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# VISCOSITY SCHEME WITH ENRICHED MAPPINGS FOR HIERARCHICAL VARIATIONAL INEQUALITIES IN CERTAIN GEODESIC SPACES

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**Abstract.** This paper presents an adaptive algorithm for solving enriched contraction variational inequality problems, using the set of fixed points of an enriched nonexpansive mapping as a constrained set. The algorithm is defined within the framework of unique geodesic spaces. In each iteration, the scheme uses only two embedded geodesic segments and does not require the computation of any metric projection. The method requires one evaluation of an enriched nonexpansive mapping $T_1$ and an enriched contraction $T_2$ at every iteration. The convergence analysis of the proposed scheme is performed in the setting of CAT(0) spaces, and a numerical example is provided to support the findings.

**Key Words and Phrases**: Enriched contraction mapping, enriched nonexpansive mapping, fixed point, geodesic convexity, hierarchical variational inequality problem, viscosity scheme.
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## 1. Introduction

The hierarchical variational inequality problem (HVI for short) is a fundamental problem in nonlinear analysis and has applications in optimization, nonlinear analysis, differential equations and related fields. A viscosity scheme is a prominent method in solving this kind of problem when a contraction mapping is involved. The scheme was initially proposed by Moudafi in [25], building on the results of [5]. Xu later studied the scheme in the context of Banach spaces [31]. One of the substantial properties of this scheme is that it solves a variational inequality problem over a set of fixed points of a given mapping without the computation of metric projection at any stage.

Consider $\mathcal{Q}$ defined in such a way that

$$\mathcal{Q}(v,w,x,y) := \frac{1}{2}\left(d^2(v,y)+d^2(w,x)-d^2(v,x)-d^2(w,y)\right)\ \forall\, v,w,x,y\in\mathcal{H}. \quad (1.1)$$

In a geodesic space $(\mathcal{H},d)$, the HVI we consider is to find $u\in \mathrm{Fix}(T_1)$ such that

$$\mathcal{Q}(u,T_2u,w,u)\geq 0 \quad \forall\, w\in \mathrm{Fix}(T_1), \quad (1.2)$$

where $T_1,T_2:\mathcal{H}\to\mathcal{H}$ are mappings and $\mathrm{Fix}(T_1)$ denotes the set of fixed points of $T_1$. The operator $\mathcal{Q}$ is what is referred to as quasilinearization in [6]. When $\mathcal{H}$ is an inner-product space with the usual norm, then HVI coincides with the problem of finding $u\in\mathrm{Fix}(T_1)$ such that

$$\langle u-T_2u, w-u\rangle\geq 0, \quad \forall\, w\in\mathrm{Fix}(T_1). \quad (1.3)$$

Moreover, it is worth noting that when $T_1$ is the metric projection onto $\mathcal{C}$ and $G\equiv I-T_2$, (1.2) is exactly the classical variational inequality problem involving $G$ with constrained $\mathcal{C}$. For the classical variational inequality, we refer to the texts [24, 18, 21].

The problem at hand involves the concepts of convexity, constraints, and differentiability. However, geodesic spaces provide a framework for viewing non-convex or constrained (and non-smooth) problems as convex or constrained (and smooth) in the sense of geodesics. As a result, many researchers are interested in geodesic spaces. In [20], the problem (1.3) was analyzed in the context of CAT(0) spaces and the existence and approximation of solutions were discussed using a quasilinearization and a nonexpansive mapping. Similar problems have also been considered in the setting of Hadamard manifolds using tangent spaces (see, e.g., [3, 26]). Huang studied the viscosity scheme using a weak contraction mapping instead of the Banach contraction and analyzed the convergence of the scheme with certain control conditions in [19]. Thereafter, a general problem is considered in [1] using $\phi$-contraction mapping. It is worth noting that CAT(0) spaces include both Hilbert spaces and Hadamard manifolds. Moreover, the problem considered provides a general framework for many nonlinear problems including certain bilevel optimization and monotone vector field inclusion problems (see, for example, [1, 4, 2] and the references therein).

Our objective in this paper is to develop a viscosity scheme with an enriched contraction mapping for solving variational inequality problems over the set of fixed points of enriched nonexpansive mappings. We provide a convergence analysis and identify the conditions required for the control parameters. The enriched classes of mappings considered herein are from [28], and the analysis is based on the setting of CAT(0) spaces. This scheme and the problem have not been studied even in the case of linear spaces. However, the results presented in this paper apply to Hilbert spaces as well since CAT(0) spaces include both Hilbert spaces and Hadamard manifolds. In this regard, the results of this paper, complement the results in [9, 8, 7].

## 2. Preliminaries

Consider $\mathcal{H}$ endowed with a metric $d$. By the definition of $\mathcal{Q}$ in (1.1), it follows that for all $(x,y,v,w,z)\in\mathcal{H}^5$;

(P1) $\mathcal{Q}(v,w,x,y)=\mathcal{Q}(x,y,v,w)$;

(P2) $\mathcal{Q}(v,w,x,y) = -\mathcal{Q}(v,w,y,x) = \mathcal{Q}(w,v,y,x)$;
(P3) $\mathcal{Q}(v,w,x,y) = \mathcal{Q}(v,w,x,z) + \mathcal{Q}(v,w,z,y)$.

A *geodesic path* from $u$ to $w$ is a map $\gamma_u^w : [0,1] \to \mathcal{H}$ such that

$$\gamma_u^w(0) = u, \quad \gamma_u^w(1) = w \quad \text{and} \quad d(\gamma_u^w(s), \gamma_u^w(t)) = |s-t| d(u,w), \ \forall\ s,t \in [0,1].$$

The image of $\gamma_u^w$ is often called a *geodesic segment* connecting $u$ and $w$. If such a segment is unique, we write $[[u,w]]$ to mean $\gamma_u^w([0,1])$. The space $(\mathcal{H}, d)$ in which every two points are connected by a geodesic segment (resp. unique geodesic segment) is called a *geodesic space* (resp. *unique geodesic space*). A set is *convex* if it contains the geodesic segment connecting any two of its points. For $u, w \in \mathcal{H}$ having unique geodesic segment and for any $t \in [0,1]$, there exists a unique point $z \in [[u,w]]$ such that

$$d(u,\ z) = td(u,\ w) \quad \text{and} \quad d(z,\ w) = (1-t)d(u,\ w). \tag{2.1}$$

We shall henceforth denote such a point $z$ by $(1-t)u \oplus tw$.

