Fixed Point Theory, 26(2025), No. 1, 235-246
DOI: 10.24193/fpt-ro.2025.1.15
http://www.math.ubbcluj.ro/~nodeacj/sfptcj.html

ABOUT SOME INTEGRAL EQUATION IN TERMS OF A METRIC AND ORDER RELATION

ION MARIAN OLARU

University "Lucian-Blaga" of Sibiu, Faculty of Sciences, Department of Mathematics
E-mail: marian.olaru@ulbsibiu.ro

Abstract. In this paper we shall study, in terms of a metric and an order relation, the existence, uniqueness and data dependence for the solutions of integral equation

$$x(t) = (g_1(t) + \int_a^t K_1(t,s,x(s))ds) \cdot (g_2(t) + \int_a^t K_2(t,s,x(s))ds),\ t \in [a,b].$$

Our results extend the results from I.M.Olaru *An Integral Equation via Weakly Picard Operators*, Fixed Point Theory, 11(2010), No. 1, pp 97-106.
Key Words and Phrases: Integral equations, Picard operators, fixed points, data dependence, ordered metric space, Gronwall lemma.
2020 Mathematics Subject Classification: 45G10, 47H10, 45D05.

1. Introduction

Let X be a nonempty set and $T : X \to X$ an operator. We denote by $T^0 := 1_X$, $T^1 := T$, $T^{n+1} := T^n \circ T$, $n \in \mathbb{N}$ the iterate operators of the operator T. We also have

$$F_T := \{x \in X \mid T(x) = x\}$$

the set of fixed point for operator T.

Definition 1.1. (see [3]) *Let X be a nonempty set and let us consider $s(X) := \{\{x_n\}_{n\in\mathbb{N}} \mid x_n \in X\}$, $c(X) \subset s(X)$ and $Lim : c(X) \to X$ an operator. We say that $(X, c(X), Lim)$ is an $L-$space (denoted also by $(X, \xrightarrow{F})$) if the following conditions are satisfied:*

(i) *if $x_n = x$ for all $n \in \mathbb{N}$ then $\{x_n\}_{n\in\mathbb{N}} \in c(X)$ and $Lim\{x_n\}_{n\in\mathbb{N}} = x$*
(ii) *if $\{x_n\}_{n\in\mathbb{N}} \in c(X)$ and $Lim\{x_n\}_{n\in\mathbb{N}} = x$ then for all subsequences $\{x_{n_i}\}_{i\in\mathbb{N}}$ of $\{x_n\}_{n\in\mathbb{N}}$ we have $\{x_{n_i}\}_{i\in\mathbb{N}} \in c(X)$ and $Lim\{x_{n_i}\}_{i\in\mathbb{N}} = x$*

Definition 1.2. *Let $(X, \xrightarrow{F})$ be an $L-$space An operator $T : X \to X$ is orbitally continuous if $x \in X$ and $T^{n(i)}(x) \to a \in X$ as $i \to \infty$ imply $T^{n(i)+1}(x) \to T(a) \in X$ as $i \to \infty$.*

Definition 1.3. (see [12]) *Let $(X, \xrightarrow{F})$ be an $L-$space An operator $T : X \to X$ is weakly Picard operator (WPO) if the sequence $(T^n(x))_{n\in N}$ converges, for all $x \in X$ and the limit (which depend on x) is a fixed point of T.*

Definition 1.4. (see [12]) *If the operator T is WPO and $F_T = \{x^*\}$ then by definition T is Picard operator.*

Notice that if T is WPO, then we define the operator $T^\infty : X \to F_T$ by

$$T^\infty(x) = \lim_{n\to\infty} T^n(x).$$

Definition 1.5. *A triple $(X, \xrightarrow{F}, \preceq)$ is called an ordered $L-$space if $(X, \xrightarrow{F})$ is an $L-$space and $\preceq$ is a partial order on X which is closed with respect to $\xrightarrow{F}$ i.e. if $\{x_n\}_{n\in\mathbb{N}}$ and $\{y_n\}_{n\in\mathbb{N}}$ are sequences in X such that $x_n \preceq y_n$ for every $n \in \mathbb{N}$ and $x_n \to x$, $y_n \to y$ as $n \to \infty$ then $x \preceq y$*

The following abstract Gronwall type lemma takes place for POs

Lemma 1.1. *Let $(X, \xrightarrow{F}, \preceq)$ be an ordered $L-$space and $T : X \to X$ be an operator. We suppose that:*

