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KORPELEVICH'S EXTRAGRADIENT METHOD FOR SPLIT COMMON FIXED POINT PROBLEMS IN HILBERT SPACES

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Abstract. In this paper, we focus on the split common fixed point problem for hemicontractive mappings. We introduce two novel iterative methods for this problem and demonstrate their weak convergence under certain mild conditions. Our methods, in comparison to existing methods, offer a larger range of stepsize parameters, thereby enhancing the method's efficiency.
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1. Introduction

The split feasibility problem (SFP) was initially introduced by Censor and Elfving in the influential paper [6]. Since its introduction in 1994, the SFP has garnered significant attention, primarily for its applications in signal processing and image reconstruction, with notable advancements in intensity-modulated radiation therapy [3, 5].

The SFP can be mathematically formulated as the task of identifying a point $x^\dagger \in H$ such that

$$x^\dagger \in C \bigcap A^{-1}(Q). \tag{1.1}$$

Here, $C \subset H$ and $Q \subset H_1$ are nonempty, closed, and convex subsets, and $A^{-1}(Q) = \{x \in H : Ax \in Q\}$ with $A : H \to H_1$ being a bounded linear mapping. Iterative methods are commonly employed to solve the SFP (1.1); see [2, 10, 19, 21, 24, 22]. Byrne [2], among others, was the first to propose the so-called CQ method, which generates a sequence $\{x_n\}$ through the recursive procedure:

$$x_{n+1} = P_C\big[x_n - \tau_n A^*(I - P_Q)Ax_n\big], \tag{1.2}$$

where A^* denotes the conjugate of A, I represents the identity mapping, the stepsize τ_n is chosen in the interval $(0, +\infty)$, and P_C and P_Q are the metric projections onto C and Q, respectively. It has been demonstrated that if τ_n is selected from the range $(0, \frac{2}{\|A\|^2})$, then (1.2) weakly converges to a solution of (1.1) whenever such a solution exists.

There are various extensions of the SFP in the literature, one notable example being the split common fixed-point problem (SCFP) introduced by Censor and Segal in [7]. The SCFP involves finding an element in a fixed point set such that its image under a linear transformation belongs to another fixed point set. Formally, it requires finding $x^\dagger \in H$ that satisfies

$$x^\dagger \in F(U) \bigcap A^{-1}(F(T)), \tag{1.3}$$

where $U : H \to H$ and $T : H_1 \to H_1$ represent two classes of nonlinear mappings, and $F(U) = \{x \in H : Ux = x\}$ and $F(T) = \{y \in H_1 : Ty = y\}$ denote the fixed point sets of U and T, respectively. In [7], Censor and Segal investigated the case of directed mappings and proposed the following method:

$$x_{n+1} = U[x_n - \tau_n A^*(I - T)Ax_n], \tag{1.4}$$

where the stepsize τ_n is chosen from the interval $(0, +\infty)$. It has been demonstrated that if τ_n is selected from the range $(0, \frac{2}{\|A\|^2})$, then (1.4) weakly converges to a solution of (1.3) whenever such a solution exists. This result has subsequently been extended to more general cases; see, for example, [20, 12, 4]. As the choice of stepsize is linked to $\|A\|$, implementing (1.4) requires computing (or at least estimating) the norm $\|A\|$, which can be challenging in practice. An alternative approach is to utilize a variable stepsize that is independent of $\|A\|$; see, for instance, [18, 8, 17, 16, 14, 15].

Hemicontraction is a fundamental class of nonlinear mappings that encompasses demicontractive mappings, quasi-nonexpansive mappings, and directed mappings as special cases. In [23], Yao et al. examined the scenario where the mappings in (1.3) are both hemicontractive and Lipschitz continuous. Specifically, they devised the following method: Starting with an initial guess $x_0 \in H$, the iteration process generates x_{n+1} according to the recursion:

$$\left[\begin{array}{l} y_n = x_n - \alpha_n A^*[I - T((1-\tau_n)I + \tau_n T)]Ax_n, \\ x_{n+1} = (1-\beta_n)y_n + \beta_n U[(1-\lambda_n)y_n + \lambda_n U y_n], \end{array} \right. \tag{1.5}$$

