Fixed Point Theory, 26(2025), No. 1, 261-274
DOI: 10.24193/fpt-ro.2025.1.17
http://www.math.ubbcluj.ro/~nodeacj/sfptcj.html

RELATIONAL CONTRACTION PRINCIPLE FOR MAPPINGS ON MENGER PM SPACES WITH APPLICATIONS

GOPI PRASAD

Department of Mathematics,
Government Degree College Rudraprayag,
246171 Uttarakhand, India.
E-mail: gopiprasad127@gmail.com

Abstract. In this paper, we prove relational analogue of the Banach contraction principle in the settings of Menger probabilistic metric spaces under a Hadžić-type t-norm. In view of such investigations we obtain a Kelisky-Rivlin type results for a class of Bernstein type special operators introduced by Deo et. al. [Appl. Math. Comput. 201, (2008), 604-612] on the spaces of continuous functions $C\big([0, \frac{k}{k+1}]\big)$. Thus, such findings enrich, modify and generalize various prominent recent fixed point results of the existing literature.
Key Words and Phrases: Fixed point, Bernstein operator, contraction mapping.
2020 Mathematics Subject Classification: 47H10, 54H25.

1. Introduction

An analogue of Banach contraction principle (Bcp) [4] in the settings of partially ordered metric spaces was presented by Turinici [26, 27] which was later revisited by several authors such as Ran and Reurings [20], Nieto and López [15], Samet and Turinici [23], Alam and Imdad [2], Alam et al. [1], and the process of this exploration is still on. Meanwhile, Jachymski [9] established an interesting metrical fixed point result by incorporating the notion of graphical contraction mapping besides presenting a variant of the Kelisky and Rivlin theorem [11] concerning Bernstein operators on the space $C[0, 1]$, and there exist detailed generalization on this theme, see for instance ([3], [17, 18, 19]).

Fixed points of lattice structures and ordered sets was investigated by several authors such as Zermelo (1908), Knaster (1928), Zorn (1935), Tarski (1955) and many others. Fixed point theorems established for ordered sets are equally important as fixed point theorems proved on metric structures discussed above. One of the main objective, adopted by many researchers in the last decade, is that to formulate a connection between these two theories. In order to study such connection, we precisely refer Petrusel and Rus [16], wherein the respective authors highlighted some interesting remarks, open questions, discussion and several conclusions for the same. They also presented some interesting applications to integral and differential equations. This research work constitutes a broader framework of the classical Bcp, leading to new fixed point results.

Among all these generalizations, we must revisit Alam and Imdad [2] and Jachymski [9] in which the authors made use of relational analogs of metrical notions of continuity, completeness and some other concepts related to metric space to acquire some interesting generalizations and conclusions of the Banach contraction principle. Noticeably, Alam et al. [1] presented a refinements of the relation-theoretic contraction principle besides highlighting the importance of the notion of d-self-closedness utilized by Alam and Imdad [2] to such settings. Moreover, the respective authors vindicated that the relation theoretic approach still remain a genuine improvement over graphical approach.

On another point of note, recently Kamran et al. [10] extended the results of Jachymski [9] to the settings of Menger probabilistic metric spaces. They introduced the class of probabilistic G-contractions and studied the existence of fixed points for such mappings. Sadeghi and Vaezpour [22] obtained generalized probabilistic analogue of the Bcp on complete Menger PM spaces and partially ordered Menger PM spaces as well. They also presented some applications to determine the solution of Volterra-type integral equations to such spaces. Most recently, some fixed point results for probabilistic α-minimum Ciric type contraction are presented by Bhandari et al. [5].

In order to understand many diversified physical problems modeled on functional analysis, the above theoretical study has been very useful. In recent times, there has been keen interest in adopting the generalizations of Bcp for the same. In light of the above research direction, the primary goal of this research work, is to investigate a new fixed point theorem for contraction mappings defined on a Menger PM spaces equipped with an arbitrary binary relation. Moreover, we furnish some non-trivial examples and establish a Kelisky-Rivlin type result for a class of Bernstein type special operators on the spaces of continuous functions, in light of the obtained results. Thus, the goal is to substantiate the usability of such exploration by establishing some new core theoretical results with some interesting applications.

2. Preliminaries

The notion of statistical metric spaces was introduced by Karl Menger in 1942. Later on the new theory of fundamental probabilistic structures was developed by many authors. In this section, firstly we recall some relevant important background study and start with some core concepts from Menger probabilistic metric spaces. For detailed study of such spaces, we refer [8, 25, 24] and references therein. Throughout this presentation, $\mathbb{N}$ indicates the set of natural numbers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ whereas, a non-empty binary relation is symbolized by $\mathcal{R}$.

A mapping $\mathcal{F} : \mathbb{R} \to [0,1]$ is called a distribution function if it satisfies the following conditions:
(d_1) $\mathcal{F}$ is nondecreasing;
(d_2) $\mathcal{F}$ is left continuous;
(d_3) $\inf_{t\in\mathbb{R}} \mathcal{F}(t) = 0$ and $\sup_{t\in\mathbb{R}} \mathcal{F}(t) = 1$.
If, in addition, we have
(d_4) $\mathcal{F}(0) = 0$, then $\mathcal{F}$ is called a distance distribution function.

Assume that $\mathcal{D}^+$ is a set defined by

$$\mathcal{D}^+ = \Big\{\mathcal{F} : \mathbb{R} \to [0,1] : \mathcal{F} \text{ is distance distribution function}, \lim_{t\to+\infty} \mathcal{F}(t) = 1\Big\}.$$

The element $\delta_0 \in \mathcal{D}^+$ defined by

$$\delta_0(t) = \begin{cases} 0 & \text{if } t \leq 0, \\ 1 & \text{if } t > 0, \end{cases}$$

is the Dirac distribution function.
Definition 2.1 [8]. A mapping $\mathcal{T} : [0,1] \times [0,1] \to [0,1]$ is said to be a triangular norm (briefly t-norm) if for every $u, v, w \in [0,1]$, we have
(t_1) $\mathcal{T}(u,v) = \mathcal{T}(v,u)$;
(t_2) $\mathcal{T}(u, \mathcal{T}(v,u)) = \mathcal{T}(\mathcal{T}(u,v), w)$;
(t_3) $\mathcal{T}(u,v) \leq \mathcal{T}(u,w) if v \leq w$;
(t_4) $\mathcal{T}(u,1) = u$.

