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A RESOLVENT COMPUTATIONAL-FREE ALGORITHM FOR SOLVING MONOTONE INCLUSION PROBLEMS WITH APPLICATIONS

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Abstract. Following Chidume (Contemp. Math. (Amer. Math. Soc.) 659, 31-41, 2016) who posed an interesting question concerning the proximal point algorithm "Can an iteration process be developed which will not involve the computation of $\big(I+\lambda_n A\big)^{-1}x_n$ at each step of the iteration process and which will guarantee strong convergence to a solution of $0 \in Au$?", in this paper, we provide an affirmative answer to the question above concerning the forward-backward algorithm. We introduce a resolvent computational-free alternative of the forward-backward algorithm for approximating zeros of sum of two monotone operators and prove a strong convergence theorem in the setting of real Hilbert spaces. Furthermore, we use our algorithm in the restoration process of some degraded images and sparse signals. In addition, numerical illustrations of our proposed algorithm in $L_2([0,1])$ is also presented. Finally, numerical comparison of the performance of our new method with some classical methods in the literature are presented.
Key Words and Phrases: Monotone, convex minimization, image restoration, signal recovery.

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1. Introduction

Let H be a real Hilbert space and $A : H \to 2^H$ be a maximal monotone operator. The following problem:

$$\text{find } \ u \in H \ \text{ such that } \ 0 \in Au \tag{1.1}$$

is a fundamental problem of interest to many authors largely due to its applicability in solving many problems arising from convex minimization, dyanamical systems, variational inequality to mention but few (see, e.g., [10, 12]). Iterative methods for solving the inclusion problem (1.1) have been proposed by many authors in the setting of real Hilbert spaces and Banach spaces (see, e.g., [28], [7], [11], [16], [17] [18]). One of the early methods proposed for solving problem (1.1) is the *celebrated proximal point algorithm (PPA)* of Martinet [21] which was defined as follows:

$$\begin{cases} x_1 \in H \\ x_{n+1} = (I + \lambda_n A)^{-1} x_n, \end{cases} \tag{1.2}$$

where $\{\lambda_n\} \subset (0, \infty)$. Weak convergence of the sequence generated by the PPA has been established by many authors (see, e.g., [28]).

We observe that the PPA involve the computation of the resolvent operator $(I+\lambda A)^{-1}$ at each step of the iteration process. It is well-known that $(I + \lambda A)^{-1}$ has high computational cost especially when the operator A is nonlinear or a matrix of higher dimension. To address this difficulty, some researchers decomposed the operator A as a sum of two operators from which the simpler decomposition (linear part) will be used in computing the resolvent. In general, the problem takes the following setting:

$$\text{find } \ u \in H \ \text{ such that } \ 0 \in (Kx + Fx), \tag{1.3}$$

where $K : H \to H$ is mostly assumed to be α-inverse strongly monotone (ism) or Lipschitz continuous and $F : H \to 2^H$ is maximal monotone.

The need for this decomposition appears naturally in applications. The fact that models arising from signal recovery, machine learning and image restoration are expressed mathematically as sum of two operators made the idea of decomposing or splitting the operator A as sum of two operators an important problem. We lifted an example from this monograph [3] given from quantum mechanics to show a practical need for the decomposition. In the example, an operator A defined by

$$Au := -a^2 + \Delta u + (g(x) + b)u(x) + u(x) \int_{\mathbb{R}^3} \frac{u^2(y)}{\|x - y\|} dy,$$

where $\Delta = \sum_{i=1}^3 \frac{\partial^2}{\partial x_i^2}$ is the Laplacian in $\mathbb{R}^3$, a and b are constants, $g(x) = g_0(x) + g_1(x)$, $g_0(x) \in L^\infty(\mathbb{R}^3)$, $g_1(x) \in L^2(\mathbb{R}^3)$ was considered. The operator A was represented as a sum of two operators, say, $A = K + F$ where K is the linear part of A (it is called Schrodinger operator) and F is defined by the last term. It is well-known (see, e.g., [19]) that there exists $b \geq 0$ such that K becomes positive (i.e., monotone). Furthermore, the operator F is the gradient of functional

$$\Phi(u) = \frac{1}{4}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{u^2(x)u^2(y)}{\|x-y\|}dxdy,\ W_1^2(\mathbb{R}^3),$$

which is proper, convex and lower semicontinuous, and $W_1^2(\mathbb{R}^3)$ is a Sobolev space. Thus, F is maximal monotone. The particular case when $g(x) = -2\|x\|^{-2}$ and $a = \frac{1}{2}$ correspond to the case of Coulomb potential in quantum mechanics, and $b \geq 2$ guarantees a positivity of the Schrodinger operator [26]. The operator A describes the atom of helium in the situation when both electrons are in the same state. Observe that zeros of $K + F$ correspond to the zeros A.

One of the famous methods for solving (1.3) is the forward-backward splitting method which was introduced by Passty [25] and studied extensively by a host of many authors. The forward-backward algorithm (FBA) generates a sequence $\{x_n\}$ on a real Hilbert space H by the recursive equation:

$$\begin{cases} x_1 \in H \\ x_{n+1} = (I + \lambda_n F)^{-1}(I - \lambda_n K)x_n, \end{cases} \tag{1.4}$$

where $\{\lambda_n\} \subset (0,\infty)$, K is α-ism, F is maximal monotone and $K + F$ is also maximal monotone. Weak convergence of the sequence generated by (1.4) has been established by many authors (see, e.g., [25], [20], and [22]).

Observe that if $K \equiv 0$ the FBA reduces to the PPA. Furthermore, since every α-ism operator is $\frac{1}{\alpha}$-Lipschitz, Tseng [30] replaced the α-ism assumption on K with monotone Lipschitz. He introduced the following algorithm:

$$\begin{cases} x_1 \in C \\ y_n = (I + \lambda_n F)^{-1}(I - \lambda_n K)x_n \\ x_{n+1} = P_C\big(y_n - \lambda_n(Ky_n - Kx_n)\big), \end{cases} \tag{1.5}$$

where $C \subset H$ is nonempty closed and convex such that $C \cap (K+F)^{-1}0 \neq \emptyset$, K is maximal monotone and Lipschitz continuous with constant $L > 0$ and F is maximal monotone. He proved weak convergence of the sequence generated by his algorithm to a solution of problem (1.3).

Since strongly convergent algorithms are more preferable in applications, Takahashi et al [29] introduced and studied a Halpern modification of the FBA that guarantees strong convergence of the iterates of the algorithm in the setting of a real Hilbert space H. Their algorithm was defined as follows:

$$\begin{cases} u, x_1 \in C \subset H, \\ x_{n+1} = \alpha_n u + (1-\alpha_n)\big(I + \lambda_n B\big)^{-1}(I - \lambda_n A)x_n, \end{cases} \tag{1.6}$$

where C is a nonempty closed and convex subset of H, $\{\lambda_n\} \subset (0, \infty)$ and $\{\alpha_n\} \subset (0, 1)$ satisfy some appropriate conditions.

