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UNIFORMLY CONVEX METRIC SPACES AND FIXED POINTS OF MONOTONE G-NONEXPANSIVE MULTIVALUED MAPPINGS

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Abstract. In this work, we introduce the UUUC property and the p-UUC property for hyperbolic metric spaces. These uniform convexity properties are more adequate than metric uniform convexities and norm uniform convexities in studying the fixed point problems for monotone nonexpansive mappings. As an application of these properties we prove a fixed point theorem for monotone G-nonexpansive multivalued mappings.
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1. Introduction

The topic of uniform convexities in metric spaces has recently become a focus of attention for many mathematicians. The first form of uniform convexity for Banach spaces was investigated by Clarkson [9]. After that Garkavi [16] introduced the notion of uniform convexity in every direction. Recently, Alfuraidan and Khamsi [2] have considered a variant form of uniform convexity in partially ordered Banach spaces. In the non-linear setting of so called CAT(0)-spaces, uniform convexity is by now well-understood (see [5], [7]). Only recently, Kuwae [25] based on the approach of Noar and Silberman [29] studied spaces with the p-uniformly convex property similar to that of Banach spaces.

In 2016, Dehaish and Khamsi [10] used uniform convexity for hyperbolic metric spaces to prove the existence of fixed points for monotone nonexpansive mappings. The function $\delta(r,\varepsilon)$, as discussed in [10], exhibits similarities to the modulus of convexity in uniformly convex Banach spaces. However, the modulus of convexity in uniformly convex Banach spaces (see Definition 1, [9]) only depends on ε. In the definition of Dehaish and Khamsi, the function $\delta(r,\varepsilon)$ (see Definition 3.1, [10]) depends not only on r and ε but also on the point a in the hyperbolic metric space X. Therefore, it will be denoted by $\delta_a(r,\varepsilon)$ in this paper. In a uniformly convex hyperbolic metric space (X,d), we have $\delta_a(r,\varepsilon) > 0$ for every $a \in X$, $r > 0$, $\varepsilon > 0$. But this condition does not guarantee that $\inf\{\delta_a(r,\varepsilon) : a \in X, r > 0, \varepsilon > 0\}$ is positive. For example, in uniformly convex spaces L_p or l_p, we have $\delta(\varepsilon) = 1 - [1-(\varepsilon/2)^p]^{1/p}$ for $p > 1$ (see [9]). If we take $\varepsilon \to 0$, then $\delta(\varepsilon) \to 0$. If $X = \mathbb{R}^2$ is furnished with one of the norm $|(a,b)| = |a| + |b|$ or $|(a,b)| = \max\{|a|,|b|\}$ then $\delta(\varepsilon) = 0$ for all $0 \le \varepsilon \le 2$. If X is a Hilbert space, then $\delta(\varepsilon) = 1 - (1-\varepsilon^2/4)^{1/2} \to 0$ as $\varepsilon \to 0$. In Section 2, we assume that $\inf\{\delta_a(r,\varepsilon) : a \in X, r \ge \alpha, \varepsilon \ge \beta\} > 0$ for every $\alpha > 0$, $\beta > 0$. From that we extend the definition of 2-uniformly convex property in [21] to p-UUC property. Our definition extends the definitions of both Noar-Silberman [29] and Kuwae [25]. By taking an approach like that of Khamsi-Khan [21], we give simple proofs of some properties in hyperbolic metric spaces.

In Section 3 we use the properties in Section 2 to show in Theorem 3.8 the existence of fixed points of monotone G-nonexpansive multivalued mappings in p-UUC hyperbolic metric spaces, where $p \ge 2$. It is a “monotone ”counterpart of the Lim theorem [27].

2. Main results

Let (X,d) be a metric space. Recall that X is said to be uniquely geodesic if any two points x, y in X are endpoints of a unique metric segment $[x,y]$ (i.e., $[x,y]$ is an isometric image of the interval $[0, d(x,y)]$). We shall denote by $\alpha x \oplus (1-\alpha)y$ a unique point z of $[x,y]$ which satisfies

$$d(x,z) = (1-\alpha)d(x,y) \quad \text{and} \quad d(z,y) = \alpha d(x,y),$$

where $\alpha \in [0,1]$. A set $C \subset X$ is convex if the metric segment $[x,y] \subset C$ for each x, $y \in C$. Moreover, if for all x_1, x_2, y_1, y_2 in X and $t \in [0,1]$ we have

$$d(tx_1 \oplus (1-t)x_2, ty_1 \oplus (1-t)y_2) \le td(x_1,y_1) + (1-t)d(x_2,y_2),$$

then X is said to be a hyperbolic metric space [30]. The space satisfying the mentioned conditions is also called a Busemann space in some references (see [24]).

Definition 2.1. Let (X,d) be a hyperbolic metric space. For any $a \in X$, $r > 0$, and $\varepsilon \ge 0$, define

$$\delta_a(r,\varepsilon) = \inf\Big\{1 - \frac{1}{r}d\Big(a, \frac{1}{2}x \oplus \frac{1}{2}y\Big) : d(x,a) \le r, d(y,a) \le r, d(x,y) \ge r\varepsilon\Big\}.$$

(i) (see Definition 3.1, [10]) We say that X is uniformly convex (UC for short) if and only if $\delta_a(r,\varepsilon) > 0$ for any $a \in X$, $r > 0$ and $\varepsilon > 0$. The space X satisfying this property is also known as weakly uniformly convex (see [26]).

(ii) (see Definition 4.1, [22]) We say that X is UUC if and only if for every $a \in X, s > 0$ and $\varepsilon \in (0,2]$, there exists $\eta_a(s,\varepsilon) > 0$ such that $\delta_a(r,\varepsilon) \ge \eta_a(s,\varepsilon) > 0$ for $r \ge s > 0$.

(iii) We say that X is UUUC if and only if for every $\alpha > 0$ and $\beta \in (0,2]$, we have $\inf\{\delta_a(r,\varepsilon) : a \in X, r \ge \alpha, \varepsilon \ge \beta\} > 0$.

The following properties follow easily from Definition 2.1.

Proposition 2.2. *The following conditions characterize relationship between the above defined notions:*

(i) *If X is UUC then X is UC.*
(ii) *If X is UUUC then X is UUC.*

We are going to prove a simple property of $\delta_a(r,\varepsilon)$ in hyperbolic metric spaces.

Lemma 2.3. *Let (X,d) be a hyperbolic metric space. Let $a \in X$, $r > 0$, and $\varepsilon \geq 0$.*

(i) *$\delta_a(r,0) = 0$ and $\delta_a(r,\varepsilon)$ is an increasing function of ε for every fixed r and a.*
(ii) *Suppose that X is UC and $t_n > 0$ for all $n \geq 1$. If $\lim\limits_{n\to\infty} \delta_a(r,t_n) = 0$ for a fixed $a \in X$ and $r > 0$, then $\inf\limits_{n\geq 1} t_n = 0$.*

Proof. It is not difficult to see (i). We are going to prove (ii). Assume that $\lim\limits_{n\to\infty} \delta_a(r,t_n) = 0$ and $\inf\limits_{n\geq 1} t_n \neq 0$. Then there exists α such that

$$0 < \alpha \leq \inf_{n\geq 1} t_n.$$

Consequently, $\alpha \leq t_n$ for all $n \geq 1$. Since the function $\delta_a(r,\varepsilon)$ is increasing of ε, we have

$$\delta_a(r,\alpha) \leq \delta_a(r,t_n), \tag{2.1}$$

for every $n \geq 1$. Taking the limit on both sides of (2.1) as $n \to \infty$, we have

$$0 < \delta_a(r,\alpha) \leq \lim_{n\to\infty} \delta_a(r,t_n).$$

It contradicts $\lim\limits_{n\to\infty} \delta_a(r,t_n) = 0$. Therefore, $\inf\limits_{n\geq 1} t_n = 0$. □