This work focuses on geodesic spaces in which the following inequality of Bruhat and Tits [13] holds. Let $u, w \in \mathcal{H}$, we have

$$d^2\left(\frac{1}{2}u \oplus \frac{1}{2}w,\ y\right) \le \frac{1}{2}d^2(w,\ y) + \frac{1}{2}d^2(w,\ y) - \frac{1}{4}d^2(u,\ w), \tag{2.2}$$

for every $y \in \mathcal{H}$. In general, this kind of space is referred to as *CAT(0) space* and a complete CAT(0) space is called a *Hadamard space*. It is well-known that Hadamard manifolds, Hilbert spaces, classical hyperbolic spaces, $\mathbb{R}$-trees, complex Hilbert balls, and Euclidean buildings are all examples of CAT(0) spaces [23, 12]. Furthermore, CAT(0) spaces are unique geodesic spaces.

It is known that CAT(0) spaces satisfy the following Cauchy Schwartz inequality:

$$\mathcal{Q}(x,y,v,w) \le d(x,y)d(v,w), \quad \forall x,y,v,w \in \mathcal{H}. \tag{2.3}$$

In fact CAT(0) spaces are characterized by (2.3) following [6, Corollary 3]. In addition to the above properties, the following inequalities are crucial in obtaining our results. Take $u, w$ arbitrary in a CAT(0) space $(\mathcal{H}, d)$ then it is known (see, e.g., [17]) that for every $z \in \mathcal{H}$,

$$d((1-t)u \oplus tw,\ z) \le (1-t)d(u,\ z) + td(w,\ z); \tag{2.4}$$

$$d^2((1-t)u \oplus tw,\ z) \le (1-t)d^2(u,\ z) + td^2(w,\ z) - t(1-t)d^2(u,\ w); \tag{2.5}$$

for every $t \in [0,1]$. From these inequalities, several other inequalities can easily be generated. For instance, for all $u, w, x, y \in \mathcal{H}$ and $t, s \in [0,1]$,

$$d\big((1-t)u \oplus tw,\ (1-t)x \oplus ty\big) \le (1-t)d(u,x) + td(w,y); \tag{2.6}$$

$$d\big((1-t)u \oplus tw,\ (1-s)u \oplus sw\big) \le |s-t| d(u,w); \tag{2.7}$$

$$d^2((1-t)u \oplus tw,\ z) \le (1-t)^2 d^2(u,\ z) + t^2 d^2(w,\ z) + 2t(1-t)\mathcal{Q}(u,z,w,z). \tag{2.8}$$

For detailed discussion on CAT(0) spaces, see for example, [11, 29].

Let $\{u_n\}$ be a bounded sequence in $\mathcal{H}$ and let $\rho(\cdot, \{u_n\}) : \mathcal{H} \to [0,\infty)$ be the function defined by $\rho(y, \{u_n\}) := \limsup_{n\to\infty} d(y, u_n),\ y \in \mathcal{H}$. Then the asymptotic radius

$R(\{u_n\})$ of $\{u_n\}$ is given by $R(\{u_n\}) := \inf_{u\in\mathcal{H}} \rho(u, \{u_n\})$ and the asymptotic center $A(\{u_n\})$ of $\{u_n\}$ is the set

$$A(\{u_n\}) := \big\{y \in \mathcal{H} \ : \ \rho(y, \{u_n\}) = R(\{u_n\})\big\}.$$

It is known (see [16, Proposition 7] ) that $A(\{u_n\})$ is a singleton set in every complete $CAT(0)$ space.

**Lemma 2.1.** [15] *If $\mathcal{C}$ is closed convex set and $\{u_n\}$ is a bounded sequence in $\mathcal{C}$, then the asymptotic centre $A(\{u_n\})$ is in $\mathcal{C}$.*

A bounded sequence $\{u_n\}$ $\Delta$-converges to $y$ in $\mathcal{H}$ if $y$ is the unique asymptotic centre for every subsequence $\{u_{n_k}\}$ of $\{u_n\}$ or equivalently, if

$$\limsup_{k\to\infty} d(u_{n_k}, y) \le \limsup_{k\to\infty} d(u_{n_k}, z),$$

for every subsequence $\{u_{n_k}\}$ of $\{u_n\}$ and for every $z \in \mathcal{H}$ [22], and $\{u_n\}$ converges strongly to $y$ in $\mathcal{H}$ if $\lim\limits_{n\to\infty} d(u_n, y) = 0$.

**Lemma 2.2.** [22, Proposition 3.6] *Every bounded sequence $\{u_n\}$ in $\mathcal{H}$ has a $\Delta$-convergent subsequence $\{u_{n_k}\}$.*

For our results, we will use the following results from [28].

**Lemma 2.3.** *Let $(\mathcal{H}, d)$ be a Hadamard space and let $T : \mathcal{H} \to \mathcal{H}$ be a mapping. For $\sigma \in (0, 1]$, let $T_\sigma$ be defined by*

$$T_\sigma u := (1-\sigma)u \oplus \sigma Tu, \ \forall u \in \mathcal{H}. \tag{2.9}$$

*Then*

*(i)* $\mathrm{Fix}(T) = \mathrm{Fix}(T_\sigma)$*;*
*(ii)* $d^2\big(T_\sigma u, T_\sigma w\big) \le (1-\sigma)^2 d^2\big(u, w\big) + \sigma^2 d^2\big(Tu, Tw\big) + 2\sigma(1-\sigma)\mathcal{Q}(u, w, Tu, Tw)$.

**Definition 2.4.** Let $(\mathcal{H}, d)$ be a metric space. A mapping $T : \mathcal{H} \to \mathcal{H}$ is called an $(\gamma, \beta)$-enriched contraction if there exist two real numbers $\gamma \in [0, +\infty)$ and $\beta \in [0, \gamma + 1)$ such that

$$d(Tu, Tw)^2 + \gamma^2 d(u, w)^2 + 2\gamma\mathcal{Q}(u, w, Tu, Tw) \le \beta^2 d(u, w)^2, \ \forall u, w \in \mathcal{H}. \tag{2.10}$$

When $\beta = \gamma + 1$, $T$ is called $\gamma$-enriched nonexpansive mapping.

For recent findings regarding enriched nonexpansive mappings, refer to [10, 27].