Remark 1.1. We remark here that the PPA, FBA and their modifications still involve the computation of the resolvent operator $(I + \lambda A)^{-1}$ at each step.

' Chidume [9] argued that by splitting or decomposing operator, the problem of the computing resolvent operator has not been addressed fully and he posed the following question:

> "Can an iteration process be developed which will not involve the computation of $\left(I + \lambda_n A\right)^{-1} x_n$ at each step of the iteration process and which will guarantee strong convergence to a solution of $0 \in Au$?"

This question was answered partially in the affirmative by Chidume [9] and eventually it was answered completely in the affirmative by Chidume et al [10].

Remark 1.2. We also remark here that the theorems of Chidume [9] and Chidume et al [10] provide a resolvent computational-free alternative of the PPA. Following Chidume's argument and looking at the nature of models arising from image restoration and signal processing its natural to ask:

Question 1. Can a resolvent computational-free alternative of the forward-backward algorithm be developed that will guarantee strong convergence to a solution of the inclusion problem (1.3)?

It is our purpose in this paper to provide an affirmative answer to Question 1 in the setting of real Hilbert spaces. Furthermore, we will use our propose method in the restoration process of some degraded images and sparse signal. In addition, some numerical illustrations of our proposed algorithm in $L_2([0, 1])$ will be presented. Finally, we will compare the numerical performance of our new method with the FBA algorithm, Tseng's algorithm and Takahashi et al's [29] algorithm.

2. Preliminaries

The basic notions of monotone, maximal monotone and other types of monotonicity can be quickly looked up by interested readers in, for example, any of these monographs [3, 4, 8].

Lemma 2.1 ([4])**.** *We will need the following famous Hilbert space identities in the proof of our main theorem:*

(i) $\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y\rangle, \ \forall x, y \in H,$

(ii) $\|\lambda x + (1 - \lambda)y\|^2 = \lambda\|x\|^2 + (1 - \lambda)\|y\|^2 - \lambda(1 - \lambda)\|x - y\|^2$, $\forall \ \lambda \in (0, 1)$ *and* $x, y \in H$.

Next is the famous lemma of Xu [31] which will be needed to establish strong convergence of the sequence generated by our proposed algorithm:

Lemma 2.2 ([31])**.** *Let* $\{a_n\}$ *be a sequence of nonnegative numbers satisfying the condition*

$$a_{n+1} \leq (1 - \alpha_n)a_n + \alpha_n\beta_n + c_n, \ n \geq 0, \tag{2.1}$$

where $\{\alpha_n\}$, $\{\beta_n\}$ *and* $\{c_n\}$ *are sequences of real numbers such that*

$$(i)\ \{\alpha_n\} \subset [0,1] \text{ and } \sum_{n=0}^{\infty} \alpha_n = \infty;\ (ii)\ \limsup_{n\to\infty} \beta_n \leq 0;\ (iii)\ c_n \geq 0,\ \sum_{n=0}^{\infty} c_n < \infty.$$

Then, $\lim_{n\to\infty} a_n = 0$.

The next lemma of Reich [27] which he proved in the setting of Banach space will be useful to establish that the limit point of the sequence generated by our algorithm is in the solution set of problem (1.3).

Lemma 2.3 ([27])**.** *Let* H *be a real Hilbert space* $T : H \to 2^H$ *be a maximal monotone operator. Let* $J_\lambda x := (I + \lambda T)^{-1}x,\ \lambda > 0$ *be the resolvent of* T *such that* $T^{-1}0 \neq \emptyset$. *Then, for each* $z \in H$ $\lim_{\lambda\to\infty} J_\lambda z \in T^{-1}0$.

Finally, the next lemma of Browder [6] is to establish a necessary and sufficient condition for us to be able to use Lemma 2.3 to conclude that the limit point of the sequence generated by our algorithm is in the solution set of problem (1.3).

Lemma 2.4 ([6])**.** *Let* H *be a real Hilbert space, and* $T_1 : H \to H$ *be a maximal monotone map and* $T_2 : H \to H$ *be a monotone hemi-continuous and bounded map. Then the mapping* $T := T_1 + T_2$ *is maximal monotone.*

3. Main result

In this section we present a strong convergence theorem concerning the sequence generated by our proposed algorithm and give its convergence analysis.

Theorem 3.1. *Let* H *be a real Hilbert space. Let* $K : H \to H$ *be a monotone Lipschitz mapping and* $F : H \to 2^H$ *be a set-valued maximal monotone and bounded mapping. Suppose* $\Omega = (K+F)^{-1}0 \neq \emptyset$. *Given* $u, x_1 \in H$, *let* $\{x_n\} \subset H$ *be a sequence generated by:*

$$x_{n+1} = x_n - \alpha_n K x_n - \alpha_n \chi_n - \alpha_n \theta_n (x_n - u), \quad \chi_n \in F x_n, \tag{3.1}$$

where $\{\alpha_n\}$ *and* $\{\theta_n\}$ *are sequences in* $(0,1)$ *that satisfy the following conditions:*

(i) $\{\theta_n\}$ *is decreasing,* $\lim_{n\to\infty} \theta_n = 0,\ \ \alpha_n \leq \theta_n^2,\ \ \sum_{n=1}^{\infty} \alpha_n \theta_n = \infty;$

(ii) $\lim_{n\to\infty} \dfrac{\left(\frac{\theta_{n-1}}{\theta_n} - 1\right)}{\alpha_n \theta_n} = 0.$

Then, $\{x_n\}$ *converges strongly to a point in* Ω.