where $\{\alpha_n\}, \{\beta_n\}, \{\tau_n\}$, and $\{\lambda_n\}$ are predefined real sequences. Moreover, assuming that U and T are L and L_1-Lipschitz continuous, the convergence of (1.5) is ensured under the conditions:

$$0 < \alpha_n < \tau_n < \frac{1}{(\sqrt{1+L_1^2}+1)\|A\|^2}, \tag{1.6}$$

$$0 < \beta_n < \lambda_n < \frac{1}{\sqrt{1+L^2}+1}. \tag{1.7}$$

In this paper, we aim to introduce and analyze iterative methods for solving problem (1.3) in cases where the involved mappings are hemicontractive. The structure of this paper is as follows: Section 2 presents essential concepts and relevant lemmas. In Section 3, we delve into key properties of hemicontractive mappings. Section 4 introduces a novel method to tackle problem (1.3) and establishes its weak convergence under mild conditions. Section 5 presents another new method for problem (1.3) and proves its weak convergence under suitable conditions. These proposed methods draw inspiration mainly from Korpelevich's extragradient method for solving variational inequalities.

2. Preliminary

In this section, assume that C is a nonempty subset of a Hilbert space H.

Definition 2.1 A mapping $T : C \to C$ is called *pseudocontractive* if for all $x, y \in C$,

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + \|(I - T)x - (I - T)y\|^2.$$

T is called *strictly pseudocontractive* if there exists $k < 1$ such that for all $x, y \in C$,

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + k\|(I - T)x - (I - T)y\|^2.$$

T is called *nonexpansive* if all $x, y \in C$,

$$\|Tx - Ty\| \leq \|x - y\|.$$

T is called *Lipschitz continuous* if exists $L > 0$ such that for all $x, y \in C$,

$$\|Tx - Ty\| \leq L\|x - y\|.$$

Definition 2.2 Let $T : C \to C$ be a mapping with $F(T) \neq \emptyset$. T is called *hemicontractive* if for all $x \in C$ and $y \in F(T)$,

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + \|x - Tx\|^2.$$

T is called *demicontractive* if for some $k < 1$, for all $x \in C$ and $y \in F(T)$,

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + k\|x - Tx\|^2.$$

T is called *quasi-nonexpansive* if for all $x \in C$ and $y \in F(T)$

$$\|Tx - y\| \leq \|x - y\|.$$

T is called *directed* if for all $x \in C$ and $y \in F(T)$

$$\|Tx - y\|^2 \leq \|x - y\|^2 - \|x - Tx\|^2.$$

It is evident that any strictly pseudocontractive mapping is hemicontractive, any quasinonexpansive mapping is demicontractive, and a demicontractive mapping is hemicontractive. However, as demonstrated by the following examples, the converse is not generally true; for further details, refer to [13].

Example 2.3 Let $H = \mathbb{R}$ with the absolute value norm and $C = [0, 1]$. Define $T : C \to C$ by

$$Tx = \begin{cases} \dfrac{1}{4}, & 0 \leq x \leq 1/3; \\ 0, & 1/3 < x \leq 1. \end{cases}$$

Then T is demicontractive with fixed point $1/4$. However, T is not quasi-nonexpansive. Indeed, let $x = 2/5, y = 1/4$. Then

$$|Tx - y| = \left|0 - \frac{1}{4}\right| = \frac{1}{4} > \frac{3}{20} = |x - y|.$$

Example 2.4 Let $H = \mathbb{R}^2$ with the usual norm. Let $C = [-1, 1] \times [-1, 1]$. Define $T : C \to C$ by

$$T(x, y) = (-y, x).$$

Then T is pseudocontractive, but not strictly pseudocontractive.