The commutativity (t_1), the monotonicity (t_3), and the boundary condition (t_4) imply that for each t-norm $\mathcal{T}$ and for each $u \in [0,1]$, we have the following boundary conditions:
$\mathcal{T}(u,1) = \mathcal{T}(1,u) = u$ and $\mathcal{T}(u,0) = \mathcal{T}(0,u) = 0$.
Typical examples of t-norms are $\mathcal{T}_M(u,v) = \min\{a,b\}$ and $\mathcal{T}_P(u,v) = uv$.

Definition 2.2 [8]. A t-norm $\mathcal{T}$ is said to be of $\mathcal{H}$-type if the family of functions $\{\mathcal{T}^k\}_{k\in\mathbb{N}}$ is equicontinuous at $t = 1$, where $\mathcal{T}^k : [0,1] \to [0,1]$ is recursively defined by

$$\mathcal{T}^1(t) = \mathcal{T}(t,t), \qquad \mathcal{T}^{k+1}(t) = \mathcal{T}\big(T^k(t), t\big); \quad t \in [0,1], k = 1, 2, \ldots.$$

A trivial example of a t-norm of $\mathcal{H}$-type is $\mathcal{T}_M = \min$, but there exist t-norms of $\mathcal{H}$-type with $\mathcal{T} \neq \mathcal{T}_M$.

Definition 2.3 [8]. A Menger probabilistic metric space or Menger PM-space is a triple $(\mathcal{X}, \mathcal{F}, \mathcal{T})$, where $\mathcal{X}$ is a nonempty set, $F : \mathcal{X} \times \mathcal{X} \to \mathcal{D}^+$, and $\mathcal{T} : [0,1] \times [0,1] \to [0,1]$ is a t-norm such that for every $u, v, w \in \mathcal{X}$, we have
(M_1) $\mathcal{F}(u,v) = \delta_0 \Leftrightarrow u = v$;
(M_2) $\mathcal{F}(u,v) = \mathcal{F}(v,u)$;
(M_3) $\mathcal{F}(u,w)(t+s) \geq \mathcal{T}(\mathcal{F}(u,v)(t), \mathcal{F}(v,w)(s))$ for all $t, s \geq 0$.

Let $(\mathcal{X}, \mathcal{F}, \mathcal{T})$ be a Menger PM-space. For $\varepsilon > 0$ and $\delta \in (0,1]$, the (ε,δ)-neighborhood of $u \in \mathcal{X}$ is denoted by $N_u(\varepsilon,\delta)$ and is defined by

$$N_u(\varepsilon,\delta) = \{y \in \mathcal{X} : \mathcal{F}(u,v)(\varepsilon) > 1-\delta\}.$$

Furthermore, if $\sup_{0<a<1} \mathcal{T}(a,a) = 1$, then the family of neighborhoods

$$\{N_u(\varepsilon,\delta) : u \in \mathcal{X}, \varepsilon > 0, \delta \in (0,1]\}$$

determines a Hausdorff topology for $\mathcal{X}$.

Definition 2.4 [8]. Assume that $(\mathcal{X}, \mathcal{F}, \mathcal{T})$ is a Menger PM-space.
(a) A sequence $\{u_n\} \subset \mathcal{X}$ converges to an element $u \in \mathcal{X}$ if for every $\varepsilon > 0$ and $\delta \in (0,1]$, there exists $k_0 \in \mathbb{N}$ such that $u_k \in N_u(\varepsilon,\delta)$ for every $k \geq k_0$.
(b) A sequence $\{u_k\} \subset \mathcal{X}$ is a Cauchy sequence if for every $\varepsilon > 0$ and $\delta \in (0,1]$, there exists $k_0 \in \mathbb{N}$ such that $\mathcal{F}(u_k,u_m)(\varepsilon) > 1-\lambda$, whenever $k,m \geq k_0$.
(c) A Menger PM-space is complete if every Cauchy sequence in $\mathcal{X}$ converges to a point in $\mathcal{X}$.
(d) A subset $\mathcal{A}$ of $\mathcal{X}$ is closed if every convergent sequence in $\mathcal{A}$ converges to an element of $\mathcal{A}$.

Some relation theoretic metrical notions:

Definition 2.5 [13, 2]. Assume that $\mathcal{X}$ is a non-empty set and $\mathcal{R}$ is a subset of $\mathcal{X} \times \mathcal{X}$. Then
(a) $\mathcal{R}$ is a binary relation and u is $\mathcal{R}$-related to v, that is, the set of ordered pair $(u,v) \in \mathcal{R}$,
(b) u and v are comparable, if either $(u,v) \in \mathcal{R}$ or $(v,u) \in \mathcal{R}$, and symbolized as $[u,v] \in \mathcal{R}$,
(c) $(u,v) \in \mathcal{R}^s$, if and only if $[u,v] \in \mathcal{R}$ where $\mathcal{R}^s$ denotes the symmetric closure of $\mathcal{R}$, that is, $\mathcal{R}^s = \mathcal{R} \cup \mathcal{R}^{-1}$.

Definition 2.6 [2]. Assume that $\mathcal{X}$ is a non-empty set and $\mathcal{R}$ is a binary relation on $\mathcal{X}$. Let f be self-mappings on $\mathcal{X}$. Then for $u,v \in \mathcal{X}$, $\mathcal{R}$ is f-closed if

$$(u,v) \in \mathcal{R} \Rightarrow (fu,fv) \in \mathcal{R}.$$

Definition 2.7 [2]. Assume that $\mathcal{X}$ is a non-empty set and $\mathcal{R}$ is a binary relation on $\mathcal{X}$. A sequence $\{u_k\} \subset \mathcal{X}$ is $\mathcal{R}$-preserving if $(u_k,u_{k+1}) \in \mathcal{R}$, $k \in \mathbb{N}_0$.

Definition 2.8. Assume that $(\mathcal{X}, \mathcal{F}, \mathcal{T})$ is a Menger PM-space equipped with a binary relation $\mathcal{R}$ on $\mathcal{X}$. If every $\mathcal{R}$-preserving Cauchy sequence converges to a point in $\mathcal{X}$. Then $(\mathcal{X}, \mathcal{F}, \mathcal{T})$ is an $\mathcal{R}$-complete Menger PM-space.

Remark 2.9. Every $\mathcal{R}$-complete Menger PM-space is complete Menger PM-space and in respect to the global relation these notions are the same.

The following notion coined by Turinici [27] is an extension of d-self-closedness of a partial order relation "$\preceq$".

