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SEVERAL NEW FIXED POINT RESULTS FOR MULTI-VALUED QUASI-CONTRACTIONS IN b-METRIC SPACES

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Abstract. In the framework of b-metric spaces, we establish three fixed point theorems concerning multi-valued contractions, which improve and generalize various well known results in the literature. Based on the result of Aydi et al. (Journal of Fixed Point Theory and Applications 2012: 88 2012), we give the first fixed point theorem for q-multi-valued quasi-contraction, where the range of the contraction constant is extended to $\left[0, \frac{1}{\sqrt[3]{s+2s^2}}\right)$ $(1 \leq s \leq 1+\sqrt{2})$ and $\left[0, \frac{1}{s}\right)$ $(s > 1+\sqrt{2})$. Also, we establish the second result which extends the theorem presented by Haghi et al. (Applied Mathematics Letters 25: 843-846 2012) from metric spaces to b-metric spaces. Furthermore, we give a unified result to improve the recent several fixed point theorems for multi-valued mappings provided by Miculescu et al. (Journal of Fixed Point Theory an Applications 19: 2153-2163 2017). Two technical lemmas are used to ensure that a Picard sequence is a Cauchy sequence. Finally, some applications are included to vindicate that the improvements are indeed genuine.
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1. Introduction

In 1969, Nadler [23] presented a fixed point theorem for multi-valued mappings with Hausdorff metric, which is an extension of Banach contraction principle. In the last decades, all kinds of fixed point results for multi-valued functions have been studied in the framework of metric spaces (see, for example, [26, 15, 27, 12, 4, 30, 5, 13, 14, 1, 28, 3, 29, 24, 22, 19] and the references therein).

Definition 1.1 ([6])**.** Let X be any nonempty set. An element x in X is said to be a fixed point of a multi-valued mapping $T : X \to 2^X$ if $x \in Tx$, where 2^X denotes the collection of all nonempty subsets of X.

Definition 1.2 ([6])**.** Let (X, d) be a metric space. Let $\mathfrak{CB}(X)$ be the collection of all nonempty closed bounded subsets of X. For $A, B \in \mathfrak{CB}(X)$, define

$$H(A, B) = \max\{\delta(A, B), \delta(B, A)\}, \tag{1.1}$$

where

$$\delta(A,B)=\sup\{d(a,B),a\in A\},\delta(B,A)=\sup\{d(b,A),b\in B\} \tag{1.2}$$

with

$$d(a,C)=\inf\{d(a,x),x\in C\},C\in\mathfrak{CB}(X). \tag{1.3}$$

Note that H is called the Hausdorff metric induced by the metric d.

On the other hand, in 1993, Czerwik [10] introduced a new class of generalized metric spaces called b-metric spaces which have been studied by numerous authors (also see [16]). In the sequel, the letters $\mathbb{R}^+$, $\mathbb{N}$ and $\mathbb{N}^*$ will denote the set of all nonnegative real numbers, the set of all natural numbers and the set of all positive integer numbers, respectively.

Definition 1.3 ([10])**.** Let X be a nonempty set and $s\geq 1$ a given real number. A mapping $d: X\times X\to\mathbb{R}^+$ is called a b-metric if

(1) $d(x,y)=0$ if and only if $x=y$;
(2) $d(x,y)=d(y,x)$, for all $x,y\in X$;
(3) $d(x,y)\leq s[d(x,z)+d(z,y)]$, for all $x,y,z\in X$.

Then, the pair (X,d) is called a b-metric space.

As a kind of the meaningful fixed point results, the theorems for multi-valued contractions have also been studied in the setting of b-metric spaces, see [6, 11, 21, 17, 2, 20, 8, 7, 25]. By combining the results of Nadler [23] and Ćirić [9], Amini-Harandi [5] initiated the concept of q-multi-valued quasi-contraction in 2011 and proved the corresponding fixed point theorem in metric spaces. After these pioneering work, Aydi et al. [6] extended Amini-Harandi's theorem and some existing results in the literature to b-metric spaces with contraction constant $q<\frac{1}{s^2+s}$. In addition, Amini-Harandi [5] put forward a question:

Does the conclusion of [5, Theorem 2.2] **remain true for any** $k\in\left[\frac{1}{2},1\right)$**?**

As the research on Amini-Harandi's question [5], Haghi et al. [14] introduced the notion of multi-valued quasi-contraction type multifunction and extended the contraction constant to $[0,1)$ for such mappings. Furthermore, Lu et al. [19] presented the fixed point theorem for q-multi-valued quasi-contraction mapping in metric spaces by extending the contraction constant from $\left[0,\frac{1}{2}\right)$ to $\left[0,\frac{1}{\sqrt[3]{3}}\right)$, which partially answers the Amini-Harandi's [5] question. Naturally, a question will arise:

Can the theorem provided by Haghi et al. [14] *and Lu et al.* [19] *be improved to b-metric spaces?*

Another remarkable generalization of Nadler's contraction principle was given by Miculescu et al. [20] in 2017. In their outstanding paper, the authors proposed three fixed point theorems for multi-valued mappings in b-metric spaces, which also improved the result due to Aydi et al. [6].

In this paper, drawing inspiration from the above mentioned works, we establish three fixed point theorems concerning multi-valued mappings by using the Hausdorff metric in b-metric spaces. The first theorem generalizes [6, Theorem 2.2] and [2, Theorem 3.3] in b-metric spaces, where the contraction constant

$$q < \begin{cases} \frac{1}{\sqrt[3]{s+2s^2}}, & 1 \le s \le 1+\sqrt{2}; \\ \frac{1}{s}, & s > 1+\sqrt{2}. \end{cases}$$

The second result extends the theorem by Haghi et al. [14] from metric spaces to b-metric spaces. As the last part of our main results, we establish an unified result of three theorems for multi-valued functions by Miculescu et al. [20]. This theorem also improves Nadler's fixed point theorem, Rus's fixed point theorem, Reich's fixed point theorem and Hardy-Rogers type fixed point theorem. The scientific novelty of our proofs lies in the application of two crucial lemmas and some skills to prove a Picard sequence is a Cauchy sequence. Finally, some related applications are given to illustrate that our results are true extensions of the existing ones.

2. Preliminaries

In this section, we present two lemmas which will be applied in later sections. Other elementary lemmas concerning Hausdorff metric refer to [6].

Lemma 2.1 ([18])**.** *Let (X, d) be a b-metric space and $\{x_n\}$ a sequence in X. If there exist $P \ge 0$ and $0 \le Q < 1$ such that*

$$d(x_n, x_{n+1}) \le PQ^{n+1}$$

for all $n \in \mathbb{N}^*$, then $\{x_n\}$ is a Cauchy sequence.

Lemma 2.2. *Let (X, d) be a b-metric space, $\{A_n\} \subset \mathfrak{CB}(X)$ be a sequence of set and $A^* \in \mathfrak{CB}(X)$. Let $\{a_n\} \subset X$ be a sequence such that $a_n \in A_n$ for all $n \in \mathbb{N}$. If*

$$\lim_{n\to\infty} H(A_n, A^*) = 0 \tag{2.1}$$

and

$$\lim_{n\to\infty} d(a_n, a^*) = 0 \tag{2.2}$$

for some $a^* \in X$, then $a^* \in A^*$.

Proof. By means of $a_n \in A_n$ for all $n \in \mathbb{N}$, we have $d(a_n, A^*) \le H(A_n, A^*)$. Due to (2.1), we can obtain that

$$\lim_{n\to\infty} d(a_n, A^*) = 0. \tag{2.3}$$

Then, there exists a sequence $\{b_n\} \subset A^*$ such that $d(a_n, b_n) \le d(a_n, A^*) + \frac{1}{n}$ for all $n \in \mathbb{N}$. From (2.3), we deduce that

$$\lim_{n\to\infty} d(a_n, b_n) = 0. \tag{2.4}$$

By the triangle inequality, we get $d(b_n, a^*) \le s[d(b_n, a_n) + d(a_n, a^*)]$. Combining (2.2) and (2.4), we conclude that $\lim_{n\to\infty} b_n = a^*$, which implies $a^* \in A^*$. □

3. Main Results

In this section, we establish and prove our main results. Before that we give two significant lemmas, which play a crucial role in the sequel.