Example 2.5 Let $H = \mathbb{R}$ with the absolute value norm and $C = [0, 1]$. Define $T : C \to C$ by

$$Tx = \begin{cases} \frac{1}{2}, & 0 \le x \le 1/2; \\ 0, & 1/2 < x \le 1. \end{cases}$$

Then clearly T is hemicontractive with fixed point $y = 1/2$. However, it is not demicontractive. Indeed, for any $k < 1$, it follows that $(k+1)^{-1} > 1/2$. Choose $1/2 < x < (k+1)^{-1}, y = 1/2$. Then $|Tx - Ty|^2 = 1/4$, whereas

$$|x-y|^2 + k|x - Tx|^2 = \left(x - \frac{1}{2}\right)^2 + kx^2 < \frac{1}{4}.$$

Definition 2.6 We say that a mapping $T : C \to C$ satisfies *the demiclosedness principle* iff $I - T$ is *demiclosed at 0*, that is, for any sequence $\{x_n\} \subset C$ and $x \in C$, the following implication relation holds:

$$\left.\begin{array}{c} x_n \rightharpoonup x \\ x_n - Tx_n \to 0 \end{array}\right] \Rightarrow Tx = x.$$

Here "$\to$" stands for strong convergence and "$\rightharpoonup$" weak convergence.

It is well known that every nonexpansive mapping and strictly pseudocontractive mapping satisfy the demiclosedness principle; for further details, refer to [1, 11].

Definition 2.7 The metric projection P_C from H onto $C \subset H$ is defined by

$$P_C x := \arg\min_{y \in C} \|x - y\|, x \in H.$$

It is well known that the metric projection P_C is characterized by the relation

$$\langle x - P_C x, z - P_C x\rangle \le 0, \forall z \in C.$$

The Féjer-monotonicity will play an important role in our convergence analysis.

Definition 2.8 A sequence $\{x_n\} \subset H$ is said to be Féjer monotone with respect to C if it satisfies the condition:

$$\|x_{n+1} - z\| \le \|x_n - z\|, \forall n \ge 0, \forall z \in C.$$

The following two lemmas are very useful in the subsequent analysis.

Lemma 2.9 [1] *Let $\{x_n\}$ be Féjer monotone with respect to C. Then $\{x_n\}$ converges weakly to an element in C iff each weak cluster point of $\{x_n\}$ belongs to C.*

Lemma 2.10 *Let A be a linear mapping from H into H_1 and let $\{x_n\} \subset H$ be such that $\{Ax_n\}$ converges to 0. If A is injective, then $\{x_n\}$ also converges strongly to 0.*

Proof. Let z be any cluster point of $\{x_n\}$. There exists a subsequence $\{x_{n_k}\}$ such that it converges strongly to z. It then follows that

$$Az = \lim_{k\to\infty} Ax_{n_k} = \lim_{n\to\infty} Ax_n = 0.$$

Since A is injective, this yields $z = 0$. Hence the cluster point of $\{x_n\}$ is unique and we conclude that $\{x_n\}$ converges strongly to 0. □

3. Properties of hemicontractive mappings

In this section, we assume that $U : H \to H$ is a hemicontractive and L-Lipschitz continuous mapping, $T : H_1 \to H_1$ is a hemicontractive and L_1-Lipschitz continuous mapping, and $A : H \to H_1$ is a linear bounded mapping. For a real number $\lambda > 0$, let us define

$$\begin{cases} U_\lambda = I - \lambda(I - U), \\ \mathcal{U}_\lambda = I - \lambda(I - U)U_\lambda, \end{cases} \tag{3.1}$$

and

$$\begin{cases} T_\lambda^A = I - \lambda A^*(I - T)A, \\ \mathcal{T}_\lambda^A = I - \lambda A^*(I - T)AT_\lambda^A. \end{cases} \tag{3.2}$$

Next, we will analyze the properties of the mapping defined above.

Lemma 3.1 *Let* $0 < \lambda(1 + L_1)\|A\|^2 < 1$ *and* A^* *be injective. Then*

$$A^{-1}(F(T)) = F(T_\lambda^A) = F(\mathcal{T}_\lambda^A).$$

Proof. It is clear that $A^{-1}(F(T)) \subseteq F(T_\lambda^A) \subseteq F(\mathcal{T}_\lambda^A)$. In order to prove the contrary, fix any $z \in F(\mathcal{T}_\lambda^A)$. Thus, $z = \mathcal{T}_\lambda^A z$. According to the definition of $\mathcal{T}_\lambda^A$, we have