Definition 2.10 [2]. Assume that $(\mathcal{X}, d)$ is a metric space and $\mathcal{R}$ is a binary relation on a non-empty set $\mathcal{X}$. Then $\mathcal{R}$ is d-self-closed if for any $\mathcal{R}$-preserving sequence $\{u_k\}$ such that $u_k \xrightarrow{d} u$, there exists a subsequence $\{u_{k_r}\}$ of $\{u_k\}$ with $[u_{k_r}, u] \in \mathcal{R}$ for all $r \in \mathbb{N}_0$.

Inspired by the above definition by Alam and Imdad [2], we define analogue of d-self-closedness in Menger PM-space.

Definition 2.11. Assume that $(\mathcal{X}, \mathcal{F}, \mathcal{T})$ is a Menger PM-space and $\mathcal{R}$ is a binary relation on a non-empty set $\mathcal{X}$. Then $\mathcal{R}$ is $d_{\mathcal{F}}$-self-closed if for any $\{u_k\} \subset \mathcal{X}$ is a sequence in $\mathcal{T}_N(f, \mathcal{R}, u_0)$, so that $u_k \xrightarrow{d} u$, there exists a subsequence $\{u_{k_r}\}$ of $\{u_k\}$ with $[u_{k_r}, u] \in \mathcal{R},\ r \in \mathbb{N}_0$.

Definition 2.12. [2]. Assume that $\mathcal{X}$ is a non-empty set and $\mathcal{R}$ is a binary relation on $\mathcal{X}$. A subset E of $\mathcal{X}$ is $\mathcal{R}$-connected if there exists a path in $\mathcal{R}$ from u to v for every $u, v \in E$.

Definition 2.13. [12]. Assume that $\mathcal{X}$ is a non-empty set and $\mathcal{R}$ be a binary relation on $\mathcal{X}$. For $u, v \in \mathcal{X}$, a path of length r ($r \in \mathbb{N}$), in $\mathcal{R}$ from u to v is a finite sequence $\{w_0, w_1, w_2, ..., w_r\} \subset \mathcal{X}$ satisfying the subsequent assumptions:
(a) $w_0 = u$ and $w_r = v$,
(b) $(w_i, w_{i+1}) \in \mathcal{R}$ for each i $(0 \leq i \leq r-1)$.
Noticeably, a path of length r has $r+1$ elements of $\mathcal{X}$, though they are not necessarily distinct. For some $u \in \mathcal{X}$, denote by $\mathcal{P}(u; r)$ the set of all $v \in \mathcal{X}$ such that there exists a path of length r from u to v, that is,

$$\mathcal{P}(u; r) = \{v \in \mathcal{X} : \text{there exists a path of length r from u to v.}\}.$$

3. Fixed points for $f_{\mathcal{R}}$-contraction mappings

In this part, at first, we establish fixed point results for $f_{\mathcal{R}}$-contraction mappings in a Menger PM-spaces. Secondly, we furnish some suitable illustrative examples and important remarks.

Definition 3.1. Assume that $(\mathcal{X}, \mathcal{F}, \mathcal{T})$ is a Menger PM-space and $f : \mathcal{X} \to \mathcal{X}$, then f is an $f_{\mathcal{R}}$-contraction if there exists $\tau \in (0, 1)$ such that

$$(u, v) \in \mathcal{R} \implies (fu, fv) \in \mathcal{R},\ \mathcal{F}(fu, fv)(\tau t) \geq \mathcal{F}(u, v)(t),\ t > 0. \tag{1}$$

Proposition 3.2. *Assume that $(\mathcal{X}, d)$ is a metric space endowed with an arbitrary binary relation $\mathcal{R}$. Let $f : \mathcal{X} \to \mathcal{X}$ is a f-closed, $f_{\mathcal{R}}$-contraction mapping. Then f is an $f_{\mathcal{R}}$-contraction mapping on the induced Menger PM-space too.*

Proof. Assume that $(\mathcal{X}, d)$ is a metric space equipped with a binary relation $\mathcal{R}$ defined on $\mathcal{X}$. Let $f : \mathcal{X} \to \mathcal{X}$ is a f-closed $f_{\mathcal{R}}$-contraction mapping. Then for $u, v \in \mathcal{X}$ with $(u, v) \in \mathcal{R}$, we have $(fu, fv) \in \mathcal{R}$ and there exists $\tau \in (0, 1)$ such that

$$d(fu, fv) \leq \tau d(u, v).$$

Now, for $t > 0$, we have

$$\begin{aligned} \mathcal{F}(fu, fv)(\tau t) &= \delta_0(\tau t - d(fu, fv)) \\ &\geq \delta_0(\tau t - \tau d(u, v)) = \mathcal{F}(u, v)(t). \end{aligned}$$

Thus f is an $f_{\mathcal{R}}$-contraction. □

Remark 3.3 From the Proposition 3.2 it is interesting to note that every relation theoretic Banach contraction mapping is also a relation theoretic probabilistic contraction mapping with the same contraction constant.

Assume that $(\mathcal{X}, \mathcal{F}, \mathcal{T})$ is a Menger PM-space equipped with a binary relation $\mathcal{R}$ defined on $\mathcal{X}$. Then $\{u^i\}_{i=0}^{l}$ is an l-directed path from $u \in \mathcal{X}$ to $v \in \mathcal{X}$ if

$$u^0 = u, \qquad u^l = v, \qquad \big(u^{i-1}, u^i\big) \in \mathcal{R} \quad \text{and} \quad d\big(u^{i-1}, u^i\big) < \infty \quad \text{for all } i = 1, 2, \ldots, l.$$

For $u \in \mathcal{X}$ and $l \in \mathbb{N}$, we denote

$$\mathcal{P}(u, l, \mathcal{R}) = \{v \in \mathcal{X} : \text{there is an } l\text{-directed path from } u \text{ to } v\}$$

and

$$\mathcal{P}(u, \mathcal{R}) = \bigcup \mathcal{P}(u, l, \mathcal{R}).$$

We prove the following lemma

Lemma 3.4. *Assume that $(\mathcal{X}, \mathcal{F}, \mathcal{T})$ be a Menger PM-space under a t-norm $\mathcal{T}$ with $\sup_{a<1} \mathcal{T}(a, a) = 1$. Let $f : \mathcal{X} \to \mathcal{X}$ be an $f_{\mathcal{R}}$-contraction mapping and $v \in \mathcal{P}(u, \mathcal{R})$, then for $t > 0$, $\mathcal{F}(f^k u, f^k v)(t) \to 1$ as $k \to \infty$. Moreover, for $w \in \mathcal{X}$, $f^k u \to w$ if and only if $f^k v \to w$ as $k \to \infty$.*

