Fixed Point Theory, 26(2025), No. 1, 217-224
DOI: 10.24193/fpt-ro.2025.1.13
http://www.math.ubbcluj.ro/~nodeacj/sfptcj.html

A NOTE ON THE CHARACTERIZATION OF STABLE MATCHINGS FOR GENERAL PREFERENCES: A FIXED POINT APPROACH

SHRAVAN LUCKRAZ

School of Economics and CeDEx China, University of Nottingham Ningbo China, Zhejiang, China
E-mail: shravan.luckraz@nottingham.edu.cn

Abstract. We represent the problem of the existence of a stable matching as a fixed point problem and show that a generalization of the no-odd ring condition is necessary and sufficient for the existence of a stable matching. Our result uses a generalization of the Abian's fixed point theorem.
Key Words and Phrases: Existence of stable matchings, roommate problem, fixed point theorem, graph theory, existence of equilibrium.
2020 Mathematics Subject Classification: 37C25, 91B68, 06B23.

1. Introduction

While the existence of a stable matching, in the context of the marriage problem, was first studied and solved by Gale and Shapley (1962) [7], the more general roommate problem was completely solved by Tan (1991) [23] who gave a necessary and sufficient condition for the existence of a stable matching when preferences are strict. When weak preferences are allowed, Chung (2000) [4] provided a sufficient condition for the existence of a stable matching. In this paper, we use a fixed point approach to completely characterize the roommate problem for the case of weak preferences and, thus, we thus generalize Chung's result. We also generalize our results to the case of a countable number of agents.

In the original marriage problem, there are n men and n women, where each has a strict preference ordering over all the individuals of the opposite sex. In this context, a stable symmetric matching is a one-to-one mapping of the men with the women such that there is no man-woman pair who prefer each other over their present mates. Gale and Shapley's simple algorithm provides a stable matching for any given arbitrary preference relation. Since then, several authors (for example, see [19], [18], [20], [23], [16], [17], [2], [3], [6], [8], [9], [11] and [12]) have provided conditions for the existence of a stable matching for some modified version of the original problem.

I would like to thank an anonymous referee for useful comments and suggestions. I would also like to thank Wong Lee Chin for her support.

While these results typically provide sufficient conditions for the existence of a stable matching, this paper is the first to characterize a general existence result for general preferences.[1]

The success of the Gale and Shapley's algorithm for the marriage problem is due to the fact that the set of agents can be partitioned into two sets such that agents in each set strictly prefer agents in the other set to agents in their own partition. However, because this "bipartite" partitioning is not always possible for the more general roommate problem, the Gale and Shapley's algorithm does not work in general settings. By ruling out odd-rings, Chung [4] showed that, in a sense, Gale and Shapley's result can be restored. This result is not surprising as it is well-known in graph theory that a graph is bipartite if and only if it does not contain any odd cycles.[2] This fact can be applied to the roommate problem by representing the set of agents (vertices) and the ring relation (edges) to which they belong as a digraph. [3]

Taking the above into account, we represent the problem of finding a stable matching as a fixed point problem of some "best-response mapping" over the space of all matchings. We show that the mapping has a fixed point if and only if no optimal bilateral deviations are possible. We then demonstrate that the existence of an odd-ring free sequence of optimal bilateral deviations is necessary and sufficient for the existence of a stable matching. Our proof is based on an extension of Abian's [1] fixed point theorem and that of Wisniewski (1973)[24] who obtained the same result for the infinite case. Abian and Wisniewski gave a fixed point theorem that does not rely on any topological properties. While Luckraz (2014) [13] relates this theorem of Abian to the cyclicity of the mapping, Luckraz (2022) [15] generalizes this result for the case of set-valued maps.

The rest of this paper is organized as follows. In the next section we describe the class of matching problems and give the main theorem of the paper, Section 3 gives the proof of the main theorem, Section 4 gives a generalization of the main theorem to the countable infinity case, while Section 5 concludes.

2. The Canonical Matching Problem

Consider a canonical one-to-one matching problem. Let I be a finite set and $\Psi \subseteq I^I$ be the set of all matchings from I to I. In particular, $\psi \in \Psi$ iff for all $i, j \in I$, $\psi(i) = j$ iff $\psi(j) = i$. The latter implies that the mapping is necessarily a bijection. Let $\{\succeq_i\}_{i \in I}$ be the usual weak preference relations (transitive and complete) of each i over I agents. Agent i is allowed to include itself in its choice set. It is well-known that such preferences admit utility functions. We denote the utility function of player i by $U_i(.)$. Pair (i, j) is said block matching ψ iff $j \succ_i \psi(i)$ and $i \succ_j \psi(j)$. Note that i can be equal to j. A matching ψ is stable iff there exist no pair (i, j) that blocks it.