Proof. First we prove that $\{x_n\}$ is bounded. To show this, we start with the following preambles: let $x^* \in \Omega$ and let $x_1, u \in H$. Then, there exists $r > 0$ such that $\|u - x^*\| \leq \frac{r}{4}$ and $\|x_1 - x^*\| \leq \frac{r}{2}$. Define $B := \{y \in H : \|y - x^*\| \leq r\}$. Since K is Lipschitz and thus bounded and F is bounded, define

$$M := \sup\{\|Kx + \chi + \theta(x - u)\|^2,\ x \in B,\ \theta \in (0,1),\ \chi \in Fx\}.$$

Observe that for r large enough, we can impose the condition that $M \leq \frac{r^2}{2}$. To prove that $\{x_n\}$ is bounded, it is enough to show that $\{x_n\} \subset B$. We prove this by induction. For $n = 1$, by construction, $x_1 \in B$. Assume $x_n \in B$, meaning $\|x_n - x^*\| \leq r$ for some $n \geq 1$ we show that $x_{n+1} \in B$. For contradiction, suppose $x_{n+1} \notin B$, i.e., $\|x_{n+1} - x^*\| > r$. Now,

$$\begin{aligned}
r^2 &< \|x_{n+1} - x^*\|^2 \\
&= \|x_n - x^* - \alpha_n(Kx_n + \chi_n + \theta_n(x_n - u))\|^2 \\
&= \|x_n - x^*\|^2 - 2\alpha_n\langle x_n - x^*, Kx_n + \chi_n + \theta_n(x_n - u)\rangle \\
&\quad + \alpha_n^2\|Kx_n + \chi_n + \theta_n(x_n - u)\|^2 \\
&\leq \|x_n - x^*\|^2 - 2\alpha_n\theta_n\langle x_n - x^*, x_n - u\rangle + \alpha_n\theta_n M \\
&= \|x_n - x^*\|^2 - 2\alpha_n\theta_n\|x_n - x^*\|^2 - 2\alpha_n\theta_n\langle x_n - x^*, x^* - u\rangle + \alpha_n\theta_n M \\
&\leq (1 - \alpha_n\theta_n)\|x_n - x^*\|^2 + \alpha_n\theta_n\|x^* - u\|^2 + \alpha_n\theta_n M \\
&\leq \Big(1 - \alpha_n\theta_n + \frac{\alpha_n\theta_n}{16} + \frac{\alpha_n\theta_n}{2}\Big)r^2 \\
&= \Big(1 - \frac{7\alpha_n\theta_n}{16}\Big)r^2 \leq r^2,
\end{aligned}$$

which leads to a contradiction. Hence, $x_{n+1} \in B$. Therefore, $\{x_n\}$ is bounded.

Next, we prove that $\{x_n\}$ converges strongly to a point in Ω. Setting $T := K + F$, it follows from Lemma 2.4 that T is maximal monotone. Furthermore, by Lemma 2.3, for $u \in H$, there exists a sequence $\{y_n\}$ defined by $y_n := \Big(I + \frac{1}{\theta_n}T\Big)^{-1}u$ such that

$$\lim_{n\to\infty} y_n = y^* \in \Omega. \tag{3.2}$$

Furthermore, for $z_n \in Ty_n = (Ky_n + Fy_n)$, from the definition of y_n, we deduce that

$$z_n + \theta_n(y_n - u) = 0. \tag{3.3}$$

We now prove that $\lim\limits_{n\to\infty} \|x_{n+1} - y_n\| = 0$. Observe that

$$\begin{aligned}
\|x_{n+1} - y_n\|^2 &= \|x_n - y_n - \alpha_n(Kx_n + \chi_n + \theta_n(x_n - u))\|^2 \\
&\leq \|x_n - y_n\|^2 - 2\alpha_n\langle x_n - y_n, Kx_n + \chi_n + \theta_n(x_n - u)\rangle + \alpha_n^2 M.
\end{aligned} \tag{3.4}$$

Next, we estimate the second term in inequality (3.4) using monotonicity of $T = K + F$, the fact that $z_n \in Ty_n$ and equation (3.3). Thus,

$$\begin{aligned}
\langle x_n - y_n, Kx_n + \chi_n + \theta_n(x_n - u)\rangle &= \langle x_n - y_n, Kx_n + \chi_n - z_n\rangle + \theta_n\|x_n - y_n\|^2 \\
&\quad + \langle x_n - y_n, z_n + \theta_n(y_n - u)\rangle \\
&\geq \theta_n\|x_n - y_n\|^2.
\end{aligned}$$

Substituting this in inequality (3.4) and using the assumption that $\alpha_n \leq \theta_n^2$, we get

$$\|x_{n+1} - y_n\|^2 \leq (1 - \alpha_n\theta_n)\|x_n - y_n\|^2 + \alpha_n\theta_n^2 M. \tag{3.5}$$

Now, using Lemma 2.1 we obtain that

$$\begin{aligned}\|x_n - y_n\|^2 &= \|x_n - y_{n-1} + y_{n-1} - y_n\|^2 \\ &\leq \|x_n - y_{n-1}\|^2 + 2\langle y_{n-1} - y_n, x_n - y_n\rangle \\ &\leq \|x_n - y_{n-1}\|^2 + 2\|y_{n-1} - y_n\|\|x_n - y_n\|. \end{aligned} \tag{3.6}$$

Next we get an estimate for $\|y_{n-1} - y_n\|$. From equation (3.3) we deduce that

$$u - y_n - \frac{1}{\theta_n} z_n = 0 \quad \text{and} \quad u - y_{n-1} - \frac{1}{\theta_{n-1}} z_{n-1} = 0.$$

By manipulating the second term, we get

$$\frac{\theta_{n-1}}{\theta_n}(u - y_{n-1}) - \frac{1}{\theta_n} z_{n-1} = 0.$$

Hence, by equating this with first term we obtain

$$\begin{aligned} u - y_n - \frac{1}{\theta_n} z_n &= \frac{\theta_{n-1}}{\theta_n}(u - y_{n-1}) - \frac{1}{\theta_n} z_{n-1} \\ y_{n-1} - y_n + \frac{1}{\theta_n}(z_{n-1} - z_n) &= \frac{\theta_{n-1}}{\theta_n}(u - y_{n-1}) - (u - y_{n-1}) \\ y_{n-1} - y_n &= -\frac{1}{\theta_n}(z_{n-1} - z_n) + \Big(\frac{\theta_{n-1}}{\theta_n} - 1\Big)(u - y_{n-1}) \end{aligned}$$

Taking the inner product of both side with $y_{n-1} - y_n$, we get

$$\begin{aligned}\|y_{n-1} - y_n\|^2 &= -\frac{1}{\theta_n}\langle y_{n-1} - y_n, z_{n-1} - z_n\rangle + \Big(\frac{\theta_{n-1}}{\theta_n} - 1\Big)\langle y_{n-1} - y_n, y_{n-1} - u\rangle \\ &\leq \Big(\frac{\theta_{n-1}}{\theta_n} - 1\Big)\|y_{n-1} - y_n\|\|y_{n-1} - u\|. \end{aligned}$$

This implies that

$$\|y_{n-1} - y_n\| \leq \Big(\frac{\theta_{n-1}}{\theta_n} - 1\Big) D, \text{ for some } \; D > 0. \tag{3.7}$$