Lemma 3.1. *Let (X, d) be a b-metric space with $s \geq 1$ and $\{x_n\}$ a sequence in X. Suppose that there exist $\beta < 1$ and a positive integer p such that*

$$d(x_n, x_{n+1}) \leq \beta \max\{d(x_i, x_{i+1}) : n-p \leq i \leq n-1\} \tag{3.1}$$

for all $n \in \mathbb{N}$ with $n \geq p$, then $\{x_n\}$ is a Cauchy sequence.

Proof. Let $\lfloor r \rfloor = \max\{n \in \mathbb{N} : n \leq r\}$ for all $r \in \mathbb{R}$. Define that

$$G = \max\{d(x_i, x_{i+1}) : 0 \leq i \leq p-1\}.$$

By (3.1), we obtain that $d(x_p, x_{p+1}) \leq \beta G < G$. Then, applying (3.1) again, we have $d(x_{p+1}, x_{p+2}) \leq \beta G$. Continuing this process, we can see that

$$d(x_{p+j}, x_{p+j+1}) \leq \beta G$$

for all $0 \leq j \leq p-1$, which implies that $\max\{d(x_{p+j}, x_{p+j+1}) : 0 \leq j \leq p-1\} \leq \beta G$. Similarly, we can obtain that $\max\{d(x_{2p+j}, x_{2p+j+1}) : 0 \leq j \leq p-1\} \leq \beta^2 G$. Proceeding inductively, we conclude that $\max\{d(x_{ip+j}, x_{ip+j+1}) : 0 \leq j \leq p-1\} \leq \beta^i G$ for all $i \in \mathbb{N}$.

For all $n \in \mathbb{N}$, by $n = \left\lfloor \frac{n}{p} \right\rfloor p + j$ for some $0 \leq j \leq p-1$, we derive that

$$\begin{aligned} d(x_n, x_{n+1}) &\leq \max\left\{d(x_{\lfloor \frac{n}{p} \rfloor p+j}, x_{\lfloor \frac{n}{p} \rfloor p+j+1}) : 0 \leq j \leq p-1\right\} \\ &\leq \beta^{\lfloor \frac{n}{p} \rfloor} G \\ &\leq \beta^{\frac{n}{p}-1} G \\ &= \beta^{\frac{n}{p}} \frac{G}{\beta} \\ &= (\sqrt[p]{\beta})^n G^*, \end{aligned}$$

where $G^* = \frac{G}{\beta}$ and for all $n \in \mathbb{N}$. From Lemma 2.1, we can obtain that $\{x_n\}$ is a Cauchy sequence in X. □

Lemma 3.2. *Let (X, d) be a b-metric space with $s > 1 + \sqrt{2}$ and $\{x_n\}$ a sequence in X. If there exist $\lambda \in \left[0, \frac{1}{s}\right)$ and a positive integer $k \geq 2$ such that*

$$d(x_n, x_{n+1}) \leq \max\{\lambda^2 d(x_{n-2}, x_n), \lambda^3 d(x_{n-3}, x_n), \cdots, \lambda^k d(x_{n-k}, x_n)\} \tag{3.2}$$

for all $n \in \mathbb{N}$ and $n \geq k$, then

$$d(x_n, x_{n+1}) \leq \frac{2s+1}{s^2} \max\{d(x_{n-1}, x_n), d(x_{n-2}, x_{n-1}), \cdots, d(x_{n-k}, x_{n-k+1})\}.$$

Proof. From (3.2) we consider the following four cases to prove the desired result.
Case 1. If

$$d(x_n, x_{n+1}) \leq \lambda^2 d(x_{n-2}, x_n), \tag{3.3}$$

then we prove

$$d(x_n, x_{n+1}) \le \frac{2}{s} \max\{d(x_{n-2}, x_{n-1}), d(x_{n-1}, x_n)\}. \tag{3.4}$$

From (3.3) and using the triangle inequality,

$$\begin{aligned} d(x_n, x_{n+1}) &\le \lambda^2 d(x_{n-2}, x_n) \\ &\le s\lambda^2[d(x_{n-2}, x_{n-1}) + d(x_{n-1}, x_n)] \\ &\le 2s\lambda^2 \max\{d(x_{n-2}, x_{n-1}), d(x_{n-1}, x_n)\}. \end{aligned}$$

Since $\lambda < \frac{1}{s}$, we conclude that

$$d(x_n, x_{n+1}) \le \frac{2}{s} \max\{d(x_{n-2}, x_{n-1}), d(x_{n-1}, x_n)\}.$$

Case 2. If

$$d(x_n, x_{n+1}) \le \lambda^3 d(x_{n-3}, x_n), \tag{3.5}$$

then we will show that

$$d(x_n, x_{n+1}) \le \frac{2s+1}{s^2} \max\{d(x_{n-3}, x_{n-2}), d(x_{n-2}, x_{n-1}), d(x_{n-1}, x_n)\}. \tag{3.6}$$

By (3.5), we obtain that

$$\begin{aligned} d(x_n, x_{n+1}) &\le \lambda^3 d(x_{n-3}, x_n) \\ &\le s\lambda^3[d(x_{n-3}, x_{n-1}) + d(x_{n-1}, x_n)] \\ &\le s^2\lambda^3[d(x_{n-3}, x_{n-2}) + d(x_{n-2}, x_{n-1})] + s\lambda^3 d(x_{n-1}, x_n) \\ &\le (2s^2\lambda^3 + s\lambda^3) \max\{d(x_{n-3}, x_{n-2}), d(x_{n-2}, x_{n-1}), d(x_{n-1}, x_n)\}. \end{aligned}$$

Similarly, owing to $\lambda < \frac{1}{s}$, we get

$$\begin{aligned} d(x_n, x_{n+1}) &\le \frac{2s^2+s}{s^3} \max\{d(x_{n-3}, x_{n-2}), d(x_{n-2}, x_{n-1}), d(x_{n-1}, x_n)\} \\ &= \frac{2s+1}{s^2} \max\{d(x_{n-3}, x_{n-2}), d(x_{n-2}, x_{n-1}), d(x_{n-1}, x_n)\}. \end{aligned}$$

Case 3. If

$$d(x_n, x_{n+1}) \le \lambda^l d(x_{n-l}, x_n), \tag{3.7}$$

where $l = 2m$ for some $m \in \{2, 3, \cdots, \lfloor \frac{k}{2} \rfloor\}$, then we prove

$$\begin{aligned} d(x_n, x_{n+1}) \le &\frac{2s^2 + 2s^3 + \cdots + 2s^{m-1} + 4s^m}{s^{2m}} \max\{d(x_{n-2m}, x_{n-2m+1}), \\ &d(x_{n-2m+1}, x_{n-2m+2}), \cdots, d(x_{n-1}, x_n)\}. \end{aligned} \tag{3.8}$$

By (3.7) and applying the triangle inequality,

$$\begin{aligned}
&d(x_n, x_{n+1}) \leq \lambda^{2m} d(x_{n-2m}, x_n)\\
\leq& s\lambda^{2m}[d(x_{n-2m}, x_{n-m}) + d(x_{n-m}, x_n)]\\
\leq& [s^2\lambda^{2m} d(x_{n-2m}, x_{n-2m+1}) + s^3\lambda^{2m} d(x_{n-2m+1}, x_{n-2m+2})\\
&+ \cdots + s^m\lambda^{2m}(d(x_{n-m-2}, x_{n-m-1}) + d(x_{n-m-1}, x_{n-m}))]\\
&+ [s^2\lambda^{2m} d(x_{n-m}, x_{n-m+1}) + s^3\lambda^{2m} d(x_{n-m+1}, x_{n-m+2})\\
&+ \cdots + s^m\lambda^{2m}(d(x_{n-2}, x_{n-1}) + d(x_{n-1}, x_n))]\\
\leq& (2s^2\lambda^{2m} + 2s^3\lambda^{2m} + \cdots + 2s^{m-1}\lambda^{2m} + 4s^m\lambda^{2m})\\
&\cdot \max\{d(x_{n-2m}, x_{n-2m+1}), d(x_{n-2m+1}, x_{n-2m+2}), \cdots, d(x_{n-1}, x_n)\}.
\end{aligned}$$

