Fixed Point Theory, 26(2025), No. 1, 247-260
DOI: 10.24193/fpt-ro.2025.1.16
http://www.math.ubbcluj.ro/~nodeacj/sfptcj.html

ASYNCHRONOUS r-CYCLIC CONTRACTIONS ON METRIC SPACES

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Abstract. Cyclic (1-cyclic) operators have been studied in a long list of papers over the last decade. r-cyclic operators were introduced recently [Păcurar, M., *Synchronous r-cyclic contractions on metric spaces*, Fixed Point Theory, **24** (2023), No. 2, 683-700] and show interesting properties. In the present paper we introduce asynchronous r-cyclic contractions condition and study the conditions under which they are Picard operators. Their behavior is compared to the one of the synchronous r-cyclic contractions. Besides generalizing known results, our approach opens a research direction with some potential, both theoretical and practical.
Key Words and Phrases: r-cyclic covering, iterative method, fixed point, Picard operator, seasonal variability.
2020 Mathematics Subject Classification: 47H10, 54H25.

1. Introduction

A considerable number of papers were written on cyclic operators satisfying various contraction conditions, see for example [1, 5, 7, 10, 11, 12, 13, 14, 15, 16, 18, 17, 20, 23, 25] for a very short list of them. Directly or indirectly this research was inspired by the results of Kirk et al. in [8], where a *Banach type cyclic contraction* was considered, namely an operator $f : X \to X$, cyclic w.r.t. $X = \bigcup_{i=1}^{m} X_i$, that is, satisfying $f(X_1) \subset X_2, f(X_2) \subset X_3, \dots, f(X_{m-1}) \subset X_m, f(X_m) \subset X_1$, for which there exists $c \in [0, 1)$ such that

$$d(f(x), f(y)) \leq c \cdot d(x, y), \forall x \in X_i, y \in X_{i+1}, 1 \leq i \leq m. \tag{1.1}$$

Theorem 1.3 in [8] states that if the metric space X is complete and all the sets $X_i, i = \overline{1, m}$ are closed, then the cyclic operator f is Picard, that is, it has a unique fixed point that can be obtained as the limit of the Picard iteration starting from any point in X.

A new general class of cyclic operators were introduced in [14], where some fixed point results were established for the so-called synchronous r-cyclic contractions. In the present paper we continue and complete this approach with the study of asynchronous r-cyclic contractions.

2. Preliminaries. Cyclic and r-cyclic operators on metric spaces

Let us start from a simple example. Consider the sets $X_i = \{x_i\}, i = \overline{1, m}$ and $X = \bigcup_{i=1}^{m} X_i$, and two operators $f, g : X \to X$ defined by

$$f(x_1) = x_2, f(x_2) = x_3, \dots, f(x_{m-1}) = x_m, f(x_m) = x_1,$$

respectively

$$g(x_1) = x_4, g(x_2) = x_5, \dots, g(x_{m-3}) = x_m,$$
$$g(x_{m-2}) = x_1, g(x_{m-1}) = x_2, g(x_m) = x_3.$$

Then f is a cyclic operator w.r.t. $\bigcup_{i=1}^{m} X_i$, while g is not, but g is a 3-cyclic operator w.r.t. the same covering.

Generally, if X is a nonempty set, $f : X \to X$ an operator and there exists a covering of $X = \bigcup_{i=1}^{m} X_i$, $m \geq 2$ such that

$$f(X_1) \subseteq X_2, f(X_2) \subseteq X_3, \dots, f(X_{m-1}) \subseteq X_m, f(X_m) \subseteq X_1,$$

then $\bigcup_{i=1}^{m} X_i$ is called a *cyclic covering* of X w.r.t. f, while f is called a *cyclic operator* w.r.t. the covering $\bigcup_{i=1}^{m} X_i$ (see [21]).

Since the union of sets is commutative and the order of the sets proves to be essential when determining if an operator is cyclic w.r.t. to $X = \bigcup_{i=1}^{m} X_i$, we introduced in [14] the notation $\overset{m}{\underset{i=1}{\uplus}} X_i = X_1 \uplus X_2 \uplus \cdots \uplus X_m$ to indicate a cyclic covering w.r.t. to an operator, which will actually say that each of the m cyclic permutations $X_1 \cup X_2 \cup X_3 \cup \cdots \cup X_{m-1} \cup X_m, X_2 \cup X_3 \cup \cdots \cup X_{m-1} \cup X_m \cup X_1, \dots, X_m \cup X_1 \cup X_2 \cup \cdots \cup X_{m-1}$ is a cyclic covering of X w.r.t. f, while generally any other permutation is not.

Considering this notation, we defined in [14] the r-cyclic operators w.r.t. to a covering:

Definition 2.1 ([14])**.** Let X be a nonempty set, $f : X \to X$ an operator and $m \geq 2$, $1 \leq r \leq m$ integers. If there is a covering $X = \overset{m}{\underset{i=1}{\uplus}} X_i$ such that

$$f(X_1) \subseteq X_{1+r}, f(X_2) \subseteq X_{2+r}, \dots, f(X_m) \subseteq X_{m+r},$$

where for $p > m$ by X_p we mean $X_{p \mod m}$, then:

i) $\overset{m}{\underset{i=1}{\uplus}} X_i$ is called a r-cyclic covering of X w.r.t. f;

ii) f is called a r-cyclic operator w.r.t. the covering $\overset{m}{\underset{i=1}{\uplus}} X_i$.