[1]Note that Chung (2000) provides a sufficient condition for this general class of weak preferences.

[2]Note that one agent can be part of many rings. But this can be represented by a vertex that has many incident edges such that the out-degree neighbours of that vertex will be the agents that are strictly preferred by the agent at the designated vertex.

[3]Note that odd cycles are ruled out in the marriage problem because an odd cycle would imply that a player prefers an agent in its own partition to some agent in the other partition, a contradiction.

For a given matching u and agents i and j, we say that $v_{u,i,j} \in \Psi$ is a *bilateral deviation* of u for players i and j if and only if

(i) $v_{u,i,j}(i) = j$, $v_{u,i,j}(j) = i$,

(ii) $v_{u,i,j}(u(i)) = u(i)$, $v_{u,i,j}(u(j)) = u(j)$,

(iii) $v_{u,i,j}(k) = u(k)$ for all $k \neq i, j, u(i), u(j)$.

We say that $v_{u,i,j}$ is a *bilateral deviation block* of u for players i and j if and only if $v_{u,i,j}$ is a bilateral deviation and the following conditions are satisfied.

$$v_{u,i,j}(i) \succ_i u(i)$$
$$v_{u,i,j}(j) \succ_j u(j)$$

For a given matching u and agent i, let $\Psi^i_{u,(i,j)}$ denote the set of all bilateral deviations blocks of u for agent i with some agent j. Note that it is possible that $\Psi^i_{u,(i,j)} = \emptyset$. When $\Psi^i_{u,(i,j)} \neq \emptyset$, we say that $v^{i*}_{u,i,j} \in \Psi^i_{u,(i,j)}$ is an *optimal bilateral deviation block* for agent i at u iff

$$v^{i*}_{u,i,j}(i) \in \underset{\left\{v_{u,i,j} \in \Psi^i_{u,(i,j)} : j \in I\right\}}{Arg\max} U_i(v_{u,i,j}(i))$$

Finally, define an *optimal bilateral deviation block set-valued map* as follows. F : $\Psi \twoheadrightarrow \Psi$ such that

$$F(u) = \begin{cases} \left\{v : v = v^{i*}_{u,i,j} \text{ for some } i\right\} \text{ if } \Psi^i_{u,(i,j)} \neq \emptyset \text{ for some } i \text{ ;} \\ \{u\} \text{ otherwise.} \end{cases}$$

Note that the above is well-defined since the finiteness of I ensures that $\underset{\left\{v_{u,i,j} \in \Psi^i_{u,(i,j)} : j \in I\right\}}{Arg\max} U_i(v_{u,i,j}(i))$ is either non-empty or empty. It can be verified that $F(u)$ is a set-valued mapping has a fixed point if and only if some stable matching exists. An *orbit* of the above set-valued map, at some point u, is an infinite sequence of iterations of the mapping starting with u. We say that an orbit is acyclical if it has no finite cyclic subsequence.

For a given u, we can also define $F^2(u)$ as follows.

$$F^2(u) = \bigcup_{v \in F(u)} F(v)$$

And more generally, we can define $F^n(u)$ for $n \geq 2$ recursively as follows.

$$F^n(u) = \bigcup_{v \in F^{n-1}(u)} F(v),$$

where we use the convention that $F^0(u) = \{u\}$, so that F^0 is the identity mapping. We can thus say that $w \in \Psi$ is on the orbit of F for some initial point u if $w \in F^n(u)$ for some n.

Note that if, at some u, player i strictly prefers itself to $u(i)$, then, in one step, it can form a blocking pair by pairing with itself. Therefore, it does not lead to any loss in generality to assume that all possible optimal blocks of player i are those that it weakly prefers to matchings in which it gets matched with itself. More formally, we define the concept of dominance as follows.

For a given $y \in \Psi$, we say that y is *dominated* if some agent i strictly prefers being alone than its match in y. We then denote the restriction of F to subset $\Omega \subseteq \Psi$, where Ω is the set of undominated elements of Ψ, by $F\mid_\Omega$.

We next give a definition of odd rings, following Chung (2000). Chung shows that if no stable matching exists, then some odd ring must exist. We consider a weaker version of the no-odd ring condition given as follows.