Proof. Consider $u \in \mathcal{X}$ and $v \in \mathcal{P}(u, \mathcal{R})$, then $\{u^i\}_{i=0}^{l}$ is a l-directed path from u to v with

$$u^0 = u, \qquad u^l = v, \qquad \big(u^{i-1}, u^i\big) \in \mathcal{R} \quad \text{and} \quad d\big(u^{i-1}, u^i\big) < \infty \quad \text{for all } i = 1, 2, \ldots, l.$$

If f is an $f_{\mathcal{R}}$-contraction, then in the light of symmetry of $\mathcal{F}$, f is $f_{\mathcal{R}^s}$-contraction. By mathematical induction, for $t > 0$ and $k \in \mathbb{N}$, we have $(f^k u^{i-1}, f^k u^i) \in \mathcal{R}^s$ and $\mathcal{F}(f^k u^{i-1}, f^k u^i)(\tau t) \geq \mathcal{F}(f^{k-1} u^{i-1}, f^{k-1} u^i)(t)$ where $i = 1, 2, ..., l$. Consequently, we obtain

$$\begin{aligned} \mathcal{F}(f^k u^{i-1}, f^k u^i)(t) &\geq \mathcal{F}(f^{k-1} u^{i-1}, f^{k-1} u^i)\left(\frac{t}{\tau}\right) \\ &\geq \mathcal{F}(f^{k-2} u^{i-1}, f^{k-2} u^i)\left(\frac{t}{\tau^2}\right) \\ &\ldots \\ &\geq \mathcal{F}(u^{i-1}, u^i)\left(\frac{t}{\tau^k}\right) \to 1 \text{ (as } k \to \infty). \end{aligned}$$

For given $\epsilon > 0$ and $t > 0$, as $\sup_{a<1} \mathcal{T}(a, a) = 1$, implies that there exists $\mu \in (0, 1)$ depending upon ϵ, so that $\mathcal{T}(1 - \mu, 1 - \mu) > 1 - \epsilon$. Select $k_0 \in \mathbb{N}$ so that for all $k \in \mathbb{N}$ and $k \geq k_0$, we have $\mathcal{F}(f^k u^0, f^k u^1)(\frac{t}{2}) > 1 - \mu$ and $\mathcal{F}(f^k u^1, f^k u^2)(\frac{t}{2}) > 1 - \mu$. For all $k \geq k_0$ and $t > 0$, we have

$$\begin{aligned} \mathcal{F}(f^k u^0, f^k u^2)(t) &\geq \mathcal{T}\Bigg(\mathcal{F}(f^k u^0, f^k u^1)\left(\frac{t}{2}\right), (f^k u^1, f^k u^2)\left(\frac{t}{2}\right)\Bigg) \\ &\geq \mathcal{T}(1 - \mu, 1 - \mu) > 1 - \epsilon, \end{aligned}$$

so that $\mathcal{F}(f^k u^0, f^k u^2)(t) \to 1$ as $k \to \infty$. Iteratively, one can easily verify that as $k \to \infty$, we have

$$\mathcal{F}(f^k u^0, f^k u^l)(t) \to 1.$$

Again, let $f^k u \to w \in \mathcal{X}$, also $\sup_{a<1} \mathcal{T}(a, a) = 1$, then there exists $\mu(\epsilon) \in (0, 1)$ so that $\mathcal{T}(1-\mu, 1-\mu) > 1-\epsilon$. Selecting $k_0 \in \mathbb{N}$, so that, we have $\mathcal{F}(f^k u, f^k v)(\frac{t}{2}) > 1-\mu$ and $\mathcal{F}(w, f^k u)(\frac{t}{2}) > 1 - \mu$ for all $k \geq k_0$. Consequently, we have

$$\begin{aligned} \mathcal{F}(w, f^k v)(t) &\geq \mathcal{T}\Bigg(\mathcal{F}(w, f^k u)\left(\frac{t}{2}\right), (f^k u, f^k v)\left(\frac{t}{2}\right)\Bigg) \\ &\geq \mathcal{T}(1 - \mu, 1 - \mu) > 1 - \epsilon. \end{aligned}$$

Thus, $f^k v \to w$ as $k \to \infty$. □

The following useful lemma is necessary to prove the main results.

Lemma 3.5 [14]. *Assume that $(\mathcal{X}, \mathcal{F}, \mathcal{T})$ be a Menger PM-space under a t-norm $\mathcal{T}$ of Hadžić type. Let $\{u_k\}$ be sequence in $\mathcal{X}$, and there exist $\tau \in (0, 1)$ such that*

$$\mathcal{F}(u_k, u_{k+1})(\tau t) \geq \mathcal{F}(u_{k-1}, u_k)(t) \text{ for all } k \in \mathbb{N} \text{ and } t > 0.$$

Then $\{u_k\}$ is a Cauchy sequence.

Theorem 3.6. *Assume that f is a self-mappings on an $\mathcal{R}$-complete Menger PM-space $(\mathcal{X}, \mathcal{F}, \mathcal{T})$ together with Hadžić-type norm. Let the subsequent assumptions hold:*
(a) *f is an $f_{\mathcal{R}}$-contraction mapping,*
(b) *there exists $u_0 \in \mathcal{X}$ such that $(u_0, fu_0) \in \mathcal{R}$,*
(c) *$\mathcal{R}$ is $d_{\mathcal{F}}$-self-closed, that is, for any $\mathcal{R}$-preserving sequence $\{u_k\} \subset \mathcal{X}$, so that $u_k \xrightarrow{d} u$, there exists a subsequence $\{u_{k_r}\}$ of $\{u_k\}$ with $[u_{k_r}, u] \in \mathcal{R}$, for all $r \in \mathbb{N}_0$. Then f has a unique fixed point.*

Proof. Since f is an $f_{\mathcal{R}}$-contraction mapping and there exists $u_0 \in \mathcal{X}$ such that $(u_0, fu_0) \in \mathcal{R}$. By mathematical induction $(f^k u_0, f^{k+1} u_0) \in \mathcal{R}$, $k \in \mathbb{N}$, we have

$$\mathcal{F}(f^k u_0, f^{k+1} u_0)(\tau t) \geq \mathcal{F}(f^{k-1} u_0, f^k u_0)(t), \ k \in \mathbb{N} \text{ with } t > 0. \tag{2}$$