Using the fact that $\lambda < \frac{1}{s}$, we deduce that

$$\begin{aligned}
d(x_n, x_{n+1}) \leq& \frac{2s^2 + 2s^3 + \cdots + 2s^{m-1} + 4s^m}{s^{2m}} \max\{d(x_{n-2m}, x_{n-2m+1}),\\
&d(x_{n-2m+1}, x_{n-2m+2}), \cdots, d(x_{n-1}, x_n).
\end{aligned}$$

Case 4. If

$$d(x_n, x_{n+1}) \leq \lambda^l d(x_{n-l}, x_n), \tag{3.9}$$

where $l = 2m + 1$ for some $m \in \left\{2, 3, \cdots, \left\lfloor \frac{k-1}{2} \right\rfloor\right\}$, then we shall prove that

$$\begin{aligned}
d(x_n,x_{n+1}) \leq& \frac{2s^2 + 2s^3 + \cdots + 2s^{m-1} + 3s^m + 2s^{m+1}}{s^{2m+1}}\\
&\cdot \max\{d(x_{n-2m-1}, x_{n-2m}), d(x_{n-2m}, x_{n-2m+1}), \cdots, d(x_{n-1}, x_n)\}. \qquad (3.10)
\end{aligned}$$

Employing (3.9) and the triangle inequality again, we derive

$$\begin{aligned}
&d(x_n, x_{n+1})\\
\leq& \lambda^{2m+1} d(x_{n-2m-1}, x_n) \leq s\lambda^{2m+1}[d(x_{n-2m-1}, x_{n-m}) + d(x_{n-m}, x_n)]\\
\leq& [s^2\lambda^{2m+1} d(x_{n-2m-1}, x_{n-2m}) + s^3\lambda^{2m+1} d(x_{n-2m}, x_{n-2m+1}) + \cdots\\
&+ s^m\lambda^{2m+1} d(x_{n-m-3}, x_{n-m-2})\\
&+ s^{m+1}\lambda^{2m+1}(d(x_{n-m-2}, x_{n-m-1}) + d(x_{n-m-1}, x_{n-m}))]\\
&+ [s^2\lambda^{2m+1} d(x_{n-m}, x_{n-m+1}) + s^3\lambda^{2m+1} d(x_{n-m+1}, x_{n-m+2})\\
&+ \cdots + s^m\lambda^{2m+1}(d(x_{n-2}, x_{n-1}) + d(x_{n-1}, x_n))]\\
\leq& (2s^2\lambda^{2m+1} + 2s^3\lambda^{2m+1} + \cdots + 2s^{m-1}\lambda^{2m+1} + 3s^m\lambda^{2m+1}\\
&+ 2s^{m+1}\lambda^{2m+1}) \max\{d(x_{n-2m-1}, x_{n-2m}), d(x_{n-2m}, x_{n-2m+1}), \cdots, d(x_{n-1}, x_n)\}.
\end{aligned}$$

Since $\lambda < \frac{1}{s}$, we conclude that

$$\begin{aligned}
d(x_n, x_{n+1}) \leq& \frac{2s^2 + 2s^3 + \cdots + 2s^{m-1} + 3s^{m-1} + 2s^{m+1}}{s^{2m+1}}\\
&\cdot \max\{d(x_{n-2m-1}, x_{n-2m}), d(x_{n-2m}, x_{n-2m+1}), \cdots, d(x_{n-1}, x_n)\}.
\end{aligned}$$

Using the fact that $s > 1+\sqrt{2}$, it is easy to calculate that

$$\frac{1}{s} < \frac{2s+1}{s^2}, \frac{2s^2+2s^3+\cdots+2s^{m-1}+4s^m}{s^{2m}} < \frac{1}{s} \left(m \in \left\{ 2,3,\cdots, \left\lfloor \frac{k}{2} \right\rfloor \right\} \right)$$

and

$$\frac{2s^2+2s^3+\cdots+2s^{m-1}+3s^m+2s^{m+1}}{s^{2m+1}} < \frac{1}{s} \left(m \in \left\{ 2,3,\cdots, \left\lfloor \frac{k-1}{2} \right\rfloor \right\} \right).$$

Combining (3.4), (3.6), (3.8) and (3.10), from (3.2), we deduce that

$$\begin{aligned}
& d(x_n, x_{n+1}) \\
\leq & \max\{\lambda^2 d(x_{n-2}, x_n), \lambda^3 d(x_{n-3}, x_n), \cdots, \lambda^k d(x_{n-k}, x_n)\} \\
\leq & \max\left\{ \frac{2s+1}{s^2} \max\{d(x_{n-1}, x_n), d(x_{n-2}, x_{n-1})\}, \right. \\
& \frac{2s+1}{s^2} \max\{d(x_{n-1}, x_n), d(x_{n-2}, x_{n-1}), d(x_{n-3}, x_{n-2})\}, \cdots, \\
& \left. \frac{2s+1}{s^2} \max\{d(x_{n-1}, x_n), d(x_{n-2}, x_{n-1}), \cdots, d(x_{n-k}, x_{n-k+1})\} \right\} \\
= & \frac{2s+1}{s^2} \max\{d(x_{n-1}, x_n), d(x_{n-2}, x_{n-1}), \cdots, d(x_{n-k}, x_{n-k+1})\}.
\end{aligned}$$

Therefore, the proof of this lemma is completed. □

Now we introduce the notion of q-multi-valued quasi-contraction in the framework of b-metric spaces and prove the corresponding theorem.

Definition 3.3 ([6])**.** Let (X, d) be a b-metric space with $s \geq 1$. The multi-valued map $T : X \to \mathfrak{CB}(X)$ is said to be a q-multi-valued quasi-contraction if for any $x, y \in X$,

$$H(Tx, Ty) \leq qM(x, y), \tag{3.11}$$

where $0 \leq q < 1$ and

$$M(x, y) = \ max\{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\}.$$

Theorem 3.4. *Let (X, d) be a complete b-metric space with $s \geq 1$ and T be a q-multi-valued quasi-contraction. Assume that*

$$q < \begin{cases} \frac{1}{\sqrt[3]{s+2s^2}}, & 1 \leq s \leq 1+\sqrt{2}; \\ \frac{1}{s}, & s > 1+\sqrt{2}, \end{cases}$$

then T has a fixed point in X, that is, there exists $u \in X$ such that $u \in Tu$.

Proof. From the assumption, there exists λ such that

$$q < \lambda < \begin{cases} \frac{1}{\sqrt[3]{s+2s^2}}, & 1 \leq s \leq 1+\sqrt{2}; \\ \frac{1}{s}, & s > 1+\sqrt{2}. \end{cases}$$

By a simple calculation, we can obtain that $\frac{1}{\sqrt[3]{s+2s^2}} \leq \frac{1}{s}$ when $1 \leq s \leq 1+\sqrt{2}$, which leading to $\lambda < \frac{1}{s}$ for $s \geq 1$. Let $x_0 \in X$ and $x_1 \in Tx_0$. If $x_0 \in Tx_0$, then x_0 is a fixed point of T. Thus, we assume that $x_0 \notin Tx_0$, which implies that $x_0 \neq x_1$ and $d(x_0, Tx_0) > 0$. Thus, there must exist $x_2 \in Tx_1$ such that

$$\begin{aligned} d(x_1, x_2) &\leq H(Tx_0, Tx_1) + (\lambda - q)M(x_0, x_1) \\ &\leq qM(x_0, x_1) + (\lambda - q)M(x_0, x_1) \\ &= \lambda M(x_0, x_1). \end{aligned}$$

Similarly, suppose that $x_1 \notin Tx_1$. Then, we can find $x_3 \in Tx_2$ such that

$$\begin{aligned} d(x_2, x_3) &\leq H(Tx_1, Tx_2) + (\lambda - q)M(x_1, x_2) \\ &\leq qM(x_1, x_2) + (\lambda - q)M(x_1, x_2) \\ &= \lambda M(x_1, x_2). \end{aligned}$$

Repeating this process infinitely, there exists a sequence $\{x_n\}$ in X such that $x_{n+1} \in Tx_n$, $x_n \notin Tx_n$ and

$$\begin{aligned} d(x_n, x_{n+1}) &\leq H(Tx_{n-1}, Tx_n) + (\lambda - q)M(x_{n-1}, x_n) \\ &\leq qM(x_{n-1}, x_n) + (\lambda - q)M(x_{n-1}, x_n) \\ &= \lambda M(x_{n-1}, x_n) \end{aligned} \tag{3.12}$$

for all $n \in \mathbb{N}^*$.