Lemma 2.4 ([14]). *Let (X,d) be a hyperbolic metric space. Assume that X is UUC. Let $r > 0$, $a \in X$.*

(i) *Assume that $t \in [\alpha,\beta]$, where $0 < \alpha \leq \beta < 1$. For any number $\varepsilon > 0$, $a,x,y \in X$ such that*
$$d(x,a) \leq r, d(y,a) \leq r, d(x,y) \geq r\varepsilon,$$
there exists $\delta(r, 2\varepsilon \min\{\alpha, 1-\beta\}) \in (0,1)$ such that
$$d(a,(1-t)x \oplus ty) \leq r\Big(1 - \delta(r, 2\varepsilon \min\{\alpha, 1-\beta\})\Big).$$
(ii) *Assume that $t_n \in [\alpha,\beta]$, where $0 < \alpha \leq \beta < 1$ and $(x_n)_n$, $(y_n)_n$ are two sequences in X such that $\limsup\limits_{n\to\infty} d(a,x_n) \leq r$, $\limsup\limits_{n\to\infty} d(a,y_n) \leq r$, $\lim\limits_{n\to\infty} d\Big(a, t_n x_n \oplus (1-t_n) y_n\Big) = r$. Then $\lim\limits_{n\to\infty} d(x_n,y_n) = 0$.*

In Theorem 2.3 [21], a metric version of the parallelogram identity is stated for the case $p = 2$, the following theorem is an extension of it when $p \geq 2$.

Theorem 2.5. *Let (X,d) be a hyperbolic metric space. Let $a \in X$, $p \geq 2$. For each $r > 0$ and $\varepsilon \geq 0$ set*

$$\psi_a(r,\varepsilon) = \inf\left\{\frac{1}{2}d^p(a,x) + \frac{1}{2}d^p(a,y) - d^p\Big(a, \frac{1}{2}x \oplus \frac{1}{2}y\Big)\right\},$$

where the infimum is taken over all $x,y \in X$ such that $d(a,x) \leq r$, $d(a,y) \leq r$ and $d(x,y) \geq r\varepsilon$. Then $\psi_a(r,\varepsilon) > 0$ for any $r,\varepsilon > 0$. Moreover, for a fixed $r > 0$, we have:

(i) $\psi_a(r,0)=0$;
(ii) $\psi_a(r,\varepsilon)$ *is a nondecreasing function of* ε;
(iii) *If* $\lim_{n\to\infty}\psi_a(r,t_n)=0$, *then* $\inf_{n\geq 1} t_n = 0$.

Proof. Assume on the contrary that $\psi_a(r,\varepsilon)=0$ for some $r>0$, $\varepsilon>0$. Then there exist two sequences $(x_n)_n$ and $(y_n)_n$ in X such that

$$\lim_{n\to\infty}\Big[\frac{1}{2}d^p(a,x_n)+\frac{1}{2}d^p(a,y_n)-d^p\Big(a,\frac{1}{2}x_n\oplus\frac{1}{2}y_n\Big)\Big]=0,$$

where $d(a,x_n)\leq r$, $d(a,y_n)\leq r$ and $d(x_n,y_n)\geq r\varepsilon$ for every $n\geq 1$. Now for $p\geq 2$, using the Clarkson's inequality

$$\Big|\frac{a+b}{2}\Big|^p+\Big|\frac{a-b}{2}\Big|^p\leq\frac{1}{2}|a|^p+\frac{1}{2}|b|^p$$

for any $a,b\in\mathbb{R}$, we get

$$\Big(\frac{d(a,x_n)+d(a,y_n)}{2}\Big)^p\leq\frac{1}{2}d^p(a,x_n)+\frac{1}{2}d^p(a,y_n)-\Big|\frac{d(a,x_n)-d(a,y_n)}{2}\Big|^p.$$

Since X is a hyperbolic metric space, it follows that

$$d^p\Big(a,\frac{1}{2}x_n\oplus\frac{1}{2}y_n\Big)\leq\frac{1}{2}d^p(a,x_n)+\frac{1}{2}d^p(a,y_n)-\Big|\frac{d(a,x_n)-d(a,y_n)}{2}\Big|^p$$

for every $n\geq 1$. Hence

$$\Big|\frac{d(a,x_n)-d(a,y_n)}{2}\Big|^p\leq\frac{1}{2}d^p(a,x_n)+\frac{1}{2}d^p(a,y_n)-d^p\Big(a,\frac{1}{2}x_n\oplus\frac{1}{2}y_n\Big)$$

for every $n\geq 1$. It implies that $\lim_{n\to\infty}|d(a,x_n)-d(a,y_n)|=0$. Since $(d(a,x_n))_n$ is a bounded sequence, we can choose a subsequence $(d(a,x_{n_k}))_k$ of $(d(a,x_n))_n$ such that $\lim_{k\to\infty}d(a,x_{n_k})=R$. By our assumptions, we have

$$\lim_{k\to\infty}d(a,y_{n_k})=R\quad\text{and}\quad\lim_{k\to\infty}d(a,\frac{1}{2}x_{n_k}\oplus\frac{1}{2}y_{n_k})=R.$$

Lemma 2.4 (ii) yields $\lim_{k\to\infty}d(x_{n_k},y_{n_k})=0$ which contradicts $d(x_{n_k},y_{n_k})\geq r\varepsilon>0$ for all $k\geq 1$. The proofs of (i)-(ii) are immediate.

By an argument analogous to that used in the proof of Lemma 2.3 (ii) we can easily prove (iii). □

Remark 2.6. Assume that $\alpha\in(0,2]$. By Theorem 2.5 (iii) we have $\inf_{\varepsilon\geq\alpha}\psi_a(r,\varepsilon)>0$.

The concept of p-uniform convexity was used extensively by Xu [31]. Its nonlinear version for $p=2$ is given by Khamsi and Khan [21]. We extend this definition in the case of $p\geq 2$ and give the definition of p-UUC property.

Definition 2.7. *Let* (X,d) *be a hyperbolic metric space. Let* $a\in X$, $p\geq 2$. *For each* $r>0$ *and* $\varepsilon>0$ *we define* $\psi_a(r,\varepsilon)$ *as in Theorem 2.5. We will say that* (X,d) *is p-UUC if*

$$c_X=\inf\Big\{\frac{\psi_a(r,\varepsilon)}{r^p\varepsilon^p}:a\in X,r>0,\varepsilon>0\Big\}>0.$$

The following proposition can be easily deduced from Definition 2.1 and 2.7.

Proposition 2.8. *If X is p-UUC then X is UUUC.*

Proof. Let $\alpha > 0$ and $\beta > 0$. Since X is p-UUC, then for every $a \in X$, $r \geq \alpha$, $\varepsilon \geq \beta$ we have $\psi_a(r, \varepsilon)/r^p \geq c_X \varepsilon^p$. Then $\psi_a(r, \varepsilon)/r^p \geq c_X \beta^p$ for every $a \in X$, $r \geq \alpha$, $\varepsilon \geq \beta$.

