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FIXED POINT RESULTS FOR NON-SELF OPERATORS ON $\mathbb{R}^m_+$-METRIC SPACES

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Abstract. The purpose of this paper is to discuss some problems of the fixed point theory for non-self operators on $\mathbb{R}^m_+$-metric spaces. The results complement and extend some known results given in the paper: A. Chis-Novac, R. Precup, I.A. Rus, Data dependence of fixed points for non-self generalized contractions, Fixed Point Theory, 10(2009), No. 1, 73–87.
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1. Introduction

1.1. **Notations.** We begin the introduction by some standard notations that will be used throughout the paper.

Let (X, d) be a $\mathbb{R}^m_+$-metric space, $Y \subset X$ a nonempty subset of X and $f : Y \to X$ an operator. In what follow we shall use the following notations:

$F_f = \{x \in Y : f(x) = x\}$ - the fixed points set of f.

$P_{cl}(X) = \{Y \subset X | Y \text{ is closed}\}$

$I(f) = \{Z \subset Y : f(Z) \subset Z, Z \neq \varnothing\}$ - the set of invariant subsets of f.

$(MI)_f = \cup I(f)$- the maximal invariant subset of f.

1.2. **Non-self operators on L-spaces.** In what follow we denote an L-space by $(X, \overset{F}{\to})$. Let $Y \subset X$ a nonempty subset of X and $f : Y \to X$ an operator. Throughout this paper we consider that $Y \in P_{cl}(X)$.

$(AB)_f(x^*) = \{x \in Y : f^n(x)$ is defined for all $n \in \mathbb{N}$ and $f^n(x) \to x^* \in F_f\}$-the attraction basin of the fixed point x^* with respect to f.

$(BA)_f = \underset{x^* \in F_f}{\cup} (AB)_f(x^*)$- the attraction basin of f.

Following [3] we have:

Definition 1.1. An operator $f : Y \to X$ is said to be a Picard operator (PO) if
(i) $F_f = \{x_f^*\}$;
(ii) $(MI)_f = (BA)_f$.

Definition 1.2. An operator $f : Y \to X$ is said to be a weakly Picard operator (WPO) if
(i) $F_f \neq \emptyset$;
(ii) $(MI)_f = (BA)_f$.

Definition 1.3. For each WPO $f : Y \to X$ we define the operator $f^\infty : (BA)_f \to (BA)_f$ by $f^\infty(x) = \lim_{n\to\infty} f^n(x)$.

Remark 1.4. It is clear that $f^\infty((BA)_f) = F_f$, so f^∞ is a set retraction of $(BA)_f$ to F_f.

Remark 1.5. In terms of weakly Picard self operators the above definitions take the following form:

$$f : Y \to X \text{ is a PO} \quad \text{iff} \quad f\,|_{(MI)_f} : (MI)_f \to (MI)_f \text{ is a PO.}$$

Remark 1.6. We have the above notions in each distance structures which induces an L-space convergence ($\mathbb{R}^m_+$-metrics, $s(\mathbb{R}_+)$ metrics, K-metrics, partial metrics, dislocated metrics, ...).

For other results on Picard and weakly Picard operators see [1], [2], [4], [5], [9], [10], [13].

1.3. **Operators on $\mathbb{R}^m_+$-metric spaces.**

Definition 1.7. An operator $f : X \to X$ is an S-contraction if there exists $S \in \mathbb{R}^{m\times m}_+$ such that:
(i) S is a convergent to zero matrix, i.e. $S^n \to 0$ as $n \to \infty$;
(ii) $d(f(x), f(y)) \leq Sd(x, y)$, for all $x, y \in X$.

The following results, in $\mathbb{R}^m_+$-metric spaces were well known.