For $s \geq 1$, we consider the following two cases.

Case 1. If $1 \leq s \leq 1 + \sqrt{2}$, for any $n \in \mathbb{N}^*$,

$$\begin{aligned} &M(x_{n-1}, x_n) \\ =&\max\{d(x_{n-1}, x_n), d(x_{n-1}, Tx_{n-1}), d(x_n, Tx_n), d(x_{n-1}, Tx_n), d(x_n, Tx_{n-1})\} \\ \leq&\max\{d(x_{n-1}, x_n), d(x_{n-1}, x_n), d(x_n, x_{n+1}), d(x_{n-1}, Tx_n), d(x_n, x_n)\} \\ =&\max\{d(x_{n-1}, x_n), d(x_n, x_{n+1}), d(x_{n-1}, Tx_n)\}. \end{aligned}$$

If $\max\{d(x_{n-1}, x_n), d(x_n, x_{n+1}), d(x_{n-1}, Tx_n)\} = d(x_n, x_{n+1})$ for some $n \in \mathbb{N}^*$. From (3.12), we find that $d(x_n, x_{n+1}) \leq \lambda M(x_{n-1}, x_n) \leq \lambda d(x_n, x_{n+1})$, which is a contradiction with $0 \leq \lambda < \frac{1}{\sqrt[3]{s+2s^2}} < 1$. Then (3.12) turns into

$$d(x_n, x_{n+1}) \leq \lambda M(x_{n-1}, x_n) \leq \max\{\lambda d(x_{n-1}, x_n), \lambda d(x_{n-1}, Tx_n)\}. \tag{3.13}$$

Using the condition (3.11), we have

$$\begin{aligned} &d(x_{n-1}, Tx_n) \leq H(Tx_{n-2}, Tx_n) \leq \lambda M(x_{n-2}, x_n) \\ =&\lambda \max\{d(x_{n-2}, x_n), d(x_{n-2}, Tx_{n-2}), d(x_n, Tx_n), d(x_{n-2}, Tx_n), d(x_n, Tx_{n-2})\} \\ \leq&\lambda \max\{d(x_{n-2}, x_n), d(x_{n-2}, x_{n-1}), d(x_n, x_{n+1}), d(x_{n-2}, Tx_n), d(x_n, x_{n-1})\}. \end{aligned} \tag{3.14}$$

Similarly, $M(x_{n-2}, x_n) \neq d(x_n, x_{n+1})$ for all $n \in \mathbb{N}$. Thus, by (3.13) and (3.14)

$$d(x_n, x_{n+1}) \leq \max\{\lambda d(x_{n-1}, x_n), \lambda^2 d(x_{n-2}, x_n), \lambda^2 d(x_{n-2}, x_{n-1}), \lambda^2 d(x_{n-2}, Tx_n)\}. \tag{3.15}$$

By (3.11), we deduce

$$\begin{aligned}&d(x_{n-2},Tx_n)\le H(Tx_{n-3},Tx_n)\le \lambda M(x_{n-3},x_n)\\ =&\lambda\max\{d(x_{n-3},x_n),d(x_{n-3},Tx_{n-3}),d(x_n,Tx_n),d(x_{n-3},Tx_n),d(x_n,Tx_{n-3})\}\\ \le&\lambda\max\{d(x_{n-3},x_n),d(x_{n-3},x_{n-2}),d(x_n,x_{n+1}),d(x_{n-3},Tx_n),d(x_n,x_{n-2})\}. \quad (3.16)\end{aligned}$$

It is similar to the process above, $M(x_{n-3},x_n)\neq d(x_n,x_{n+1})$ for all $n\in\mathbb{N}$. By (3.15) and (3.16), we get

$$\begin{aligned}d(x_n,x_{n+1})\le \max\{&\lambda d(x_{n-1},x_n),\lambda^2 d(x_{n-2},x_n),\lambda^2 d(x_{n-2},x_{n-1}),\\ &\lambda^3 d(x_{n-3},x_n),\lambda^3 d(x_{n-3},x_{n-2}),\lambda^3 d(x_{n-3},Tx_n)\}. \quad (3.17)\end{aligned}$$

Applying the condition (3.11),

$$\begin{aligned}&d(x_{n-3},Tx_n)\le H(Tx_{n-4},Tx_n)\le \lambda M(x_{n-4},x_n)\\ =&\lambda\max\{d(x_{n-4},x_n),d(x_{n-4},Tx_{n-4}),d(x_n,Tx_n),d(x_{n-4},Tx_n),d(x_n,Tx_{n-4})\}\\ \le&\lambda\max\{d(x_{n-4},x_n),d(x_{n-4},x_{n-3}),d(x_n,x_{n+1}),d(x_{n-4},Tx_n),d(x_n,x_{n-3})\}. \quad (3.18)\end{aligned}$$

Note that if

$$d(x_{n-3},Tx_n)\le \lambda d(x_{n-4},Tx_n)\le s\lambda[d(x_{n-4},x_{n-3})+d(x_{n-3},Tx_n)],$$

then

$$d(x_{n-3},Tx_n)\le \frac{s\lambda}{1-s\lambda}d(x_{n-4},x_{n-3}). \quad (3.19)$$

From (3.17), (3.18) and (3.19),

$$\begin{aligned}&d(x_n,x_{n+1})\\ \le&\max\Big\{\lambda d(x_{n-1},x_n),\lambda^2 d(x_{n-2},x_n),\lambda^2 d(x_{n-2},x_{n-1}),\\ &\lambda^3 d(x_{n-3},x_n),\lambda^3 d(x_{n-3},x_{n-2}),\lambda^4 d(x_{n-4},x_n),\\ &\lambda^4 d(x_{n-4},x_{n-3}),\frac{s\lambda^4}{1-s\lambda}d(x_{n-4},x_{n-3})\Big\}\\ \le&\max\Big\{\lambda d(x_{n-1},x_n),s\lambda^2[d(x_{n-2},x_{n-1})+d(x_{n-1},x_n)],\\ &s^2\lambda^3[d(x_{n-3},x_{n-2})+d(x_{n-2},x_{n-1})]+s\lambda^3 d(x_{n-1},x_n),\\ &s^2\lambda^4[d(x_{n-4},x_{n-3})+d(x_{n-3},x_{n-2})+d(x_{n-2},x_{n-1})+d(x_{n-1},x_n)],\\ &\frac{s\lambda^4}{1-s\lambda}d(x_{n-4},x_{n-3})\Big\}\\ \le&\max\Big\{\lambda,2s\lambda^2,2s^2\lambda^3+s\lambda^3,4s^2\lambda^4,\frac{s\lambda^4}{1-s\lambda}\Big\}\cdot\max\{d(x_{n-4},x_{n-3}),\\ &d(x_{n-3},x_{n-2}),d(x_{n-2},x_{n-1}),d(x_{n-1},x_n)\}\\ =&\beta_1\max\{d(x_{n-4},x_{n-3}),d(x_{n-3},x_{n-2}),d(x_{n-2},x_{n-1}),d(x_{n-1},x_n)\},\end{aligned}$$

where $\beta_1 = \max\{\lambda, 2s\lambda^2, 2s^2\lambda^3 + s\lambda^3, 4s^2\lambda^4, \frac{s\lambda^4}{1-s\lambda}\}$. Since $\lambda < \frac{1}{\sqrt[3]{s+2s^2}}$, it is easy to verify that $\beta_1 < 1$. From Lemma 3.1, we deduce that $\{x_n\}$ is a Cauchy sequence in X.