On the other hand, taking any $a \in X$, $r \geq \alpha$, $\varepsilon \geq \beta$, we get

$$\psi_a(r,\varepsilon) \leq \frac{1}{2}d^p(a,x) + \frac{1}{2}d^p(a,y) - d^p(a, \frac{1}{2}x \oplus \frac{1}{2}y) \leq r^p - d^p(a, \frac{1}{2}x \oplus \frac{1}{2}y)$$

for every $x, y \in X$ such that $d(a,x) \leq r$, $d(a,y) \leq r$ and $d(x,y) \geq r\varepsilon$. Hence

$$\psi_a(r,\varepsilon)/r^p \leq 1 - \Big(\frac{d(a, \frac{1}{2}x \oplus \frac{1}{2}y)}{r}\Big)^p \leq p\Big(1 - \frac{1}{r}d(a, \frac{1}{2}x \oplus \frac{1}{2}y)\Big).$$

It implies that

$$c_X \beta^p/p \leq 1 - \frac{1}{r}d(a, \frac{1}{2}x \oplus \frac{1}{2}y)$$

for every $x, y \in X$ such that $d(a,x) \leq r$, $d(a,y) \leq r$ and $d(x,y) \geq r\varepsilon$. Thus we have $c_X\beta^p/p \leq \delta_a(r,\varepsilon)$ for every $a \in X$, $r \geq \alpha$, $\varepsilon \geq \beta$. Therefore, X is UUUC. □

Example 2.9. A geodesic metric space X is said to be a CAT(0) space (the term is due to M. Gromov-see, e.g., [7], page 159) if every geodesic triangle in X is at least as "thin" as its comparison triangle in the Euclidean plane.

In 2017, Khamsi and Shukri [23] have extended the Gromov geometric definition of CAT(0) spaces to the case when the comparison triangles belong to a general Banach space. In particular, to the case when the Banach space is l_p, $p \geq 2$.

Recall that a geodesic triangle $\triangle(x_1, x_2, x_3)$ in a geodesic metric space (X, d) consists of three points x_1, x_2, x_3 in X (the vertices of $\triangle$) and a geodesic segment between each pair of vertices (the edges of $\triangle$). A comparison triangle for geodesic triangle $\triangle(x_1, x_2, x_3)$ in (X, d) is a triangle $\overline{\triangle}(x_1, x_2, x_3) := \triangle(\overline{x_1}, \overline{x_2}, \overline{x_3})$ in the Banach space l_p, for $p \geq 2$, such that $\|\overline{x}_i - \overline{x}_j\|$ for $i, j \in \{1, 2, 3\}$. A point $\overline{x} \in [\overline{x}_1, \overline{x}_2]$ is called a comparison point for $x \in [x_1, x_2]$ if $d(x_1, x_2) = d(\overline{x}_1, \overline{x}_2)$.

Let (X, d) be a geodesic metric space. X is said to be a $\mathrm{CAT}_p(0)$ space, for $p \geq 2$, if for any geodesic triangle $\triangle$ in X, there exists a comparison triangle $\overline{\triangle}$ in l_p such that the comparison axiom is satisfied, i.e., for all $x, y \in \triangle$ and all comparison points $\overline{x}, \overline{y} \in \overline{\triangle}$, we have

$$d(x,y) \leq \|\overline{x} - \overline{y}\|.$$

It is obvious that l_p, $p > 2$, is a $\mathrm{CAT}_p(0)$ space which is not a CAT(0) space [23]. Let x, y_1, y_2 be in $\mathrm{CAT}_p(0)$, and $\frac{1}{2}y_1 \oplus \frac{1}{2}y_2$ be the midpoint of the geodesic $[y_1, y_2]$, then the comparison axiom implies

$$d^p(x, \frac{1}{2}y_1 \oplus \frac{1}{2}y_2) \leq \frac{1}{2}d^p(x, y_1) + \frac{1}{2}d^p(x, y_2) - \frac{1}{2^p}d^p(y_1, y_2).$$

This inequality is the (CN_p) inequality of Khamsi and Shukri [23]. The (CN_p) inequality implies that

$$\psi_a(r,\varepsilon) \geq \frac{r^p\varepsilon^p}{2^p} \quad \text{for every } a \in \mathrm{CAT}_p(0).$$

This clearly implies that any $\mathrm{CAT}_p(0)$ space is p-UUC with $c_X \geq \frac{1}{2^p}$.

Example 2.10. Every p-uniformly convex space as defined in [25, 29] is p-UUC.

In [19], Kell introduced the notion of uniform p-convexity (where $p \in [1, \infty]$) in metric spaces admitting midpoints. A complete metric space (X, d) admits midpoints if for every $x, y \in X$ there is an $m(x, y) \in X$ such that $d(x, m(x, y)) = d(y, m(x, y)) = 1/2d(x, y)$. Obviously each such space is a geodesic space. Let us recall the definition of uniform p-convexity.

Definition 2.11 (see Definition 1.3, [19]). *A metric space X admitting midpoints is called uniformly p-convex (UpC for short) if for every $\varepsilon > 0$ there exists $\rho_p(\varepsilon) \in (0, 1)$ such that for all $x, y, a \in X$ satisfying $d(x, y) > \varepsilon \mathcal{M}^p(d(x, a), d(y, a))$ for $p > 1$ and $d(x, y) > |d(x, a) - d(y, a)| + \varepsilon \mathcal{M}^1(d(x, a), d(y, a))$ for $p = 1$, we have that*

$$d(a, m(x, y)) \leq (1 - \rho_p(\varepsilon))\mathcal{M}^p(d(x, a), d(y, a)),$$

where $\mathcal{M}^p(a, b) = \Big(a^p/2 + b^p/2\Big)^{1/p}$ and $\mathcal{M}^\infty(a, b) = \max\{a, b\}$.

We show the connection between properties UpC and p-UUC.

Lemma 2.12. *If X is p-UUC then X is UpC for all $p \geq 2$.*

Proof. We put

$$c_X = \inf\Big\{\frac{\psi_a(r, \varepsilon)}{r^p \varepsilon^p} : a \in X, r > 0, \varepsilon > 0\Big\}.$$

Assume that $\varepsilon > 0$ and put $\rho(\varepsilon) := (1 - c_X \varepsilon^p)^{1/p}$. Take any triples $x, y, z \in X$ satisfying

$$d(x, y) > \varepsilon\Big(\frac{1}{2}d^p(x, z) + \frac{1}{2}d^p(y, z)\Big)^{1/p}.$$

Thus

$$d(x, z) < \frac{\sqrt[p]{2}}{\varepsilon}d(x, y),\ d(y, z) < \frac{\sqrt[p]{2}}{\varepsilon}d(x, y) \text{ and } d(x, y) = \frac{\sqrt[p]{2}}{\varepsilon}d(x, y).\frac{\varepsilon}{\sqrt[p]{2}}.$$

Since X is p-UUC, we have

$$\begin{aligned} c_X &\leq \frac{1}{d^p(x, y)}\Big(\frac{1}{2}d^p(x, z) + \frac{1}{2}d^p(y, z) - d^p(z, \frac{1}{2}x \oplus \frac{1}{2}y)\Big) \\ &\leq \frac{1}{\varepsilon^p\Big(\frac{1}{2}d^p(x, z) + \frac{1}{2}d^p(y, z)\Big)}\Big(\frac{1}{2}d^p(x, z) + \frac{1}{2}d^p(y, z) - d^p(z, \frac{1}{2}x \oplus \frac{1}{2}y)\Big) \end{aligned}$$

It implies that

$$d(z, \frac{1}{2}x \oplus \frac{1}{2}y) \leq (1 - c_X \varepsilon^p)^{1/p}\Big(\frac{1}{2}d^p(x, z) + \frac{1}{2}d^p(y, z)\Big)^{1/p}.$$

Therefore X is UpC. □

Next, we show some properties of hyperbolic metric space with UUUC and p-UUC properties.