(a) *T is a PO with respect to $\to$(we denote by x_T^* its unique fixed point);*
(b) *T is increasing with respect to $\preceq$;*

Then we have:

(i) *$x \in X$, $x \preceq T(x)$ implies $x \preceq x_T^*$;*
(ii) *$x \in X$, $T(x) \preceq x$ implies $x_T^* \preceq x$.*

Definition 1.6. (see [9]) *A nonempty ordered set $(X, \preceq)$ is said to be generalized directed set if for each pair of elements $x, y \in X$ there exists $z \in X$ such that (x, z) and (y, z) are in $X_\preceq$ where $X_\preceq := \{(x, y) \in X \times X \mid x \preceq y \text{ or } y \preceq x\}$.*

Definition 1.7. (see [9]) *Let us consider $(X, \preceq)$ an ordered set and $T : X \to X$ an operator. Then T is called a generalized monotone operator if $(T \times T)(X_\preceq) \subset X_\preceq$, where $(T \times T))(x, y) := (T(x), T(y))$ for $(x, y) \in X \times X$.*

Definition 1.8. (see [11]) *A function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is a comparison function if the following conditions are satisfied:*

(i) *φ is increasing;*
(ii) *the sequence $\varphi^n(t) \to 0$ as $n \to \infty$, for every $t > 0$.*

Definition 1.9. (see [11]) *A comparison function is a strong comparison function if*

$$\sum_{k\geq 0} \varphi^k(t) < \infty, \text{ for any } t > 0.$$

Next result will be used by us in order to study the existence and uniqueness of solution for the above mentioned integral equation.

Lemma 1.2. (see [9]) *Let X be a nonempty set, d be a metric on X and $\preceq$ an order relation on X. We consider an operator $T : X \to X$ having the generalized monotone property. We suppose that:*

(i) *$(X, \preceq)$ is a generalized directed set;*
(ii) *if $(x, y) \in X_{\preceq}$, then x and y are asymptotically equivalent;*
(iii) *the set $X_T = \{x \in X \mid T(x) \preceq x$ or $x \preceq T(x)\}$ is not empty and $T : X_T \to X_T$ is WPO.*

Then $T : X \to X$ is a PO.

More results about generalized contraction in partially ordered complete metric spaces can be found in [2], [1], [5], [4], [6], [8], [10], [9]

2. Existence and uniqueness results

Let us consider denote

$$X = C([a, b], \mathbb{R}_+) := \{x : [a, b] \to \mathbb{R}_+ \mid x \text{ is continuous}\}.$$

We consider on X the following norms:

$$\|x\|_\infty = \max_{t\in[a,b]} |x(t)|, \ \|x\|_\tau = \max_{t\in[a,b]} |x(t)| \cdot e^{-\tau(t-a)}, \tau > 0$$

and the standard order relation

$$x \preceq y \Longleftrightarrow x(t) \le y(t), \ (\forall) t \in [a, b].$$

We get the following Banach lattices $(X, +, \mathbb{R}, \|\cdot\|_\tau, \preceq)$ and $(X, +, \mathbb{R}, \|\cdot\|_\infty, \preceq)$. From the above definition we notice that $(X, \preceq)$ is a generalized directed set. Next we shall study, in the above defined Banach lattices, the existence and uniqueness, data dependence for the solution of the following integral equation:

$$x(t) = (g_1(t) + \int_a^t K_1(t, s, x(s))ds) \cdot (g_2(t) + \int_a^t K_2(t, s, x(s))ds), \ t \in [a, b] \tag{2.1}$$

The equation (2.1) is equivalent with the following fixed point problem

$$x = T(x) \tag{2.2}$$

where:

$$T : C([a, b], \mathbb{R}_+) \to C([a, b], \mathbb{R}_+)$$

$$T(x)(t) = (g_1(t) + \int_a^t K_1(t, s, x(s))ds) \cdot (g_2(t) + \int_a^t K_2(t, s, x(s))ds) \tag{2.3}$$

Our first result is the following one, where we get the existence and uniqueness, in the Banach lattices $(X, +, \mathbb{R}, \|\cdot\|_\tau, \preceq)$, for the solution of equation (2.1)