Case 2. If $s > 1+\sqrt{2}$. On account of $s\lambda < 1$, there exists $r \in \mathbb{N}$ such that $s\lambda < r < 1$. Taking $n_0 \in \mathbb{N}$ such that $\lambda^{n_0} < \frac{1-r}{s}$. By the same proof of (3.13), (3.15) and (3.17), we can deduce that

$$\begin{aligned} d(x_n, x_{n+1}) \leq \max\{&\lambda d(x_{n-1}, x_n), \lambda^2 d(x_{n-2}, x_n), \lambda^2 d(x_{n-2}, x_{n-1}), \lambda^3 d(x_{n-3}, x_n), \\ &\lambda^3 d(x_{n-3}, x_{n-2}), \lambda^4 d(x_{n-4}, x_n), \lambda^4 d(x_{n-4}, x_{n-3}), \cdots, \lambda^{n_0-1} d(x_{n-n_0+1}, x_n), \\ &\lambda^{n_0-1} d(x_{n-n_0+1}, x_{n-n_0+2}), \lambda^{n_0-1} d(x_{n-n_0+1}, Tx_n)\}. \end{aligned} \tag{3.20}$$

Applying (3.11), we obtain that

$$\begin{aligned} &d(x_{n-n_0+1}, Tx_n) \leq H(Tx_{n-n_0}, Tx_n) \leq \lambda M(x_{n-n_0}, x_n) \\ =&\lambda \max\{d(x_{n-n_0}, x_n), d(x_{n-n_0}, Tx_{n-n_0}), d(x_n, Tx_n), d(x_{n-n_0}, Tx_n), d(x_n, Tx_{n-n_0})\} \\ \leq&\lambda \max\{d(x_{n-n_0}, x_n), d(x_{n-n_0}, x_{n-n_0+1}), d(x_n, x_{n+1}), d(x_{n-n_0}, Tx_n), \\ &d(x_n, x_{n-n_0+1})\}. \end{aligned} \tag{3.21}$$

Note that if

$$d(x_{n-n_0+1}, Tx_n) \leq \lambda d(x_{n-n_0}, Tx_n) \leq s\lambda[d(x_{n-n_0}, x_{n-n_0+1}) + d(x_{n-n_0+1}, Tx_n)],$$

then

$$d(x_{n-n_0+1}, Tx_n) \leq \frac{s\lambda}{1-s\lambda} d(x_{n-n_0}, x_{n-n_0+1}). \tag{3.22}$$

Combining (3.20), (3.21) and (3.22), we derive

$$\begin{aligned} d(x_n, x_{n+1}) \leq \max\Big\{&\lambda d(x_{n-1}, x_n), \lambda^2 d(x_{n-2}, x_{n-1}), \cdots, \\ &\lambda^{n_0} d(x_{n-n_0}, x_{n-n_0+1}), \lambda^2 d(x_{n-2}, x_n), \lambda^3 d(x_{n-3}, x_n), \\ &\lambda^{n_0} d(x_{n-n_0}, x_n), \frac{s\lambda^{n_0}}{1-s\lambda} d(x_{n-n_0}, x_{n-n_0+1})\Big\}. \end{aligned} \tag{3.23}$$

Owing to (3.23) and Lemma 3.2, since $\lambda < \frac{1}{s}$, we deduce

$$\begin{aligned} &d(x_n, x_{n+1}) \\ \leq& \max\Big\{\lambda d(x_{n-1}, x_n), \lambda^2 d(x_{n-2}, x_{n-1}), \cdots, \lambda^{n_0} d(x_{n-n_0}, x_{n-n_0+1}), \\ &\frac{2s+1}{s^2} \max\{d(x_{n-1}, x_n), d(x_{n-2}, x_{n-1}), \cdots, d(x_{n-n_0}, x_{n-n_0+1})\}, \\ &\frac{s\lambda^{n_0}}{1-s\lambda} d(x_{n-n_0}, x_{n-n_0+1})\Big\} \\ \leq& \max\Big\{\frac{2s+1}{s^2} \max\{d(x_{n-1}, x_n), d(x_{n-2}, x_{n-1}), \cdots, d(x_{n-n_0}, x_{n-n_0+1})\}, \\ &\frac{s\lambda^{n_0}}{1-s\lambda} d(x_{n-n_0}, x_{n-n_0+1})\Big\} \\ \leq&\beta_2 \max\{d(x_{n-1}, x_n), d(x_{n-2}, x_{n-1}), \cdots, d(x_{n-n_0}, x_{n-n_0+1})\}, \end{aligned}$$

where $\beta_2 = \max\left\{\frac{2s+1}{s^2}, \frac{s\lambda^{n_0}}{1-s\lambda}\right\}$. Note that $\beta_2 < 1$. From Lemma 3.1, we obtain that $\{x_n\}$ is a Cauchy sequence in X.

Following the above discussion, since (X, d) is complete, there exists $x^* \in X$ such that $\{x_n\}$ converges to x^*. Then we shall show that x^* is a fixed point of T. By (3.11), we have

$$\begin{aligned} H(Tx_n, Tx^*) &\leq \lambda M(x_n, x^*) \\ &= \lambda \max\{d(x_n, x^*), d(x_n, Tx_n), d(x^*, Tx^*), d(x_n, Tx^*), d(x^*, Tx_n)\} \\ &\leq \max\{d(x_n, x^*), d(x_n, x_{n+1}), d(x^*, Tx^*), d(x_n, Tx^*), d(x^*, x_{n+1})\}. \end{aligned}$$

If $M(x_n, x^*) = d(x^*, Tx^*)$ for some $n \in \mathbb{N}$, then we get

$$H(Tx_n, Tx^*) \leq \lambda d(x^*, Tx^*) \leq s\lambda[d(x^*, Tx_n) + H(Tx_n, Tx^*)],$$

which implies that

$$H(Tx_n, Tx^*) \leq \frac{s\lambda}{1-s\lambda} d(x^*, Tx_n) \leq \frac{s\lambda}{1-s\lambda} d(x^*, x_{n+1}).$$

Similarly, if $M(x_n, x^*) = d(x_n, Tx^*)$ for some $n \in \mathbb{N}$, then we obtain that

$$H(Tx_n, Tx^*) \leq \frac{s\lambda}{1-s\lambda} d(x_n, Tx_n) \leq \frac{s\lambda}{1-s\lambda} d(x_n, x_{n+1}).$$

Thus, for every $n \in \mathbb{N}$ we can see that

$$\begin{aligned} H(Tx_n, Tx^*) \leq \lambda \max\Big\{& d(x_n, x^*), d(x_n, x_{n+1}), \\ & \frac{s}{1-s\lambda} d(x^*, x_{n+1}), \frac{s}{1-s\lambda} d(x_n, x_{n+1}), d(x^*, x_{n+1}) \\ = \max&\left\{\lambda d(x_n, x^*), \frac{s\lambda}{1-s\lambda} d(x^*, x_{n+1}), \frac{s\lambda}{1-s\lambda} d(x_n, x_{n+1})\right\}, \end{aligned} \tag{3.24}$$

Letting $n \to \infty$, we obtain that $H(Tx_n, Tx^*) \to 0$ as $n \to \infty$. Note that $x_{n+1} \in Tx_n$ for all $n \in \mathbb{N}$. Thus, from Lemma 2.2, we conclude that $x^* \in Tx^*$. Therefore, x^* is a fixed point of T. □

Remark 3.5. Our Theorem 3.4 generalizes [6, Theorem 2.2] and [2, Theorem 3.3], where the contraction constant of the former theorem is $q \in \left[0, \frac{1}{s+s^2}\right)$ and the constant of the other one is $q \in \left[0, \frac{1}{2s^2}\right)$. Obviously, the range of the contraction constant of Theorem 3.4 is wider than theirs. Indeed, we can see that

$$\begin{aligned} &\text{if } 1 \leq s \leq 1+\sqrt{2}, \text{then } \frac{1}{2s^2} < \frac{1}{s+s^2} < \frac{1}{\sqrt[3]{s+2s^2}}, \\ &\text{if } s > 1+\sqrt{2}, \text{then } \frac{1}{2s^2} < \frac{1}{s+s^2} < \frac{1}{s}. \end{aligned}$$

Next, we present the concept of multi-valued quasi-contraction type multifunctions in the setting of b-metric spaces and investigate the existence of the fixed point of such a mapping. The obtained result is an extension of [14, Theorem 2.2] and [19, Theorem 3.4] from metric spaces to b-metric spaces.

Definition 3.6. (see [14]) Let (X,d) be a b-metric space with $s \geq 1$. The multi-valued map $T : X \to \mathfrak{CB}(X)$ is said to be a multi-valued quasi-contraction type if there exists $\lambda \in \left[0, \frac{1}{s}\right)$ such that

$$H(Tx,Ty) \leq \lambda N(x,y), \tag{3.25}$$

for all $x, y \in X$, where $N(x,y) = \max\{d(x,Tx), d(y,Ty), d(x,Ty), d(y,Tx)\}$.