An odd ring is an ordered subset of agents $\langle x_1, ...x_k\rangle \pmod k$ for $k \geq 3$ satisfying the following.

$$x_{i+1} \succ_i x_{i-1} \succeq_i x_i \; 1 \leq i \leq k$$
$$\text{and } k \text{ is odd.}$$

We define a $k-$general ring as an infinite sequence of agents denoted by $\langle x_1, ...x_k, ...\rangle \pmod k$. We call $k-$general ring $\langle x_1, ...x_k, x_{k+1}, ..\rangle \pmod k$ (where $x_{k+1} = x_1$) an *optimal* $k-$*general ring* for some cyclical orbit M of $F\mid_\Omega$ at some $u \in \Omega$ if M has a finite subsequence segment of length k, denoted by $M' = \langle m_j\rangle_{j=1}^k$, satisfying

(i) $m_1 = u$,

(ii) $x_1 = m_k(x_k)$ and

(iii) $x_i = m_{i-1}(x_{i-1})$ for $i = 2, .., k$, where $x_i \neq x_{i-1}$ for each i and each m_i is on the orbit of $F\mid_\Omega$ for initial point u.

We say that an optimal $k-$general ring is *odd (even)* if k is odd (even). We say that some orbit M of $F\mid_\Omega$ is optimal and odd rings free if for all u along M, there exist no optimal odd k $-$general rings at u.

The following theorem is the main result of our paper.

Theorem 1. *A stable matching of Ψ exists iff $F\mid_\Omega$ has some orbit that is optimal and odd rings free.*

Theorem 1 completely characterizes the existence of stable matchings for general preferences. It strengthens the existing result of [4], who showed that the no-odd ring condition is sufficient for the existence of a stable matching for this class of problems. The following example shows that even if the no-odd ring condition is violated, a stable matching can exist as long as the no optimal odd k $-$general rings is satisfied.

Example 1. Consider a matching problem with four agents: A,B,C,D with the following preferences.

Agent A: $D \succ_A B \succ_A C \succeq_A A$

Agent B: $C \succ_B D \succ_B A \succeq_B B$

Agent C: $A \succ_C B \succ_C D \succeq_C C$

Agent D: $A \succ_D C \succ_D B \succeq_D D$

Matching u that maps A with D and B with C is stable. Note that the no-odd ring condition is violated as the following odd ring can be constructed.

$$\langle x_1, ...x_k\rangle \pmod k \text{ for } k = 3,$$
$$\text{where } x_1 = A, \;\; x_2 = B \text{ and } x_3 = C.$$

However, it can be verified that the above odd ring is not an optimal odd k $-$general ring since only mappings that map A with D are optimal.

3. Proof of Theorem 1

In this section we construct a proof of the main result of the paper. Our strategy will be to show that $F\mid_{\Omega}$ has a fixed point iff the no odd k −general rings condition is satisfied. We will use the [13]'s adaptation of the Abian's fixed point theorem. We therefore first state the Abian's theorem.

Abian Theorem. *Let X be a non-empty finite set and let $f : X \to X$. Then, f has a fixed point iff X cannot be partitioned into three sets A, B and C such that $A \cap F(A) = B \cap F(B) = C \cap F(C) = \emptyset$.*

Since Theorem 1 is based on a set-valued map, a generalization of the Abian's theorem is needed. However, [14] observed that the partitioning method does not work if the mapping is set-valued. We first state the Abian's theorem for set-valued maps.

Claim 1. Let $F : X \twoheadrightarrow X$ be a set-valued mapping. Then F has a fixed point iff X cannot be partitioned into three sets A, B and C such that $A \cap F(A) = B \cap F(B) = C \cap F(C) = \emptyset$.

The above is equivalent to the statement that the mapping has no fixed points iff such a partitioning exists. The following counter-example shows that the claim is false.

Example 2. Let $X = \{a, b, c, d\}$, $F(a) = \{b, c\}$, $F(b) = \{c\}$, $F(c) = \{b, d\}$ and $F(d) = \{a, b\}$. Then clearly, F has no fixed points. Then by the negation of Claim 1, we should be able to find a partition of X into sets A, B and C such that that $A \cap F(A) = B \cap F(B) = C \cap F(C) = \emptyset$. Suppose such three sets exist, then from the given set-valued map F, a, b and c need to be in three different sets. But d cannot be in any of these three sets, which is a counter-example to Claim 1.

We show that even with $n \geq 3$ partitions, Abian's theorem does not generalize. The following definition and example follow from [14].