$$\begin{aligned}
&(1 - \lambda(1 + L_1)\|A\|^2)\|T_\lambda^A z - z\| \\
&= \|T_\lambda^A z - z\| - \lambda(1 + L_1)\|A\|^2\|T_\lambda^A z - z\| \\
&= \|T_\lambda^A z - \mathcal{T}_\lambda^A z\| - \lambda(1 + L_1)\|A\|^2\|T_\lambda^A z - z\| \\
&= \lambda\|A^*(I - T)AT_\lambda^A z - A^*(I - T)Az\| - \lambda(1 + L_1)\|A\|^2\|T_\lambda^A z - z\| \\
&\le \lambda(1 + L_1)\|A\|^2\|T_\lambda^A z - z\| - \lambda(1 + L_1)\|A\|^2\|T_\lambda^A z - z\| = 0,
\end{aligned}$$

where the inequality follows form the Lipschitz continuity of T. Since $\lambda(1+L)\|A\|^2 < 1$, this implies that $z \in F(T_\lambda^A)$, and thus $F(\mathcal{T}_\lambda^A) \subseteq F(T_\lambda^A)$. □

Now let z be any fixed point of T_λ^A. Hence $A^*(I - T)Az = 0$. By our hypothesis on A^*, $T(Az) = Az$, and thus $F(T_\lambda^A) \subseteq A^{-1}(F(T))$.

To sum up, we conclude the desired conclusion.

Lemma 3.2 *For any* $(x, z) \in H \times A^{-1}(F(T))$*, it follows that*

$$\|\mathcal{T}_\lambda^A x - z\|^2 \le \|x - z\|^2 - [1 - \lambda^2(1 + L_1)^2\|A\|^4]\|x - T_\lambda^A x\|^2.$$

Proof. Take any $z \in A^{-1}(F(T))$. Since $I - T$ is monotone, this yields

$$\langle AT_\lambda^A x - Az, (I-T)AT_\lambda^A x\rangle \geq 0. \tag{3.3}$$

It then follows from inequality (3.3) and the Lipschitz continuity of T that

$$\begin{aligned}
&-2\lambda\langle Ax - Az, (I-T)AT_\lambda^A x\rangle \\
&= -2\lambda\langle Ax - AT_\lambda^A x, (I-T)AT_\lambda^A x\rangle - 2\lambda\langle AT_\lambda^A x - Az, (I-T)AT_\lambda^A x\rangle \\
&\leq -2\lambda\langle Ax - AT_\lambda^A x, (I-T)AT_\lambda^A x\rangle = -2\lambda^2\langle A^*(I-T)Ax, A^*(I-T)AT_\lambda^A x\rangle \\
&\leq \lambda^2\left((1+L_1)^2\|A\|^4\|x - T_\lambda^A x\|^2 - \|A^*(I-T)Ax\|^2 - \|A^*(I-T)AT_\lambda^A x\|^2\right).
\end{aligned}$$

Note that $\|x - T_\lambda^A\| = \lambda\|A^*(I-T)Ax\|$. Hence, we have

$$\begin{aligned}
&-2\lambda\langle Ax - Az, (I-T)AT_\lambda^A x\rangle \\
&\leq \lambda^2(1+L_1)^2\|A\|^4\|x - T_\lambda^A x\|^2 - \|x - T_\lambda^A x\|^2 - \lambda^2\|A^*(I-T)AT_\lambda^A x\|^2 \\
&= (\lambda^2(1+L_1)^2\|A\|^4 - 1)\|x - T_\lambda^A x\|^2 - \lambda^2\|A^*(I-T)AT_\lambda^A x\|^2.
\end{aligned}$$

From this, it then follows that

$$\begin{aligned}
\|\mathcal{T}_\lambda^A x - z\|^2 &= \left\|x - z - \lambda A^*(I-T)AT_\lambda^A x\right\|^2 \\
&= \|x - z\|^2 - 2\lambda\langle Ax - Az, (I-T)AT_\lambda^A x\rangle + \lambda^2\left\|A^*(I-T)AT_\lambda^A x\right\|^2 \\
&\leq \|x - z\|^2 - [1 - \lambda^2(1+L_1)^2\|A\|^4]\|x - T_\lambda^A x\|^2.
\end{aligned}$$

Hence the proof is complete. □

Lemma 3.3 *Let $0 < \lambda(1+L_1)\|A\|^2 < 1$ and A^* be injective. If T satisfies the demiclosedness principle, then T_λ^A and $\mathcal{T}_\lambda^A$ also satisfy this property.*