**Proposition 2.5.** *Let $T$ be a mapping and $\alpha, \beta$ be such that* (2.10) *is satisfied. Then*

$$d\left(\frac{\gamma}{\gamma+1}u \oplus \frac{1}{\gamma+1}Tu, \frac{\gamma}{\gamma+1}w \oplus \frac{1}{\gamma+1}Tw\right) \le \frac{\beta}{(\gamma+1)}d(u, w)$$

*Proof.* For $\sigma = \dfrac{1}{\gamma+1}$, Lemma 2.3 (ii) gives

$$\begin{aligned} d^2\left(\frac{\gamma}{\gamma+1}u \oplus \frac{1}{\gamma+1}Tu, \frac{\gamma}{\gamma+1}w \oplus \frac{1}{\gamma+1}Tw\right) &\leq \frac{\gamma^2}{(\gamma+1)^2}d^2(u,w) + \frac{1}{(\gamma+1)^2}d^2(Tu,Tw) \\ &\quad + 2\frac{\gamma}{(\gamma+1)^2}\mathcal{Q}(u,w,Tu,Tw) \\ &\leq \frac{\beta^2}{(\gamma+1)^2}d^2(u,w). \end{aligned}$$

This completes the proof. □

The following substantial lemma can be found in [30, Lemma 2.5].

**Lemma 2.6.** *Let $\{\theta_n\}$ be a sequence in $[0,+\infty) \subset \mathbb{R}$ with*

$$\theta_{n+1} \leq (1-\sigma_n)\theta_n + \sigma_n\phi_n + \gamma_n, \quad n \geq 1, \tag{2.11}$$

*where $\{\sigma_n\}$, $\{\phi_n\}$ and $\{\gamma_n\}$ satisfy the following conditions:*

$$\{\sigma_n\} \subset [0,1],\ \sum_{n=1}^{\infty}\sigma_n = \infty,\ \limsup_{n\to\infty}\phi_n \leq 0, \{\gamma_n\} \subset [0,\infty) \text{ and } \sum_{n=1}^{\infty}\gamma_n < \infty.$$

*Then* $\lim_{n\to\infty}\theta_n = 0$.

## 3. Viscosity scheme and its convergence analysis

For the proposed algorithm, we adopt the 'oplus' notation from [14], in the sense that for $u, v, w \in \mathcal{H}$ and $t_1, t_2, t_3 \in [0,1]$ such that $t_1 + t_2 + t_3 = 1$,

$$t_1u \oplus t_2v \oplus t_3w := t_1u \oplus (1-t_1)\left(\frac{t_2}{1-t_1}v \oplus \frac{t_3}{1-t_1}w\right). \tag{3.1}$$

**Algorithm 1** Viscosity Scheme

Choose an arbitrary element $u_1 \in \mathcal{H}$, define a sequence $\{u_n\}$ by

$$\begin{aligned} w_n &= \left(1 - \frac{\alpha_n}{1+\gamma_1}\right)u_n \oplus \frac{\alpha_n}{1+\gamma_1}T_1u_n, \\ u_{n+1} &= (1-\beta_n)w_n \oplus \frac{\gamma_2\beta_n}{1+\gamma_2}u_n \oplus \frac{\beta_n}{1+\gamma_2}T_2u_n, \quad n \geq 1; \end{aligned}$$

where $\{\alpha_n\}, \{\beta_n\}$ are sequences in $(0,1]$ and $\gamma_1$, $\gamma_2$ are non-negative real numbers.

We consider a class of mappings satisfying the conditions below.

(A1) $T_1 : \mathcal{H} \to \mathcal{H}$ is $\gamma_1$-enriched nonexpansive mapping;
(A2) $T_2 : \mathcal{H} \to \mathcal{H}$ is $(\gamma_2, \beta)$-enriched contraction mapping.

**Lemma 3.1.** *Let $(\mathcal{H}, d)$ be a CAT(0) space. Suppose that $T_1$ and $T_2$ satisfy (A1) and (A2), and* $\mathrm{Fix}(T_1) \neq \emptyset$. *Then $\{u_n\}$ generated by Algorithm 1 is bounded.*

*Proof.* Let $p \in \mathrm{Fix}(T_1)$. It follows from (2.4) that

$$\begin{aligned}
d\left(u_{n+1}, p\right) &= d\left((1-\beta_n)w_n \oplus \beta_n\left(\frac{\gamma_2}{1+\gamma_2}u_n \oplus \frac{1}{1+\gamma_2}T_2u_n\right), p\right) \\
&\leq (1-\beta_n)d(w_n,p) + \beta_n d\left(\frac{\gamma_2}{1+\gamma_2}u_n \oplus \frac{1}{1+\gamma_2}T_2u_n, p\right) \\
&\leq (1-\beta_n)d(w_n,p) + \beta_n d\left(\frac{\gamma_2}{1+\gamma_2}u_n \oplus \frac{1}{1+\gamma_2}T_2u_n, \frac{\gamma_2}{1+\gamma_2}p \oplus \frac{1}{1+\gamma_2}T_2p\right) \\
&\quad + \beta_n d\left(\frac{\gamma_2}{1+\gamma_2}p \oplus \frac{1}{1+\gamma_2}T_2p, p\right) \\
&= (1-\beta_n)d(w_n,p) + \beta_n d\left(\frac{\gamma_2}{1+\gamma_2}u_n \oplus \frac{1}{1+\gamma_2}T_2u_n, \frac{\gamma_2}{1+\gamma_2}p \oplus \frac{1}{1+\gamma_2}T_2p\right) \\
&\quad + \frac{\beta_n}{1+\gamma_2}d\left(T_2p, p\right).
\end{aligned}$$

Since $T_2$ is $(\gamma_2, \beta)$-enriched contraction, it follows from Proposition 2.5 that

$$d\left(u_{n+1}, p\right) \leq (1-\beta_n)d(w_n,p) + \beta_n \frac{\beta}{1+\gamma_2}d\left(u_n, p\right) + \frac{\beta_n}{1+\gamma_2}d\left(T_2p, p\right). \tag{3.2}$$