Theorem 2.1. *We suppose that*

(i) $g_i \in C([a, b], \mathbb{R}_+)$, $K_i \in C([a, b] \times [a, b] \times \mathbb{R}_+, \mathbb{R}_+)$, $i = \overline{1, 2}$;
(ii) $K_i(t, s, \cdot) : \mathbb{R}_+ \to \mathbb{R}_+$ *is increasing for every* $t, s \in [a, b]$, $i = \overline{1, 2}$;
(iii) *there exists* $L_{K_i} > 0$ *such that*

$$|K_i(t, s, u) - K_i(t, s, v)| \le L_{K_i} |u - v|,$$

for all $t, s \in [a, b]$, $u, v \in \mathbb{R}_+$, $u \le v$, $i = \overline{1, 2}$;

(iv) *there exists* $M_{K_i} > 0$ *such that*

$$|K_i(t,s,u)| \leq M_{K_i},$$

for all $t,s \in [a,b]$, $u,v \in \mathbb{R}_+$, $i = \overline{1,2}$;

(v) *the set* X_T *is not empty;*

Then

(a) *the equation (2.1) has a unique solution* $x^\star \in C([a,b],\mathbb{R}_+)$*;*

(b) *if* $x \in X$ *is such that* $x \preceq T(x)$ *then* $x \preceq x^\star$*;*

(c) *if* $x \in X$ *is such that* $T(x) \preceq x$ *then* $x^\star \preceq x$*;*

Proof. (a) First of all we remark that the condition (ii) leads us to the fact that the operator T defined by equation (2.3) is increasing and consequently it is a generalized monotone operator. On the other hand for all $x,y \in X_{\preceq}$ we have that

$$|T(x)(t) - T(y)(t)| \leq$$

$$\leq |(g_1(t) + \int_a^t K_1(t,s,x(s))ds) \cdot (g_2(t) + \int_a^t K_2(t,s,x(s))ds) -$$

$$-(g_1(t) + \int_a^t K_1(t,s,y(s))ds) \cdot (g_2(t) + \int_a^t K_2(t,s,y(s))ds)| =$$

$$= |(g_1(t) + \int_a^t K_1(t,s,x(s))ds) \cdot (g_2(t) + \int_a^t K_2(t,s,x(s))ds) -$$

$$-(g_1(t) + \int_a^t K_1(t,s,x(s))ds) \cdot (g_2(t) + \int_a^t K_2(t,s,y(s))ds) +$$

$$+(g_1(t) + \int_a^t K_1(t,s,x(s))ds) \cdot (g_2(t) + \int_a^t K_2(t,s,y(s))ds) -$$

$$-(g_1(t) + \int_a^t K_1(t,s,y(s))ds) \cdot (g_2(t) + \int_a^t K_2(t,s,y(s))ds)| \leq$$

$$\Big(\frac{L_{K_2} M(g_1,K_1)}{r} + \frac{L_{K_1} M(g_2,K_2)}{\tau}\Big)\|x-y\|_\tau e^{\tau(t-a)},$$

where

$$M(g_1,K_1) := \max_{t\in[a,b]} |g_1(t)| + M_{K_1}(b-a)$$

$$M(g_2,K_2) := \max_{t\in[a,b]} |g_2(t)| + M_{K_2}(b-a)|.$$

Then, for any $x,y \in X_{\preceq}$ we have that

$$\|T(x) - T(y)\|_\tau \leq \Big(\frac{L_{K_2} M(g_1,K_1)}{\tau} + \frac{L_{K_1} M(g_2,K_2)}{\tau}\Big)\|x-y\|_\tau.$$

Thus, for any $x \in X_T$ we get that

$$|T(x) - T^2(x)\|_\tau \leq \big(\frac{L_{K_2}M(g_1, K_1)}{\tau} + \frac{L_{K_1}M(g_2, K_2)}{\tau}\big)\|x - T(x)\|_\tau$$

and consequently $T : X_T \to X_T$ is a graphic $L-$contraction. and taking into account that

$$O_T(z) := \{T^n(z) \mid n \in \mathbb{N}\} \subset X_T$$

we get that $T^n(z) \to T^\infty(z) \in F_T$, for any $z \in X_T$. Hence T is a WPO on X_T. Since the operator T is a $L-$contraction on X_T we obtain that

$$d(T^n(x), T^n(y)) \leq L^n \cdot d(x, y) \to 0,$$

as $n \to \infty$ for any $x, y \in X_\preceq$ and consequently x, y are asymptotic equivalent. Lemma 1.2 leads us to the conclusion that operator T is PO and therefore the equation (2.1) has a unique solution in $C([a, b], \mathbb{R}_+)$.