Theorem 2.13. *Let (X, d) be a hyperbolic metric space. Assume that X is UUUC.*

Let $p \geq 2$, $r > 0$. Assume that $x_0, y_0 \in X$ such that $x_0 \neq y_0$ and $\overline{B}(x_0, r) \cap \overline{B}(y_0, r) \neq \emptyset$. Then

$$\psi_{x_0,y_0}(r) := \inf\{\psi(z) : d(x_0, z) \leq r, d(y_0, z) \leq r\} > 0,$$

where $\psi(z) = \frac{1}{2}d^p(z, x_0) + \frac{1}{2}d^p(z, y_0) - d^p\Big(z, \frac{1}{2}x_0 \oplus \frac{1}{2}y_0\Big)$.

Proof. It is not difficult to see that $\psi_{x_0,y_0}(r) \geq 0$. Assume that $\psi_{x_0,y_0}(r) = 0$, then there exists a sequence $(z_n)_n$ in X such that

$$\lim_{n\to\infty} \Big[\frac{1}{2}d^p(z_n, x_0) + \frac{1}{2}d^p(z_n, y_0) - d^p\Big(z_n, \frac{1}{2}x_0 \oplus \frac{1}{2}y_0\Big)\Big] = 0,$$

where $d(z_n, x_0) \leq r$, $d(z_n, y_0) \leq r$ for every $n \geq 1$. Similarly, we get

$$\Big|\frac{d(z_n, x_0) - d(z_n, y_0)}{2}\Big|^p \leq \frac{1}{2}d^p(z_n, x_0) + \frac{1}{2}d^p(z_n, y_0) - d^p\Big(z_n, \frac{1}{2}x_0 \oplus \frac{1}{2}y_0\Big).$$

Thus

$$\lim_{n\to\infty} \Big|\frac{d(z_n, x_0) - d(z_n, y_0)}{2}\Big|^p = 0.$$

Since the sequence $(d(z_n, x_0))_n$ is bounded, passing to a subsequence if necessary, we can assume that $\lim\limits_{n\to\infty} d(z_n, x_0) = r_0$. Hence $\lim\limits_{n\to\infty} d(z_n, x_0) = \lim\limits_{n\to\infty} d(z_n, y_0)$. It implies that

$$\lim_{n\to\infty} d(z_n, x_0) = \lim_{n\to\infty} d(z_n, y_0) = \lim_{n\to\infty} d\Big(z_n, \frac{1}{2}x_0 \oplus \frac{1}{2}y_0\Big) = r_0.$$

Obviously, $0 < r_0 \leq r$. Let $\alpha = 1/2r_0$ and for every $n \geq 1$, put

$$r_n = \max\{d(z_n, x_0), d(z_n, y_0)\}.$$

Clearly, $\lim\limits_{n\to\infty} r_n = r_0$. Thus there exists $n_0 \geq 1$ such that $r_n \geq \alpha$ for every $n \geq n_0$. We have

$$\delta_{z_n}(r_n, \frac{d(x_0, y_0)}{r_n}) \leq 1 - \frac{1}{r_n}d\Big(z_n, \frac{1}{2}x_0 \oplus \frac{1}{2}y_0\Big)$$

for every $n \geq n_0$, where $\delta_{z_n}(r_n, \frac{d(x_0,y_0)}{r_n}) = \inf\Big\{1 - \frac{1}{r_n}d\Big(z_n, \frac{1}{2}x \oplus \frac{1}{2}y\Big)\Big\}$, the infimum is taken over all $x, y \in X$ such that $d(z_n, x) \leq r_n$, $d(z_n, y) \leq r_n$, $d(x, y) \geq r_n\frac{d(x_0,y_0)}{r_n}$.

Thus we have

$$\lim_{n\to\infty} \delta_{z_n}(r_n, \frac{d(x_0, y_0)}{r_n}) = 0,$$

which contradicts the fact that for every $n \geq n_0$, $\delta_{z_n}(r_n, \frac{d(x_0,y_0)}{r_n}) \geq \beta$ where $\beta = \inf\{\delta_a(r, \varepsilon) : a \in X, r \geq \alpha, \varepsilon \geq \frac{d(x_0,y_0)}{r}\} > 0$ since X is UUUC. □

By using the inequality $\psi_{x_0,y_0}(r) \leq pr^p\delta_{x_0,y_0}(r)$, where $p \geq 2$, it is not difficult to derive the following lemma.

Lemma 2.14. *Let (X, d) be a hyperbolic metric space. Assume that X is UUUC. Let $r > 0$, $x_0, y_0 \in X$ such that $x_0 \neq y_0$ and $\overline{B}(x_0, r) \cap \overline{B}(y_0, r) \neq \emptyset$. Then*

$$\delta_{x_0,y_0}(r) = \inf\Big\{1 - \frac{1}{r}d\Big(z, \frac{1}{2}x_0 \oplus \frac{1}{2}y_0\Big) : d(x_0, z) \leq r, d(y_0, z) \leq r\Big\} > 0.$$

Lemma 2.15. *Let (X,d) be a hyperbolic metric space. Assume that X is p-UUC. Let $(y_n)_n$, $(z_n)_n$ be two sequences in X such that $\lim_{n\to\infty} d(y_n,z_n) = l$. Assume that there exist $R \geq 0$ and a sequence $(x_m)_m$ in X such that*

$$\limsup_{m\to\infty} d(x_m,y_n) \leq R, \quad \limsup_{m\to\infty} d(x_m,z_n) \leq R, \quad \limsup_{m\to\infty} d(x_m, \frac{1}{2}y_n \oplus \frac{1}{2}z_n) = R$$

for every $n \geq 1$. Then $l = 0$.

Proof. If $R = 0$, then $\limsup_{m\to\infty} d(x_m,y_n) = 0$, $\limsup_{m\to\infty} d(x_m,z_n) = 0$. It is not difficult to see that $l = 0$.

Otherwise, assume that $R > 0$ and $l \neq 0$. Fix $\varepsilon > 0$. There exists $N \geq 1$ such that

$$d(x_m,y_n) \leq R+\varepsilon, \quad d(x_m,z_n) \leq R+\varepsilon, \quad d(y_n,z_n) > \varepsilon$$

for any $m \geq N$, $n \geq N$. Using Theorem 2.13, we have

$$\psi_{y_n,z_n}(R+\varepsilon) \leq \frac{1}{2}d^p(x_m,y_n) + \frac{1}{2}d^p(x_m,z_n) - d^p\Big(x_m, \frac{1}{2}y_n \oplus \frac{1}{2}z_n\Big)$$

for every $m \geq N$, $p \geq 2$. Letting $m \to \infty$ we have $\psi_{y_n,z_n}(R+\varepsilon) = 0$ for all $n \geq N$ which contradicts $l \neq 0$. Therefore, $l = 0$. $\square$

In the case of convex modular spaces, uniform convexity in every direction was introduced by Mostafa Bachar and Osvaldo Méndez [4]. We have a similar notion in hyperbolic metric spaces.

Definition 2.16. *A hyperbolic metric space X is said to be uniformly convex in every direction (UCED for short) if and only if for any $y,z \in X$, $y \neq z$ and $R > 0$, there exists $\delta = \delta(y,z,R) > 0$ such that if $d(x,y) \leq R$, $d(x,z) \leq R$, then*

$$d(x, \frac{1}{2}y \oplus \frac{1}{2}z) \leq R(1-\delta).$$

From Lemma 2.14 we have the following proposition.