Theorem 3.7. *Let (X,d) be a complete b-metric space and $T : X \to \mathfrak{CB}(X)$ a multi-valued quasi-contraction type multifunctions. Then T has a fixed point.*

Proof. Since $\lambda < \frac{1}{s}$, there exists α such that $\lambda < \alpha < \frac{1}{s}$. Let $x_0 \in X$ and $x_1 \in Tx_0$. It is similar to the proof of Theorem 3.3, we can construct a sequence $\{x_n\}$ such that $x_{n+1} \in Tx_n$, $x_n \notin Tx_n$ and

$$d(x_n, x_{n+1}) \leq \alpha N(x_{n-1}, x_n). \tag{3.26}$$

Next, we prove that $\{x_n\}$ is a Cauchy sequence in X. Due to the fact that $\alpha < \frac{1}{s}$, we can find a positive integer $r \in \mathbb{N}$ such that

$$\frac{s\alpha^{r+1}}{1 - s\alpha} < \frac{1}{s}. \tag{3.27}$$

Put $n \in \mathbb{N}$ such that $n \geq r + 1$. Combining (3.26) and $x_{n+1} \in Tx_n$, we can see that

$$\begin{aligned} &d(x_n, x_{n+1}) \leq \alpha N(x_{n-1}, x_n) \\ =&\alpha \max\{d(x_{n-1}, Tx_{n-1}), d(x_n, Tx_n), d(x_{n-1}, Tx_n), d(x_n, Tx_{n-1})\} \\ \leq&\alpha \max\{d(x_{n-1}, x_n), d(x_n, x_{n+1}), d(x_{n-1}, Tx_n), d(x_n, Tx_{n-1})\}. \end{aligned} \tag{3.28}$$

If $d(x_n, x_{n+1}) \leq \alpha d(x_n, x_{n+1})$, then we conclude $x_n = x_{n+1}$, which is a contradiction with the fact $x_{n+1} \in Tx_n$ and $x_n \notin Tx_n$. Thus, we obtain that

$$d(x_n, x_{n+1}) \leq \alpha \max\{d(x_{n-1}, x_n), d(x_{n-1}, Tx_n), d(x_n, Tx_{n-1})\}. \tag{3.29}$$

Owing to (3.25),

$$\begin{aligned} &d(x_{n-1}, Tx_n) \leq H(Tx_{n-2}, Tx_n) \leq \lambda N(x_{n-2}, x_n) \leq \alpha N(x_{n-2}, x_n) \\ =&\alpha \max\{d(x_{n-2}, Tx_{n-2}), d(x_n, Tx_n), d(x_{n-2}, Tx_n), d(x_n, Tx_{n-2})\} \\ \leq&\alpha \max\{d(x_{n-2}, x_{n-1}), d(x_n, x_{n+1}), d(x_{n-2}, Tx_n), d(x_n, Tx_{n-2})\}. \end{aligned} \tag{3.30}$$

From (3.25), we have

$$\begin{aligned} &d(x_n, Tx_{n-1}) \leq H(Tx_{n-1}, Tx_{n-1}) \leq \lambda N(x_{n-1}, x_{n-1}) \leq \alpha N(x_{n-1}, x_{n-1}) \\ =&\alpha \max\{d(x_{n-1}, Tx_{n-1}), d(x_{n-1}, Tx_{n-1}), d(x_{n-1}, Tx_{n-1}), d(x_{n-1}, Tx_{n-1})\} \\ \leq&\alpha \max\{d(x_{n-1}, x_n), d(x_{n-1}, Tx_{n-1})\}. \end{aligned} \tag{3.31}$$

Combining (3.29), (3.30) and (3.31), we obtain that

$$\begin{aligned} d(x_n, x_{n+1}) \leq \max\{&\alpha d(x_{n-1}, x_n), \alpha d(x_{n-2}, x_{n-1}), \alpha^2 d(x_{n-2}, Tx_n), \\ &\alpha^2 d(x_n, Tx_{n-2}), \alpha^2 d(x_{n-1}, Tx_{n-1})\}. \end{aligned} \tag{3.32}$$

Employing (3.25) again, we have

$$\begin{aligned} d(x_{n-2}, Tx_n) &\le H(Tx_{n-3}, Tx_n) \le \alpha N(x_{n-3}, x_n) \\ =&\alpha \max\{d(x_{n-3}, Tx_{n-3}), d(x_n, Tx_n), d(x_{n-3}, Tx_n), d(x_n, Tx_{n-3})\} \\ \le&\alpha \max\{d(x_{n-3}, x_{n-2}), d(x_n, x_{n+1}), d(x_{n-3}, Tx_n), d(x_n, Tx_{n-3})\}. \end{aligned} \tag{3.33}$$

Similarly,

$$\begin{aligned} d(x_n, Tx_{n-2}) \le \alpha \max\{&d(x_{n-3}, x_{n-2}), d(x_{n-2}, x_{n-1}), \\ &d(x_{n-1}, Tx_{n-2}), d(x_{n-2}, Tx_{n-1})\}, \\ d(x_{n-1}, Tx_{n-1}) \le \alpha \max\{&d(x_{n-2}, x_{n-1}), d(x_{n-1}, x_n), \\ &d(x_{n-2}, Tx_{n-1}), d(x_{n-1}, Tx_{n-2})\}. \end{aligned} \tag{3.34}$$

Thus, we obtain that

$$\begin{aligned} d(x_n, x_{n+1}) \le \max\{&\alpha d(x_{n-1}, x_n), \alpha d(x_{n-2}, x_{n-1}), \alpha d(x_{n-3}, x_{n-2}), \\ &\alpha^3 d(x_{n-3}, Tx_n), \alpha^3 d(x_n, Tx_{n-3}), \alpha^3 d(x_{n-2}, Tx_{n-1}), \alpha^3 d(x_{n-1}, Tx_{n-2})\}. \end{aligned} \tag{3.35}$$

Repeating the above process r times, we deduce that

$$d(x_n, x_{n+1}) \le \max C_r \cup D_r, \tag{3.36}$$

where

$$C_r = \{\alpha d(x_{n-i}, x_{n-i+1}) : 1 \le i \le r\}$$

and

$$D_r = \max\{\alpha^r d(x_{n-i}, Tx_{n-j}) : 0 \le i, j \le r \text{ and } i + j = r\}.$$

Then, using (3.25),

$$\begin{aligned} d(x_{n-i}, Tx_{n-j}) &\le H(Tx_{n-i-1}, Tx_{n-j}) \le \alpha N(x_{n-i-1}, x_{n-j}) \\ =&\alpha \max\{d(x_{n-i-1}, Tx_{n-i-1}), d(x_{n-j}, Tx_{n-j}), d(x_{n-i-1}, Tx_{n-j}), d(x_{n-j}, Tx_{n-i-1})\} \\ \le&\alpha \max\{d(x_{n-i-1}, x_{n-i}), d(x_{n-j}, x_{n-j+1}), d(x_{n-i-1}, Tx_{n-j}), d(x_{n-j}, Tx_{n-i-1})\}. \end{aligned}$$

If $H(Tx_{n-i-1}, Tx_{n-j}) \le \alpha d(x_{n-i-1}, Tx_{n-j})$, then we deduce

$$H(Tx_{n-i-1}, Tx_{n-j}) \le s\alpha[d(x_{n-i-1}, Tx_{n-i-1}) + H(Tx_{n-i-1}, Tx_{n-j})],$$

which implies that

$$H(Tx_{n-i-1}, Tx_{n-j}) \le \frac{s\alpha}{1 - s\alpha} d(x_{n-i-1}, Tx_{n-i-1}) \le \frac{s\alpha}{1 - s\alpha} d(x_{n-i-1}, x_{n-i}).$$

If $H(Tx_{n-i-1}, Tx_{n-j}) \le \alpha d(x_{n-j}, Tx_{n-i-1})$, then we derive

$$H(Tx_{n-i-1}, Tx_{n-j}) \le s\alpha[d(x_{n-j}, Tx_{n-j}) + H(Tx_{n-j}, Tx_{n-i-1})].$$

It follows that

$$H(Tx_{n-i-1}, Tx_{n-j}) \le \frac{s\alpha}{1 - s\alpha} d(x_{n-j}, Tx_{n-j}) \le \frac{s\alpha}{1 - s\alpha} d(x_{n-j}, x_{n-j+1}).$$