Proof. We first show the demiclosedness of T_λ^A. To this end, let $\{x_n\} \subset H$ be such that $x_n \rightharpoonup x$ and $x_n - T_\lambda^A x_n \to 0$, and so it remains to check $\tilde{x} \in A^{-1}(F(T))$. Indeed, from the definition of T_λ^A, it follows that

$$A^*(I-T)Ax_n = \frac{1}{\lambda}(x_n - T_\lambda^A x_n) \to 0,$$

which from Lemma 2.10 yields $(I-T)Ax_n \to 0$ as $n \to \infty$. Since $Ax_n \rightharpoonup A\tilde{x}$ and T satisfies the demiclosedness principle, this indicates $A\tilde{x} = T(A\tilde{x})$. Moreover, from Lemma 3.1, we have $\tilde{x} \in F(T_\lambda^A)$.

We next show the demiclosedness of $\mathcal{T}_\lambda^A$. Let $\{x_n\} \subset H$ be such that $x_n \rightharpoonup \tilde{x}$ and $x_n - \mathcal{T}_\lambda^A x_n \to 0$. So it remains to check $\tilde{x} \in F(\mathcal{T}_\lambda^A)$. By our definition (3.2), we have

$$\begin{aligned}
\|x_n - T_\lambda^A x_n\| &\leq \|x_n - \mathcal{T}_\lambda^A x_n\| + \|T_\lambda^A x_n - \mathcal{T}_\lambda^A x_n\| \\
&\leq \|x_n - \mathcal{T}_\lambda^A x_n\| + \lambda\|A^*(I-T)Ax_n - A^*(I-T)AT_\lambda^A x_n\| \\
&\leq \|x_n - \mathcal{T}_\lambda^A x_n\| + \lambda(1+L_1)\|A\|^2\|x_n - T_\lambda^A x_n\|,
\end{aligned}$$

which implies

$$\|x_n - T_\lambda^A x_n\| \leq \frac{\|x_n - \mathcal{T}_\lambda^A x_n\|}{1 - \lambda(1+L_1)\|A\|^2} \to 0.$$

Since we have shown that T_λ^A satisfies the demiclosedness principle, this indicates $\tilde{x} \in F(T_\lambda^A)$. Furthermore, by Lemma 3.1, we get $\tilde{x} \in F(\mathcal{T}_\lambda^A)$. □

Similarly, it is easy to draw the following conclusions.

Lemma 3.4 *Assuming* $0 < \lambda(1+L) < 1$*, the following properties hold:*

(1) $F(U) = F(U_\lambda) = F(\mathcal{U}_\lambda)$;

(2) *If* $I - U$ *is demiclosed at* 0*, then so are* $I - U_\lambda$ *and* $I - \mathcal{U}_\lambda$*;*

(3) *For any* $(x, z) \in H \times F(U)$*, it follows that*

$$\|\mathcal{U}_\lambda x - z\|^2 \leq \|x - z\|^2 - [1 - \lambda^2(1+L)^2]\|x - U_\lambda x\|^2.$$

4. Iterative method I

To establish an iterative method, we revisit the variational inequality problem (VIP), which aims to find a point $x^* \in C$ such that

$$\langle f(x^*), x - x^* \rangle \geq 0, \forall x \in C.$$

Here, $f : C \to H$ is a monotone mapping, implying that

$$\langle f(x) - f(y), x - y \rangle \geq 0$$

for all $x, y \in C$. This problem is fundamental in optimization theory. One of the well-known methods for solving VIP is Korpelevich's extragradient method [9]. For any initial guess x_0, Korpelevich's extragradient method generates a sequence $\{x_n\}$ using the recursive formula:

$$\left[\begin{array}{l} y_n = P_C\left(x_n - \tau f(x_n)\right), \\ x_{n+1} = P_C\left(x_n - \tau f(y_n)\right), \end{array}\right.$$

where τ is a positive number. Weak convergence has been proven under the assumptions of Lipschitz continuity and pseudo-monotonicity.