Also,

$$\begin{aligned}
d(w_n,p) &= d\left((1-\alpha_n)u_n \oplus \alpha_n\left(\frac{\gamma_1}{1+\gamma_1}u_n \oplus \frac{1}{1+\gamma_1}T_1u_n\right), p\right) \\
&\leq (1-\alpha_n)d\left(u_n, p\right) + \alpha_n d\left(\frac{\gamma_1}{1+\gamma_1}u_n \oplus \frac{1}{1+\gamma_1}T_1u_n, p\right) \\
&= (1-\alpha_n)d\left(u_n, p\right) + \alpha_n d\left(\frac{\gamma_1}{1+\gamma_1}u_n \oplus \frac{1}{1+\gamma_1}T_1u_n, \frac{\gamma_1}{1+\gamma_1}p \oplus \frac{1}{1+\gamma_1}T_1p\right).
\end{aligned}$$

Since $T_1$ is $\gamma_1$-enriched nonexpansive mapping, we get from Proposition 2.5 that

$$d(w_n,p) \leq (1-\alpha_n)d\left(u_n,p\right) + \alpha_n d\left(u_n,p\right) \leq d(u_n,p). \tag{3.3}$$

It follows from (3.2) and (3.3) that

$$\begin{aligned}
d\left(u_{n+1}, p\right) &\leq (1-\beta_n)d(u_n,p) + \beta_n \frac{\beta}{1+\gamma_2}d\left(u_n, p\right) + \frac{\beta_n}{1+\gamma_2}d\left(T_2p, p\right) \\
&\leq \left(1-\beta_n\left(1-\frac{\beta}{1+\gamma_2}\right)\right)d\left(u_n,p\right) + \frac{\beta_n}{1+\gamma_2}d\left(T_2p,p\right) \\
&\leq \max\left\{d(u_n,p), \frac{d\left(T_2p,p\right)}{1+\gamma_2-\beta}\right\} \\
&\;\;\vdots \\
&\leq \max\left\{d(u_1,p), \frac{d\left(T_2p,p\right)}{1+\gamma_2-\beta}\right\}.
\end{aligned}$$

This implies that $\{u_n\}$ is bounded. □

In the next result, we may require the following control conditions:

(C1) $\lim_{n\to\infty} \alpha_n = 1$; (C2) $\lim_{n\to\infty} \beta_n = 0$; (C3) $\sum_{n\geq 1} \beta_n = +\infty$;

(C4) $\lim_{n\to\infty} \dfrac{|\alpha_n - \alpha_{n-1}|}{\beta_n} = 0$; (C5) $\lim_{n\to\infty} \dfrac{|\beta_n - \beta_{n-1}|}{\beta_n} = 0$.

**Lemma 3.2.** *Let $\mathcal{H}$, $T_1$ and $T_2$ be as in Lemma 3.1. Suppose that $\{u_n\}$ is generated by Algorithm 1 such that $\{\alpha_n\}$ and $\{\beta_n\}$ satisfied (C1)-(C3). Then*

$$\lim_{n\to\infty} d(u_{n+1}, u_n) = \lim_{n\to\infty} d(u_n, w_n) = \lim_{n\to\infty} d(w_n, G_1 u_n) = 0. \tag{3.4}$$

*Proof.* From Lemma 3.1, we can easily deduce that $\{w_n\}$, $\{T_1 u_n\}$ and $\{T_2 u_n\}$ are all bounded. Now, consider the mapping $G_1$ in which $u \mapsto \frac{\gamma_1}{1+\gamma_1} u \oplus \frac{1}{1+\gamma_1} T_1 u$. From (2.6) and (2.7), we have that

$$\begin{aligned} d(w_n, w_{n-1}) &= d((1-\alpha_n)u_n \oplus \alpha_n G_1 u_n, (1-\alpha_{n-1})u_{n-1} \oplus \alpha_{n-1} G_1 u_{n-1}) \\ &\leq d((1-\alpha_n)u_n \oplus \alpha_n G_1 u_n, (1-\alpha_n)u_{n-1} \oplus \alpha_n G_1 u_{n-1}) \\ &\quad + d((1-\alpha_n)u_{n-1} \oplus \alpha_n G_1 u_{n-1}, (1-\alpha_{n-1})u_{n-1} \oplus \alpha_{n-1} G_1 u_{n-1}) \\ &\leq (1-\alpha_n) d(u_n, u_{n-1}) + \alpha_n d(G_1 u_n, G_1 u_{n-1}) \\ &\quad + |\alpha_n - \alpha_{n-1}|\, d(u_{n-1}, G u_{n-1}). \end{aligned}$$

Since $T_1$ is $\gamma_1$-enriched nonexpansive mapping, we have

$$\begin{aligned} d(w_n, w_{n-1}) &\leq (1-\alpha_n) d(u_n, u_{n-1}) + \alpha_n d(u_n, u_{n-1}) + |\alpha_n - \alpha_{n-1}|\, d(u_{n-1}, G u_{n-1}) \\ &\leq d(u_n, u_{n-1}) + |\alpha_n - \alpha_{n-1}|\, d(u_{n-1}, G u_{n-1}) \\ &\leq d(u_n, u_{n-1}) + |\alpha_n - \alpha_{n-1}|\, M_1, \end{aligned} \tag{3.5}$$

where $M_1 := \sup\{d(u_{n-1}, G_1 u_{n-1}) : n \geq 1\}$. Note that $\{G_1 u_{n-1}\}$ is bounded by the boundedness of $\{u_n\}$ and that of $\{T_1 u_n\}$.

Now, let $G_2$ be such that $u \mapsto \frac{\gamma_2}{1+\gamma_2} u \oplus \frac{1}{1+\gamma_2} T_2 u$. Then we have from (2.6) and (2.7) that

$$\begin{aligned} d(u_{n+1}, u_n) &= d((1-\beta_n)w_n \oplus \beta_n G_2 u_n, (1-\beta_{n-1})w_{n-1} \oplus \beta_{n-1} G_2 u_{n-1}) \\ &\leq d((1-\beta_n)w_n \oplus \beta_n G_2 u_n, (1-\beta_n)w_{n-1} \oplus \beta_n G_2 u_{n-1}) \\ &\quad + d((1-\beta_n)w_{n-1} \oplus \beta_n G_2 u_{n-1}, (1-\beta_{n-1})w_{n-1} \oplus \beta_{n-1} G_2 u_{n-1}) \\ &\leq (1-\beta_n) d(w_n, w_{n-1}) + \beta_n d(G_2 u_n, G_2 u_{n-1}) \\ &\quad + |\beta_n - \beta_{n-1}|\, d(w_{n-1}, G_2 u_{n-1}). \end{aligned}$$