Substituting inequality (3.6) in (3.5) and using inequality (3.7), the fact that $\{x_n\}$ and $\{y_n\}$ are bounded, we get

$$\begin{aligned}\|x_{n+1} - y_n\|^2 &\leq (1 - \alpha_n\theta_n)\|x_n - y_{n-1}\|^2 + 2(1 - \alpha_n\theta_n)\Big(\frac{\theta_{n-1}}{\theta_n} - 1\Big) D\|x_n - y_n\| + \alpha_n\theta_n^2 M \\ &\leq (1 - \alpha_n\theta_n)\|x_n - y_{n-1}\|^2 + 2\Big(\frac{\theta_{n-1}}{\theta_n} - 1\Big) M^* + \alpha_n\theta_n^2 M \text{ (for some } \; M^* > 0) \\ &= (1 - \alpha_n\theta_n)\|x_n - y_{n-1}\|^2 + \alpha_n\theta_n\Bigg(\frac{\Big(\frac{\theta_{n-1}}{\theta_n} - 1\Big)}{\alpha_n\theta_n} 2M^* + \theta_n M\Bigg). \end{aligned}$$

By Lemma 2.2 $\lim\limits_{n\to\infty} \|x_{n+1} - y_n\| = 0$. This completes the proof. □

4. Numerical illustrations and applications

4.1. **Numerical implementation.** In this subsection, we present numerical examples in $L_2([0,1])$ to illustrate the performance of our proposed algorithm (3.1) in comparison to the performance of the FBA (1.4), Tseng's algorithm (1.5) and Takahashi et al's algorithm (1.6).

Example 4.1.

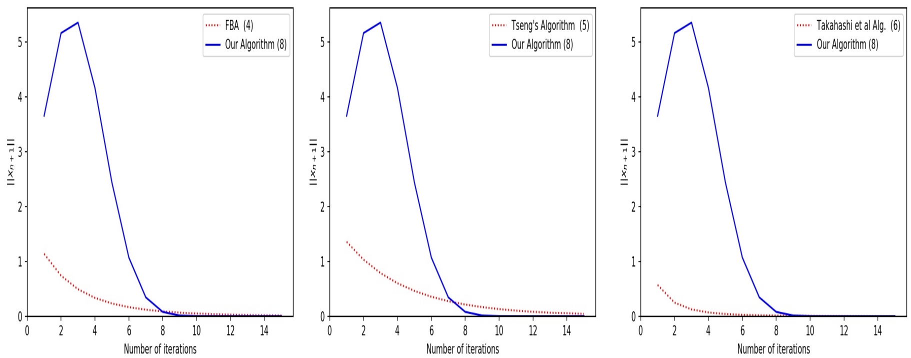
We consider the infinite dimensional real Hilbert space $H = L_2([0,1])$ and the functions $F, K : H \to H$ were define as follows:

$$Fx := \sin tx(t) \quad \text{then} \quad (I + \lambda F)^{-1} = \frac{x(t)}{1 + \lambda \sin t}, \qquad Kx := 2(t+1)x(t),$$

for the experiment. Obviously, F is maximal monotone and K is Lipschitz. Furthermore, the function $x(t) = 0$, $\forall\, t \in [0,1]$ is the only point in $\Omega = (K+F)^{-1}0$. For the control parameters, in algorithms (1.4) and (1.5) we choose $\lambda_n = 0.1$ in algorithm (1.6), we choose $\lambda_n = 0.1$ and $\alpha_n = \frac{1}{n+1}$ and in our proposed algorithm (3.1), we choose $\alpha_n = \frac{1}{(n+1)^{\frac{2}{3}}}$ and $\theta_n = \frac{1}{(n+1)^{\frac{1}{4}}}$. We started the iteration process at $x_1 = \exp(t)$ and $u(t) \equiv 0$. The iteration process for each algorithm was terminated when $\|x_{n+1}\| > 10^{-8}$ or $n > 15$. The results obtained from the experiment are illustrated in Table 1 and Figure 1.

Table 1. Numerical simulations for Example 4.1

The iterates obtained choosing $x_1 = \exp(t)$				
	Algorithm (1.4)	Algorithm (1.5)	Algorithm (1.6)	Algorithm (3.1)
n	$\|x_{n+1}\|$	$\|x_{n+1}\|$	$\|x_{n+1}\|$	$\|x_{n+1}\|$
2	1.1364	1.3551	0.5682	3.6521
3	0.7379	0.7848	0.1225	5.3533
4	0.3321	0.5997	0.0665	4.1612
5	0.2307	0.4597	0.0384	2.4378
6	0.1632	0.3535	0.0233	1.0682
7	0.1173	0.2727	0.0146	0.3423
8	0.0856	0.2112	0.0095	0.0766
9	0.0632	0.1641	0.0063	0.0111
10	0.0471	0.1280	0.0042	8.31e-4
11	0.0354	0.1002	0.0029	1.53e-5
12	0.0258	0.0787	0.0021	5.74e-7
13	0.0204	0.0621	0.0014	1.03e-7
14	0.0155	0.0532	0.0011	5.16e-8
15	0.0119	0.0394	7.47e-4	3.02e-8



(A) Graphical representation of the iterates in Table 1

FIGURE 1. Graphical comparison of the iterates of algorithms (1.4), (1.5) and (1.6) with those of (3.1)

Remark 4.2. We observe that in this experiment, algorithms (1.4), (1.5) and (1.6) failed to satisfy the stopping criterion before the specified maximum number of iterations was exhausted. However, even though in the first few iterates our algorithm (3.1) was moving away from the solution, it later took a turn moving towards the solution and satisfied the stopping before the prescribed maximum number iterations.

4.2. **Applications to signal processing.** In this subsection, we will use our proposed algorithm (3.1) and algorithms (1.4), (1.5) and (1.6) to reconstruct a *sparse signal* of length M and observations N degraded by a random noise. It is no news that the famous l_1-regularization problem

$$\arg\min_{x\in\mathbb{R}^n}\left\{\frac{1}{2}\|Ax-b\|_2^2+\lambda_n\|x\|_1\right\} \tag{4.1}$$

popularly known as LASSO problem can be used to restore a sparse signal $x \in \mathbb{R}^N$ from the following observation model:

$$y = Ax + b, \tag{4.2}$$

where A is an $M \times N$ sensing matrix with random entries and $M << N$, and $b \in \mathbb{R}^M$ is noise and $y \in \mathbb{R}^M$ is the observed or measured data. In the literature, several authors have shown that problem (4.1) can be transformed into the inclusion problem (1.3) (see, e.g., [1], [23] [15]). Thus, algorithms proposed for solving problem (1.3) can be used to find equivalent solutions of the LASSO problem (4.1).