Proposition 2.17. *Let (X,d) be a hyperbolic metric space.*

(i) *If X is UUUC then X is UCED.*
(ii) *If X is UCED then X is also strictly convex (see* [15]*) i.e., whenever $x,y,z \in X$ with $x \neq y$ if $d(z,x) \leq R$ and $d(z,y) \leq R$ then $d(z, \frac{1}{2}x \oplus \frac{1}{2}y) < R$.*
iii) *Assume that X is uniformly convex defined in* [13]*, i.e., for any $r > 0$ and $\varepsilon \in (0,2]$, there exists $\delta \in (0,1]$ such that for all $a,x,y \in X$ with $d(x,a) \leq r$, $d(y,a) \leq r$ and $d(x,y) \geq r\varepsilon$, we have $d(a, \frac{1}{2}x \oplus \frac{1}{2}y) \leq (1-\delta)r$. Then X is UCED.*

The type function was introduced by Bin Dehaish and Khamsi [10]. Recently, there appeared many fixed point theorems using the type function in their proofs. First we recall two important properties of a hyperbolic metric space satisfying UC property that we apply to obtain the properties of the type function.

Theorem 2.18 ([21]). *Let (X,d) be a hyperbolic metric space. Assume that X is*

UC. Let C be a nonempty closed convex subset of X. Then the following properties hold.

(i) *Let $a \in X$ with $d(a, C) = \inf\{d(a, x) : x \in C\} < \infty$. Then there exists a unique $c \in C$ such that $d(a, c) = d(a, C)$.*

(ii) *Assume that $(C_n)_{n \geq 1}$ is a decreasing sequence of nonempty bounded closed convex subsets of X. Then $\bigcap_{n \geq 1} C_n \neq \emptyset$.*

By taking the same approach as Dehaish and Khamsi (see Lemma 3.1, [10]), we have the following counterpart in hyperbolic metric spaces with the UUUC property.

Theorem 2.19. *Let C be a nonempty closed convex subset of a hyperbolic metric space (X, d). Assume that X is UUUC. Let $\tau : C \to [0, \infty)$ be a type function, that is, there exists a bounded sequence $(x_n)_n \subset X$ such that*

$$\tau(x) = \limsup_{n \to \infty} d(x_n, x)$$

for any $x \in C$. Then τ is continuous. Since X is hyperbolic, τ is convex, that is the subset $\{x \in C : \tau(x) \leq r\}$ is convex for every $r \geq 0$. Moreover, there exists a unique minimum point $c \in C$ such that

$$\tau(c) = \inf\{\tau(x) : x \in C\}.$$

Proof. It is not difficult to prove the continuity and convexity of the function τ. We are going to show the existence of the minimum point of τ. Set $\tau_0 = \inf\{\tau(x) : x \in C\}$. Then for any $n \geq 1$,

$$C_n = \{x \in C : \tau \leq \tau_0 + \frac{1}{n}\}$$

is not empty, closed, convex subset of C. Obviously, $(C_n)_{n \geq 1}$ is a decreasing sequence. It follows from Theorem 2.18 that $C_\infty = \bigcap_{n \geq 1} C_n \neq \emptyset$. Clearly, $C_\infty = \{z \in C : \tau(z) = \tau_0\}$. We are going to prove that C_∞ consists of one point. Assume that $z_1, z_2 \in C$ with $d(z_1, z_2) > 0$. We have $\tau_0 > 0$. Take $\alpha \in (0, \tau_0)$. Then there exists $n_0 \geq 1$ such that for any $n \geq n_0$ we have

$$d(x_n, z_1) \leq \tau_0 + \alpha \quad \text{and} \quad d(x_n, z_2) \leq \tau_0 + \alpha.$$

Since X is UUUC and $d(z_1, z_2) \geq (\tau_0 + \alpha) d(z_1, z_2)/2\tau_0$, there exists $\delta_{x_n}(\tau_0 + \alpha, \frac{d(z_1, z_2)}{2\tau_0})$ and $\gamma = \inf\{\delta_x(r, \varepsilon) : x \in X, r \geq \tau_0, \varepsilon \geq d(z_1, z_2)/2\tau_0\}$ such that $\delta_{x_n}(\tau_0 + \alpha, \frac{d(z_1, z_2)}{2\tau_0}) \geq \gamma > 0$. Hence

$$d(x_n, \frac{1}{2} z_1 \oplus \frac{1}{2} z_2) \leq (\tau_0 + \alpha)\Big(1 - \delta_{x_n}(\tau_0 + \alpha, \frac{d(z_1, z_2)}{2\tau_0})\Big) \leq (\tau_0 + \alpha)(1 - \gamma).$$

Letting limsup as $n \to +\infty$, we get

$$\tau(\frac{1}{2} z_1 \oplus \frac{1}{2} z_2) \leq (\tau_0 + \alpha)(1 - \gamma),$$

and letting $\alpha \to 0$, we have

$$\tau(\frac{1}{2} z_1 \oplus \frac{1}{2} z_2) \leq \tau_0 (1 - \gamma).$$

Since C_∞ is convex, $\frac{1}{2}z_1 \oplus \frac{1}{2}z_2 \in C$. Hence

$$\tau_0 \leq \tau(\frac{1}{2}z_1 \oplus \frac{1}{2}z_2) \leq \tau_0(1-\gamma)$$

which is a contradiction. The proof is complete. □

Theorem 2.20. *Let C be a nonempty, closed, convex and bounded subset of a complete hyperbolic metric space (X, d). Assume that X is p-UUC and τ is a type function on C. Then any minimizing sequence of τ converges to $c \in C$ and there exists $M > 0$ such that*

$$\tau^p(c) + 2Md^p(x,c) \leq \tau^p(x)$$

for all $x \in C$.

Proof. Suppose that $(x_m)_m$ is a sequence in C such that for any $x \in C$,

$$\tau(x) = \limsup_{m\to\infty} d(x_m, x).$$

Let $\tau_0 = \inf\{\tau(x) : x \in C\}$. Then there exists a minimizing sequence $(y_n)_n$ of τ, that is, $\lim_{n\to\infty} \tau(y_n) = \tau_0$. Since C is bounded, there exists $r > 0$ such that $d(x, y) \leq r$ for every $x, y \in C$. We are going to prove that $(y_n)_n$ is a Cauchy sequence. In the contrary case, there exists $\varepsilon > 0$ and two subsequences $(y_{n_k})_k$, $(y_{n_l})_l$ of $(y_n)_n$ such that

$$d(y_{n_l}, y_{n_k}) \geq \varepsilon, \quad \text{for every } l > k \geq 1.$$

We have $d(x_m, y_{n_l}) \leq r$, $d(x_m, y_{n_k}) \leq r$, for every $m \geq 1$, $l > k \geq 1$. Hence for every $m \geq 1$,

$$\psi_{x_m}(r, \frac{\varepsilon}{r}) \leq \frac{1}{2}d^p(x_m, y_{n_l}) + \frac{1}{2}d^p(x_m, y_{n_k}) - d^p(x_m, \frac{1}{2}y_{n_l} \oplus \frac{1}{2}y_{n_k}),$$

where

$$\psi_{x_m}(r, \frac{\varepsilon}{r}) = \inf\Big\{\frac{1}{2}d^p(x_m, x) + \frac{1}{2}d^p(x_m, y) - d^p\Big(x_m, \frac{1}{2}x \oplus \frac{1}{2}y\Big)\Big\},$$

the infimum taken over all $x, y \in X$ such that $d(x_m, x) \leq r$, $d(x_m, y) \leq r$, $d(x, y) \geq r\varepsilon/r$. Since X is p-UUC, $\psi_{x_m}(r, \frac{\varepsilon}{r}) \geq c_X > 0$, where

$$c_X = \inf\Big\{\frac{\psi_a(r_1, \varepsilon_1)}{r_1^p \varepsilon_1^p} : a \in X, r_1 > 0, \varepsilon_1 > 0\Big\}.$$