Consequently, (3.5) and (A2) yield

$$\begin{aligned} d\left(u_{n+1}, u_n\right) \leq & (1-\beta_n) d\left(w_n, w_{n-1}\right)+\frac{\beta_n \beta}{1+\gamma_2} d\left(u_n, u_{n-1}\right)) \\ & +\left|\beta_n-\beta_{n-1}\right| d\left(w_{n-1}, G_2 u_{n-1}\right) \\ \leq & (1-\beta_n) d(u_n, u_{n-1})+(1-\beta_n)\left|\alpha_n-\alpha_{n-1}\right| M_1 \\ & +\frac{\beta_n \beta}{1+\gamma_2} d\left(u_n, u_{n-1}\right)+\left|\beta_n-\beta_{n-1}\right| d\left(w_{n-1}, G_2 u_{n-1}\right) \\ \leq & \left(1-\beta_n\left(1-\frac{\beta}{1+\gamma_2}\right)\right) d(u_n, u_{n-1})+\left|\alpha_n-\alpha_{n-1}\right| M_1 \\ & +\left|\beta_n-\beta_{n-1}\right| M_2 \\ \leq & \left(1-\beta_n \tilde{\beta}\right) d(u_n, u_{n-1})+\left|\alpha_n-\alpha_{n-1}\right| M_1+\left|\beta_n-\beta_{n-1}\right| M_2, \end{aligned}$$

where $M_2 := \sup\left\{d\left(w_{n-1}, G_2 u_{n-1}\right) : n \geq 1\right\}$ and $\tilde{\beta} = \left(1-\frac{\beta}{1+\gamma_2}\right)$. The last inequality is equivalent to

$$d\left(u_{n+1}, u_n\right) \leq \left(1-\beta_n \tilde{\beta}\right) d(u_n, u_{n-1})+\beta_n \tilde{\beta}\left(\frac{\left|\alpha_n-\alpha_{n-1}\right|}{\beta_n \tilde{\beta}} M_1+\frac{\left|\beta_n-\beta_{n-1}\right|}{\beta_n \tilde{\beta}} M_2\right).$$

This, together with Lemma 2.6 and the conditions (C3)-(C5), imply that

$$\lim_{n \to \infty} d\left(u_{n+1}, u_n\right)=0. \tag{3.6}$$

It follows from (2.1) and (C1)-(C2) that

$$d\left(u_{n+1}, w_n\right)=d\left((1-\beta_n) w_n \oplus \beta_n G_2 u_n, w_n\right)=\beta_n d\left(G_2 u_n, w_n\right) \to 0;$$

and

$$d\left(w_n, G_1 u_n\right)=d\left((1-\alpha_n) u_n \oplus \alpha_n G_1 u_n, G_1 u_n\right)=(1-\alpha_n) d(u_n, G_1 u_n) \to 0.$$

□

**Theorem 3.3.** *Let* $(\mathcal{H}, d)$ *be a complete CAT(0) space. Assume that* $T_1$ *and* $T_2$ *satisfy (A1) and (A2), and* $\operatorname{Fix}(T_1) \neq \emptyset$. *Suppose that* $\{u_n\}$ *is generated by Algorithm 1 such that* $\{\alpha_n\}$ *and* $\{\beta_n\}$ *satisfied (C1)-(C5). Then* $\{u_n\}$ *strongly converges to a solution of the problem* (1.2).

*Proof.* Let $p^*$ be a solution of the problem (1.2). We claim that

$$\limsup_{n \to \infty} \mathcal{Q}(p^*, G_2 p^*, w_n, p^*) \geq 0.$$

To show the claim, we start by observing that $\{\mathcal{Q}(p^*, G_2 p^*, w_n, p^*)\}$ is bounded by the boundedness of $\{w_n\}$. Thus the upper limit exists. Moreover, without loss of generality, we may choose a subsequence $\{w_{n_k}\}$ of $\{w_n\}$ that $\Delta$-converges to $w^*$ and

$$\limsup_{n \to \infty} \mathcal{Q}\left(p^*, G_2 p^*, w_n, p^*\right)=\lim_{k \to \infty} \mathcal{Q}(p^*, G_2 p^*, w_{n_k}, p^*\right). \tag{3.7}$$

Moreover,

$$\begin{aligned} d\left(w_{n_k}, G_1 w^*\right) &\leq d\left(w_{n_k}, G_1 u_{n_k}\right) + d\left(G_1 u_{n_k}, G_1 w^*\right) \\ &\leq d\left(w_{n_k}, G_1 u_{n_k}\right) + d\left(u_{n_k}, w^*\right) \\ &\leq d\left(w_{n_k}, G_1 u_{n_k}\right) + d\left(u_{n_k}, u_{n_k+1}\right) + d\left(u_{n_k+1}, w_{n_k}\right) + d\left(w_{n_k}, w^*\right). \end{aligned}$$

This together with (3.6) and (3.4) yield

$$\limsup_{k\to\infty} d\left(w_{n_k}, G_1 w^*\right) \leq \limsup_{k\to\infty} d\left(w_{n_k}, w^*\right).$$

The uniqueness of asymptotic center implies that $w^* = G_1 w^*$. Consequently, we have that $w^* \in \mathrm{Fix}(T_1)$. Since $p^*$ is a solution of (1.2), it follows that

$$\mathcal{Q}\left(p^*, G_2 p^*, w^*, p^*\right) \geq 0.$$

Thus

$$\limsup_{n\to\infty} \mathcal{Q}\left(p^*, G_2 p^*, w_n, p^*\right) = \lim_{k\to\infty} \mathcal{Q}\left(p^*, G_2 p^*, w_{n_k}, p^*\right) = \mathcal{Q}\left(p^*, G_2 p^*, w^*, p^*\right) \geq 0,$$

that is,

$$\limsup_{n\to\infty} \mathcal{Q}\left(p^*, G_2 p^*, p^*, w_n\right) \leq 0. \tag{3.8}$$

Let $p^*$ be a solution of the problem (1.2). Then

$$\begin{aligned} & d^2\left(u_{n+1}, p^*\right) = d^2\left((1-\beta_n) w_n \oplus \beta_n G_2 u_n, p^*\right) \\ & \leq (1-\beta_n)^2 d^2\left(w_n, p^*\right) + \beta_n^2 d^2\left(G_2 u_n, p^*\right) + 2\beta_n(1-\beta_n)\mathcal{Q}(w_n, p^*, G_2 u_n, p^*) \\ & = (1-\beta_n)^2 d^2\left(w_n, p^*\right) + \beta_n^2 d^2\left(G_2 u_n, p^*\right) \\ & \quad + 2\beta_n(1-\beta_n)\mathcal{Q}(w_n, p^*, G_2 u_n, G_2 p^*) + 2\beta_n(1-\beta_n)\mathcal{Q}(p^*, G_2 p^*, p^*, w_n). \end{aligned}$$