Theorem 1.8. *(Saturated Perov-Schröder Theorem) Let (X, d) be a complete $\mathbb{R}^m_+$-metric space. We suppose that, $f : X \to X$ and $S \in \mathbb{R}^{m\times m}_+$ are such that:*
(1) *S is a matrix convergent to matrix 0;*
(2) *$d(f(x), f(y)) \leq Sd(x, y)$, $\forall x, y \in X$.*

Then:
(i) $F_f = F_{f^n} = \{x^*\}$, $\forall n \in \mathbb{N}^*$;
(ii) *f is Picard mapping, i.e., $f^n(x) \to x^*$ as $n \to \infty$, $\forall x \in X$;*
(iii) $d(x, x^*) \leq (I - S)^{-1} d(x, f(x))$, $\forall x \in X$.
(iv) *the fixed point equation, $x = f(x)$ is Ulam-Hyers stable;*
(v) *if $x_n \in X$, $d(x_n, f(x_n)) \to 0$ as $n \to \infty$, then, $x_n \to x^*$ as $n \to \infty$, i.e. the fixed point problem for f is well posed;*

(vi) *if* $x_n \in X, n \in \mathbb{N}$ *are such that*

$$d(x_{n+1}, f(x_n)) \to 0 \text{ as } n \to \infty,$$

then for all $x \in X$ *we have*

$$d(x_n, x^*) \to 0 \text{ as } n \to \infty,$$

i.e. f *has the Ostrowski property.*

For this result and other results on fixed point theory in a $\mathbb{R}^m_+$-metric space see [7], [8], [12], [11].

The aim of this paper is to complement and extend the mentioned results in the case of non-self operators.

2. Metric conditions on non-self operators on $\mathbb{R}^m_+$-metrics and fixed points

Let (X, d) be a $\mathbb{R}^m_+$-metric space, $Y \subset X$ a nonempty subset and $f : Y \to X$ an operator.

Definition 2.1. The operators f is an S-contraction if $S \in \mathbb{R}^{m\times m}_+$ and f is such that

(i) $S^n \to 0$ as $n \to \infty$, i.e. S is a matrix convergent to 0;
(ii) $d(f(x), f(y)) \leq Sd(x, y)$, for all $x, y \in Y$.

Definition 2.2. The operator f is a graphic S-contraction if $S \in \mathbb{R}^{m\times m}_+$ is convergent to 0 and

$$d(f(x), f^2(x)) \leq Sd(x, f(x)), \forall x \in Y.$$

Definition 2.3. The operator f is a quasi S-contraction if $F_f = \{x^*\},\ S \in \mathbb{R}^{m\times m}_+$ is convergent to 0 and

$$d(f(x), x^*) \leq Sd(x, x^*), \forall x \in Y.$$

Definition 2.4. The operator f satisfies a retraction-displacement condition if $F_f = \{x^*\}$ and there exists an increasing function $\psi : \mathbb{R}^m_+ \to \mathbb{R}^m_+, \psi(0) = 0$, and is continuous in 0, such that

$$d(x, x^*) \leq \psi\left(d(x, f(x))\right), \forall x \in Y.$$

Definition 2.5. Let ψ defined as in Definition 2.4. By definition, f is ψ-PO if f is PO with respect to $\overset{d}{\to}$, and

$$d(x, x^*) \leq \psi\left(d(x, f(x))\right), \forall x \in (MI)_f.$$

In the terms of the above metric conditions we have the following results.

Theorem 2.6. *Let* (X, d) *be an* $\mathbb{R}^m_+$*-metric space,* $Y \subset X$ *a nonempty subset and* $f : Y \to X$ *be a* S*-contraction with* $F_f = \{x^*\}$*. Then:*

(i) $d(x, x^*) \leq (I - S)^{-1}d(x, f(x)), \forall x \in Y$*;*
(ii) *if* $g : Y \to X$ *is such that* $d(f(x), g(x)) \leq \eta, \forall x \in Y$, *for some* $\eta \in \left(\mathbb{R}^*_+\right)^m$, *then*

$$d(y^*, x^*) \leq (I - S)^{-1}\eta,\ \forall y^* \in F_g;$$

(iii) *if* $y \in Y$ *is such that*

$$d(y, f(y)) \le \varepsilon,$$

for some $\varepsilon \in \left(\mathbb{R}_+^*\right)^m$, *then*

$$d(y, x^*) \le (I - S)^{-1}\varepsilon,$$

i.e., the equation $x = f(x)$ *is Ulam-Hyers stable;*

(iv) $x_n \in Y,\ n \in \mathbb{N},\ d(x_n, f(x_n)) \to 0$ *as* $n \to \infty$ *implies that* $x_n \to x^*$ *as* $n \to \infty$, *i.e. the fixed point problem for* f *is well posed.*