Obviously, any cyclic operator is 1-cyclic in view of this definition.

Note that in several papers (see for example [11]) the term p-cyclic is used to indicate an operator that is actually 1-cyclic or simply cyclic, but p denotes the number of sets in the cyclic covering, which is denoted in this paper by m. This is still essentially different from what we understand by r-cyclic operators.

In [14] are presented some examples and properties of r-cyclic operators. At this point we have to note that visual representations play an important role in understanding the behavior of r-cyclic operators, since the rigorous mathematical notation gets somewhat hairy in stating and proving the results. Let us take as an example the case $m = 10$:

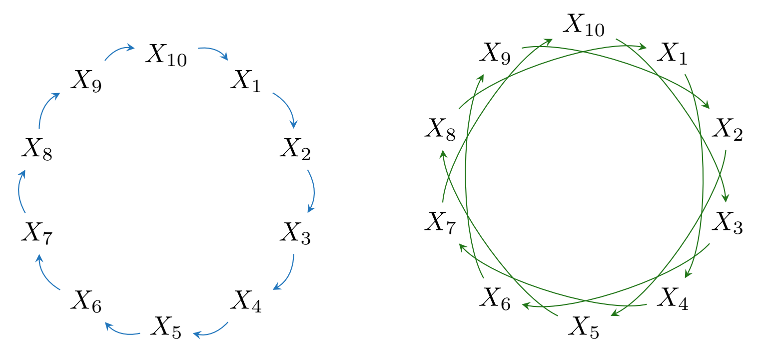

FIGURE 2.1. Cyclic coverings for $m = 10$ and $r = 1$, respectively $r = 3$

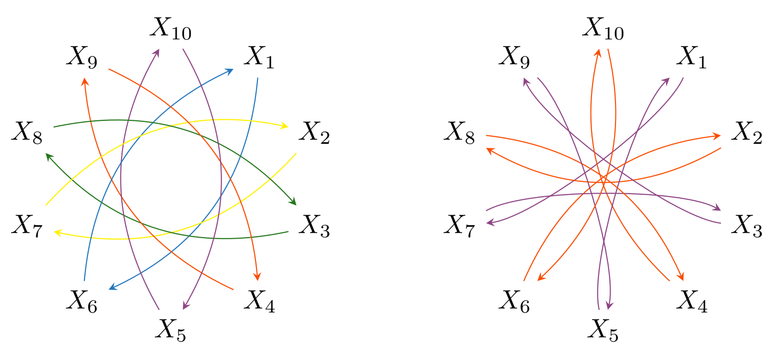

FIGURE 2.2. Cyclic coverings for $m = 10$ and $r = 5$, respectively $r = 6$

We pictured here only four of all possible cases, namely when the operator f defined on a covering with 10 sets $X = X_1 \cup X_2 \cup \cdots \cup X_{10}$ is 1-cyclic, 3-cyclic, 5-cyclic, and 6-cyclic, respectively. It is depicted how the sets of the covering are mapped by f, according to the definition of an r-cyclic operator.

Although not a necessity, the color code used in these representations facilitates understanding the properties of r-cyclic operators on a given covering. For $r = 1$, the classical case of cyclic operators, no matter where the starting point, the Picard iteration $\{f^n(x)\}_{n\geq 0}$ will pass infinitely many times through all the sets of the covering. The same happens in the case $r = 3$, but the order of the sets is not the same.

A different phenomenon is to be observed if $r = 5$. In this case the Picard iteration starting in X_1 will have all its terms in $X_1 \cup X_6$. The one starting in X_2 will remain in $X_2 \cup X_7$ and so on. In this manner one can see that 5 closed circuits appear with respect to the Picard iteration. Similarly in the case $r = 6$, one can easily spot the 2 closed circuits in which the Picard iteration will take its successive values.

The intention of these few examples is to illustrate two essentially different cases, depending on the value of $\gcd(m, r)$, that is, the greatest common divisor of m and r: first when $\gcd(m, r) = 1$ and second when $\gcd(m, r) = k > 1$. They are summed up in the following two lemmas, which we recall from [14], for the sake of readability of this paper.

Lemma 2.2. *Let $f : X \to X$ be r-cyclic w.r.t. $X = \overset{m}{\underset{i=1}{\uplus}} X_i$, $m \geq 2$, $1 \leq r < m$ integers. If $\gcd(m, r) = k > 1$, then there exists a covering $X = \overset{k}{\underset{j=1}{\uplus}} Y_j$ with the following properties:*

i) The k subcoverings Y_j, $1 \leq j \leq k$ are given by

$$\begin{aligned}
&Y_1 = X_1 \uplus X_{1+r} \uplus \cdots \uplus X_{1+(\frac{m}{k}-1)r},\\
&Y_2 = X_2 \uplus X_{2+r} \uplus \cdots \uplus X_{2+(\frac{m}{k}-1)r},\\
&\vdots\\
&Y_k = X_k \uplus X_{k+r} \uplus \cdots \uplus X_{k+(\frac{m}{k}-1)r},
\end{aligned}$$

or generally $Y_j = \overset{\frac{m}{k}-1}{\underset{i=0}{\uplus}} X_{j+i\cdot r}, 1 \leq j \leq k$.

ii) The subcoverings (circuits) Y_j, $1 \leq j \leq k$ are invariant for f.

iii) For each $j \in \{1, \ldots, k\}$, the corresponding restriction $f|_{Y_j}$ is a 1-cyclic operator w.r.t. Y_j.