This together with the Cauchy-Schwartz inequality, (A2) and (3.3), yield

$$\begin{aligned} & d^2\left(u_{n+1}, p^*\right) \leq (1-\beta_n)^2 d^2\left(w_n, p^*\right) + \beta_n^2 d^2\left(G_2 u_n, p^*\right) \\ & \quad + 2\beta_n(1-\beta_n) d(w_n, p^*) d\left(G_2 u_n, G_2 p^*\right) + 2\beta_n(1-\beta_n)\mathcal{Q}(p^*, G_2 p^*, p^*, w_n) \\ & \leq (1-\beta_n)^2 d^2\left(u_n, p^*\right) + \beta_n^2 d^2\left(G_2 u_n, p^*\right) \\ & \quad + 2\frac{\beta\beta_n(1-\beta_n)}{1+\gamma_2} d^2(u_n, p^*) + 2\beta_n(1-\beta_n)\mathcal{Q}(p^*, G_2 p^*, p^*, w_n) \\ & \leq (1-2\beta_n(1-\kappa)) d^2\left(u_n, p^*\right) + \beta_n^2\left(d^2\left(G_2 u_n, p^*\right) + d^2\left(u_n, p^*\right)\right) \\ & \quad + 2\beta_n(1-\beta_n)\mathcal{Q}(p^*, G_2 p^*, p^*, w_n), \end{aligned}$$

where $\kappa = \dfrac{\beta}{1+\gamma}$. Hence, we have

$$d^2\left(u_{n+1}, p^*\right) \leq (1-\theta_n) d^2\left(u_n, p^*\right) + \theta_n \varphi_n, \tag{3.9}$$

where $\varphi_n = \dfrac{1}{2(1-k)}\left[\beta_n d^2\left(G_2 u_n, p^*\right) + \beta_n d^2\left(u_n, p^*\right) + 2(1-\beta_n)\mathcal{Q}(p^*, G_2 p^*, p^*, w_n)\right]$ and $\theta_n = 2\beta_n(1-\kappa)$. Observe that $\limsup_{n\to\infty} \varphi_n \leq 0$. Consequently, Lemma 2.6 and (3.9) guarantee that $\lim_{n\to\infty} d\left(u_n, p^*\right) = 0$. □

The following result follows from the fact that every nonexpansive mapping is 0-enriched nonexpansive.

**Corollary 3.4.** *Let $(\mathcal{H}, d)$ be a complete CAT(0) space. Assume that $T_1$ is a nonexpansive mapping with a fixed point and let $T_2$ satisfies (A2). Suppose that $\{u_n\}$ is generated by*

$$\begin{aligned} w_n &= (1-\alpha_n)\, u_n \oplus \alpha_n T_1 u_n, \\ u_{n+1} &= (1-\beta_n) w_n \oplus \frac{\gamma_2 \beta_n}{1+\gamma_2} u_n \oplus \frac{\beta_n}{1+\gamma_2} T_2 u_n, \quad n \geq 1; \end{aligned}$$

*where $\{\alpha_n\}, \{\beta_n\}$ are sequences in $(0,1]$ that satisfied (C1)-(C5). Then $\{u_n\}$ strongly converges to a solution of the problem* (1.2).

Another substantial result that follows from Theorem 3.3 is the case when $T_2$ is Banach contraction mapping. Recall that every $\kappa$-contraction mapping is $(0, \kappa)$-enriched contraction mapping.

**Corollary 3.5.** *Let $(\mathcal{H}, d)$ be a complete CAT(0) space. Assume that $T_1$ satisfied (A1) with a nonempty fixed points set and let $T_2$ be a contraction mapping. Suppose that $\{u_n\}$ is generated by*

$$\begin{aligned} w_n &= \left(1 - \frac{\alpha_n}{1+\gamma_1}\right) u_n \oplus \frac{\alpha_n}{1+\gamma_1} T_1 u_n, \\ u_{n+1} &= (1-\beta_n) w_n \oplus \beta_n T_2 u_n, \quad n \geq 1; \end{aligned}$$

*where $\{\alpha_n\}, \{\beta_n\}$ are sequences in $(0,1]$ that satisfied (C1)-(C5). Then $\{u_n\}$ strongly converges to a solution of the problem* (1.2).

Next corollary follows from Corollary 3.4 using the Banach contraction mapping.

**Corollary 3.6.** *Let $(\mathcal{H}, d)$ be a CAT(0) space. Assume that $T_1$ is a nonexpansive mapping with a fixed point and let $T_2$ be a contraction mapping. Suppose that $\{u_n\}$ is generated by*

$$\begin{aligned} w_n &= (1-\alpha_n)\, u_n \oplus \alpha_n T_1 u_n, \\ u_{n+1} &= (1-\beta_n) w_n \oplus \beta_n T_2 u_n, \quad n \geq 1; \end{aligned}$$

*where $\{\alpha_n\}, \{\beta_n\}$ are sequences in $(0,1]$ that satisfied (C1)-(C5). Then $\{u_n\}$ strongly converges to a solution of the problem* (1.2).

**Remark 3.7.** *It is important to note that setting $\alpha_n \equiv 1$ and taking $(\mathcal{H}, d)$ as Hilbert space, the algorithm of the Corollary 3.6 coincides with the Moudafi's viscosity iteration.*

By the fact that CAT(0) spaces contained Hilbert spaces, we have the following results based on Theorem 3.3 and Corollaries 3.4-3.6.

**Corollary 3.8.** *Let $(\mathcal{H}, \|\cdot\|)$ be a Hilbert space. Assume that $T_1$ and $T_2$ satisfy (A1) and (A2), and $\mathrm{Fix}(T_1) \neq \emptyset$. Suppose that $\{u_n\}$ is generated by*

$$w_n = \left(1 - \frac{\alpha_n}{1+\gamma_1}\right) u_n + \frac{\alpha_n}{1+\gamma_1} T_1 u_n,$$
$$u_{n+1} = (1-\beta_n)w_n + \frac{\gamma_2\beta_n}{1+\gamma_2} u_n + \frac{\beta_n}{1+\gamma_2} T_2 u_n, \quad n \geq 1;$$

*where $\{\alpha_n\}, \{\beta_n\}$ are sequences in $(0,1]$ that satisfied (C1)-(C5). Then $\{u_n\}$ strongly converges to a solution of the problem* (1.2).