(b)+(c) It follows from Lemma 1.1 applied to the operator T. □

Theorem 2.2. *We suppose that $g_1, g_2,\ K_1, K_2$, verify the conditions $(i) - (v)$ from the Theorem 2.1 and $(L_{K_1} \cdot M(g_2, K_2) + L_{K_2} \cdot M(g_1, K_1)) \cdot (b - a) < 1$ where*

$$M(g_1, K_1) := \max_{t\in[a,b]} |g_1(t)| + M_{K_1}(b - a)$$

$$M(g_2, K_2) := \max_{t\in[a,b]} |g_2(t)| + M_{K_2}(b - a)|.$$

Then

(a) *the equation (2.1) has a unique solution $x^\star \in C([a, b], \mathbb{R}_+)$;*
(b) *if $x \in X$ is such that $x \preceq T(x)$ then $x \preceq x^\star$;*
(c) *if $x \in X$ is such that $T(x) \preceq x$ then $x^\star \preceq x$.*

Proof. (a) By using the same arguments as in the proof of Theorem 2.1 we have that

$$\|T(x) - T(y)(t)\|_\infty \leq M(g_1, g_2, K_1, K_2)(b - a)\|x - y\|_\infty,$$

for all $x, y \in X_\preceq$.

$$M(g_1, g_2, K_1, K_2) := L_{K_1}M(g_2, K_2) + L_{K_2}M(g_1, K_1).$$

Now, the conclusions follows from Lemma 1.2

(b)-(e) Analogous with the proof or Theorem2.1. □

Theorem 2.3. *We suppose that*

(i) *$g_1, g_2,\ K_1, K_2$, verify the conditions $(i), (ii),\ (iv)$ and (iv) from the Theorem 2.1;*
(ii) *there exists $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ a strong comparison function such that*

$$|K_i(t, s, u) - K_i(t, s, v)| \leq \varphi(|u - v|),$$

for all $t, s \in [a, b],\ u, v \in \mathbb{R}_+, u \leq v\ i = \overline{1, 2}$;
(iii) $(M(g_1, K_1) + M(g_2, K_2))(b - a) \leq 1$
(iv) *X_T is not empty and $T : X \to X$ is orbitally continuous;*

Then

(a) *the equation (2.1) has a unique solution $x^\star \in C([a, b], \mathbb{R}_+)$;*

(b) *if* $x \in X$ *is such that* $x \preceq T(x)$ *then* $x \preceq x^\star$*;*
(c) *if* $x \in X$ *is such that* $T(x) \preceq x$ *then* $x^\star \preceq x$*.*

Proof. (a) First of all we remark that the condition (ii) leads us to the fact that the operator T has the generalized monotone property. On the other hand for all $x, y \in X_{\preceq}$ we have that

$$|T(x)(t) - T(y)(t)| \leq$$
$$\leq |(g_1(t) + \int_a^t K_1(t,s,x(s))ds) \cdot (g_2(t) + \int_a^t K_2(t,s,x(s))ds) -$$
$$-(g_1(t) + \int_a^t K_1(t,s,y(s))ds) \cdot (g_2(t) + \int_a^t K_2(t,s,y(s))ds)| =$$
$$= |(g_1(t) + \int_a^t K_1(t,s,x(s))ds) \cdot (g_2(t) + \int_a^t K_2(t,s,x(s))ds) -$$
$$-(g_1(t) + \int_a^t K_1(t,s,x(s))ds) \cdot (g_2(t) + \int_a^t K_2(t,s,y(s))ds) +$$
$$+(g_1(t) + \int_a^t K_1(t,s,x(s))ds) \cdot (g_2(t) + \int_a^t K_2(t,s,y(s))ds) -$$
$$-(g_1(t) + \int_a^t K_1(t,s,y(s))ds) \cdot (g_2(t) + \int_a^t K_2(t,s,y(s))ds)| \leq$$
$$(M(g_1,K_1) + M(g_2,K_2)) \cdot (b-a)\varphi(\|x-y\|_\infty),$$

where

$$M(g_1,K_1) := \max_{t\in[a,b]} |g_1(t)| + M_{K_1}(b-a)$$
$$M(g_2,K_2) := \max_{t\in[a,b]} |g_2(t)| + M_{K_2}(b-a)|.$$

Then, for any $x, y \in X_{\preceq}$ we have that

$$\|T(x) - T(y)\|_\infty \leq (M(g_1,K_1) + M(g_2,K_2)) \cdot (b-a) \cdot \varphi(\|x-y\|_\infty) \leq \varphi(\|x-y\|_\infty).$$