Lemma 2.3. *Let $f : X \to X$ be r-cyclic w.r.t. $X = \overset{m}{\underset{i=1}{\uplus}} X_i$, where $m \geq 2$, $1 \leq r < m$. If $\gcd(m, r) = 1$, then for any $x \in X$ the sequence $\{f^n(x)\}_{n \geq 0}$ has infinitely many terms in each X_i, $1 \leq i \leq m$.*

3. Asynchronous r-cyclic contractions and fixed points

Analyzing condition (1.1), it is immediate that for r-cyclic operators two directions have to be investigated: first if the inequality holds for any $x \in X_i, y \in X_{i+r}$, and second if it holds for any $x \in X_i, y \in X_{i+1}$. In the first case the operators are called synchronous r-cyclic contractions and were investigated in [14], in the latter they are called *asynchronous r-cyclic contractions* and are the object of the present paper, therefore we introduce the following:

Definition 3.1. Let (X, d) be a metric space, $f : X \to X$ and $X = \overset{m}{\underset{i=1}{\uplus}} X_i$ a r-cyclic covering w.r.t. f, where $m \geq 2$ and $1 \leq r < m$ are integers. If there exists $c \in [0, 1)$

such that for any $x \in X_i, y \in X_{i+1}, 1 \le i \le m$,

$$d(f(x), f(y)) \le c \cdot d(x, y),$$

then f is called *asynchronous r-cyclic contraction* w.r.t. $\overset{m}{\underset{i=1}{\cup}} X_i$.

Because for $r = 1$ the discussion would reduce to the cyclic contraction condition in [8], we shall continue the discussion for $r \ge 2$. Consequently we shall take $m \ge 3$. The behavior of the asynchronous r-cyclic contractions is somehow intuitive, but if one wishes to keep the notation rigorous for the general case, it becomes complicated. That is why we shall first analyze the case $r = 2$. The discussion is dictated by the greatest common divisor of m and r. For the case $r = 2$ this means m odd or even, and both situations are covered in the next result:

Theorem 3.2. *Let (X, d) be a complete metric space, $X = \overset{m}{\underset{i=1}{\cup}} X_i$, $m \ge 3$ a covering such that X_i closed, $1 \le i \le m$, and $c \in [0, 1)$ such that $f : X \to X$ is asynchronous 2-cyclic contraction with constant c w.r.t. $\overset{m}{\underset{i=1}{\cup}} X_i$.*

Then f is a Picard operator.

Proof. Let us start by resuming what the hypothesis of the theorem actually implies. Since f is 2-cyclic operator on $\overset{m}{\underset{i=1}{\cup}} X_i$, we have that

$$f(X_1) \subset X_3, f(X_2) \subset X_4, \dots,$$
$$f(X_{m-2}) \subset X_m, f(X_{m-1}) \subset X_{m+1}, f(X_m) \subset X_{m+2},$$

while $X_p = X_{p \bmod m}$, for any $p > m$. The fact that f is asynchronous cyclic contraction means that

$$d(f(x), f(y)) \le c \cdot d(x, y),$$

for any $x \in X_i, y \in X_{i+1}, i = \overline{1, m}$.

In view of the general result that will be stated later on, we will separate the proof in two cases, m odd, respectively m even, although in this case $r = 2$ the two arguments partially coincide.

Case 1. If m is odd, so $\gcd(m, 2) = 1$, we consider two orbits starting from two different points x_0 and y_0 belonging to successive sets of the cyclic covering. Without loss of generality, we may assume that $x_0 \in X_1$ and $y_0 \in X_2$ and define $x_n = f^n(x_0), n \ge 0$ and $y_n = f^n(y_0), n \ge 0$.

The sequence $\{y_n\}_{n\ge0}$ plays a secondary role in the next reasoning, it only helps proving the convergence of $\{x_n\}_{n\ge0}$, although the same can be shown for $\{y_n\}_{n\ge0}$, too.

The sequences $\{x_n\}_{n\ge0}$ and $\{y_n\}_{n\ge0}$ have their terms in $X_1, X_3, X_5, \dots$ and, respectively, in $X_2, X_4, X_6, \dots$. It follows that

$$d(x_1, y_1) = d(f(x_0), f(y_0)) \le c \cdot d(x_0, y_0)$$
$$d(y_1, x_2) = d(f(y_0), f(x_1)) \le c \cdot d(y_0, x_1),$$

then

$$d(x_2, y_2) = d(f(x_1), f(y_1)) \leq c^2 \cdot d(x_0, y_0)$$
$$d(y_2, x_3) = d(f(y_1), f(x_2)) \leq c^2 \cdot d(y_0, x_1),$$

and so on

$$d(x_n, y_n) \leq c^n \cdot d(x_0, y_0) \tag{3.1}$$
$$d(y_n, x_{n+1}) \leq c^n \cdot d(y_0, x_1).$$