As $\mathcal{T}$ is of Hadžić-type, therefore in view of Lemma 3.5 it can be concluded that $\{f^k u_0\}$ is a Cauchy sequence in $\mathcal{X}$. By the $\mathcal{R}$-completeness of the Menger PM-space $(\mathcal{X}, \mathcal{F}, \mathcal{T})$ there exists $\sigma \in \mathcal{X}$, such that

$$\lim_{k \to \infty} f^k u_0 = \sigma. \tag{3}$$

Now we shall show that σ is a fixed point of f. Let the assumption (c) holds, then there exists a subsequence $\{f^{k_r}u_0\}$ of $\{f^k u_0\}$ with $[f^{k_r}u_0, \sigma] \in \mathcal{R},$ for all $r \in \mathbb{N}_0$. Note that $\{u_0, fu_0, f^2u_0, ..., f^{k_1}u_0, ..., f^{k_N}u_0, \sigma\}$ is a path in $\mathcal{R}$ and therefore in $\mathcal{R}^s$ from u_0 to σ, consequently, $\sigma \in \mathcal{P}(u_0, \mathcal{R}^s)$.
Since f is an $f_{\mathcal{R}}$-contraction and $[f^{k_r}u_0, \sigma] \in \mathcal{R},$ for all $r \geq N$. For $t > 0$ and $r \geq N$, we have

$$\begin{aligned}\mathcal{F}(f^{k_r+1}u_0, f\sigma)(t) &\geq \mathcal{F}(f^{k_r+1}u_0, f\sigma)(\tau t)\\ &\geq \mathcal{F}(f^{k_r}u_0, \sigma)(t) \to 1, \text{ as } r \to \infty.\end{aligned}$$

So, we have

$$\lim_{r\to\infty} f^{k_r+1}u_0 = f\sigma. \tag{4}$$

Thus, we obtain $f\sigma = \sigma$. Now if $v \in \mathcal{P}(u_0, \mathcal{R}^s)$ then in light of the Lemma 3.4, we have

$$\lim_{r\to\infty} f^k v = \sigma. \tag{5}$$

Next we prove the uniqueness of a fixed point, suppose $\sigma^* \in \mathcal{P}(u_0, \mathcal{R}^s) = \mathcal{P}(\sigma, \mathcal{R}^s)$ so that $f\sigma^* = \sigma^*$. Owing to Lemma 3.4, we have

$$\mathcal{F}(\sigma, \sigma^*)(t) = \mathcal{F}(f\sigma, f\sigma^*)(t) \to 1, \text{ as } k \to \infty. \tag{6}$$

Therefore $\sigma^* = \sigma$, that is, f has a unique fixed point. □

Now, we provide a non-trivial example to substantiate the assumptions of Theorem 3.6, which assure the existence of unique fixed point.

Example 3.7 Assume that $\mathcal{X} = \mathbb{R}^+$ with the metric $d(u, v) = |u - v|$. Define

$$\mathcal{F}(u, v)(t) = \frac{t}{t + d(u, v)},$$

then $(\mathcal{X}, \mathcal{F}, \mathcal{T})$ is a complete Menger PM-space with $\mathcal{T}$, the t-norm of $\mathcal{H}$-type. So $(\mathcal{X}, \mathcal{F}, \mathcal{T})$ is an $\mathcal{R}$-complete Menger PM space. Let the binary relation $\mathcal{R} = \{(u, v) : u, v \in [0, 1]\}$. Define $f : \mathcal{X} \to \mathcal{X}$ by

$$fu = \begin{cases} \frac{u}{2(u+1)}, & \text{if } u \in [0, 1],\\ \\ 2u^2 + 1, & \text{if } u > 1.\end{cases}$$

Now, we prove that f is an $f_{\mathcal{R}}$-contraction, that is, for $(u, v) \in \mathcal{R}$ implies $(fu, fv) \in \mathcal{R}$, such that

$$\begin{aligned}\mathcal{F}(fu, fv)(\tau t) = \mathcal{F}\bigg(\frac{u}{2(u+1)}, \frac{v}{2(v+1)}\bigg)\bigg(\frac{t}{2}\bigg) &= \frac{\frac{t}{2}}{\frac{t}{2} + |\frac{v}{2(v+1)} - \frac{u}{2(u+1)}|}\\ &= \frac{t}{t + \frac{1}{2}|\frac{v-u}{(v+1)(u+1)}|}\\ &\geq \frac{t}{t + |u - v|} = \mathcal{F}(u, v)(t),\end{aligned}$$

for all $t > 0$. This implies that f is an $f_{\mathcal{R}}$-contraction. Thus, assumption (a) of Theorem 3.6 holds for $\tau = \frac{1}{2}$. For $u \in [0,1]$ we have $fu \in [0,1]$ this implies that $(u, fu) \in \mathcal{R}$. Therefore, taking any $u_0 \in [0,1]$ we have $(u_0, fu_0) \in \mathcal{R}$. This verifies the assumption (b) of Theorem 3.6. Also if we take $\{u_k\}$ such that $u_k \leq 1$ for all $k \in \mathbb{N}_0$ then $u_k \in \mathcal{T}_N(f, \mathcal{R}, u_0)$, such that $u_k \longrightarrow u$. Therefore by the definition of the $\mathcal{R}$, we have $u_k \in [0,1]$ for all $k \in \mathbb{N}_0$ and there exists a subsequence $\{u_{k_r}\}$ of $\{u_k\}$ with $[u_{k_r}, u] \in \mathcal{R}$, for all $r \in \mathbb{N}_0$. This implies that $\mathcal{X}$ satisfies the assumption (c) of Theorem 3.6. Since all the assumptions of the Theorem 3.6 satisfied for $\tau \in (0,1)$, and $t > 0$ which ensures the survival of a unique fixed point. Namely, 0 is the fixed point of f.
Notice that, if we take $u = 1,\ v = 2$ then $fu = \frac{1}{4},\ fv = 9$, we have

$$\mathcal{F}(\frac{1}{4},\ 9)(\tau t) = \frac{\tau t}{\tau t + |\frac{1}{4} - 9|} = \frac{t}{t + \frac{35}{4\tau}} \geq \frac{t}{t + |1-2|}$$

implies that $\tau \geq \frac{35}{4}$, this implies that there is no $\tau < 1$ such that

$$\mathcal{F}(fu, fv)(\tau t) \geq \mathcal{F}(u,v)(t) \text{ for all } t > 0.$$

This implies that f does not satisfy the contraction assumption. However if $(u, v) \in \mathcal{R}$ then the assumption is satisfied for all such $u,\ v \in \mathcal{X}$.