From the above inequality we get that $T : X_T \to X_T$ is a $\varphi-$contraction and consequently

$$d(Tx, T^2(x)) \leq \varphi(d(x, T(x))),$$

for every $x \in X_T$. Then we obtain

$$d(T^n(x), T^{n+1}(x)) \leq \varphi^n(d(x, T(x))) \to 0,$$

as $n \to \infty$, for every $x \in X_T$. Therefore, for any $n \in \mathbb{N}$ and $p \geq 1$ we have that

$$d(T^n(x), T^{n+p}(x)) \leq \sum_{k=0}^{p-1} \varphi^{n+k}(d(x, T(x)))$$

and taking into consideration the strong comparison assumption on φ we get that, for each $x \in X_T$, the sequence $\{T^n(x)\}_{n\in\mathbb{N}}$ is Cauchy. Since $T \mid X_T$ is orbitally continuous it follows that $T^n(x) \to T^\infty(x) \in F_T$ as $n \to \infty$, for each $x \in X_T$ and consequently $T : X_T \to X_T$ is WPO.

On the other hand the operator T being a $\varphi-$contraction on X_T we obtain that

$$d(T^n(x), T^n(y)) \leq \varphi^n(d(x, y)) \to 0,$$

as $n \to \infty$ for any $x, y \in X_\preceq$ and consequently x, y are asymptotic equivalent. By using Lemma 1.2 we conclude that the equation (2.1) has a unique solution in $C([a, b], \mathbb{R}_+)$.

(b)+(c) It follows from Lemma 1.1 applied to the operator T. □

3. Data dependence: Continuity

Consider the equation (2.1) and let us denote by $x(\cdot; g_1, g_2, K_1, K_2)$ the solution of this equation. We have

Theorem 3.1. *Let $g_1^j, g_2^j, K_1^j, K_2^j$, $j = 1, 2$ be as in the Theorem 2.2. We suppose that*

(a) *there exists $\eta_i > 0$ such that*

$$|g_i^1(t) - g_i^2(t)| \leq \eta_i,$$

for all $t \in [a, b]$, $i = 1, 2$;

(b) *there exists $\mu_i > 0$ such that*

$$|K_i^1(t, s, u) - K_i^2(t, s, u)| \leq \mu_i,$$

for all $t, s \in [a, b]$, $u \in \mathbb{R}_+$, $i = 1, 2$.

Then

$$\|x(\cdot; g_1^1, g_2^1, K_1^1, K_2^1) - x(\cdot; g_1^2, g_2^2, K_1^2, K_2^2)\|_\infty \leq$$
$$\leq \frac{M(g_1^1, K_1^1)(\eta_2 + \mu_2(b-a)) + M(g_2^2, K_2^2)(\eta_1 + \mu_1(b-a))}{1 - \frac{\alpha}{\tau}},$$

where

$$\alpha = \max_{j=1,2}\{L_{K_2^j} M(g_1^j, K_1^j) + L_{K_1^j} M(g_2^j, K_2^j)\}.$$

Proof. For $j = \overline{1, 2}$ we consider the operators $T_j : C([a, b], \mathbb{R}_+) \to C([a, b], \mathbb{R}_+)$ defined by

$$T_j(x)(t) = (g_1^j(t) + \int_a^t K_1^j(t, s, x(s))ds) \cdot (g_2^j(t) + \int_a^t K_2^j(t, s, x(s))ds).$$

According with Theorem 2.2 the above operators are POs and additionally

$$\|T_1(x) - T_2(x)\|_\infty \leq M(g_1^1, K_1^1)(\eta_2 + \mu_2(b-a)) + M(g_2^2, K_2^2)(\eta_1 + \mu_1(b-a)),$$

for all $x \in C([a, b], \mathbb{R}_+)$. Now the proof follows from the well known data dependence result (see [12], Theorem 10.2.1 pp.122) □