So for $n \geq 1$ we have that

$$d(x_n, x_{n+1}) \leq d(x_n, y_n) + d(y_n, x_{n+1}) \leq c^n \cdot [d(x_0, y_0) + d(y_0, x_1)]. \tag{3.2}$$

If we denote by $A = d(x_0, y_0) + d(y_0, x_1) \geq 0$, we have that $d(x_n, x_{n+1}) \leq c^n \cdot A, n \geq 1$. For $p \geq 1$ we obtain that

$$d(x_n, x_{n+p}) \leq c^n \cdot \frac{1-c^p}{1-c} \cdot A,$$

which leads to the conclusion that $\{x_n\}_{n\geq 0}$ is a Cauchy sequence in the complete metric space (X, d), so there exists its limit $\overline{x} \in X$. Since $\gcd(m, 2) = 1$, the sequence $\{x_n\}_{n\geq 0}$ has infinitely many terms in each X_i, $i = \overline{1, m}$ (see Lemma 2.3), so from each X_i one can extract a subsequence of $\{x_n\}_{n\geq 0}$ that converges to $\overline{x}$. As X_i, $i = \overline{1, m}$ are all closed, it follows that $\overline{x} \in \bigcap_{i=1}^{m} X_i$.

Case 2. If m is even, so $\gcd(m, 2) = 2$, there arise two closed circuits $X_1 \cup X_3 \cup \cdots \cup X_{m-1}$ and $X_2 \cup X_4 \cup \cdots \cup X_m$ which are invariant for f (see Lemma 2.2). We consider two sequences $\{x_n\}_{n\geq 0}$ and $\{y_n\}_{n\geq 0}$ defined as in Case 1. Note that $\{x_n\}_{n\geq 0}$ and $\{y_n\}_{n\geq 0}$ do not have infinitely many terms in each X_i, $i = \overline{1, m}$. Similarly to the previous case we come to the conclusion that $\{x_n\}_{n\geq 0}$ converges, but in $\bigcup_{i=1}^{\frac{m}{2}} X_{2i-1}$, to $\overline{x} \in \bigcap_{i=1}^{\frac{m}{2}} X_{2i-1}$ and $\{y_n\}_{n\geq 0}$ converges in $\bigcup_{i=1}^{\frac{m}{2}} X_{2i}$ to $\overline{y} \in \bigcap_{i=1}^{\frac{m}{2}} X_{2i}$. Still the inequality $d(x_n, y_n) \leq c^n \cdot d(x_0, y_0), n \geq 0$, valid also in this case, ensures that $\{x_n\}_{n\geq 0}$ and $\{y_n\}_{n\geq 0}$ have the same limit $\overline{x} = \overline{y} \in \bigcap_{i=1}^{m} X_i$.

So for both cases m odd and m even we know for sure that $\bigcap_{i=1}^{m} X_i \neq \emptyset$. Then the restriction $f|_{\bigcap_{i=1}^{m} X_i}$ is a Banach contraction with constant c on $\bigcap_{i=1}^{m} X_i$, having a unique fixed point $x^* \in \bigcap_{i=1}^{m} X_i$ which can be obtained by means of the Picard iteration starting from any point in $\bigcap_{i=1}^{m} X_i$.

We still have to see if x^* can be obtained starting from any initial point in X. Therefore we take an arbitrary $x \in X$. Then there is some $l \in \{1, 2, \ldots, m\}$ such that $x \in X_l$. As $x^* \in X_{l+1}$, the following holds: $d(f(x), f(x^*)) \leq c \cdot d(x, x^*)$. Since f is 2-cyclic, it follows that $f(x) \in X_{l+2}$. But $x^* \in X_{l+3}$ as well, so

$$d(f^2(x), x^*) = d(f(f(x)), f(x^*)) \leq c \cdot d(f(x), x^*) \leq c^2 \cdot d(x, x^*).$$

In this manner we obtain that

$$d(f^n(x), x^*) \le c^n \cdot d(x, x^*), n \ge 1.$$

Now this implies that $f^n(x) \to x^*$, as $n \to \infty$, for any $x \in X$. In conclusion f has a unique fixed point, no matter if m is odd or even and this fixed point may be obtained as the limit of the Picard iteration starting from any point in $X = \overset{m}{\underset{i=1}{\uplus}} X_i$.

□

Remark 3.3. The proof of the previous theorem is essentially based on considering the sequence $\{y_n\}_{n\ge0}$. We need a closer look at its role. Since f was assumed to be 2-cyclic, we had to consider two Picard iterations starting from two different points, belonging to consecutive sets of the covering $\overset{m}{\underset{i=1}{\uplus}} X_i$.

Due to the asynchronous cyclic contraction condition, one obtains the inequalities (3.1), involving terms of both sequences. Further on $\{y_n\}_{n\ge0}$ behaves like the opposite part of a "zipper" which, by closing, gets to the desired inequality (3.2).

Therefore such a sequence will be called a *zipper sequence*. As we shall outline in the next Remark, the number and kind of zipper sequences needed for proving a fixed point result for asynchronous r-cyclic contractions depend on r and the relation between m and r.