Hence

$$C_X \leq \frac{1}{2}d^p(x_m, y_{n_l}) + \frac{1}{2}d^p(x_m, y_{n_k}) - d^p(x_m, \frac{1}{2}y_{n_l} \oplus \frac{1}{2}y_{n_k}) \tag{2.2}$$

for every $1 \leq k < l$, where $C_X = c_X\varepsilon^p > 0$. Taking lim sup as $m \to \infty$ of (2.2), we get

$$C_X \leq \frac{1}{2}\tau^p(y_{n_l}) + \frac{1}{2}\tau^p(y_{n_k}) - \tau^p\Big(\frac{1}{2}y_{n_l} \oplus \frac{1}{2}y_{n_k}\Big)$$

for every $1 \leq k < l$. Since C is convex, $\frac{1}{2}y_{n_l} \oplus \frac{1}{2}y_{n_k} \in C$. Hence

$$C_X \leq \frac{1}{2}\tau^p(y_{n_l}) + \frac{1}{2}\tau^p(y_{n_k}) - \tau_0^p \tag{2.3}$$

for every $1 \leq k < l$. Letting $k \to \infty$ in (2.3), we have $C_X \leq 0$. It is a contraction. Thus $(y_n)_n$ is a Cauchy sequence. Hence there exists $c \in C$ such that $\lim_{n\to\infty} y_n = c$. Clearly, $\tau_0 = \tau(c)$ since τ is a continuous function.

Furthermore, take an arbitrary $x \in C$, $x \neq c$. Since X is p-UUC, we have

$$\begin{aligned} d^p\Big(x_m, \frac{1}{2}x \oplus \frac{1}{2}c\Big) &\leq \frac{1}{2}d^p(x_m, x) + \frac{1}{2}d^p(x_m, c) - \psi_{x_m}\Big(r, \frac{d(x,c)}{r}\Big) \\ &\leq \frac{1}{2}d^p(x_m, x) + \frac{1}{2}d^p(x_m, c) - c_X d^p(x, c) \end{aligned}$$

for any $m \geq 1$. Letting $m \to \infty$, we get

$$\limsup_{m\to\infty} d^p\Big(x_m, \frac{1}{2}x \oplus \frac{1}{2}c\Big) \leq \frac{1}{2}\limsup_{m\to\infty} d^p(x_m, x) + \frac{1}{2}\limsup_{m\to\infty} d^p(x_m, c) - c_X d^p(x, c).$$

It implies that

$$\limsup_{m\to\infty} d^p(x_m, c) \leq \frac{1}{2}\limsup_{m\to\infty} d^p(x_m, x) + \frac{1}{2}\limsup_{m\to\infty} d^p(x_m, c) - c_X d^p(x, c).$$

Hence

$$\limsup_{m\to\infty} d^p(x_m, c) + 2c_X d^p(x, c) \leq \limsup_{m\to\infty} d^p(x_m, x).$$

It follows that $\tau^p(c) + 2c_X d^p(x, c) \leq \tau^p(x)$. This inequality is also clearly true when $x = c$. Therefore, the proof is complete. □

The minimizer of $\tau(x)$ is called the asymptotic center. With this definition we can define the weak sequential convergence as follows.

Definition 2.21 (Weak sequential convergence). *We say that a sequence $(x_n)_n$ converges weakly sequentially to a point c if c is the asymptotic center for each subsequence of $(x_n)_n$. We denote this by $x_n \overset{w}{\to} x$.*

Proposition 2.22. *Let X be a hyperbolic metric space. Assume that X is UUUC. Then each bounded sequence $(x_n)_n$ has a subsequence $(x_{n_k})_k$ such that $x_{n_k} \overset{w}{\to} x$.*

Proof. The proof can be found in ([5], Proposition 3.1.2) in the case of Hadamard spaces. Since it is rather technical we leave it out. □

Assume that $(x_n)_n$ is a bounded sequence in a hyperbolic metric space (X, d), K is a nonempty subset of X. We denote

$$r(K, (x_n)) = \inf\Big\{\limsup_{n\to\infty} d(x_n, x) : x \in K\Big\}.$$

In what follows, we show an analogue of Lemma 15.2 [18] for hyperbolic metric spaces. It will be used in the proof of Theorem 3.8.

Lemma 2.23. *Let X be a hyperbolic metric space, K a nonempty subset of X and $(x_n)_n$ a bounded sequence in X. Then there exists a subsequence $(x_{n_k})_k$ of $(x_n)_n$ such that for every subsequence $(x_{n_l})_l$ of $(x_{n_k})_k$,*

$$r(K, (x_{n_k})) = r(K, (x_{n_l})).$$

Proof. If $(y_n)_n$ is a subsequence of $(x_n)_n$, we will use the notation $(y_n) \prec (x_n)$. Denote

$$r_0 := \inf\Big\{r(K,(y_n)) : (y_n) \prec (x_n)\Big\},$$

Then we can choose $(y_n^1) \prec (x_n)$ such that

$$r(K,(y_n^1)) < r_0 + 1.$$

Denote

$$r_1 := \inf\Big\{r(K,(z_n)) : (z_n) \prec (y_n^1)\Big\}.$$

Now select $(y_n^2) \prec (y_n^1)$ such that

$$r(K,(y_n^2)) < r_1 + \frac{1}{2}.$$

Continuing this process, we can construct sequences (y_n^i) and

$$r_i := \inf\Big\{r(K,(z_n)) : (z_n) \prec (y_n^i)\Big\}$$

such that $(y_n^i) \prec (y_n^{i-1})$ and

$$r(K,(y_n^{i+1})) < r_i + \frac{1}{i+1}$$

for any $i \geq 1$. Since $(r_i)_i$ is nondecreasing and bounded from above by $r(K,(x_n))$, it has a limit, say r. Hence $\lim\limits_{i\to\infty} r(K,(y_n^{i+1})) = r$.

Now take the diagonal sequence (y_n^n) and denote $\overline{r} = r(K,(y_n^n))$. Then $(y_n^n) \prec (y_n^i)$, and hence $\overline{r} \geq r_i$. On the other hand, we have $(y_n^n) \prec (y_n^{i+1})$, which gives $\overline{r} \leq r_i + \frac{1}{i+1}$. Thus $\overline{r} = r$.

Since any subsequence (z_n) of (y_n^n) also satisfies (for the same reasons) the inequalities

$$r(K,(u_n) \geq r_i \quad \text{and} \quad r(K,(u_n)) \leq r_i + \frac{1}{i+1}$$

for any $i \geq 1$, one gets $r(K,(u_n)) = r$. We conclude that $(y_n^n)_n$ is the desired subsequence. □

Proposition 2.24. *Let (X,d) be a uniformly convex hyperbolic metric space, C be a nonempty closed convex subset of X. Assume that $T : C \to X$ is a nonexpansive mapping. If $(x_n)_n$ is a sequence in C such that $x_n \stackrel{w}{\to} x$ and $\lim\limits_{n\to\infty} d(x_n, Tx_n) = 0$, then $x \in C$ and $Tx = x$.*

Proof. Notice that

$$\begin{aligned}\tau(Tx) = \limsup_{n\to\infty} d(x_n, Tx) &\leq \limsup_{n\to\infty} d(x_n, Tx_n) + \limsup_{n\to\infty} d(Tx_n, Tx)\\ &\leq \limsup_{n\to\infty} d(x_n, Tx_n) + \limsup_{n\to\infty} d(x_n, x)\end{aligned}$$

and hence $\tau(Tx) \leq \tau(x)$. Since x is the unique minimizer of τ, we have $Tx = x$. □

3. Monotone multivalued nonexpansive mappings

Let $(X, \|.\|)$ be a Banach space, C be a nonempty weakly compact convex subset of X and $P_{cp}(C)$ be the family of nonempty compact subsets of C equipped with the Hausdorff metric $H(.,.)$. A multivalued mapping $T : C \to P_{cp}(C)$ is said to be nonexpansive if for each $x, y \in C$, $H(T(x), T(y)) \leq \|x - y\|$. In 1968, Markin [28] established a fixed point theorem for such mappings in Hilbert spaces. Later, Browder [8] proved a similar result for spaces with weakly continuous duality mapping, and Lami Dozo [12] proved it for spaces satisfying Opial's condition. Assad and Kirk [3] then generalized Lami Dozo's result. In 1974, Lim [27] established a fixed point theorem by considering a closed convex subset of a uniformly convex Banach space. It is natural to extend these fixed point results to the case of monotone nonexpansive mappings. In this section we are going to prove the existence of fixed points of monotone G-nonexpansive multivalued mappings in hyperbolic metric spaces.