Therefore, having in mind $\alpha < \frac{s\alpha}{1-s\alpha}$, we conclude that

$$\begin{aligned} d(x_{n-i}, Tx_{n-j}) &\le H(Tx_{n-i-1}, Tx_{n-j}) \\ &\le \frac{s\alpha}{1 - s\alpha} \max\{d(x_{n-i-1}, x_{n-i}), d(x_{n-j}, x_{n-j+1})\}. \end{aligned} \tag{3.37}$$

Hence, by (3.27), (3.36) and (3.37), we can deduce that

$$d(x_n, x_{n+1}) \leq \gamma \max\{d(x_{n-i}, x_{n-i+1}) : 1 \leq i \leq r+1\},$$

where $\gamma = \max\left\{\alpha, \frac{s\alpha^{r+1}}{1-s\alpha}\right\} < \frac{1}{s}$. According to Lemma 3.1 with $p = r+1$, we conclude that $\{x_n\}$ is a Cauchy sequence in X. Since b-metric space (X, d) is complete, there exists $x^* \in X$ such that $x_n \to x^*$ as $n \to \infty$. Note that

$$\begin{aligned} H(Tx_n, Tx^*) &\leq \lambda N(x_n, x^*) \\ &= \lambda \max\{d(x_n, Tx_n), d(x^*, Tx^*), d(x_n, Tx^*), d(x^*, Tx_n)\} \\ &\leq \lambda \max\{d(x_n, x_{n+1}), d(x^*, Tx^*), d(x_n, Tx^*), d(x^*, x_{n+1})\}. \end{aligned}$$

If $H(Tx_n, Tx^*) \leq \lambda d(x^*, Tx^*)$ for some $n \in \mathbb{N}$, we can see that

$$H(Tx_n, Tx^*) \leq s\lambda[d(x^*, Tx_n) + H(Tx_n, Tx^*)],$$

which implies that

$$H(Tx_n, Tx^*) \leq \frac{s\lambda}{1-s\lambda} d(x^*, Tx_n) \leq \frac{s\lambda}{1-s\lambda} d(x^*, x_{n+1}). \tag{3.38}$$

If $H(Tx_n, Tx^*) \leq \lambda d(x_n, Tx^*)$ for some $n \in \mathbb{N}$, we obtain that

$$H(Tx_n, Tx^*) \leq s\lambda[d(x_n, Tx_n) + H(Tx_n, Tx^*)].$$

It follows that

$$H(Tx_n, Tx^*) \leq \frac{s\lambda}{1-s\lambda} d(x_n, Tx_n) \leq \frac{s\lambda}{1-s\lambda} d(x_n, x_{n+1}). \tag{3.39}$$

By inequalities (3.38) and (3.39), since $\lambda < \frac{1}{s}$, we get

$$H(Tx_n, Tx^*) \leq \frac{s\lambda}{1-s\lambda} \max\{d(x_n, x_{n+1}), d(x^*, x_{n+1})\}.$$

Note that $d(x_n, x_{n+1}) \to 0$ and $d(x^*, x_{n+1}) \to 0$ as $n \to \infty$. Then, we can see that $(Tx_n, Tx^*) \to 0$ as $n \to \infty$. Observing the fact $x_{n+1} \in Tx_n$ for all $n \in \mathbb{N}$ and from Lemma 2.2, we conclude that $x^* \in Tx^*$. Therefore, x^* is a fixed point of T. □

Inspired by the results of Miculescu et al. [20], we propose a new theorem to generalize and improve their three fixed point theorems for multi-valued functions in b-metric spaces as follows.

Theorem 3.8. *A function $T : X \to \mathfrak{CB}(X)$, where (X, d) is a complete b-metric space of constant s. Suppose that there exists $\lambda \in [0, 1)$ such that*

$$H(Tx, Ty) \leq \lambda M_1(x, y)$$

for all $x, y \in X$, where

$$M_1(x, y) = \max\left\{d(x, y), \frac{1}{s} d(x, Tx), \frac{1}{s} d(y, Ty), \frac{1}{2s}[d(x, Ty) + d(y, Tx)]\right\}. \tag{3.40}$$

Then T has a fixed point in X.

Proof. Since $\lambda < 1$, there exists θ such that $\lambda < \theta < 1$. Let $x_0 \in X$ and $x_1 \in Tx_0$. From a similar argument in the proof of Theorem 3.3, we can obtain a sequence $\{x_n\}$ such that $x_{n+1} \in Tx_n$, $x_n \notin Tx_n$ and

$$\begin{aligned} &d(x_n, x_{n+1}) \leq \theta M_1(x_{n-1}, x_n) \\ =&\theta \max\left\{ d(x_{n-1}, x_n), \frac{1}{s} d(x_{n-1}, Tx_{n-1}), \frac{1}{s} d(x_n, Tx_n), \right. \\ &\left. \frac{1}{2s}[d(x_{n-1}, Tx_n) + d(x_n, Tx_{n-1})] \right\} \\ &\leq \theta \max\left\{ d(x_{n-1}, x_n), \frac{1}{s} d(x_{n-1}, x_n), \frac{1}{s} d(x_n, x_{n+1}), \right. \\ &\left. \frac{1}{2s}[d(x_{n-1}, x_{n+1}) + d(x_n, x_n)] \right\} \\ &\leq \theta \max\left\{ d(x_{n-1}, x_n), \frac{1}{s} d(x_n, x_{n+1}), \frac{1}{2}[d(x_{n-1}, x_n) + d(x_n, x_{n+1})] \right\} \\ &\leq \theta \max\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\}. \end{aligned}$$

If $d(x_n, x_{n+1}) \leq \theta d(x_n, x_{n+1})$, we have $d(x_n, x_{n+1}) = 0$, which is a contraction with the fact that $x_{n+1} \in Tx_n$ and $x_n \notin Tx_n$. Thus, by induction we deduce that

$$d(x_n, x_{n+1}) \leq \theta d(x_{n-1}, x_n) \leq \theta^2 d(x_{n-2}, x_{n-1}) \leq \cdots \leq \theta^n d(x_0, x_1).$$

Combining the assumption $\theta < 1$ and Lemma 2.1, we conclude that $\{x_n\}$ is a Cauchy sequence in X. Since (X, d) is complete, there exists $x^* \in X$ such that $\{x_n\}$ converges to x^*. Then we show that x^* is a fixed point of T. By (3.40), we have

$$\begin{aligned} &H(Tx_n, Tx^*) \leq \lambda M_1(x_n, x^*) \\ =&\lambda \max\left\{ d(x_n, x^*), \frac{1}{s} d(x_n, Tx_n), \frac{1}{s} d(x^*, Tx^*), \frac{1}{2s}[d(x_n, Tx^*) + d(x^*, Tx_n)] \right\} \\ \leq&\lambda \max\left\{ d(x_n, x^*), \frac{1}{s} d(x_n, x_{n+1}), \frac{1}{s} d(x^*, Tx^*), \frac{1}{2s}[d(x_n, Tx^*) + d(x^*, Tx_n)] \right\} \end{aligned}$$

If $M_1(x_n, x^*) = \frac{1}{s} d(x^*, Tx^*)$ for some $n \in \mathbb{N}$, then we obtain that

$$H(Tx_n, Tx^*) \leq \frac{\lambda}{s} d(x^*, Tx^*) \leq \lambda[d(x^*, Tx_n) + H(Tx_n, Tx^*)],$$

which implies that

$$H(Tx_n, Tx^*) \leq \frac{\lambda}{1-\lambda} d(x^*, Tx_n) \leq \frac{\lambda}{1-\lambda} d(x^*, x_{n+1}).$$

Similarly, if $M_1(x_n, x^*) = \frac{1}{2s}[d(x_n, Tx^*) + d(Tx_n, x^*)]$ for some $n \in \mathbb{N}$, then we derive

$$\begin{aligned} H(Tx_n, Tx^*) &\leq \frac{\lambda}{2s}[d(x_n, Tx^*) + d(Tx_n, x^*)] \leq \frac{\lambda}{2s}[d(x_n, Tx^*) + d(x_{n+1}, x^*)] \\ &\leq \frac{\lambda}{2}[d(x_n, Tx_n) + H(Tx_n, Tx^*)] + \frac{\lambda}{2s} d(x_{n+1}, x^*). \end{aligned}$$