Analyzing the proof of Theorem 3.2 and having in view Lemmas 2.2 and 2.3, it is not difficult to imagine how the same works for any $r \ge 2$, where $m \ge 3, r \le m$. In the general case, three different situations arise:

Case 1. If $k = \gcd(m, r) = 1$, then one needs to employ r zipper sequences in the proof:

$$\begin{aligned}
&\{x_n^1\} \subset X_1 \uplus X_{1+r} \uplus \cdots \uplus X_{1+(m-1)r},\\
&\{x_n^2\} \subset X_2 \uplus X_{2+r} \uplus \cdots \uplus X_{2+(m-1)r},\\
&\vdots\\
&\{x_n^r\} \subset X_r \uplus X_{2r} \uplus \cdots \uplus X_{mr}.
\end{aligned}$$

Each of them has infinitely many terms in each X_i, $i = \overline{1,m}$ and converges to the unique fixed point of f.

Case 2. If $k = \gcd(m, r) = r$, then one needs to consider again r zipper sequences, but each of them will be contained in one of the r closed circuits, such that they start in successive sets of the cyclic covering:

$$\begin{aligned}
&\{x_n^1\} \subset X_1 \uplus X_{1+r} \uplus \cdots \uplus X_{1+m-r},\\
&\{x_n^2\} \subset X_2 \uplus X_{2+r} \uplus \cdots \uplus X_{2+m-r},\\
&\vdots\\
&\{x_n^r\} \subset X_r \uplus X_{2r} \uplus \cdots \uplus X_{mr}.
\end{aligned}$$

Similar to Case 2 in the previous proof, each of them will converge to the same (unique) fixed point of f.

Case 3. If $1 < k = \gcd(m,r) < r$, then one still needs to consider r zipper sequences, but each of them will be contained in one of the k closed circuits, such that they start in successive sets of the cyclic covering and each of the k circuits will be "traversed" r/k times:

$$\begin{aligned}
&\{x_n^1\} \subset X_1 \cup X_{1+r} \cup \cdots \cup X_{1+(\frac{m}{k}-1)r},\\
&\{x_n^2\} \subset X_2 \cup X_{2+r} \cup \cdots \cup X_{2+(\frac{m}{k}-1)r},\\
&\vdots\\
&\{x_n^k\} \subset X_k \cup X_{k+r} \cup \cdots \cup X_{k+(\frac{m}{k}-1)r},\\
&\{x_n^{k+1}\} \subset X_{k+1} \cup X_{k+1+r} \cup \cdots \cup X_{k+1+(\frac{m}{k}-1)r},\\
&\qquad\qquad \text{which coincides with } X_1 \cup X_{1+r} \cup \cdots \cup X_{1+(\frac{m}{k}-1)r},\\
&\vdots\\
&\{x_n^r\} \subset X_r \cup X_{2r} \cup \cdots \cup X_{r+(\frac{m}{k}-1)r},\\
&\qquad\qquad \text{which coincides with } X_k \cup X_{k+r} \cup \cdots \cup X_{k+(\frac{m}{k}-1)r}.
\end{aligned}$$

Due to the contraction condition, they will all converge to the unique fixed point of f.

Having in view the above discussion, we may state now the following general result without a proof, which would imply the same technique as in the proof of Theorem 3.2, but a more complicated notation. In the case $r = 1$, the next theorem reduces to the result due to Kirk et al. [8].

Theorem 3.4. *Let (X,d) be a complete metric space, $X = \bigcup_{i=1}^{m} X_i$, $m \geq 3$ a covering of X such that $X_i, i = \overline{1,m}$ are closed.*

If $f : X \to X$ is an asynchronous r-cyclic contraction w.r.t. $\bigcup_{i=1}^{m} X_i$, $2 \leq r < m$, then f is a Picard operator.

Remark 3.5. We have to note an important difference between synchronous and, respectively, asynchronous r-cyclic contractions.

In the case of synchronous r-cyclic contractions, see [14], the operator f has a different behavior conditional on $\gcd(m,r)$. If $\gcd(m,r) = 1$, f is a Picard operator. If $\gcd(m,r) > 1$, then f is a 1-cyclic contraction on each of the closed circuits which arise, and consequently it is a weakly Picard operator.

But in the case of asynchronous r-cyclic contractions, as Theorem 3.4 asserts, f is a Picard operator, no matter if $\gcd(m,r) = 1$ or $\gcd(m,r) > 1$. Note that f is not a cyclic contraction on the closed circuits which arise in case that $\gcd(m,r) > 1$. The proof is based on different arguments, and the zipper sequences play the key role.

Example. Consider a cyclic covering with $m = 10$ and let us take some simple cases.

If $r = 2$, so $\gcd(10,2) = 2$, then two zipper sequences $\{x_n\} \subset X_1 \cup X_3 \cup X_5 \cup X_7 \cup X_9$, $\{y_n\} \subset X_2 \cup X_4 \cup X_6 \cup X_8 \cup X_{10}$ are needed in the proof, while $d(x_n, y_n) \to 0$ as $n \to \infty$, see Fig. 3.1.