Remark 3.8 Noticeably, in light of the above example we point out the fact that f is not a probabilistic contraction and therefore the corresponding Theorem 3 of ([25], Sehgal and Bharucha-Reid) and Corollary 1 of ([7], Fang) are not applicable here which indicate the usability of such generalizations over the corresponding several prominent recent fixed point results on this settings.

4. Applications: Kelisky-Rivlin type result for Bernstein operators

In this section we establish a Kelisky-Rivlin type result for a class of Bernstein type of special operators introduced by Deo et. al. [6]. Thus, the aim is to substantiate the utility of such exploration by establishing some new core theoretical results with some interesting applications and important remarks on this theme.
We start with the proof of result related to the convergence of successive approximations for class of Bernstein operators.

Theorem 4.1. *Assume that E is a group with respect to addition and $\mathcal{X} \subseteq E$ equipped with a metric d such that $(\mathcal{X}, d)$ is complete. Let $\mathcal{X}_0 \subseteq \mathcal{X}$ be a closed subset of $\mathcal{X}$ such that $\mathcal{X}_0$ is a subgroup of E. Let $f : \mathcal{X} \to \mathcal{X}$ such that*

$$(u,v) \in \mathcal{X} \times \mathcal{X}, u - v \in \mathcal{X}_0 \implies d(fu, fv) \leq \tau d(u,v),$$

where $\tau \in (0,1)$ is a constant. Assume that

$$u - fu \in \mathcal{X}_0 \quad \text{for all } u \in \mathcal{X}. \tag{7}$$

Then we have
(a) for every $u \in \mathcal{X}$, the Picard sequence $\{f^k u\}$ converges to a fixed point of f,

(b) for every $u \in \mathcal{X}, (u+\mathcal{X}_0) \cap \mathrm{Fix}\, f = \{\lim_{k\to\infty} f^k u\}$, *where Fix* f *denotes the set of fixed points of* f.

Proof. Consider $\mathcal{F} : \mathcal{X} \times \mathcal{X} \to \mathcal{D}^+$ defined by

$$\mathcal{F}(u,v)(t) = \delta_0\big(t - d(u,v)\big) \quad \text{for all } u, v \in \mathcal{X}, t > 0,$$

where δ_0 is the Dirac distribution function. Consider the arbitrary binary relation $\mathcal{R} \subset \mathcal{X} \times \mathcal{X}$ such that

$$\mathcal{R} = \big\{(u,v) \in \mathcal{X} \times \mathcal{X} : u - v \in \mathcal{X}_0\big\}.$$

Owing to (7), we have

$$\begin{aligned}(u,v) \in \mathcal{R} \Longrightarrow u - v \in \mathcal{X}_0 \Longrightarrow\ & fu - fv = (fu-u) + (y - fy) + (u-v) \in \mathcal{X}_0 \\ \Longrightarrow\ & (fu, fv) \in \mathcal{R}.\end{aligned}$$

Then by the definition of δ_0, we have

$$(u,v) \in \mathcal{R} \quad \Longrightarrow \quad (fu,fv) \in \mathcal{R}, \mathcal{F}(fu,fv)(\tau t) \geq \mathcal{F}(u,v)(t),\ t > 0,$$

which implies that f is an $f_{\mathcal{R}}$-contraction mapping. Also a sequence $\{u_k\} \subset \mathcal{X}$ converges to $u \in \mathcal{X}$ with respect to d if and only if $\{u_k\}$ converges to u with respect to the Menger PM-space. Let $u_0 \in \mathcal{X}$ be an arbitrary point. By (7), we have $u_0 - fu_0 \in \mathcal{X}_0$, that is, $(u_0, fu_0) \in \mathcal{R}$, which implies that $fu_0 \in \mathcal{P}(u_0, \mathcal{R})$.
Now suppose that $\{f^k u_0\}$ converges to $u \in \mathcal{X}$ with respect to $(\mathcal{X}, \mathcal{F}, \mathcal{T}_M)$, that is, Menger PM-space. Then $\{f^k u_0\}$ converges to u with respect to the metric d.
On the other hand, we have $fu_0 = (fu_0 - u_0) + u_0 \in \mathcal{X}_0$. Again, we have $f^2u_0 = (f^2u_0 - fu_0) + fu_0 \in \mathcal{X}_0$. Continuing in this process, we have $f^k u_0 \in \mathcal{X}_0$ for every $k \in \mathbb{N}$. As $\mathcal{X}_0$ is closed, then $u \in \mathcal{X}_0$. As a result, we have $(f^k u_0, u) \in \mathcal{R}$ for every $k \geq 1$. Finally, in view of Theorem 3.6, the proof of (a) accomplished.

Now, to prove (b) let $u \in \mathcal{X}$ be any arbitrary point. From (a), we know that $\{f^k u\}$ converges with respect to the metric d to some $u^* \in \mathcal{X}_0$, a fixed point of f. Moreover, from the proof of (a), we have $f^k u_0 - u \in \mathcal{X}_0$ for all $k \in \mathbb{N}$. Since $\mathcal{X}_0$ is closed, we have $u^* - u \in \mathcal{X}_0$, that is, $u^* \in u + \mathcal{X}_0$. On the other hand, suppose that $u_1, u_2 \in (u+\mathcal{X}_0) \cap \mathrm{Fix}\, f$, with $u_1 \neq u_2$. Since $u_1 - u, u_2 - u \in \mathcal{X}_0$, then $d(u_1, u_1) = d(fu_1, fu_2) \leq \tau d(u_1, u_2)$, which is a contradiction. This completes the proof of (b). □

Remark 4.2. Theorem 4.1 recovers Theorem 4.1 in [9], where $\mathcal{X}$ was supposed to be a Banach space and f was supposed to be a linear operator.