Definition. Let X be a non-empty finite set and let $F : X \to P(X)$ be a non-empty valued set-valued map. Then collection $\mathcal{X} = \{X_i\}_{i\in\{1,..,k\}}$ is called a trivial partition of X iff $k = |X|$ and each X_i is a singleton.

Example 3. Let X be a non-empty finite set and let $\mathcal{F}$ denote the set of all set-valued maps from X to X. Then, there exist some $F \in \mathcal{F}$ that has no fixed points and that does not satisfy Abian's partitioning condition for any number of sets strictly less than $|X|$. Indeed, since X is finite it can be enumerated as follows: $X = \{x_i\}_{i\in\{1,..,|X|\}}$. For each pair i, j, let $x_i \in F(x_j)$ iff $i \neq j$. Then clearly, X cannot be partitioned into n sets such that $n < |X|$ and that satisfy Abian's condition.

Since the Abian's theorem does not generalize, we make use of [15]'s idea of cycles to study the fixed points of the mappings. The following can be derived from Theorem 4 of [15].

Result 1. [15] Let F be a set-valued map from a finite set D into itself. Then, F has a fixed point iff F has an acyclical orbit.

We are now ready to prove Theorem 1.

Proof of Theorem 1.

(If) We prove by contradiction. Suppose that $F\mid_{\Omega}$ has some orbit that is free from optimal odd k −general rings but $F\mid_{\Omega}$ has no fixed points. Then, by the finiteness of the number of agents, the acyclicity of the preference relation and Result 1, all orbits M must be cyclic. But the definition of optimal k −general rings implies that all cyclic orbits have subsequence segments that form optimal k −general rings since, starting from any point of some M, some agent will be matched to its initial match at some point along M. Hence, each term of each M is the initial point of some subsequence that generates some optimal k −general ring (even or odd).

Next, we claim that the above implies that all generated optimal k −general rings must be odd. Suppose not. Then, there must exist some optimal k −general ring generated by terms of some M that is even. Then, one can consider some optimal even k −general ring (generated by some term of some M), denoted by ω, and let matching $v^* \in \Omega$ be such that agent i is mapped to agent $i+1$ for each $i < k$ along ω. Thus, the k agents on ω are arranged into $k/2$ pairs via v^*, where for each pair, each agent weakly prefers her match to being alone (by the definition of Ω). Since each $z \in \Omega$ is part of some cyclical orbit (because $F\mid_\Omega$ has no fixed points), there must exist some orbit N and some optimal k' −general ring denoted by μ for some subsequence segment N' of N with initial point v^*. But on μ at v^*, there must exist at least one agent that we could designate as agent 1 on μ so that $x_1 = m_{k'}(x_{k'})$ cannot hold since the optimal choice of x_1 at v^* (subject to the other agents willingness to form a bilateral deviation block with it) is its current match $x_2 = v^*(x_1)$. As a result, we have a contradiction to the fact that at least one optimal k −general ring must be even. Hence, we obtain the claim that all generated optimal k −general rings must be odd. However, this contradicts the hypothesis of the theorem that $F\mid_\Omega$ has some orbit that is optimal odd k −general rings free.

(Only if) We prove by contradiction. Suppose that a fixed point exists and that each orbit M of $F\mid_\Omega$ contains at least one point $u \in \Omega$ so that u is the initial point of some finite subsequence segment M' (of M) that generates some optimal odd k −general ring, for some k. Then, the definition of orbits and that of $F\mid_\Omega$ would imply that all orbits of $F\mid_\Omega$ are cyclical, a contradiction to Result 1.□

4. Generalizations to the countable case

For single-valued mappings on infinite sets, Wisniewski (1973)[4] introduced a fixed point theorem that is in similar vein of the Abian's Theorem. Using similar arguments as in Claim 1, Example 2 and Example 3 from the previous sections, one can show that the fixed point theorem cannot be applied in a straightforward manner to set-valued maps since in the matching problem at hand, $F\mid_\Omega$ need not be single-valued.

For set-valued maps defined on countable domains, [15] (Remark 1 on page 4), gave a characterization of fixed points that is closely related to the acyclicity of the mapping. In order to apply [15]'s result to the problem at hand when the agent set can be countably infinite, we first need to assume the following.

(A1) $\underset{\left\{v_{u,i,j} \in \Psi^i_{u,(i,j)} : j \in I\right\}}{Arg\max} U_i(v_{u,i,j}(i))$ is well-defined.

[4]We thank an anonymous referee for suggesting this reference.