In the experiment, a signal of length $N = 4096$ and $M = 2048$ observations is constructed and was degraded by a random noise. The number of nonzero components in the constructed signal were varied to be 100 and 200 (spikes). In the experiment we vary the parameters for algorithms (1.4), (1.5), (1.6) and (3.1), then we chose the parameters that best recover the original signal and satisfies the stopping criterion (10^{-8}) with less number of iterations. For algorithms (1.4) and (1.5), we choose $\lambda_n = 0.001$. In algorithm (1.6), we choose $\lambda_n = 0.001$ and $\alpha_n = \frac{1}{(n+1)^2}$ and in our proposed algorithm (3.1), we choose $\alpha_n = \frac{1}{(n+1)^{0.01}}$ and $\theta_n = \frac{1}{(n+1)^3}$. We started the iteration process at $x_1 = ((Ax + b)^T A)^T$ and $u = 0$. The iteration process for each algorithm was terminated when $\|x_{n+1} - x_n\| > 10^{-8}$ or $n > 1000$. The results obtained from the experiment are illustrated in Table 1 and Figure 1.

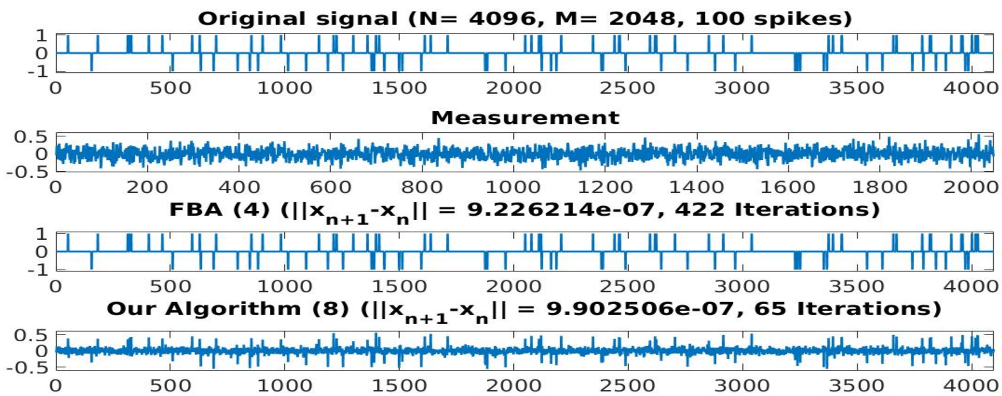


Figure 2. Recovery of the sparse signal with 100 spikes via algorithms (1.4) and (3.1)

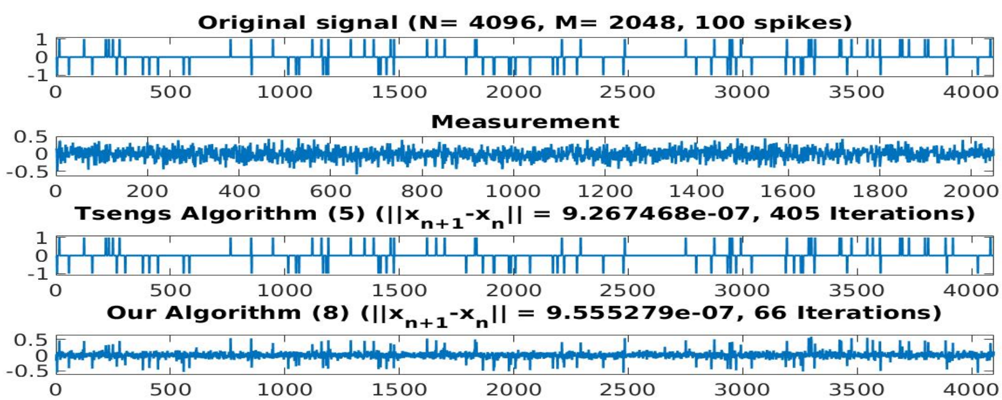


Figure 3. Recovery of the sparse signal with 100 spikes via algorithms (1.5) and (3.1)

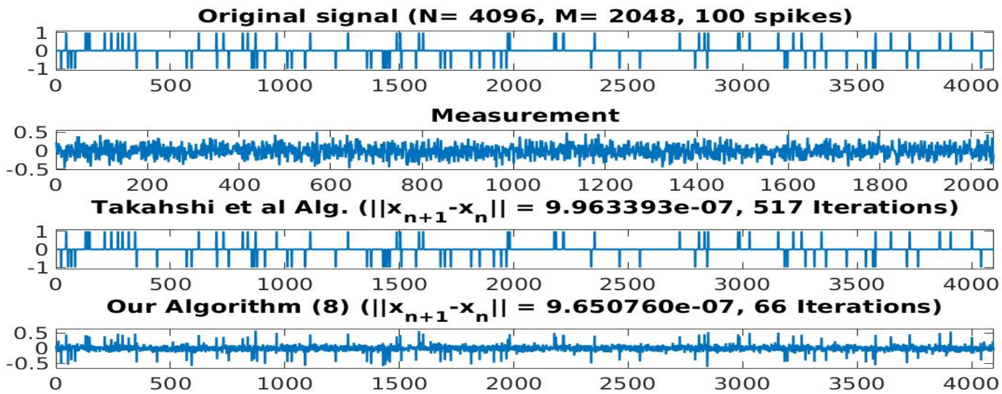
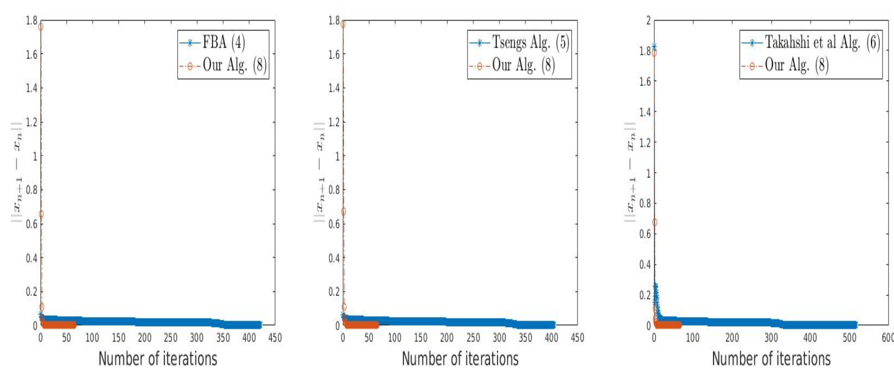


FIGURE 4. Recovery of the sparse signal with 100 spikes via algorithms (1.6) and (3.1)