The Bernstein operator on $f \in C([0,1])$, the space of all continuous real functions on the interval $[0,1]$, is defined by

$$(\mathcal{B}_k f)(u) = \sum_{r=0}^{k} f\left(\frac{r}{k}\right)\binom{k}{r} u^r (1-u)^{k-r}, \quad f \in C\big([0,1]\big), u \in [0,1], k = 1, 2, \ldots.$$

Kelisky and Rivlin [11] proved that each Bernstein operator $\mathcal{B}_k$ is a weak operator. Moreover, for any k and $f \in C([0,1])$,

$$\lim_{j\to\infty}\big(\mathcal{B}_k^j f\big)(u) = f(0) + \big(f(1) - f(0)\big)u, \quad u \in [0,1].$$

The proof given by Kelisky and Rivlin is based on linear algebra involving the Stirling numbers of the second kind, and eigenvalues and eigenvectors of some matrices. In view of these findings I. A. Rus [21] presented a simple and elegant proof utilizing contraction principle to study the iterates of Bernstein operators. Jachymski [9] (see Theorem 4.1) introduced yet more analogous and simpler proof via contraction mapping to produce fixed points for linear operators on Banach spaces.

Now, we are interested in establishing Kelisky and Rivlin type results for the class of Bernstein type of special operators introduced by Deo et. al. [6] as follows:

If $f(u)$ is a function defined on $[0, \frac{k}{k+1}]$

$$(\mathcal{V}_k f)(u) = \sum_{r=0}^{k} p_{k,r}(u) f\left(\frac{r}{k}\right),$$

where

$$p_{k,r}(u) = \left(1 + \frac{1}{k}\right)^k \binom{k}{r} u^r \left(\frac{k}{k+1} - u\right)^{k-r}, \text{ for } \frac{k}{k+1} \geq u.$$

Let

$$\mathcal{X} = \big\{f \in C\big(\big[0, \frac{k}{k+1}\big]\big) : f(0) \geq 0,\ f\big(\frac{k}{k+1}\big) \geq 0\big\}.$$

Clearly, $\mathcal{V}_k(\cdot) : \mathcal{X} \to \mathcal{X}$ is well defined.
We have the following result.

Theorem 4.3. *Assume that $k \in \mathbb{N}$. Then, for every $f \in \mathcal{X}$, the Picard sequence $\{\mathcal{V}_k^j(f)\}_{j\in\mathbb{N}}$ converges to a fixed point of $\mathcal{V}_k(\cdot)$. Moreover, for every $f \in \mathcal{X}$, we have*

$$\lim_{j\to\infty} \max_{u\in[0,\frac{k}{k+1}]} \big|\mathcal{V}_k^j(f)(u) - \omega(u)\big| = 0,$$

where $\omega(u) = f(0)\big(\frac{k}{k+1} - u\big) + f\big(\frac{k}{k+1}\big)u,\ u \in [0, \frac{k}{k+1}]$.

Proof. Let $\mathbb{E} = C\big(\big[0, \frac{k}{k+1}\big]\big)$. We endow $\mathcal{X}$ with the metric defined by

$$d(\mathcal{U}, \mathcal{V}) = \max_{u\in[0,\frac{k}{kk+1}]} \big|\mathcal{U}(u) - \mathcal{V}(u)\big|, \quad \mathcal{U}, \mathcal{V} \in \mathcal{X}.$$

Clearly, $(\mathcal{X}, d)$ is a complete metric space. Let

$$X_0 = \big\{\mathcal{U} \in \mathbb{E} : \mathcal{U}(0) = \mathcal{U}\big(\frac{k}{k+1}\big) = 0\big\}.$$

Then $\mathcal{X}_0 \subset \mathcal{X}$ is a closed subgroup of $\mathbb{E}$. Let $f, g \in \mathcal{X}$ such that $f - g \in \mathcal{X}_0$. Let $u \in \big[0, \frac{k}{k+1}\big]$, then we have

$$\begin{aligned}\big|\mathcal{V}_k(f)(u) - \mathcal{V}_k(g)(u)\big| &= \left|\sum_{r=0}^{k}\left(f\left(\frac{r}{k}\right) - g\left(\frac{r}{k}\right)\right)\left(1+\frac{1}{k}\right)^k \binom{k}{r} u^r \left(\frac{k}{k+1} - u\right)^{k-r}\right| \\ &\le \sum_{r=0}^{k}\left|(f-g)\left(\frac{r}{k}\right)\right|\left(1+\frac{1}{k}\right)^k \binom{k}{r} u^r \left(\frac{k}{k+1} - u\right)^{k-r} \\ &\le \sum_{r=1}^{k-1}\left(1+\frac{1}{k}\right)^k \binom{k}{r} u^r \left(\frac{k}{k+1} - u\right)^{k-r} d(f,g).\end{aligned}$$

Note that

$$\sum_{r=0}^{k}\left(1+\frac{1}{k}\right)^k \binom{k}{r} u^r \left(\frac{k}{k+1} - u\right)^{k-r} = 1.$$

Then it is easy to observe that

$$\begin{aligned}\sum_{r=1}^{k-1}\left(1+\frac{1}{k}\right)^k \binom{k}{r} u^r \left(\frac{k}{k+1} - u\right)^{k-r} &\le 1 - \left(1+\frac{1}{k}\right)^k u^k - \left(1+\frac{1}{k}\right)^k \left(\frac{k}{k+1} - u\right)^k \\ &\le 1 - \frac{1}{2^{k-1}}.\end{aligned}$$

As a consequence, we have

$$f, g \in \mathcal{X}, f - g \in \mathcal{X}_0 \quad \Longrightarrow \quad d\big(\mathcal{V}_k(f), \mathcal{V}_k(g)\big) \le \left(1 - \frac{1}{2^{k-1}}\right) d(f,g).$$

Now, let $f \in \mathcal{X}$. For any $u \in [0, \frac{k}{k+1}]$, we have

$$f(u) - \mathcal{V}_k(f) = \sum_{r=0}^{k}\left(f(u) - f\left(\frac{r}{k}\right)\right)\left(1+\frac{1}{k}\right)^k \binom{k}{r} u^r \left(\frac{k}{k+1} - u\right)^{k-r}.$$

Observe that $f(0) - \mathcal{V}_k(f)(0) = f(\frac{k}{k+1}) - \mathcal{V}_k(f)(\frac{k}{k+1}) = 0$. Then, for every $f \in \mathcal{X}$, we have

$$f - \mathcal{V}_k(f) \in \mathcal{X}_0.$$

By Theorem 4.1, we deduce that for every $f \in \mathcal{X}$, the Picard sequence $\{\mathcal{V}_k^j(f)\}_{j \in \mathbb{N}}$ converges to a fixed point of $\mathcal{V}_k(\cdot)$ and

$$(f + \mathcal{X}_0) \cap \operatorname{Fix} \mathcal{V}_k(\cdot) = \Big\{\lim_{j\to\infty} \mathcal{V}_k^j(f)\Big\}.$$