4. Smooth dependence on parameter

Next we consider the following integral equation

$$x(t,\lambda)=(g_1(t,\lambda)+\int_a^t K_1(t,s,x(s,\lambda),\lambda)ds)\cdot(g_2(t,\lambda)+\int_a^t K_2(t,s,x(s,\lambda),\lambda)ds), \quad (4.1)$$

for all $t\in[a,b],\lambda\in J\subset\mathbb{R}$. We assume that

(H_1) $J\subset\mathbb{R}$ an compact interval;
(H_2) $g_i\in C^1([a,b]\times J,\mathbb{R}_+)$, $K_i\in C^1([a,b]\times[a,b]\times\mathbb{R}_+\times J,\mathbb{R}_+)$, $i=1,2$;
(H_3) there exists $L_{K_i}>0$ such that:

$$|K_i(t,s,u)-K_i(t,s,v)|\le L_{K_i}\cdot|u-v|,$$

for all $t,s\in[a,b]$, $u,v\in\mathbb{R}_+,u\le v$, $\lambda\in J$, $i=1,2$;
(H_4) there exists $M_{K_i}>0$ such that

$$|K_i(t,s,u,\lambda)|\le M_{K_i}$$
$$\Big|\frac{\partial K_i}{\partial u}(t,s,u,\lambda)\Big|\le M_{K_i}$$

for all $t,s\in[a,b]$, $u\in\mathbb{R}_+$, $\lambda\in J$, $i=1,2$;
(H_5) $K_i(t,s,\cdot,\lambda):\mathbb{R}_+\to\mathbb{R}_+$ is increasing for every $t,s\in[a,b]$, $\lambda\in J$, $i=1,2$.

We define the operator

$$B:C([a,b]\times J,\mathbb{R}_+)\to C([a,b]\times J,\mathbb{R}_+),$$
$$B(x)(t,\lambda)=$$
$$=[g_1(t,\lambda)+\int_a^t K_1(t,s,x(s,\lambda),\lambda)ds]\cdot[g_2(t,\lambda)+\int_a^t K_2(t,s,x(s,\lambda),\lambda)ds]$$

According with Theorem 2.1, under hypothesis $(H_1)-(H_5)$, the operator B is PO. Let $x^\star(\cdot,\lambda)$ the unique fixed point of operator B. Then

$$x^\star(t,\lambda)=$$
$$[g_1(t,\lambda)+\int_a^t K_1(t,s,x^\star(s,\lambda),\lambda)ds]\cdot[g_2(t,\lambda)+\int_a^t K_2(t,s,x^\star(s,\lambda),\lambda)ds], \quad (4.2)$$

for all $t\in[a,b],\lambda\in J\subset\mathbb{R}$. We suppose that there exists $\frac{\partial x^\star}{\partial\lambda}$. Then from relation (4.2) we obtain that

$$\frac{\partial x^\star}{\partial\lambda}=$$
$$=[\frac{\partial g_1}{\partial\lambda}(t,\lambda)+\int_a^t\frac{\partial K_1}{\partial u}(t,s,x^\star(s,\lambda),\lambda)\cdot\frac{\partial x^\star}{\partial\lambda}(s,\lambda)ds+\int_a^t\frac{\partial K_1}{\partial\lambda}(t,s,x^\star(s,\lambda),\lambda)ds]\cdot$$
$$[g_2(t,\lambda)+\int_a^t K_2(t,s,x^\star(s,\lambda),\lambda)ds]+[g_1(t,\lambda)+\int_a^t K_1(t,s,x^\star(s,\lambda),\lambda)ds]\cdot$$

$$[\frac{\partial g_2}{\partial \lambda}(t,\lambda)+\int_a^t \frac{\partial K_2}{\partial u}(t,s,x^\star(s,\lambda),\lambda)\cdot\frac{\partial x^\star}{\partial \lambda}(s,\lambda)ds+\int_a^t \frac{\partial K_2}{\partial \lambda}(t,s,x^\star(s,\lambda),\lambda)ds].$$

This relation suggest us to consider the following operator

$$C: C([a,b]\times J,\mathbb{R}_+)\times C([a,b]\times J,\mathbb{R}_+)\to C([a,b]\times J,\mathbb{R}_+),$$

$$C(x,y)(t,\lambda):=$$

$$\big[\frac{\partial g_1}{\partial \lambda}(t,\lambda)+\int_a^t \frac{\partial K_1}{\partial u}(t,s,x(s,\lambda),\lambda)\cdot y(s,\lambda)ds+\int_a^t \frac{\partial K_1}{\partial \lambda}(t,s,x(s,\lambda),\lambda)ds\big]\cdot$$

$$\big[g_2(t,\lambda)+\int_a^t K_2(t,s,x(s,\lambda),\lambda)ds\big]+$$

$$\big[g_1(t,\lambda)+\int_a^t K_1(t,s,x(s,\lambda),\lambda)ds\big]\cdot$$

$$\big[\frac{\partial g_2}{\partial \lambda}(t,\lambda)+\int_a^t \frac{\partial K_2}{\partial u}(t,s,x(s,\lambda),\lambda)\cdot y(s,\lambda)ds+\int_a^t \frac{\partial K_2}{\partial \lambda}(t,s,x(s,\lambda),\lambda)ds\big].$$