We say that $F\mid_\Omega$ contains an infinite cycle if it has some orbit of infinite length with all distinct elements. We can now restate Theorem 1 of the previous section as follows.

Theorem 2. Suppose that A1 holds and I is countable. Then, Ψ has a stable matching iff $F\mid_\Omega$ has some orbit that is not an infinite cycle and that is optimal and odd rings free.

The proof follows from [15]'s result and from Theorem 1.

5. Conclusion

This paper gave a necessary and sufficient condition for the existence of a stable matching when agents are allowed to have general weak preferences. The no-odd ring condition of Chung (2000) was generalized to a no-optimal odd k −general ring condition. The proof was constructed by representing the existence problem as a fixed point problem and by using an extension of the Abian's and Wisniewski's theorems.

References

[1] A. Abian, *A fixed point theorem*, Nieuw Arch. Wiskd, **16**(1968),184-185.

[2] D. Abraham, P. Biró, D. Manlove, *Almost stable matchings in the roommates problem*, Lecture Notes in Computer Science, **3879**(2006), 1–14.

[3] P. Biró, K. Cechlarova, T. Fleiner, *The dynamics of stable matchings and half-matchings for thes table marriage and roommates problems*, International Journal of Game Theory, **36**(2008), 333–352.

[4] K. Chung, *On the existence of stable roommate matchings*, Games and Economic Behavior, **33**(2000), 206–230.

[5] E. Diamantoudi, E. Miyagawa, L. Xue, *Random paths to stability in the roommate problem*, Games and Economic Behavior, **48**(2004), 18–28.

[6] L. Ehlers, *Von Neumann–Morgenstern stable sets in matching problems*, Journal of Economic Theory, **134**(2007), 537–547.

[7] D. Gale, L. Shapley, *College admissions and the stability of marriage*, American Mathematical Monthly, **69**(1962), 9–15.

[8] R. Gusfield, R. Irving, *The Stable Marriage Problem: Structure and Algorithms*, MIT Press, Cambridge, 1989.

[9] E. Inarra, C. Larrea, E. Molis, *Random paths to P-stability in the roommate problem*, International Journal of Game Theory, **36**(2008), 461–471.

[10] Inarra, E., Larrea, C., Molis, *Absorbing Sets in Roommate Problem*, Games and Economic Behavior, **81**(2013), 165-178.

[11] A. Kelso, V. Crawford, *Job matching,coalition formation and gross substitutes*, Econometrica, **50**(1982), 1483–1504.

[12] B. Klaus, F. Klijn, M. Walzl, *Stochastic stability for roommates markets*, Journal of Economic Theory, **145**(2010), 2218–2240.

[13] S. Luckraz, *A note on the relationship between the isotone assumption of the Abian-Brown fixed point theorem and Abian's most basic fixed point theorem*, Fixed Point Theory and Applications, **129**(2014), 1-7.

[14] S. Luckraz, *Two Remarks on the Infinite Approximation of a Finite World in Economic Models*, Journal of Mathematics, **2457746**(2022), 1-5.

[15] S. Luckraz, *On a fixed point theorem for general multivalued mappings on finite sets with applications in game theory*, Journal of Mathematics, **2477124**(2022), 1-9.

[16] D. McVitie, D.L. Wilson, *Stable marriage assignments for unequal sets*, BIT, **10**(1970) 295–309.

[17] B. Pittel, R. Irving, *An upper bound for the solvability probability of a random stable roommates instance*, Random Structures Algorithms, **5**(1994), 465–486.

[18] A. Roth, *Subsolutions and the super core of cooperative games*, Mathematics of Operations Research, **1**(1976), 43–49.
[19] A. Roth, *The college admissions problem is not equivalent to the marriage problem*, Journal of Economic Theory, **36**(1985), 277–288.
[20] A. Roth, M. Sotomayor, *Two-Sided Matching: A Study in Game-Theoretic Modeling and Analysis*, Econometric Society Monograph, 18, Cambridge University Press, Cambridge, 1990.
[21] I.A. Rus, A. Petruşel, G. Petruşel, *Fixed Point Theory*, Cluj Univ. Press, 2008.
[22] J. Tan, *A maximum stable matching for the roommates problem*, BIT, **29**(1990), 631–640.
[23] J. Tan, *A necessary and sufficient condition for the existence of a complete stable matching*, Journal of Algorithms, **12**(1991), 154–178.
[24] K. Wisniewski, *On functions without fixed points*, Ann. Soc. Math. Pol, **17**(1973), 227-228.

Received: January 24, 2022; Accepted: October 26, 2023.