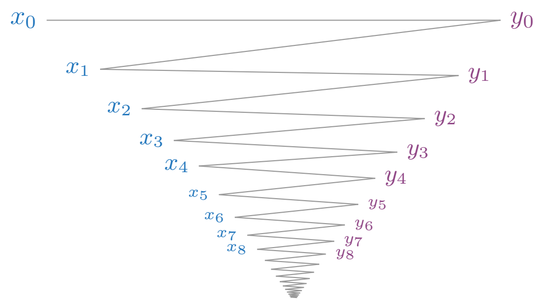

FIGURE 3.1. Convergence if the two zipper sequences are involved

If $r = 3$, so $\gcd(10, 3) = 1$, then three zipper sequences are needed:

$$\{x_n\} \subset X_1 \cup X_4 \cup X_7 \cup X_{10} \cup X_3 \cup X_6 \cup X_9 \cup X_2 \cup X_5 \cup X_8$$
$$\{y_n\} \subset X_2 \cup X_5 \cup X_8 \cup X_1 \cup X_4 \cup X_7 \cup X_{10} \cup X_3 \cup X_6 \cup X_9$$
$$\{z_n\} \subset X_3 \cup X_6 \cup X_9 \cup X_2 \cup X_5 \cup X_8 \cup X_1 \cup X_4 \cup X_7 \cup X_{10},$$

while $d(x_n, y_n) \to 0$ and $d(y_n, z_n) \to 0$ as $n \to \infty$, so $d(x_n, z_n) \to 0$ as $n \to \infty$, see Figure 3.2.

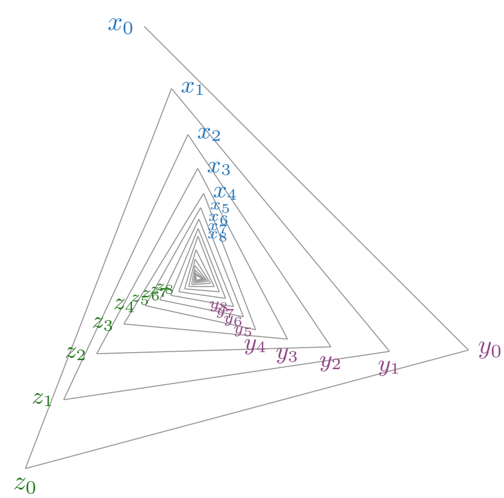

FIGURE 3.2. Convergence if three zipper sequences are involved

If $r = 4$, so $\gcd(10, 4) = 2$, then four zipper sequences are needed:

$$\{x_n\} \subset X_1 \cup X_5 \cup X_9 \cup X_3 \cup X_7; \quad \{y_n\} \subset X_2 \cup X_6 \cup X_{10} \cup X_4 \cup X_8$$
$$\{z_n\} \subset X_3 \cup X_7 \cup X_1 \cup X_5 \cup X_9; \quad \{t_n\} \subset X_4 \cup X_8 \cup X_2 \cup X_6 \cup X_{10}.$$

while $d(x_n, y_n) \to 0, d(y_n, z_n) \to 0, d(z_n, t_n) \to 0$ as $n \to \infty$, so $d(t_n, x_n) \to 0$ as $n \to \infty$, see Fig. 3.3.

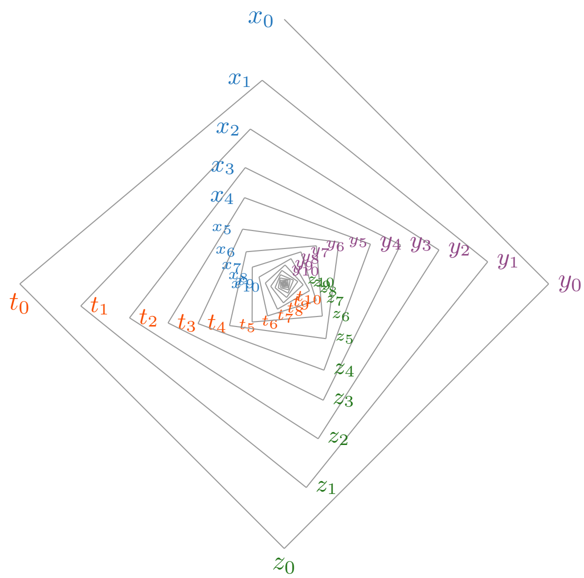

FIGURE 3.3. Convergence if four zipper sequences are involved

4. Related results

In [14] it is shown that, given a covering $X = \dot{\bigcup}_{i=1}^{m} X_i$, the set of all r-cyclic operators w.r.t. $\dot{\bigcup}_{i=1}^{m} X_i$, $1 \le r \le m$, denoted

$$CycG_m = \{f_r \text{ is } r\text{-cyclic w.r.t. } \dot{\bigcup}_{i=1}^{m} X_i | 1 \le r \le m\}, \tag{4.1}$$

is an abelian group relative to the composition of functions denoted by $\circ$.