It is important to note that a mapping T is pseudocontractive if and only if $I - T$ is monotone. Motivated by Korpelevich's extragradient method, we can propose the first method for solving problem (1.3). Specifically, our method begins with an arbitrary initial guess $x_0 \in H$ and generates x_{n+1} according to the following recursion process:

$$\left[\begin{array}{l} y_n = x_n - \tau_n A^*(I - T)AT_{\tau_n}^A x_n, \\ x_{n+1} = y_n - \lambda_n(I - U)U_{\lambda_n} y_n, \end{array}\right. \tag{4.1}$$

where $\{\tau_n\}$ and $\{\lambda_n\}$ are two real sequences. Here, the mappings U_{λ_n} and $T_{\tau_n}^A$ are defined as in (3.1) and (3.2). In problem (1.3), the following fundamental assumptions are needed:

(c1) U is hemicontractive, L-Lipschitz continuous, and satisfies the demiclosedness property;

(c2) T is hemicontractive, L_1-Lipschitz continuous, and satisfies the demiclosedness property;

(c3) The solution set S for problem (1.3) is guaranteed to be nonempty.

Theorem 4.1 *Assuming* A^* *is injective, for a sufficiently small* $\varepsilon > 0$*, let the parameters satisfy the following conditions:*

(a1) $\sqrt{\varepsilon} \le \tau_n(1+L_1)\|A\|^2 \le \sqrt{1-\varepsilon}$,
(a2) $\sqrt{\varepsilon} \le \lambda_n(1+L) \le \sqrt{1-\varepsilon}$.

Under these conditions, the sequence $\{x_n\}$ generated by (4.1) weakly converges to a solution of problem (1.3).

Proof. We first show that $\{x_n\}$ is Féjer monotone with respect to S. To see this, fix any $z \in S$. It follows from Lemma 3.2 that

$$\begin{aligned}\|y_n - z\|^2 &\le \|x_n - z\|^2 - [1-\tau_n^2(1+L_1)^2\|A\|^4]\|x_n - T^A_{\tau_n}x_n\|^2\\ &\le \|x_n - z\|^2 - \varepsilon\|x_n - T^A_{\tau_n}x_n\|^2.\end{aligned}$$

On the other hand, it follows from Lemma 3.4 that

$$\begin{aligned}\|x_{n+1} - z\|^2 &\le \|y_n - z\|^2 - [1-\lambda_n^2(1+L)^2]\|y_n - U_{\lambda_n}y_n\|^2\\ &\le \|y_n - z\|^2 - \varepsilon\|y_n - U_{\lambda_n}y_n\|^2.\end{aligned}$$

Combining the above two inequalities, we can get

$$\|x_{n+1} - z\|^2 \le \|x_n - z\|^2 - \varepsilon\big(\|y_n - U_{\lambda_n}y_n\|^2 + \|x_n - T^A_{\tau_n}x_n\|^2\big). \tag{4.2}$$

In particular, $\|x_{n+1}-z\| \le \|x_n - z\|, \forall z \in S$. This implies that $\{x_n\}$ is Féjer monotone with respct to S. Moreover, we deduce from (4.2) that

$$\varepsilon\big(\|y_n - U_{\lambda_n}y_n\|^2 + \|x_n - T^A_{\tau_n}x_n\|^2\big) \le \|x_n - z\|^2 - \|x_{n+1} - z\|^2.$$

By induction, this implies

$$\varepsilon(\sum_{k=0}^{n} \|(I-U_{\lambda_k})y_k\|^2 + \|x_n - T^A_{\tau_k}x_k\|^2) \le \|x_0 - z\|^2,$$

so that $\sum_{n=0}^{\infty}(\|(I-U_{\lambda_n})y_n\|^2 + +\|x_n - T^A_{\tau_n}x_n\|^2) < \infty$. In particular,

$$\lim_{n\to\infty}\|(I-U_{\lambda_n})y_n\| = \lim_{n\to\infty}\|x_n - T^A_{\tau_n}x_n\| = 0. \tag{4.3}$$

We next show that each weak cluster point of $\{x_n\}$ belongs to S. To see this, we deduce from our definition that

$$\begin{aligned}\|y_n - T^A_{\tau_n}x_n\| &= \tau_n\|A^*(I-T)AT^A_{\tau_n}x_n - A^*(I-T)Ax_n\|\\ &\le \tau_n\|A\|^2(1+L_1)\|T^A_{\tau_n}x_n - x_n\|\\ &\le \sqrt{1-\varepsilon}\|T^A_{\tau_n}x_n - x_n\| \to 0;\end{aligned}$$

which combined with (4.3) implies that

$$\|y_n - x_n\| \le \|y_n - T^A_{\tau_n}x_n\| + \|x_n - T^A_{\tau_n}x_n\| \to 0. \tag{4.4}$$