We start this section with recalling some basic notions in graph theory (see [6, 11]).

Definition 3.1. *A graph G is a pair $(V(G), E(G))$, where the elements of a nonempty set $V(G)$ are called vertices of G, and $E(G)$ is a set of paired vertices called edges. If a direction is imposed on each edge, we call the graph a directed graph or a digraph.*

Definition 3.2. *Assume that $G = (V(G), E(G))$ is a digraph.*

(i) *G is reflexive if for each $x \in V(G)$, $(x, x) \in E(G)$.*
(ii) *G is transitive if for every $x, y, z \in V(G)$ with $(x, y), (y, z) \in E(G)$, we have $(x, z) \in E(G)$.*
(iii) *We call (V', E') a subgraph of G if $V' \subseteq V(G)$, $E' \subseteq E(G)$, and $x, y \in V'$ whenever $(x, y) \in E'$.*
(iv) *For $a, b \in V(G)$, we define G-intervals as follows:*

$$\begin{aligned} [a, \rightarrow) &= \{x \in V(G) : (a, x) \in E(G)\}, \\ (\leftarrow, b] &= \{x \in V(G) : (x, b) \in E(G)\}, \\ [a, b] &= [a, \rightarrow) \cap (\leftarrow, b]. \end{aligned}$$

Definition 3.3. ([20]) *Let (X, d) be a hyperbolic metric space. A graph G on X is said to be convex if and only if for any $x, y, z, t \in X$ and $\alpha \in [0, 1]$, we have*

$$(x, z), (y, t) \in E(G) \Rightarrow \Big(\alpha x \oplus (1 - \alpha) y, \alpha z \oplus (1 - \alpha) t\Big) \in E(G)$$

We note that this property implies any G-interval is convex.

Definition 3.4. ([1]) *Let (X, d) be a metric space with a digraph G and C a nonempty subset of X. We say that a mapping $T : C \to C$ is G-monotone if*

$$\forall (x, y) \in C \quad ((x, y) \in E(G) \quad \Rightarrow \quad (Tx, Ty) \in E(G)).$$

Definition 3.5. ([1]) *Let (X, d) be a metric space with a digraph G, C be a nonempty subset of X, and $P(C)$ be a family of nonempty subsets of C. A multivalued mapping*

$T : C \to P(C)$ is said to be monotone G-nonexpansive if for any $x, y \in C$ with $(x, y) \in E(G)$ and any $u \in Tx$, there exists $v \in Ty$ such that

$$(u, v) \in E(G), \quad \textit{and} \quad d(u, v) \leq d(x, y).$$

A point $x \in C$ is called a fixed point of T if and only if $x \in Tx$. The set of all fixed points of a mapping T is denoted by $Fix(T)$.

Definition 3.6. *Let (X, d) be a metric space and $T : X \to P(X)$ be a multivalued mapping. A sequence $(x_n)_n$ is called an approximate fixed point sequence if for any $n \geq 1$, there exists $y_n \in Tx_n$ such that $\lim_{n\to\infty} d(x_n, y_n) = 0$.*

Let C be a nonempty closed convex bounded subset of a hyperbolic metric space X with a reflexive, transitive digraph G and $T : C \to P(C)$ be a monotone G-nonexpansive multivalued mapping. Assume that all G-intervals are closed and convex, and $(x_0, y_0) \in E(G)$ for some $y_0 \in T(x_0)$. Put $x_1 = \frac{1}{2}x_0 \oplus \frac{1}{2}y_0$. Since G-intervals are convex, we have $(x_0, x_1), (x_1, y_0) \in E(G)$. Since T is a monotone G-nonexpansive multivalued mapping, there is $y_1 \in T(x_1)$ such that

$$(y_0, y_1) \in E(G) \quad \text{and} \quad d(y_1, y_0) \leq d(x_1, x_0).$$

Continuing in this manner, we get a sequence $(x_n)_n, (y_n)_n$ in C defined by

$$x_{n+1} = \frac{1}{2}x_n \oplus \frac{1}{2}y_n, \quad y_n \in T(x_n) \quad \text{for all } n \geq 0. \tag{3.1}$$

By induction, we have

$$(x_n, x_{n+1}), (x_{n+1}, y_n), (y_n, y_{n+1}) \in E(G)$$

and

$$d(y_{n+1}, y_n) \leq d(x_{n+1}, x_n) = \frac{1}{2}d(x_n, y_n)$$

for any $n \geq 0$. We have

$$\begin{aligned} d(x_{n+1}, y_{n+1}) &\leq d(x_{n+1}, y_n) + d(y_{n+1}, y_n) \\ &\leq \frac{1}{2}d(x_n, y_n) + d(x_{n+1}, x_n) = \frac{1}{2}d(x_n, y_n) + \frac{1}{2}d(x_n, y_n) = d(x_n, y_n). \end{aligned}$$

Hence $d(x_{n+1}, y_{n+1}) \leq d(x_n, y_n)$ for every $n \geq 0$.

The following important proposition is a consequence of the result of Goebel and Kirk [17].

Proposition 3.7. *Let (X, d) be a hyperbolic metric space. Let $(x_n)_n$ and $(y_n)_n$ be two sequences in (X, d) such that*

$$x_{n+1} = \frac{1}{2}x_n \oplus \frac{1}{2}y_n$$

for any $n \in \mathbb{N}$. Suppose that

$$d(y_n, y_{n+1}) \leq d(x_n, x_{n+1}), \quad n \in \mathbb{N}.$$

Then we have

$$(1 + \frac{n}{2})d(x_i, y_i) \leq d(x_i, y_{i+n}) + 2^n \Big(d(x_i, y_i) - d(x_{i+n}, y_{i+n})\Big), \tag{3.2}$$

for every $i, n \in \mathbb{N}$. In particular, $\lim_{n\to\infty} d(x_n, y_n) = 0$.

Next we give the main result of this section. This theorem presents a "monotonic"counterpart of the Lim's theorem [27].

Theorem 3.8. *Let (X, d) be a complete hyperbolic metric space with a reflexive, transitive digraph G. Assume that X is p-UUC and G-intervals are closed and convex. Let C be a nonempty, closed, convex and bounded subset of X. Let $T : C \to P_{cp}(C)$ be a monotone G-nonexpansive multivalued mapping. If there exists $x_0 \in C$ such that $(x_0, y_0) \in E(G)$ for some $y_0 \in Tx_0$ then $Fix(T) \neq \emptyset$.*

Proof. The argument above and Proposition 3.7 yield that there are two sequences $(x_n)_n, (y_n)_n$ in C such that

$$x_{n+1} = \frac{1}{2}x_n \oplus \frac{1}{2}y_n, \quad y_n \in T(x_n),$$

$d(x_{n+1}, y_{n+1}) \leq d(x_n, y_n)$ for all $n \geq 0$ and $\lim_{n\to\infty} d(x_n, y_n) = 0$.