(v) $x_n \in Y,\ n \in \mathbb{N},\ d(x_{n+1}, f(x_n)) \to 0$ *as* $n \to \infty$ *implies that* $x_n \to x^*$ *as* $n \to \infty$, *i.e.* f *has the Ostrowski property.*

Proof. (i) We have for $x \in Y$,

$$d(x, x^*) \le d(x, f(x)) + d(f(x), x^*) \le d(x, f(x)) + Sd(x, x^*).$$

From this it follows

$$(I - S)d(x, x^*) \le d(x, f(x)).$$

Since S is a matrix convergent to 0, there exists $(I-S)^{-1}$ and $(I-S)^{-1} \ge 0$, i.e., the corresponding function, $(I-S)^{-1} : \mathbb{R}_+^{m\times m} \to \mathbb{R}_+^m,\ \eta \mapsto (I-S)^{-1}\eta$ is increasing, with value 0 at 0 and continuous. So we have (i), a retraction-displacement condition.

(ii) From (i),

$$\begin{aligned} d(y^*, x^*) &\le (I - S)^{-1}d(y^*, f(y^*)) = (I - S)^{-1}d(g(y^*), f(y^*)) \\ &\le (I - S)^{-1}\eta. \end{aligned}$$

(iii) From (i),

$$d(y, x^*) \le (I - S)^{-1}d(y, f(y)) \le (I - S)^{-1}\varepsilon.$$

(iv) The proof follows directly from (i).

(v) Since $f : Y \to X$ is an S-contraction with $F_f = \{x^*\}$ it follows that, f is a quasi S-contraction. So, we have,

$$\begin{aligned} &d(x_{n+1}, x^*) \le \\ &\le d(x_{n+1}, f(x_n)) + d(f(x_n), x^*) \le \\ &\le d(x_{n+1}, f(x_n)) + Sd(x_n, x^*) \le \\ &\le d(x_{n+1}, f(x_n)) + Sd(x_n, f(x_{n-1})) + S^2d(x_{n-1}, x^*) \le \\ &\le \ldots \le \\ &\le d(x_{n+1}, f(x_n)) + Sd(x_n, f(x_{n-1})) + \ldots + S^n d(x_1, f(x_0)) + S^{n+1}d(x_0, x^*). \end{aligned}$$

So, $d(x_{n+1}, x^*) \to 0$ as $n \to \infty$, by a Cauchy-Toeplitz lemma (see [14]). □

Theorem 2.7. *Let* (X, d) *be an* $\mathbb{R}_+^m$*-metric space,* $Y \subset X$ *a nonempty subset and* $f : Y \to X$ *be an operator such that,*

$$d(f(x), f(y)) \le Pd(x, f(x)) + Qd(y, f(y)) + Rd(x, y), \tag{2.1}$$

for all $x, y \in Y$, *where* $P, Q, R \in \mathbb{R}_+^{m\times m}$. *We suppose that,* $F_f = \{x^*\}$ *and* $(I-R)^{-1} \ge 0$. *Then we have that:*

(i) $d(x, x^*) \le Cd(x, f(x)),\ \forall x \in Y$, *where* $C := (I - R)^{-1}(I + P)$;