Now let $\tilde{x}$ be any weak cluster point of $\{x_n\}$. There exists a subsequence $\{x_{n_k}\}$ such that it weakly converges to $\tilde{x}$. Note that by formula (4.2) at this time, we can get

$$\varepsilon\|y_n - Uy_n\| \le \|\lambda_n(y_n - Uy_n)\| = \|U_{\lambda_n}y_n - y_n\| \to 0.$$

By hypothesis (c1), we conclude $\tilde{x} \in F(U)$. On the other hand, observe

$$\varepsilon\|A^*(I-T)Ay_n\| \le \|\tau_n A^*(I-T)Ay_n\| = \|T^A_{\tau_n}y_n - y_n\| \to 0.$$

From Lemma 2.10, this yields $\|(I-T)Ay_n\| \to 0$. Note that, by (4.4), and $\{Ay_{n_k}\}$ also converges weakly to $A\tilde{x}$. By hypothesis (c1), $\tilde{x} \in A^{-1}(F(T))$. To sum up, we get $\tilde{x} \in S$.

We finally deduce from Lemma 2.9 that the sequence $\{x_n\}$ converges weakly to a solution of problem (1.3). This is because we have shown that $\{x_n\}$ is Féjer-monotone and every weak cluster point of $\{x_n\}$ belongs to the solution set. □

Remark 4.2 In comparison to the existing method (1.5), our approach enables a wider range of the stepsize parameter. Specifically, we enhance the upper bound of τ_n from $1/(1+\sqrt{1+L_1^2})\|A\|^2$ to $1/(1+L_1)\|A\|^2$, and the upper bound of λ_n from $1/(1+\sqrt{1+L^2})$ to $1/(1+L)$.

5. Iterative method II

In the previous section, we need the injective assumption of the linear mapping to guarantee its convergence. Such a condition maybe restrictive in some particular cases. In this section we propose another method for solving problem (1.3). However, its convergence does not require any assumption on the linear mapping A. More precisely, our method starts with an arbitrary initial guess $x_0 \in H$ and generates x_n according to the recursion process:

$$\left[\begin{array}{l} y_n = x_n - \alpha_n A^*(I-\mathcal{T}_\tau)Ax_n, \\ x_{n+1} = y_n - \lambda_n (I-U)U_{\lambda_n} y_n, \end{array}\right. \tag{5.1}$$

where $\{\alpha_n\}$ and $\{\lambda_n\}$ are two real sequences. Here the mappings U_{λ_n} and $\mathcal{T}_\tau$ are defined as in (3.1) and (3.2).

Theorem 5.1 *Assuming a sufficiently small $\varepsilon > 0$, let the parameters satisfy the following conditions:*

(b1) $0 < \tau < 1/(1+L_1)$;
(b2) $\sqrt{\varepsilon} \le \alpha_n \le (1-\sqrt{\varepsilon})/\|A\|^2$;
(b3) $\sqrt{\varepsilon} \le \lambda_n \le \sqrt{1-\varepsilon}/(1+L)$.

Under these conditions, the sequence $\{x_n\}$ generated by (5.1) weakly converges to a solution of problem (1.3).

Proof. Fix any $z \in S$. It follows from Lemma 3.1 and 3.2 that $Az \in F(\mathcal{T}_\tau)$ and $\|\mathcal{T}_\tau Ax - Az\| \le \|Ax - Az\|$ for all $x \in H$. This clearly implies

$$\begin{aligned} &2\langle Ax_n - Az, (I-\mathcal{T}_\tau)Ax_n\rangle \\ &= \|(I-\mathcal{T}_\tau)Ax_n\|^2 + \|Ax_n - Az\|^2 - \|\mathcal{T}_\tau Ax_n - Az\|^2 \\ &\ge \|(I-\mathcal{T}_\tau)Ax_n\|^2. \end{aligned}$$

