_{n\in\mathbb{N}}$ is unique, hence $Tx^* = x^*$.

Suppose there are $x^*, y^* \in X$ such that $Tx^* = x^*$ and $Ty^* = y^*$. Then

$$d(x^*, y^*) = d(Tx^*, Ty^*) \leq b[d(x^*, Ty^*) + bd(y^*, Tx^*)] = 2bd(x^*, y^*).$$

Therefore $d(x^*, y^*) = 0$, i.e. $x^* = y^*$. □

Remark 3.3. For f as in Example 2.1, 1), the sequence $(d(T^{[n]}x, x))_{n\in\mathbb{N}}$ is bounded for any $x \in X$.

Indeed we have

$$\begin{aligned} d(T^{[n]}x, T^{[n+1]}x) &\leq b[d(T^{[n-1]}x, T^{[n+1]}x) + d(T^{[n]}x, T^{[n]}x)] \\ &\leq b[d(T^{[n-1]}x, T^{[n]}x) + d(T^{[n]}x, T^{[n+1]}x)], \end{aligned}$$

for all $n \in \mathbb{N}$, hence

$$d(T^{[n]}x, T^{[n+1]}x) \leq \frac{b}{1-b} d(T^{[n-1]}x, T^{[n]}x), \text{ for all } n \in \mathbb{N}.$$

Inductively we obtain

$$d(T^{[n]}x, T^{[n+1]}x) \leq \left(\frac{b}{1-b}\right)^n d(Tx, x), \text{ for all } n \in \mathbb{N}.$$

Consequently

$$\begin{aligned} d(T^{[n]}x, x) &\leq d(T^{[n]}x, T^{[n-1]}x) + d(T^{[n-1]}x, T^{[n-2]}x) + \cdots + d(Tx, x) \\ &\leq \left[\left(\frac{b}{1-b}\right)^{n-1} + \left(\frac{b}{1-b}\right)^{n-2} + \cdots + 1\right] d(Tx, x) \\ &\leq \frac{1 - \left(\frac{b}{1-b}\right)^n}{1 - \frac{b}{1-b}} d(Tx, x) \leq \frac{1}{1 - \frac{b}{1-b}} d(Tx, x), \end{aligned}$$

for all $n \in \mathbb{N}$. We conclude that $(d(T^{[n]}x, x))_{n\in\mathbb{N}}$ is bounded for any $x \in X$.

Therefore Theorem 3.4 yields the well known Chatterjea fixed point theorem. □

4. Conclusion

In this study we introduced the notion of f-metrics and the corresponding f-metric spaces as a generalization of the concept of b-metric spaces. In the sequel, we proved that Matkowski's fixed point theorem still holds true in this, more general, context. Likewise, we extended Kannan's and Chatterjea's fixed point results, albeit with some restrictions, in the case of complete f-metric spaces.

It remains to be seen if the above mentioned restrictions, namely the condition imposed on the function f in Theorem 3.3 and the boundness condition required in Theorem 3.4, are necessary.

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