Remark 4.1. Considering the set $SCycG_m$ of all *synchronous* r-cyclic contractions on $\dot{\bigcup}_{i=1}^{m} X_i$, this is generally not a subgroup of $(CycG_m, \circ)$. Note that for some r the contraction condition holds for elements $x \in X_i$ and $y \in X_{i+r}$, $i = \overline{1, m}$, with a certain constant $c_r \in [0, 1)$. But even if $f_{r_1} \circ f_{r_2} \in CycG_m$ for some given $r_1, r_2 \in \{1, 2, \ldots, m\}$, the contraction condition cannot be guaranteed anymore in the general case.

Remark 4.2. We denote by $ACycG_m$ the set of all *asynchronous* r-cyclic contractions on $\dot{\bigcup}_{i=1}^{m} X_i$. Then $(ACycG_m, \circ)$ is an abelian subgroup of $(CycG_m, \circ)$. Moreover, it is cyclic in the algebraic sense, that is, all elements in $ACycG_m$ are generated by f_1 (so 1-cyclic contraction, which is both synchronous and asynchronous, see (4.1)), up to a contraction constant C, taken as the maximum of all contraction constants of the elements in $ACycG_m$.

Next are some other interesting properties.

Remark 4.3. Let $X = \overset{m}{\underset{i=1}{\uplus}} A_i = \overset{m}{\underset{i=1}{\uplus}} B_i$ be two coverings of the nonempty set X. If $f : X \to X$ is r-cyclic w.r.t. $\overset{m}{\underset{i=1}{\uplus}} A_i$ and, respectively, w.r.t. $\overset{m}{\underset{i=1}{\uplus}} B_i$, then f is r-cyclic w.r.t. $\overset{m}{\underset{i=1}{\uplus}} (A_i \cup B_i)$.

There remains the open question what happens if $X = \overset{m}{\underset{i=1}{\uplus}} A_i = \overset{n}{\underset{j=1}{\uplus}} B_j$, and f is r-cyclic w.r.t. $\overset{m}{\underset{i=1}{\uplus}} A_i$, $1 \leq r \leq m$ and q-cyclic w.r.t. $\overset{n}{\underset{j=1}{\uplus}} B_j$, $1 \leq q \leq n$, while $m \neq n$. An answer for a particular case is given by the next example.

Example. Let $A_i, i = \overline{1,6}$ and $B_j, j = \overline{1,3}$ nonempty sets such that f is 2-cyclic w.r.t. $\overset{6}{\underset{i=1}{\uplus}} A_i$ and 1-cyclic w.r.t. $\overset{3}{\underset{j=1}{\uplus}} B_j$.

Then one can check that f is 1-cyclic w.r.t. $(B_1 \cup A_1) \uplus (B_2 \cup A_3) \uplus (B_3 \cup A_5)$, respectively w.r.t. $(B_1 \cup A_2) \uplus (B_2 \cup A_4) \uplus (B_3 \cup A_6)$.

Remark 4.4. Things change if we consider additional properties on the cyclic operators, in our case if they satisfy contraction conditions.

1) If f is r-cyclic contraction (either synchronous or asynchronous) w.r.t. $\overset{m}{\underset{i=1}{\uplus}} A_i$ and w.r.t. $\overset{m}{\underset{i=1}{\uplus}} B_i$, then generally f does not satisfy the same cyclic contraction condition w.r.t. $\overset{m}{\underset{i=1}{\uplus}} (A_i \cup B_i)$.
2) If f is 1-cyclic contraction w.r.t. $\overset{m}{\underset{i=1}{\uplus}} A_i$ and w.r.t. $\overset{m}{\underset{i=1}{\uplus}} B_i$, then it is a synchronous 2-cyclic contraction w.r.t. $\overset{m}{\underset{i=1}{\uplus}} (A_i \uplus B_i)$. This can be generalized.

Another remark refers to the case of two operators.

Remark 4.5. Let $X = \overset{m}{\underset{i=1}{\uplus}} X_i$ a covering of the nonempty set X and $Y = \overset{m}{\underset{i=1}{\uplus}} Y_i$ a covering of the nonempty set Y. Let $f : X \longrightarrow X$ and $g : Y \longrightarrow Y$ two r-cyclic operators w.r.t. the two considered coverings, respectively. Let $U = \overset{m}{\underset{i=1}{\uplus}} (X_i \times Y_i)$.

Then the restriction of the Cartesian product $f \times g : X \times Y \longrightarrow X \times Y$ to U is a r-cyclic operator w.r.t. U, see [16] for the case $r = 1$.

Remark 4.6. Other interesting facts about synchronous and asynchronous r-cyclic contractions on a covering $\overset{m}{\underset{i=1}{\uplus}} X_i$ are the following:

1) in case $\gcd(m, r) = 1$, both synchronous and asynchronous r-cyclic contractions are Picard operators.
2) in case $\gcd(m, r) = k > 1$, only asynchronous r-cyclic contractions are Picard operators, while synchronous r-cyclic contractions are weakly Picard operators, having at most k fixed points.