In this way we have the triangular operator

$$A: C([a,b]\times J,\mathbb{R}_+)\times C([a,b]\times J,\mathbb{R}_+)\to C([a,b]\times J,\mathbb{R}_+)\times C([a,b]\times J,\mathbb{R}_+),$$

$$A(x,y)(t,\lambda)=(B(x)(t,\lambda),C(x,y)(t,\lambda)).$$

We remark that for each $x\in C([a,b]\times J,\mathbb{R}_+)$ we have

$$\big|C(x,y)(t,\lambda)-C(x,z)(t,\lambda)\big|=$$

$$\big|\big[\frac{\partial g_1}{\partial \lambda}(t,\lambda)+\int_a^t \frac{\partial K_1}{\partial u}(t,s,x(s,\lambda),\lambda)\cdot y(s,\lambda)ds+\int_a^t \frac{\partial K_1}{\partial \lambda}(t,s,x(s,\lambda),\lambda)ds\big]\cdot$$

$$\big[g_2(t,\lambda)+\int_a^t K_2(t,s,x(s,\lambda),\lambda)ds\big]$$

$$+\big[g_1(t,\lambda)+\int_a^t K_1(t,s,x(s,\lambda),\lambda)ds\big]\cdot$$

$$\big[\frac{\partial g_2}{\partial \lambda}(t,\lambda)+\int_a^t \frac{\partial K_2}{\partial u}(t,s,x,\lambda)\cdot y(s,\lambda)ds+\int_a^t \frac{\partial K_2}{\partial \lambda}(t,s,x,\lambda)ds\big]$$

$$-\big[\frac{\partial g_1}{\partial \lambda}(t,\lambda)+\int_a^t \frac{\partial K_1}{\partial u}(t,s,x(s,\lambda),\lambda)\cdot z(s,\lambda)ds+\int_a^t \frac{\partial K_1}{\partial \lambda}(t,s,x(s,\lambda),\lambda)ds\big]\cdot$$

$$\big[g_2(t,\lambda)+\int_a^t K_2(t,s,x(s,\lambda),\lambda)ds\big]$$

$$-\big[g_1(t,\lambda)+\int_a^t K_1(t,s,x(s,\lambda),\lambda)ds\big]\cdot$$

$$\big[\frac{\partial g_2}{\partial\lambda}(t,\lambda)+\int_a^t \frac{\partial K_2}{\partial u}(t,s,x(s,\lambda),\lambda)\cdot z(s,\lambda)ds+\int_a^t \frac{\partial K_2}{\partial\lambda}(t,s,x(s,\lambda),\lambda)ds\big]\big|$$

$$\leq \big[\int_a^t \big|\frac{\partial K_1}{\partial u}(t,s,x(s,\lambda),\lambda)\big|\cdot|y(s,\lambda)-z(s,\lambda)|ds\big]\cdot\big[|g_2(t,\lambda)|+\int_a^t |K_2(t,s,x(s,\lambda),\lambda)|ds\big]+$$

$$\big[\int_a^t \big|\frac{\partial K_2}{\partial u}(t,s,x(s,\lambda),\lambda)\big|\cdot|y(s,\lambda)-z(s,\lambda)|ds\big]\cdot\big[|g_1(t,\lambda)|+\int_a^t |K_1(t,s,x(s,\lambda),\lambda)|ds\big]\leq$$

$$\frac{M_{K_1}\cdot M(g_2,K_2)+M_{K_2}\cdot M(g_1,K_1)}{\tau}\cdot e^{\tau\cdot(t-a)}\cdot\|y-z\|_\tau$$

where

$$M(g_1,K_1):=\max_{(t,\lambda)\in[a,b]\times J}|g_1(t,\lambda)|+M_{K_1}(b-a)$$

$$M(g_2,K_2):=\max_{(t,\lambda)\in[a,b]\times J}|g_2(t,\lambda)|+M_{K_2}(b-a)|.$$