(A) Graphical representation of $\|x_{n+1} - x_n\|$ for 100 spikes

FIGURE 5. Error norm for algorithms (1.4), (1.5), (1.6) and (3.1) for 100 spikes

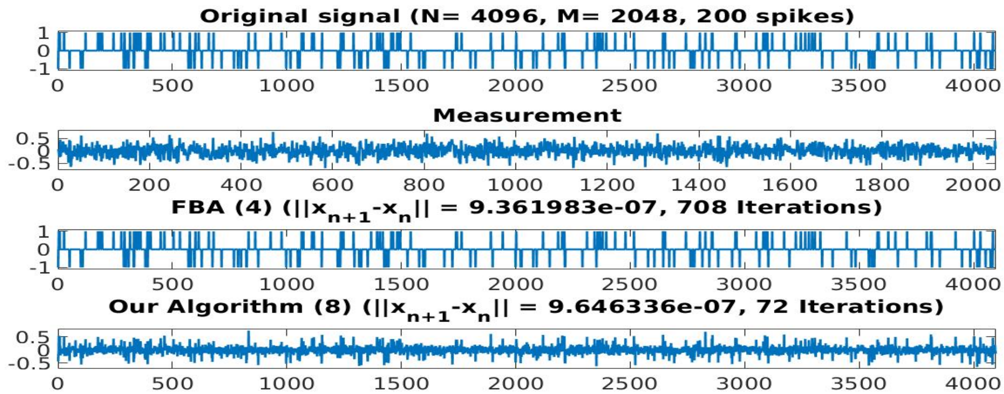


FIGURE 6. Recovery of the sparse signal with 200 spikes via algorithms (1.4) and (3.1)

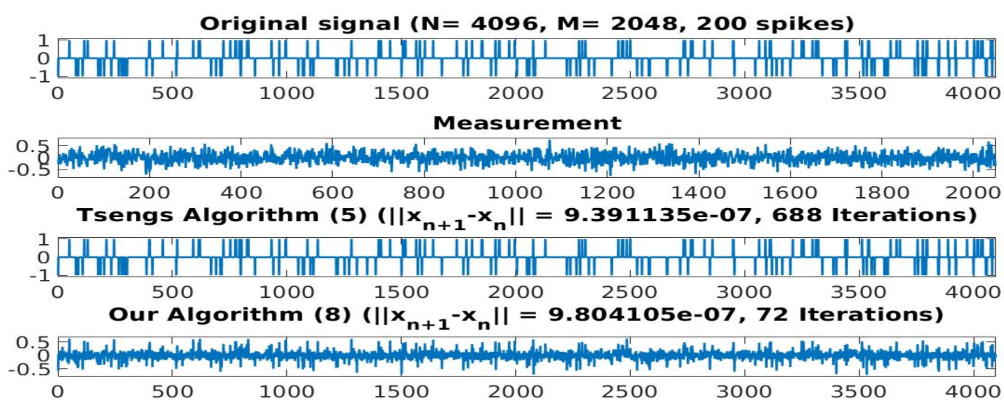


FIGURE 7. Recovery of the sparse signal with 200 spikes via algorithms (1.5) and (3.1)

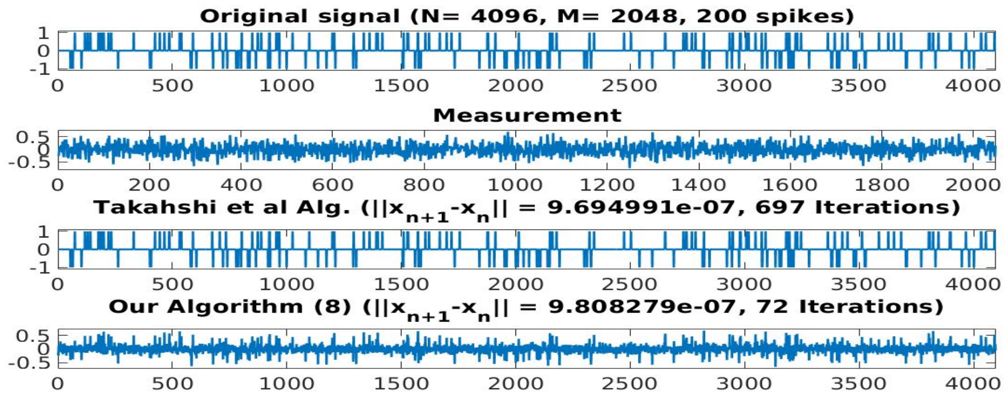
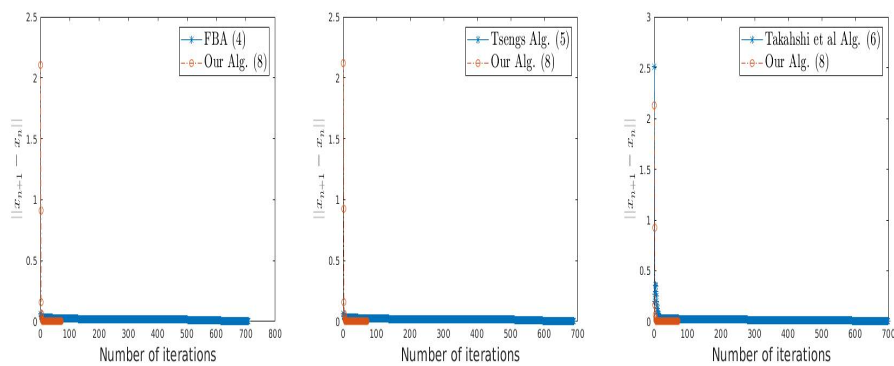


FIGURE 8. Recovery of the sparse signal with 200 spikes via algorithms (1.6) and (3.1)



(A) Graphical representation of $\|x_{n+1} - x_n\|$ for 200 spikes

FIGURE 9. Error norm for algorithms (1.4), (1.5), (1.6) and (3.1) for 200 spikes

Remark 4.3. We observe that from the experiment our proposed algorithm (3.1) satisfies the stopping criterion in fewer iterations compared to algorithms (1.4), (1.5) and (1.6). However, the restored signals vai algorithms (1.4), (1.5) and (1.6) is better than the restored signal via our proposed (3.1).

4.3. **Applications to image restoration.** In this subsection, we will use our proposed algorithm (3.1) and algorithms (1.4), (1.5) and (1.6) to restore some personal images degraded by motion blur and random noise. In the literature several authors

have shown that the convex minimization problem arising from image restoration can be transformed as the inclusion problem (1.3). Interested readers may see, for example, any of these papers [5], [13] for the formulation.
In the experiment, we considered four personal images of Yodjai (Y), Duangkamon (D), Abubakar (A) and Poom (P) and degraded them with MATLAB blur function "P=fspecial('motion',20,30)" and added random noise. For the algorithms that required the initial vector u, we set u to be zeros and x_1 to be ones in R^n. For algorithms (1.4) and (1.5), we choose $\lambda_n = 0.001$. In algorithm (1.6), we choose $\lambda_n = 0.001$ and $\alpha_n = \frac{1}{(n+1)}$ and in our proposed algorithm (3.1), we choose $\alpha_n = \frac{1}{(n+1)^{0.01}}$ and $\theta_n = \frac{1}{(n+1)^3}$. The iteration process for each algorithm was terminated when $\frac{\|x_{n+1}-x_n\|}{\|x_{n+1}\|} > 10^{-4}$ or $n > 150$. The results obtained from the experiment are illustrated in Table 1 and Figure 1.