Let $f \in \mathcal{X}$. It is not difficult to observe that $\omega(u) = f(0)\big(\frac{k}{k+1} - u\big) + f\big(\frac{k}{k+1}\big)u \in \operatorname{Fix} \mathcal{V}_k(\cdot)$. We have also

$$\omega(u) = f(u) + \theta(u),$$

where

$$\theta(u) = f(0)\big(\frac{k}{k+1} - u\big) + f\big(\frac{k}{k+1}\big)u - f(u).$$

Observe that $\theta(0) = \theta\left(\frac{k}{k+1}\right) = 0$, which implies that $\theta \in \mathcal{X}_0$. This completes the proof of Theorem 4.3. □

Conclusion

In this research work, we studied problems modeled on non-linear functional analysis and approximation theory that have been very useful for future research directions. We investigated a variant of Banach contraction principle for mappings defined on a Menger PM spaces equipped with an arbitrary binary relation. Indeed, we presented a variant of prominent recent results on the relational metric settings as the underlying linear contraction is assumed to satisfy for elements belonging to an arbitrary binary relation. Further, we presented non-trivial example vindicating that the claims are novel and original. In addition, to annotate the utility of such newly obtained results, we established Kelisky-Rivlin type result for the class of Bernstein type special operators on the spaces of continuous functions. Thus, these findings supply yet another view on fixed point results. In fact, we substantiated the usability of such exploration by establishing some new core theoretical results with applications.

Acknowledgement. The referees have reviewed the paper very carefully. The author express their deep thanks for the comments.

References

[1] A. Alam, R. George, M. Imdad, *Refinements to relation-theoretic contraction principle*, Axioms, **11**(136)(2022).

[2] A. Alam, M. Imdad, *Relation-theoretic contraction principle*, J. Fixed Point Theory Appl., **17**(2015), no. 4, 693-702.

[3] H. Argoubi, M. Jleli, B. Samet, *The study of fixed points for multivalued mappings in a Menger probabilistic metric space endowed with a graph*, Fixed Point Theory Appl., **2015**(113)(2015).

[4] S. Banach, *Sur les operations dans les ensembles abstraits et leur application aux equations intgrales*, Fund. Math., **3**(1922), 133-181.

[5] S.K. Bhandari, D. Gopal, P. Konar, *Probabilistic α-min Ciric type contraction results using a control function*, AIMS Mathematics, **5**(2020), no. 2, 1186-1198.

[6] N. Deo, M.A. Noor, M.A. Siddiqui, *On approximation by a class of new Bernstein type operators*, Appl. Math. Comput., **201**(2008), 604-612.

[7] J.X. Fang, *A note on fixed point theorems of Hadžić*, Fuzzy Sets Syst., **48**(1992), 391-395.

[8] O. Hadžić, E. Pap, *Fixed Point Theory in Probabilistic Metric Spaces*, Kluwer Academic, Dordrecht, 2001.

[9] J. Jachymski, *The contraction principle for mappings on a metric space with a graph*, Proc. Am. Math. Soc., **136**(2008), 1359-1373.

[10] T. Kamran, M. Samreen, N. Shahzad, *Probabilistic G-contractions*, Fixed Point Theory Appl., **2013**(223)(2013).

[11] R.P. Kelisky, T.J. Rivlin, *Iterates of Bernstein polynomials*, Pac. J. Math., **21**(1967), 511-520.

[12] B. Kolman, R.C. Busby, S. Ross, *Discrete Mathematical Structures*, Third Edition, PHI Pvt. Ltd., New Delhi, 2000.

[13] S. Lipschutz, *Schaum's Outlines of Theory and Problems of Set Theory and Related Topics*, McGraw-Hill, New York, 1964.

[14] D. Mihet, *A generalization of a contraction principle in probabilistic metric spaces*, Part II, Int. J. Math. Math. Sci., **5**(2005), 729-736.

[15] J.J. Nieto, R. Rodríguez-López, *Contractive mapping theorems in partially ordered sets and applications to ordinary differential equations*, Order, **22**(2005), 223-239.

[16] A. Petruşel, I.A. Rus, *Fixed point theory in terms of a metric and of an order relation*, Fixed Point Theory, **20**(2019), no. 2, 601-622.

[17] G. Prasad, *Coincidence points of relational Ψ-contractions and an application*, Afrika Mathematica, **32**(2021), no. 6-7, 1475-1490.

[18] G. Prasad, *Fixed points of Kannan contractive mappings in relational metric spaces*, J. Anal., **29**(2021), no. 3, 669-684.

[19] G. Prasad, H. Işık, *On solution of boundary value problems via weak contractions*, J. Funct. Spaces, **2022**, Article ID 6799205, (2022).

[20] A.C.M. Ran, M.C.B. Reurings, *A fixed point theorem in partially ordered sets and some applications to matrix equations*, Proc. Am. Math. Soc., **132**(2004), no. 5, 1435-1443.

[21] I.A. Rus, *Iterates of Bernstein operators, via contraction principle*, J. Math. Anal. Appl., **282**(2004), 259-261.

[22] Z. Sadeghi, S.M. Vaezpour, *Fixed point theorems for multivalued and single-valued contractive mappings on Menger PM spaces with applications*, J. Fixed Point Theory Appl., **20**(114)(2018).

[23] B. Samet, M. Turinici, *Fixed point theorems on a metric space endowed with an arbitrary binary relation and applications*, Commun. Math. Anal., **13**(2012), no. 2, 82-97.

[24] B. Schweizer, A. Sklar, *Probabilistic Metric Spaces*, North-Holland, New York, 1983.

[25] V.M. Sehgal, A.T. Bharucha-Reid, *Fixed points of contraction mappings on PM-spaces*, Math. Syst. Theory, **6**(1972), 97-102.

[26] M. Turinici, *Fixed points for monotone iteratively local contractions*, Demonstr. Math., **19**(1986), no. 1, 171-180.

[27] M. Turinici, *Ran and Reuring's theorems in ordered metric spaces*, J. Indian Math. Soc., **78**(2011), 207-214.

Received: June 16, 2023; Accepted: December 12, 2023.