It follows that

$$\begin{aligned}H(Tx_n, Tx^*) &\leq \frac{\frac{\lambda}{2}}{1-\frac{\lambda}{2}} d(x_n, Tx_n) + \frac{\frac{\lambda}{2s}}{1-\frac{\lambda}{2}} d(x_{n+1}, x^*) \\ &\leq \frac{\frac{\lambda}{2}}{1-\frac{\lambda}{2}} d(x_n, x_{n+1}) + \frac{\frac{\lambda}{2s}}{1-\frac{\lambda}{2}} d(x_{n+1}, x^*).\end{aligned}$$

Thus, for every $n \in \mathbb{N}$ we can see that

$$\begin{aligned}H(Tx_n, Tx^*) \leq \lambda \max \Bigg\{ &d(x_n, x^*), \frac{1}{s} d(x_n, x_{n+1}), \frac{1}{1-\lambda} d(x^*, x_{n+1}), \\ &\frac{\frac{1}{2}}{1-\frac{\lambda}{2}} d(x_n, x_{n+1}) + \frac{\frac{1}{2s}}{1-\frac{\lambda}{2}} d(x_{n+1}, x^*) \Bigg\}. \end{aligned} \tag{3.41}$$

Letting $n \to \infty$ in the above inequality, we obtain that $H(Tx_n, Tx^*) \to 0$ as $n \to \infty$. Note that $x_{n+1} \in Tx_n$ for all $n \in \mathbb{N}$. Hence, from Lemma 2.2, we conclude that $x^* \in Tx^*$. Therefore, x^* is a fixed point of T. □

Remark 3.9. Compared with the main results in [20], it is obvious that Theorem 3.8 gives several improvements. Actually, the condition that the mapping T is closed from [20, Theorem 3.1] and the condition that d is $*$-continuous from [20, Theorem 3.2] are omitted. Moreover, the range of contraction constant in [20, Theorem 3.3] is smaller than ours.

Remark 3.10. In Theorem 3.7 and Theorem 3.8, the proofs that $\{x_n\}$ is a Cauchy sequence are still valid for $\lambda \in [0, 1)$. However, under the above conditions we can not obtain the fixed points in corresponding theorems. Therefore, we present such results.

4. Applications

In what follows we discuss some consequences of the above theorems in the context of metric spaces and b-metric spaces. (In the following corollaries, we always assume that $M_1(x, y)$ is same as Theorem 3.8 if not stated otherwise.)

Corollary 4.1 ([19])**.** *Let (X, d) be a complete metric space and $T : X \to \mathfrak{CB}(X)$ be a multi-valued quasi-contraction with constant λ. If there exists $\lambda \in \left[0, \frac{1}{\sqrt[3]{3}}\right)$ such that*

$$H(Tx, Ty) \leq \lambda \{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\}.$$

Then T has a fixed point in X, that is, there exists $u \in X$ such that $u \in Tu$.

Proof. In Theorem 3.4, if we take $s = 1$, then Theorem 3.1 in [19] is obtained. □

Corollary 4.2 (see [23])**.** *Let (X, d) be a complete b-metric space with coefficient $s \geq 1$ and $T : X \to \mathfrak{CB}(X)$ be a mapping. Suppose that there exists $\alpha \in [0, 1)$ such that*

$$H(Tx, Ty) \leq \alpha d(x, y) \tag{4.1}$$

for all $x, y \in X$. Then T has a fixed point in X.

Proof. Note that

$$H(Tx,Ty) \leq \alpha d(x,y) \leq \alpha M_1(x,y)$$

for all $x, y \in X$. By applying Theorem 3.8, it is easy to obtain the desired result. □

Remark 4.3. It is generally known that a b-metric space is a generalized metric space and thus Corollary 4.2 is more general than the result of Nadler [23] in metric spaces.

Corollary 4.4 (see [27]). *Let (X,d) be a complete b-metric space with coefficient $s \geq 1$. $T : X \to \mathfrak{CB}(X)$ be a mapping. Suppose that there exist $\beta, \gamma \in [0,1)$ such that*

$$H(Tx,Ty) \leq \beta d(x,y) + \gamma d(y,Ty).$$

Assume that $\beta + \gamma < \frac{1}{s}$, then T has a fixed point in X.

Proof. Observing that

$$\begin{aligned}\beta d(x,y) + \gamma d(y,Ty) &\leq (\beta+\gamma)\max\{d(x,y), d(y,Ty)\}\\ &= \lambda_0 \max\left\{\frac{1}{s}d(x,y), \frac{1}{s}d(y,Ty)\right\} \leq \lambda_0 M_1(x,y),\end{aligned}$$

where $\lambda_0 \in [0,1)$. Thus, by Theorem 3.8, we can obtain that T has a fixed point in X. □

Remark 4.5. Taking $s = 1$ in Corollary 4.4, we improve and simplify the main result of Rus [27] by deleting the condition "T is a closed multi-valued operator".

Corollary 4.6. *Let (X,d) be a complete b-metric space with coefficient $s \geq 1$ and $T : X \to \mathfrak{CB}(X)$ be a mapping. If there exist $\alpha, \beta, \gamma \in [0,1)$ such that*

$$H(Tx,Ty) \leq \alpha d(x,y) + \beta d(x,Tx) + \gamma d(y,Ty).$$

Assume that $\alpha + \beta + \gamma < \frac{1}{s}$, then T has a fixed point in X.

Proof. It is notice that

$$\begin{aligned}\alpha d(x,y) + \beta d(x,Tx) + \gamma d(y,Ty) &\leq (\alpha+\beta+\gamma)\max\{d(x,y), d(x,Tx), d(y,Ty)\}\\ &= \lambda_1 \max\left\{\frac{1}{s}d(x,y), \frac{1}{s}d(x,Tx), \frac{1}{s}d(y,Ty)\right\} \leq \lambda_1 M_1(x,y),\end{aligned}$$

where $\lambda_1 \in [0,1)$. Therefore, by applying Theorem 3.8, we must have T has a fixed point in X. □

Remark 4.7. Letting $s = 1$ in Corollary 4.6, we obtain the result of Reich [26]. Therefore, Corollary 4.6 is more general than theirs.

The last corollary is the fixed point theorem for Hardy-Rogers type multi-valued contractions in b-metric spaces.

Corollary 4.8. *Let (X,d) be a complete b-metric space with coefficient $s \geq 1$. $T : X \to \mathfrak{CB}(X)$ such that*

$$H(Tx,Ty) \leq a_1 d(x,y) + a_2 d(x,Tx) + a_3 d(y,Ty) + a_4 d(x,Ty) + a_5 d(y,Tx) \quad (4.2)$$

for all $x, y \in X$, where a_1, a_2, a_3, a_4, a_5 are nonnegative constants such that $\sum_{i=1}^{5} a_i < \frac{1}{s}$. Then T has a fixed point in X.

Proof. By virtue of (4.2), we can obtain

$$H(Ty,Tx) \leq a_1 d(y,x) + a_2 d(y,Ty) + a_3 d(x,Tx) + a_4 d(y,Tx) + a_5 d(x,Ty). \quad (4.3)$$

Adding (4.2) to (4.3), we deduce that

$$\begin{aligned}
& H(Tx,Ty) \\
\leq & a_1 d(x,y) + \frac{a_2+a_3}{2}[d(x,Tx)+d(y,Ty)] + \frac{a_4+a_5}{2}[d(x,Ty)+d(y,Tx)] \\
= & a_1 d(x,y) + (a_2+a_3)\frac{d(x,Tx)+d(y,Ty)}{2} + (a_4+a_5)\frac{d(x,Ty)+d(y,Tx)}{2} \\
\leq & a_1 d(x,y) + (a_2+a_3)\max\{d(x,Tx),d(y,Ty)\} + (a_4+a_5)\frac{d(x,Ty)+d(y,Tx)}{2} \\
\leq & (a_1+a_2+a_3+a_4+a_5) \\
& \cdot \max\left\{d(x,y), d(x,Tx), d(y,Ty), \frac{d(x,Ty)+d(y,Tx)}{2}\right\} \\
= & \lambda_2 \max\left\{\frac{1}{s}d(x,y), \frac{1}{s}d(x,Tx), \frac{1}{s}d(y,Ty), \frac{d(x,Ty)+d(y,Tx)}{2s}\right\} \leq \lambda_2 M_1(x,y),
\end{aligned}$$

where $\lambda_2 \in [0,1)$. Hence, by Theorem 3.8, it can be proved that T has a fixed point in X. □

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