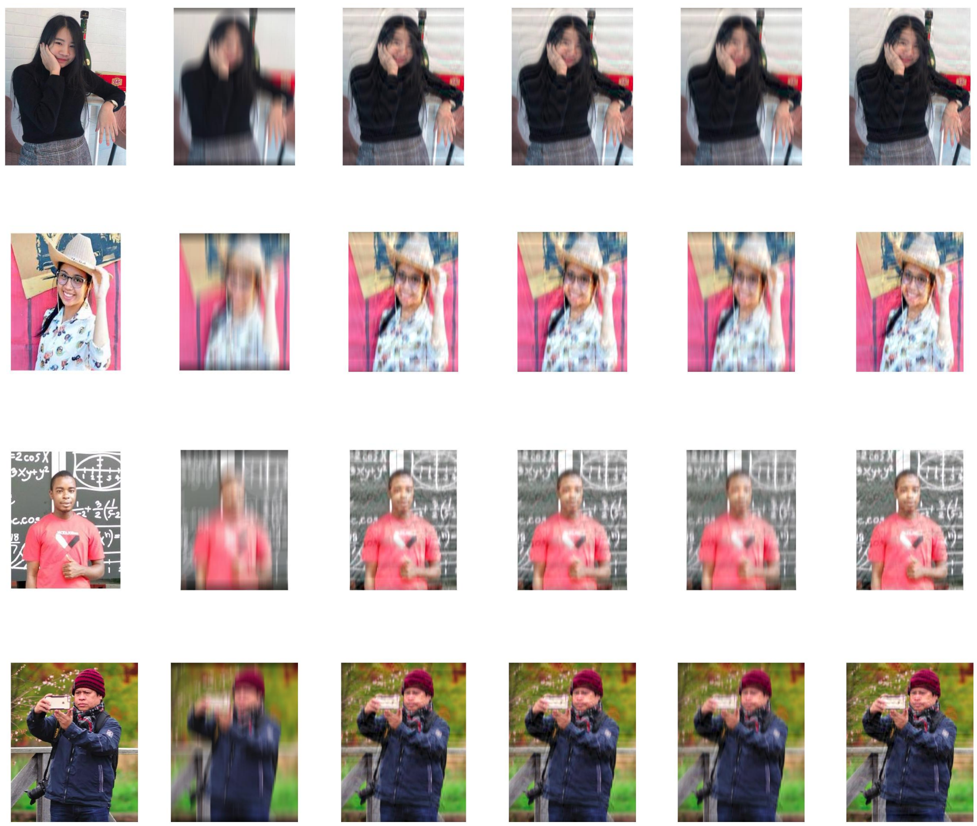


FIGURE 10. Restoration process via algorithms

Key to Figure 10. In Figure 10, the images displayed in the first column are the original test images next is their degraded version via random noise and motion blur.

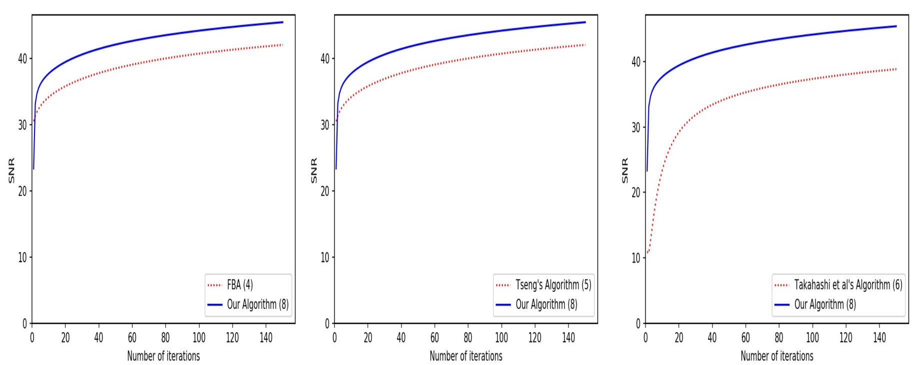
The next columns are the restored test images via algorithms (1.4), (1.5), (1.6) and (3.1), respectively.

Remark 4.4. We observe from Figure 10, the restored images in the last column appear to be closest to the original image compared to those in the 3rd, 4th and 5th column. To support this deduction, we use a well known metric called signal to noise ratio (SNR) to compare the quality of the restored images (see, e.g., [2] for how to compute SNR value of an image).

In Table 2 and Figures 11, 12, 13, 14, we illustrate the SNR values for the restored test images in Figure 9. Note that higher SNR value indicates that restored image has good quality.

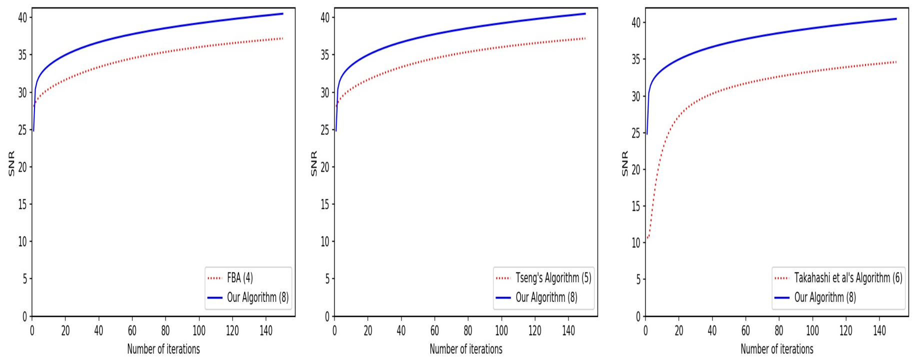
Table 2. SNR of the restored (Y), (D), (A), and (P) images in Figure 10