**Corollary 3.9.** *Let $(\mathcal{H}, d)$ be a complete CAT(0) space. Assume that $T_1$ is a nonexpansive mapping with a fixed point and let $T_2$ satisfies (A2). Suppose that $\{u_n\}$ is generated by*

$$w_n = (1-\alpha_n) u_n + \alpha_n T_1 u_n,$$
$$u_{n+1} = (1-\beta_n)w_n + \frac{\gamma_2\beta_n}{1+\gamma_2} u_n + \frac{\beta_n}{1+\gamma_2} T_2 u_n, \quad n \geq 1;$$

*where $\{\alpha_n\}, \{\beta_n\}$ are sequences in $(0,1]$ that satisfied (C1)-(C5). Then $\{u_n\}$ strongly converges to a solution of the problem* (1.2).

**Corollary 3.10.** *Let $(\mathcal{H}, d)$ be a complete CAT(0) space. Assume that $T_1$ satisfied (A1) with a nonempty fixed points set and let $T_2$ be a contraction mapping. Suppose that $\{u_n\}$ is generated by*

$$w_n = \left(1 - \frac{\alpha_n}{1+\gamma_1}\right) u_n + \frac{\alpha_n}{1+\gamma_1} T_1 u_n,$$
$$u_{n+1} = (1-\beta_n)w_n + \beta_n T_2 u_n, \quad n \geq 1;$$

*where $\{\alpha_n\}, \{\beta_n\}$ are sequences in $(0,1]$ that satisfied (C1)-(C5). Then $\{u_n\}$ strongly converges to a solution of the problem* (1.2).

**Corollary 3.11.** *Let $(\mathcal{H}, d)$ be a complete CAT(0) space. Assume that $T_1$ is a nonexpansive mapping with a fixed point and let $T_2$ be a contraction mapping. Suppose that $\{u_n\}$ is generated by*

$$w_n = (1-\alpha_n) u_n + \alpha_n T_1 u_n,$$
$$u_{n+1} = (1-\beta_n)w_n + \beta_n T_2 u_n, \quad n \geq 1;$$

*where $\{\alpha_n\}, \{\beta_n\}$ are sequences in $(0,1]$ that satisfied (C1)-(C5). Then $\{u_n\}$ strongly converges to a solution of the problem* (1.2).

## 4. Numerical example

**Example 4.1.** *Let $m \geq 2$ be a fixed natural number. Consider $\mathbb{R}^m$ endowed with the metric $d$ defined by*

$$d(u,w) = \sqrt{\sum_{i=1}^{m-1} (u_i - w_i)^2 + (u_{m-1}^2 + w_m - u_m - w_{m-1}^2)^2},$$

*for all* $u = (u_1, u_2, \ldots, u_m) \in \mathbb{R}^m$ *and* $w = (w_1, w_2, \ldots, w_m) \in \mathbb{R}^m$. *For* $u, w \in \mathbb{R}^m$, *consider* $\gamma_u^w : [0,1] \to \mathbb{R}^m$ *such that*

$$\gamma_u^w(t) = \big(\gamma_1(t), \gamma_2(t), \ldots, \gamma_m(t)\big),$$

*where* $\gamma_i(t) = (1-t)u_i + tw_i$ *for all* $i = 1, 2, \cdots, m-1$ *and*

$$\gamma_m(t) = u_m + t(w_m - u_m) - t(1-t)(w_{m-1} - u_{m-1})^2.$$

*It follows that* $\gamma_u^w$ *is the geodesic connecting* $u$ *and* $w$, *and* $(\mathbb{R}^m, d)$ *is a complete CAT(0) space. In fact, using the identity that*

$$|tx + (1-t)y|^2 = tx^2 + (1-t)y^2 - t(1-t)|x-y|^2, \quad \forall x, y \in \mathbb{R},$$

*the inequality* (2.5) *can be shown to be equality by a simple computation. However,* $(\mathbb{R}^m, d)$ *is not a Hilbert space.*

*Now, take any* $a > 1$ *and consider* $T_2 : \mathbb{R}^m \to \mathbb{R}^m$ *by*

$$T_2 u = (-au_1, -au_2, \ldots, -au_{m-1}, a^2 u_{m-1}^2).$$

*Clearly* $T_2$ *is not contraction and not nonexpansive mapping. However,* $T_2$ *is an enriched contraction mapping with respect to* $(\mathbb{R}^m, d)$. *Indeed for all* $u, w \in \mathbb{R}^m$, *we have*

$$d^2(T_2u, T_2w) = a^2 \sum_{i=1}^{m-1} (u_i - w_i)^2$$

*and*

$$\begin{aligned}
2\mathcal{Q}(u, w, T_2u, T_2w) &= d^2(u, T_2w) + d^2(w, T_2u) - d^2(u, T_2u) - d^2(w, T_2w) \\
&= \left[\sum_{i=1}^{m-1} (u_i + aw_i)^2 + (u_{m-1}^2 - u_m)^2\right] + \left[\sum_{i=1}^{m-1} (w_i + au_i)^2 + (w_{m-1}^2 - w_m)^2\right] \\
&- \left[\sum_{i=1}^{m-1} (u_i + au_i)^2 + (u_{m-1}^2 - u_m)^2\right] - \left[\sum_{i=1}^{m-1} (w_i + aw_i)^2 + (w_{m-1}^2 - w_m)^2\right] \\
&= \sum_{i=1}^{m-1} \left[(u_i + aw_i)^2 + (w_i + au_i)^2 - (u_i + au_i)^2 - (w_i + aw_i)^2\right] \\
&= -2a \sum_{i=1}^{m-1} (u_i - w_i)^2 .
\end{aligned}$$