From the above inequality we get that

$$\|C(x,y)-C(x,z)\|_\tau\leq\frac{M_{K_1}\cdot M(g_2,K_2)+M_{K_2}\cdot M(g_1,K_1)}{\tau}\cdot\|y-z\|_\tau.$$

for each $x,y,z\in C([a,b]\times J,\mathbb{R}_+)$. Therefore the operator

$$C(x,\cdot):C([a,b]\times J,\mathbb{R}_+)\to C([a,b]\times J,\mathbb{R}_+)$$

is a $\alpha-$ contraction with $\alpha:=\dfrac{M(g_2,K_2)M_{K_1}+M(g_1,K_1)M_{K_2}}{\tau}$. From the theorem of fiber contraction (see I.A. Rus [12] Theorem 10.5.1, pp.125) we have that the operator A is Picard operator. So, the sequences

$$x_{n+1}=B(x_n),n\in\mathbb{N}$$

$$y_{n+1}=C(x_n,y_n)$$

converges uniformly to $(x^\star,y^\star)\in F_A$, for all $x_0,y_0\in C([a,b]\times J,\mathbb{R}_+)$.

If we take $x_0,y_0\in C([a,b]\times J,\mathbb{R}_+)$ such that $y_0=\frac{\partial x_0}{\partial\lambda}$ then $y_1=\frac{\partial x_1}{\partial\lambda}$ and by induction we prove that $y_n=\frac{\partial x_n}{\partial\lambda}$, for all $n\in\mathbb{N}^\star$.

Thus

$$x_n\to x^\star,\quad uniform\ \ as\ n\to\infty$$

$$\frac{\partial x_n}{\partial\lambda}\to y^\star,\quad uniform\ \ as\ n\to\infty$$

These imply that there exists $\frac{\partial x^\star}{\partial\lambda}$ and $\frac{\partial x^\star}{\partial\lambda}=y^\star$

From the above considerations, we have that

Theorem 4.1. *We consider the integral equation (4.1) in the hypothesis* $(H_1)-(H_5)$. *Then*

(i) *the equation (4.1) has, in* $C([a,b]\times J,\mathbb{R}_+)$, *a unique solution* $x^\star(t,\cdot)$;
(ii) $x^\star(t,\cdot)\in C^1(J,\mathbb{R}_+)$, *for all* $t\in[a,b]$.

References

[1] C.-M. Chen, J.-C. Lee, C.-H. Chen, *Fixed point theory of weak contractions in partially ordered metric spaces*, Journal of Applied Mathematics, **2013**, Article ID 302438(2013), 1-7.
[2] L.B. Ciric, N. Cakic, M. Rajovic, J.S. Ume, *Monotone generalized nonlinear contractions in partially ordered metric spaces*, Fixed Point Theory Appl., **2008**, Article ID 131294 (2009), 1-11.
[3] M. Frechet, *Les Espaces Abstraits*, Gauthier-Villars, Paris, 1928.
[4] J. Harjani, F. Sabetghadam, K. Sadarangani, *Generalized contractions in partially ordered metric spaces and applications to ordinary differential equations*, Nonlinear Analysis: Theory, Methods and Applications, **72**(2010), No. 3, 1188-1197.
[5] J. Harjani, K. Sadarangani, *Fixed point theorems for weakly contractive mappings in partially ordered sets*, Nonlinear Analysis: Theory, Methods and Applications, **71**(2009), No. 7-8, 3403-3410.
[6] J. Harjani, K. Sadarangani, *Fixed point theorems for monotone generalized contractions in partially ordered metric spaces and applications to integral equations*, J. Convex Anal., **19**(2012), No. 3, 853-864.
[7] I.M. Olaru, *An integral equation via weakly Picard operators*, Fixed Point Theory, **11**(2010), No. 1, 97-106.
[8] A. Petruşel, I.A. Rus, *Fixed point theorems in ordered L-Spaces*, Proc. Amer. Math. Soc., **134**(2005), No. 2, 411-418.
[9] A. Petruşel, I.A. Rus, *Fixed point theory in terms of a metric and an order relation*, Fixed Point Theory, **20**(2019), No. 2, 601-622.
[10] A.C.M. Ran, M.C. Reurings, *A fixed point theorem in partially ordered sets and some applications to matrix equations*, Proc. Amer. Math. Soc., **132**(2004), No. 5, 1435-1443.
[11] I.A. Rus, *Generalized Contractions and Applications*, Cluj University Press, Cluj-Napoca, Romania, 2001.
[12] I.A. Rus, A. Petruşel, G. Petruşel, *Fixed Point Theory*, Cluj Univ. Press, 2008.

Received: June 3, 2022; Accepted: September 17, 2023.