	FBA (1.4)				Tseng's Algorithm (1.5)				Takahashi et al's Algorithm (1.6)				Our Algorithm (3.1)			
n	Y	D	A	P	Y	D	A	P	Y	D	A	P	Y	D	A	P
1	30.48	28.06	23.28	25.14	30.48	28.06	23.28	25.14	10.91	10.75	10.13	10.51	23.34	24.86	27.73	27.21
10	34.13	30.42	25.01	27.03	34.13	30.42	25.01	27.03	23.33	22.44	19.81	21.21	37.68	33.56	27.73	27.21
20	35.77	31.66	26.01	28.22	35.77	31.66	26.01	28.22	29.22	27.23	23.16	25.03	39.43	34.98	28.91	28.87
30	36.89	32.59	26.77	29.12	36.89	32.59	26.77	29.12	31.87	29.18	24.48	26.55	40.57	35.95	29.76	29.99
40	37.75	33.36	27.41	29.84	37.75	33.36	27.41	29.84	33.44	30.31	25.25	27.45	41.41	36.68	30.43	30.85
50	38.45	33.99	27.93	30.44	38.45	33.99	27.93	30.44	34.52	31.11	25.82	28.10	42.07	37.27	31.00	31.55
60	39.04	34.53	28.39	30.95	39.04	34.53	28.39	30.95	35.33	31.71	26.26	28.61	42.61	37.77	31.48	32.14
70	39.54	34.99	28.79	31.39	39.54	34.99	28.79	31.39	35.98	32.22	26.64	29.03	43.08	38.19	31.91	32.65
80	39.97	35.39	29.15	31.78	39.97	35.39	29.15	31.78	36.52	32.65	26.96	29.39	43.48	38.57	32.29	33.11
90	40.35	35.73	29.46	32.12	40.35	35.73	29.46	32.12	36.98	33.02	27.25	29.71	43.84	38.91	32.64	33.51
100	40.70	36.04	29.75	32.43	40.70	36.04	29.75	32.43	37.38	33.35	27.51	30.00	44.16	39.23	32.95	33.86
110	41.01	36.32	30.01	32.71	41.01	36.32	30.01	32.71	37.74	33.65	27.75	30.26	44.46	39.52	33.24	34.20
120	41.29	36.56	30.25	32.97	41.29	36.56	30.25	32.97	38.06	33.93	27.97	30.50	44.73	39.79	33.52	34.51
130	41.55	36.79	30.47	33.21	41.55	36.79	30.47	33.21	38.35	34.18	28.18	30.72	44.98	40.04	33.77	34.79
140	41.79	37.00	30.68	33.43	41.79	37.00	30.68	33.43	38.62	34.41	28.37	30.92	45.22	40.28	34.01	35.05
150	42.02	37.19	30.87	33.64	42.02	37.19	30.87	33.64	38.86	34.62	28.54	31.12	45.44	40.51	34.23	35.30



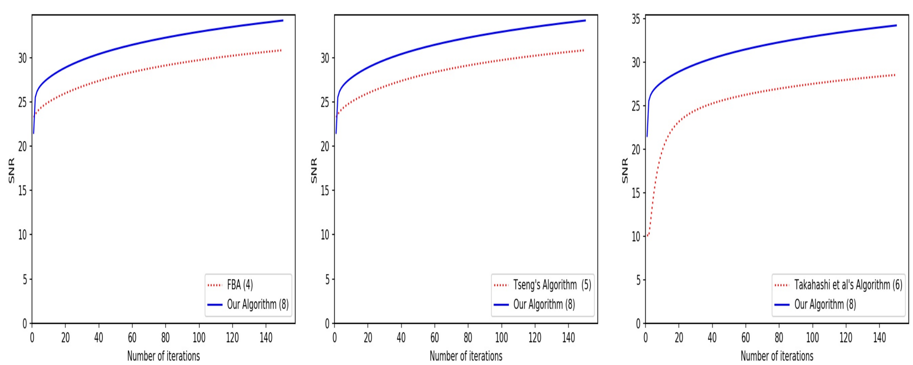
(A) Plots of SNR values

Figure 11. Comparison of the quality of the restored Yodjai's image via algorithms (1.4), (1.5) and (1.6) with algorithm (3.1)



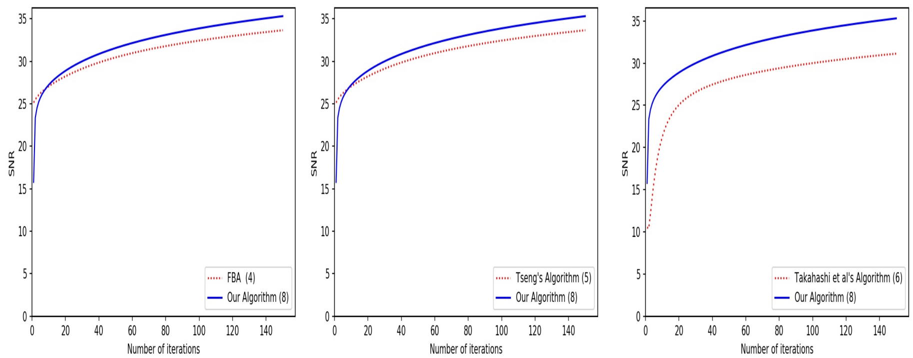
(A) Plots of SNR values

FIGURE 12. Comparison of the quality of the restored Duangkamon's image via algorithms (1.4), (1.5) and (1.6) with algorithm (3.1)



(A) Plots of SNR values

FIGURE 13. Comparison of the quality of the restored Abubakar's image via algorithms (1.4), (1.5) and (1.6) with algorithm (3.1)



(A) Plots of SNR values

FIGURE 14. Comparison of the quality of the restored Poom's image via algorithms (1.4), (1.5) and (1.6) with algorithm (3.1)

Remark 4.5. We observe from the SNR plots in Figures 11, 12, 13 and 14 the restored images via algorithm (3.1) have better quality compare to the restored images via algorithms (1.4), (1.5) and (1.6).

Remark 4.6. For the numerical illustrations presented in subsections 4.2 and 4.3 we used the open access software, SOFT_THERESHOLD (https://lts2.epfl.ch/unlocbox/doc/utils/soft_threshold.php) for computing $(I+\lambda_n F)^{-1}x$.

5. CONCLUSION

This paper presents an alternative algorithm to compliment the existing forward-backward algorithm and its modifications for approximating solutions of problem (1.3). A key and interesting property of the new algorithm is the fact that it does not require the computation of $(I+\lambda_n F)^{-1}$ at any step of the iteration process. Computing this inverse is a shortcoming of the FBA and its modifications. A practical example is in the reformulation of the LASSO problem

$$\arg\min_{x\in\mathbb{R}^n}\left\{\frac{1}{2}\|Ax-b\|_2^2+\lambda_n\|x\|_1\right\}$$

as the inclusion problem (1.3), where the function F mentioned in (1.3) is taking to be $\partial\|x\|_1$. Then,

$$(I+\lambda_n F)^{-1}x=\begin{cases}\frac{\|y\|+\lambda_n}{\|y\|}, & x\neq 0,\ y=(I+\lambda_n F)^{-1}x\\ 0, & x=0.\end{cases}$$

Clearly, it will not be easy to implement this numerically. However, a lot of researchers try to avoid the computation of $(I+\lambda_n F)^{-1}x$ by using the proximity operator (see, e.g., [14] for some examples). Finally, from the numerical simulations presented in section 4, our proposed algorithm (3.1) appears to be competitive and promising as it outperforms the well-known FBA and some of its modifications in the problems considered.

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