*Thus, we get*

$$\begin{aligned}
d(T_2u, T_2w)^2 &+ \alpha^2 d(u,w)^2 + 2\alpha\mathcal{Q}(u,w,T_2u,T_2w) \\
&= a^2 \sum_{i=1}^{m-1} (u_i - w_i)^2 + \alpha^2 \sum_{i=1}^{m-1} (u_i - w_i)^2 \\
&\quad + \alpha^2 (u_{m-1}^2 + w_m - u_m - w_{m-1}^2)^2 - 2a\alpha \sum_{i=1}^{m-1} (u_i - w_i)^2 \\
&= (a-\alpha)^2 \sum_{i=1}^{m-1} (u_i - w_i)^2 + \alpha^2 (u_{m-1}^2 + w_m - u_m - w_{m-1}^2)^2.
\end{aligned}$$

*Hence, for any* $\alpha \geq 0$ *such that* $|a - \alpha| \leq \alpha + \frac{1}{2}$, *we get that*

$$d^2(T_2u, T_2w) + \alpha^2 d^2(u,w) + 2\alpha\mathcal{Q}(u,w,T_2u,T_2w) \leq \left(\alpha + \frac{1}{2}\right)^2 d^2(u,w).$$

*This implies that for* $\alpha = \frac{2a-1}{4}$, $T_2$ *is* $(\alpha,\beta)$*-enriched contraction with* $\beta = \alpha + \frac{1}{2}$.

*Consider,* $T_1 : \mathbb{R}^m \to \mathbb{R}^m$ *by*

$$T_1 u = (-5u_1, -5u_2, \ldots, -5u_{m-1}, 25u_{m-1}^2).$$

*Following similar fashion as in the case of* $T_2$*, we obtain that*

$$\begin{aligned}
d(T_1u, T_1w)^2 &+ \alpha^2 d(u,w)^2 + 2\alpha\mathcal{Q}(u,w,T_1u,T_1w) \\
&= (5-\alpha)^2 \sum_{i=1}^{m-1} (u_i - w_i)^2 + \alpha^2 (u_{m-1}^2 + w_m - u_m - w_{m-1}^2)^2.
\end{aligned}$$

*Hence, for any* $\alpha \geq 0$ *such that* $|5 - \alpha| \leq \alpha + 1$, *we get that*

$$d^2(T_1u, T_1w) + \alpha^2 d^2(u,w) + 2\alpha\mathcal{Q}(u,w,T_1u,T_1w) \leq (\alpha+1)^2 d^2(u,w).$$

*Hence for* $\alpha = 3$, *we have that* $T_1$ *is* $\alpha$*-enriched nonexpansive mapping with respect to* $(\mathbb{R}^m, d)$. *Observe that* $\mathrm{Fix}(T_2) = \mathrm{Fix}(T_1) = \{0\}$.

*For the numerical sake, we take* $\alpha_n = \frac{n^2-1}{n^2}$ *and* $\beta_n = \frac{1}{n}$. *Clearly,* $\alpha_n$ *and* $\beta_n$ *satisfy conditions (C1)-(C5). Also, we take* $\gamma_1 = 3$ *and* $\gamma_2 = \frac{2a-1}{4}$ *since* $T_1$ *is* 2*-enriched nonexpansive mapping and* $T_2$ *is* $\left(\frac{2a-1}{4}, \frac{2a+1}{4}\right)$*-enriched contraction mapping. Then Algorithm 1 yield the results in Table 1 for different values of* $a$ *and* $m$.

Table 1. Report of Numerical Experiments for Example 4.1

| $n$ | $d(u_{n+1}, u_n)$ | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | $m = 5, \quad u_1 = [10, 10, \cdots, 10]$ | | | | $m = 100, \quad u_1 = [5, 5, \cdots, 5]$ | | | |
| | $a = 2$ | $a = 10$ | $a = 50$ | $a = 100$ | $a = 2$ | $a = 10$ | $a = 50$ | $a = 100$ |
| 1 | 1410.684 | 2147.071 | 2385.722 | 2418.652 | 87.78738 | 112.1003 | 120.1704 | 121.278 |
| 2 | 364.8119 | 422.508 | 916.794 | 997.4766 | 32.73974 | 51.06676 | 72.51798 | 76.235 |
| 3 | 521.243 | 716.1982 | 662.1826 | 646.0764 | 29.59942 | 39.01646 | 37.12957 | 36.57538 |
| 4 | 299.9425 | 701.892 | 854.9075 | 875.459 | 17.25447 | 36.98129 | 44.28359 | 45.25116 |
| 5 | 153.4569 | 485.9599 | 666.3377 | 694.6907 | 9.087196 | 25.81621 | 34.74674 | 36.14387 |
| 6 | 77.3291 | 304.5363 | 452.2964 | 477.0773 | 4.748873 | 16.38064 | 23.81465 | 25.05638 |
| 7 | 39.35134 | 183.7841 | 290.2926 | 308.9324 | 2.526705 | 10.05299 | 15.4762 | 16.4206 |
| ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ |
| 95 | 7.70E-12 | 7.04E-11 | 1.44E-10 | 1.59E-10 | 1.71E-12 | 1.56E-11 | 3.19E-11 | 3.53E-11 |
| 96 | 5.75E-12 | 5.29E-11 | 1.08E-10 | 1.19E-10 | 1.28E-12 | 1.18E-11 | 2.40E-11 | 2.65E-11 |
| 97 | 4.29E-12 | 3.97E-11 | 8.13E-11 | 8.98E-11 | 9.54E-13 | 8.82E-12 | 1.81E-11 | 2.00E-11 |
| 98 | 3.20E-12 | 2.98E-11 | 6.11E-11 | 6.76E-11 | 7.12E-13 | 6.62E-12 | 1.36E-11 | 1.50E-11 |
| 99 | 2.39E-12 | 2.24E-11 | 4.60E-11 | 5.09E-11 | 5.32E-13 | 4.97E-12 | 1.02E-11 | 1.13E-11 |
| 100 | 1.79E-12 | 1.68E-11 | 3.46E-11 | 3.83E-11 | 3.97E-13 | 3.73E-12 | 7.69E-12 | 8.50E-12 |

**Concluding remarks.** In this work, we have developed a viscosity scheme for variational inequality problems involving enriched contraction mappings, with the constrained set being the set of fixed points of an enriched nonexpansive mapping. We have analyzed certain essential properties of the proposed scheme and stated control conditions upon which the sequence generated from the scheme is shown to converge strongly to a solution of the problem. Finally, we provide a numerical example in a non-linear setting to demonstrate the possibility of implementing the scheme. The results presented herein hold for Hadamard manifolds, $\mathbb{R}$-trees, Hilbert spaces, Hilbert balls, and all CAT($\kappa$) spaces for $\kappa \leq 0$.

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