It then follows from this and Lemma 3.2 that

$$\begin{aligned}\|y_n - z\|^2 &= \|x_n - z - \alpha_n A^*(I - \mathcal{T}_\tau)Ax_n\|^2 \\ &= \|x_n - z\|^2 - 2\alpha_n \langle x_n - z, A^*(I - \mathcal{T}_\tau)Ax_n\rangle + \alpha_n^2 \|A^*(I - \mathcal{T}_\tau)Ax_n\|^2 \\ &= \|x_n - z\|^2 - 2\alpha_n \langle Ax_n - Az, (I - \mathcal{T}_\tau)Ax_n\rangle + \alpha_n^2 \|A^*(I - \mathcal{T}_\tau)Ax_n\|^2 \\ &\le \|x_n - z\|^2 - \alpha_n \|(I - \mathcal{T}_\tau)Ax_n\|^2 + \alpha_n^2 \|A^*(I - \mathcal{T}_\tau)Ax_n\|^2 \\ &\le \|x_n - z\|^2 - \alpha_n \|(I - \mathcal{T}_\tau)Ax_n\|^2 + \alpha_n^2 \|A\|^2 \|(I - \mathcal{T}_\tau)Ax_n\|^2 \\ &= \|x_n - z\|^2 - \alpha_n(1 - \alpha_n\|A\|^2)\|(I - \mathcal{T}_\tau)Ax_n\|^2,\end{aligned}$$

which together with condition (b2) implies that

$$\|y_n - z\|^2 \le \|x_n - z\|^2 - \varepsilon\|(I - \mathcal{T}_\tau)Ax_n\|^2.$$

On the other hand, it follows from Lemma 3.4 that

$$\begin{aligned}\|x_{n+1} - z\|^2 &\le \|y_n - z\|^2 - \lambda_n(1 - \lambda_n(1 + L_1))\|y_n - U_{\lambda_n} y_n\|^2 \\ &\le \|y_n - z\|^2 - \varepsilon\|y_n - U_{\lambda_n} y_n\|^2.\end{aligned}$$

Combining the last two inequalities, we can get

$$\|x_{n+1} - z\|^2 \le \|x_n - z\|^2 - \varepsilon\big(\|y_n - U_{\lambda_n} y_n\|^2 + \|(I - \mathcal{T}_\tau)Ax_n\|^2\big). \tag{5.2}$$

In particular, we have

$$\|x_{n+1} - z\| \le \|x_n - z\|, \forall z \in S.$$

This implies that $\{x_n\}$ is Féjer monotone with respct to S.

We next show that each weak cluster point of $\{x_n\}$ belongs to S. To see this, let $\tilde{x}$ be any weak cluster point of $\{x_n\}$. Thus there exists a subsequence $\{x_{n_k}\}$ such that it weakly converges to $\tilde{x}$. Analogously, we can deduce from (5.2) that

$$\lim_{n\to\infty} \|(I - U_{\lambda_n})y_n\| = \lim_{n\to\infty} \|(I - \mathcal{T}_\tau)Ax_n\| = 0. \tag{5.3}$$

It is readily seen that $\{Ax_{n_k}\}$ converges weakly to $A\tilde{x}$. By (5.3) and condition (c1), we get $A\tilde{x} \in F(T)$. On the other hand, by the definition, we have

$$\|y_n - x_n\| = \alpha_n\|A^*(I - \mathcal{T}_\tau)Ax_n\| \le \|A\|\|(I - \mathcal{T}_\tau)Ax_n\| \to 0. \tag{5.4}$$

This shows that $\{y_{n_k}\}$ also weakly converges to $\tilde{x}$. From (5.2) it follows that

$$\sqrt{\varepsilon}\|(I - U)y_n\| \le \|\lambda_n(I - U)y_n\| = \|(I - U_{\lambda_n})y_n\| \to 0,$$

which combined with (c1) yields $\tilde{x} \in F(U)$. To sum up, we get $x \in S$; the desired conclusion follows.

Finally, we deduce from Lemma 2.9 that the sequence $\{x_n\}$ converges weakly to a solution of problem (1.3). This is because we have shown that $\{x_n\}$ is Féjer-monotone and every weak cluster point of $\{x_n\}$ belongs to the solution set.

Remark 5.2 Our second method also enables a broader range of the stepsize parameter. Furthermore, it does not require any additional assumptions on the linear mapping.

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