By the properties of (x_n) and transitivity of G, $\{[x_n, \to), n \geq 0\}$ is a nonincreasing sequence of nonempty convex and closed subsets of X. It follows from Theorem 2.18 that

$$C_\infty = \bigcap_{n\geq 0} [x_n, \to) \cap C = \bigcap_{n\geq 0} \{x \in C : (x_n, x) \in E(G)\} \neq \emptyset.$$

Now Lemma 2.23 implies the existence of a subsequence $(x_{n_k})_k$ of $(x_n)_n$ such that for each subsequence $(x_{n_l})_l$ of $(x_{n_k})_k$ we have

$$r(C_\infty, (x_{n_l})) = r(C_\infty, (x_{n_k})).$$

From Theorem 2.19 there exists a unique $c \in C_\infty$ such that

$$\limsup_{k\to\infty} d(x_{n_k}, c) = \inf\{\limsup_{k\to\infty} d(x_{n_k}, x) : x \in C_\infty\} = r(C_\infty, (x_{n_k})).$$

Thus we have $(x_{n_k}, c) \in E(G)$ for any $k \geq 0$. Since T is a monotone G-nonexpansive multivalued mapping, there exists $c_{n_k} \in T(c)$ such that

$$(y_{n_k}, c_{n_k}) \in E(G) \quad \text{and} \quad d(y_{n_k}, c_{n_k}) \leq d(x_{n_k}, c)$$

for any $k \geq 0$. Since Tc is compact, there exists a subsequence $(c_{n_l})_l$ of $(c_{n_k})_k$ such that $\lim_{l\to\infty} c_{n_l} = c' \in T(c)$. First we prove that $c' \in C_\infty$. Indeed, it is not difficult to see that

$$\emptyset \neq \bigcap_{k\geq 0} [y_{n_k}, \to) \subseteq \bigcap_{k\geq 0} [x_{n_k}, \to) = \bigcap_{n\geq 0} [x_n, \to) = C_\infty.$$

For each $m \geq 0$ and $n_l \geq m$, we get

$$(y_m, y_{m+1}), (y_{n_l}, c_{n_l}) \in E(G).$$

It implies that $(c_{n_l})_{n_l \geq m}$ is in $[y_m, \to)$. Hence $c' \in [y_m, \to)$ for every $m \geq 0$. Thus $c' \in \cap_{m\geq 0}[y_m, \to)$ and therefore,

$$c' \in \bigcap_{m\geq 0} [x_m, \to).$$

Now, we are going to prove that $c = c'$. Assume that $\varepsilon = d(c, c') > 0$. We have $d(x_{n_l}, c) \leq r$, $d(x_{n_l}, c') \leq r$, where $(x_{n_l})_l$ is a subsequence of $(x_n)_n$, $r = \text{diam}(C)$ and hence

$$d^p(x_{n_l}, \frac{1}{2}c \oplus \frac{1}{2}c') \leq \frac{1}{2}d^p(x_{n_l}, c) + \frac{1}{2}d^p(x_{n_l}, c') - \psi_{x_{n_l}}(r, \varepsilon/r).$$

Since X is p-UUC, we have

$$d^p(x_{n_l}, \frac{1}{2}c \oplus \frac{1}{2}c') \leq \frac{1}{2}d^p(x_{n_l}, c) + \frac{1}{2}d^p(x_{n_l}, c') - c_X\varepsilon^p$$

for every $k \geq 1$, where

$$c_X = \inf\Big\{\frac{\psi_a(r_1, \varepsilon_1)}{r_1^p \varepsilon_1^p} : a \in X, r_1 > 0, \varepsilon_1 > 0\Big\}.$$

Taking limsup as $l \to \infty$, we have

$$\begin{aligned}
\limsup_{l\to\infty} d^p(x_{n_l}, \frac{1}{2}c \oplus \frac{1}{2}c') &\leq \frac{1}{2}\limsup_{l\to\infty} d^p(x_{n_l}, c) + \frac{1}{2}\limsup_{l\to\infty} d^p(x_{n_l}, c') - c_X\varepsilon^p \\
&\leq \frac{1}{2}\limsup_{l\to\infty} d^p(x_{n_l}, c) + \frac{1}{2}\limsup_{l\to\infty} \Big[d(x_{n_l}, y_{n_l}) + d(y_{n_l}, c_{n_l}) + d(c_{n_l}, c')\Big]^p - c_X\varepsilon^p \\
&\leq \frac{1}{2}\limsup_{l\to\infty} d^p(x_{n_l}, c) + \frac{1}{2}\limsup_{l\to\infty} d^p(y_{n_l}, c_{n_l}) - c_X\varepsilon^p \\
&\leq \limsup_{l\to\infty} d^p(x_{n_l}, c) - c_X\varepsilon^p \\
&\leq \limsup_{k\to\infty} d^p(x_{n_k}, c) - c_X\varepsilon^p = r(C_\infty, (x_{n_k})) - c_X\varepsilon^p.
\end{aligned}$$

On the other hand, since C_∞ is nonempty bounded, closed and convex, we have

$$\limsup_{l\to\infty} d^p(x_{n_l}, \frac{1}{2}c \oplus \frac{1}{2}c') \geq r(C_\infty, (x_{n_l})).$$

Thus we have

$$r(C_\infty, (x_{n_k})) = r(C_\infty, (x_{n_l})) \leq r(C_\infty, (x_{n_k})) - c_X\varepsilon^p.$$

This is a contradiction since $c_X\varepsilon^p > 0$. Thus $c = c'$. Therefore, $c \in T(c)$, i.e, c is a fixed point of T. □

Our first corollary is an application of Theorem 3.8 to the case of a partial order $\leq := E(G)$. We recall that on $(X, \leq)$, order intervals are sets of the forms $[a, \to) = \{x \in X : a \leq x\}$, $(\leftarrow, b] = \{x \in X : x \leq b\}$ and $[a, b] = [a, \to) \cap (\leftarrow, b]$ for some $a, b \in X$.

Corollary 3.9. *Let $(X, d, \leq)$ be a complete hyperbolic metric space with a partial order $\leq$. Assume (X, d) is p-UUC and all order intervals are closed and convex. Let C be a nonempty convex closed bounded subset of X. Let $T : C \to P_{cp}(C)$ be a monotone nonexpansive multivalued mapping. Assume that there exists $x_0 \in C$ such that $x_0 \leq y_0$ for some $y_0 \in Tx_0$. Then there exists $c \in X$ such that $T(c) = c$.*

In the case of single-valued mappings, we recall that a mapping $T : X \to X$ is said to be monotone (or increasing) if $T(x) \leq T(y)$ whenever $x, y \in X$ such that $x \leq y$, and T is monotone nonexpansive if T is monotone and for every $x, y \in X$ such that $x \leq y$, $d(Tx, Ty) \leq d(x, y)$. Thus we have the following corollary.

Corollary 3.10. *Let $(X, d, \leq)$ be a complete hyperbolic metric space with a partial order $\leq$. Assume that (X, d) is p-UUC and all order intervals are closed and convex. Let C be a nonempty convex closed bounded subset of X. Let $T : C \to C$ be a monotone nonexpansive mapping. If there exists $x_0 \in C$ such that $x_0 \leq Tx_0$ then T has a fixed point.*

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