Based on these observations, we can prove a generalization of an interesting result in [19]. Lemma 1.3.3 in [19] asserts that, assuming X is a nonempty set and $f : X \to X$ a mapping, if there exists $n \in \mathbb{N}$ such that $F_{f^n} = \{x^*\}$, then $F_f = \{x^*\}$. Consequently, if (X, d) is a complete metric space and there exists $n \in \mathbb{N}$ such that f^n is a contraction, then $F_f = \{x^*\}$, see Theorem 1.3.2 in [19]. Now we may state the following:

Theorem 4.7. *Let (X, d) be a complete metric space and $f : X \to X$. Assume there exist a nonempty strict subset $Y \subset X$ and an integer $n_0 > 0$ such that*

$$X = Y \cup f(Y) \cup f^2(Y) \cup \cdots \cup f^{n_0-1}(Y),$$

while $f^{n_0}(Y) \subseteq Y$.

1) *If there exists $l \in \{1, 2, \ldots, n_0\}$ such that f^l is a synchronous l-cyclic contraction w.r.t. $\bigcup_{i=0}^{n_0-1} f^i(Y)$, then $F_f \neq \emptyset$.*

   Moreover, if $\gcd(n_0, l) = 1$, then $F_f = \{x^*\}$. If $\gcd(n_0, l) > 1$, then f has at least one and at most $\gcd(n_0, l)$ fixed points.
2) *If there exists $k \in \{1, 2, \ldots, n_0\}$ such that f^k is an asynchronous k-cyclic contraction w.r.t. $\bigcup_{i=0}^{n_0-1} f^i(Y)$, then $F_f = \{x^*\}$.*

Proof. Denoting by $X_i = f^{i-1}(Y), i = 1, 2, \ldots, n_0 - 1$, it is immediate that f^k is k-cyclic operator w.r.t. $\bigcup_{i=1}^{n_0} X_i$, for any $k \in \{1, 2, \ldots, n_0\}$, see Example 3.5 in [14].

1) If there is $l \in \{1, 2, \ldots, n_0\}$ such that f^l is a synchronous l-cyclic contraction w.r.t. $\bigcup_{i=1}^{n_0} X_i$, then there are two cases:

   If $\gcd(n_0, l) = 1$, then by Theorem 5.1 in [14] it follows that f^l is a Picard operator. Then by Lemma 1.3.3 in [19] $F_f = \{x^*\}$.

   If $\gcd(n_0, l) > 1$, then by Theorem 5.2 in [14] it follows that f^l is a weakly Picard operator, so $F_f \neq \emptyset$.
2) If there is $k \in \{1, 2, \ldots, n_0\}$ such that f^l is an asynchronous k-cyclic contraction w.r.t. $\bigcup_{i=1}^{n_0} X_i$, then by Theorem 3.4 it follows that $F_{f^k} = \{x^*\}$, so consequently $F_f = \{x^*\}$.

□

We end this section by applying the general Theorem 3.4 to give a short new proof of the result of Kirk et al. in [8], that we recall in its original notation.

Theorem 4.8 ([8]). *Let $\{A_i\}_{i=1}^p$ be nonempty closed subsets of a complete metric space and suppose $F : \bigcup_{i=1}^p A_i \to \bigcup_{i=1}^p A_i$ satisfies the following conditions (where $A_{p+1} = A_1$):*

(1) *$F(A_i) \subseteq A_{i+1}$, for $1 \leq i \leq p$;*
(2) *$\exists k \in (0, 1)$ such that $d(F(x), F(y)) \leq k \cdot d(x, y)$, $\forall x \in A_i, y \in A_{i+1}$, for $1 \leq i \leq p$.*

Then F has a unique fixed point.

Proof. Since F is cyclic (1-cyclic) contraction with constant k w.r.t. $\overset{p}{\underset{i=1}{\cup}} A_i$, it follows that for any $n \in \mathbb{N}^*$, F^n is a n-cyclic asynchronous contraction with constant $k^n \in (0,1)$ w.r.t. $\overset{p}{\underset{i=1}{\cup}} A_i$.

By Theorem 3.4 it follows that $F_F = F_{F^n} = \{x^*\}, n \in \mathbb{N}^*$. □

5. Conclusions

The results presented in this paper continue and complete our approach in [14] regarding r-cyclic operators in general and r-cyclic contractions in particular. We introduced the asynchronous r-cyclic contractions and we established the existence and uniqueness of the fixed point, which can be obtained by means of the Picard iteration starting from any point in the definition domain. Compared to the synchronous r-cyclic contractions, the asynchronous ones show a different behavior, which is discussed in the paper.

Based on these new notions and results, one can continue exploring other classes of cyclic operators satisfying a generalized contraction condition, both in a synchronous and in an asynchronous manner.

Being a rather new topic, there is not much evidence of practical applications yet. For example, recent research in fields like systems biology, climatology, meteorology, oceanology, etc. involves very complex numeric models based on some measurable data in order to describe, understand, diagnose or predict the evolution of certain processes. One of the challenges in such studies is finding a way to include seasonal changes of certain indicators in models in which only annual means are usually employed. It is reported, for example, that some errors, which further propagate, come from "holding the seasonal cycles of specific carbonate systems variables constant over time" [2] or that "seasonal variability is of limited use in diagnosing the inter-annual variability of carbon fluxes" [22]. Researchers in specific fields use different methods to partially overcome such drawbacks.

We think that reevaluating the mathematical instruments that lie beneath the models, such that the indicators reported to have small seasonal variations can be taken into account, could lead to more accurate and realistic models, in which our results could probably be of some use.

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Received: November 28, 2023; Accepted: March 14, 2024.