(ii) *if* $g : Y \to X$ *is such that* $d(f(x), g(x)) \le \eta, \forall x \in Y$, *for some* $\eta \in \left(\mathbb{R}_+^*\right)^m$, *then*

$$d(y^*, x^*) \le C\eta, \ \forall y^* \in F_g;$$

(iii) *if* $y \in Y$ *is such that*

$$d(y, f(y)) \le \varepsilon,$$

for some $\varepsilon \in \left(\mathbb{R}_+^*\right)^m$, *then*

$$d(y, x^*) \le C\varepsilon,$$

i.e., the equation $x = f(x)$ *is Ulam-Hyers stable;*

(iv) $x_n \in Y, \ n \in \mathbb{N}, \ d(x_n, f(x_n)) \to 0$ *as* $n \to \infty$ *implies that* $x_n \to x^*$ *as* $n \to \infty$, *i.e. the fixed point problem for* f *is well posed.*

Proof. (i) We have for $x \in Y$,

$$\begin{aligned} d(x, x^*) &\le d(x, f(x)) + d(f(x), x^*) \le d(x, f(x)) + d(f(x), f(x^*)) \\ &\le d(x, f(x)) + Pd(x, f(x)) + Qd(x^*, f(x^*)) + Rd(x, x^*). \end{aligned}$$

From this it follows

$$(I - R)d(x, x^*) \le (I + P)d(x, f(x)), \forall x \in Y.$$

So, $d(x, x^*) \le (I - R)^{-1}(I + P)d(x, f(x))$.

(ii) By applying (i) to $y^* \in F_g$

$$\begin{aligned} d(y^*, x^*) &\le (I - R)^{-1}(I + P)d(y^*, f(y^*)) \\ &= (I - R)^{-1}(I + P)d(g(y^*), f(y^*)) \\ &\le (I - R)^{-1}(I + P)\eta, \forall y^* \in F_g. \end{aligned}$$

(iii) We apply again (i) and obtain

$$d(y, x^*) \le (I - R)^{-1}(I + P)d(y, f(y)) = (I - R)^{-1}(I + P)\varepsilon, \forall y \in Y.$$

(iv) Let

$$\begin{aligned} d(x_n, x^*) &\le d(x_n, f(x_n)) + d(f(x_n), f(x^*)) \\ &\le d(x_n, f(x_n)) + Pd(x_n, f(x_n)) + Qd(x^*, f(x^*)) + Rd(x_n, x^*). \end{aligned}$$

We have

$$(I - R)d(x_n, x^*) \le (I + P)d(x_n, f(x_n)),$$

so, this implies that $x_n \to x^*$as $n \to \infty$, i.e. the fixed point problem for f is well posed. □

Theorem 2.8. *Let* (X, d) *be an* $\mathbb{R}_+^m$*-metric space,* $Y \subset X$ *a nonempty subset and* $f : Y \to X$ *be an operator such that (2.1) holds, for all* $x, y \in Y$, *where* $P, Q, R \in \mathbb{R}_+^m$. *We suppose that,* $(I - P)^{-1} \ge 0$ *and the matrix* $(I - P)^{-1}(P + R)$ *is convergent to* 0. *Then we have that:*

(i) $d(f(x), x^*) \le (I-P)^{-1}(P+R)d(x, x^*), \ \forall x \in Y$, *i.e.,* f *is a quasi contraction;*

(ii) $x_n \in Y, \ n \in \mathbb{N}, \ d(x_{n+1}, f(x_n)) \to 0$ *as* $n \to \infty$ *implies that* $x_n \to x^*$ *as* $n \to \infty$.

Proof. (i)

$$\begin{aligned} d(f(x),x^*) = d(f(x),f(x^*)) &\leq Pd(x,f(x)) + Qd(x^*,f(x^*)) + Rd(x,x^*) \\ &\leq P[d(x,x^*) + d(x^*,f(x))] + Rd(x,x^*). \\ (I-P)d(f(x),x^*) &\leq (P+R)d(x,x^*). \end{aligned}$$

Thus, follows the conclusion.

(ii) Let

$$\begin{aligned} d(x_{n+1},x^*) &\leq d(x_{n+1},f(x_n)) + d(f(x_n),x^*) \\ &\leq d(x_{n+1},f(x_n)) + (I-P)^{-1}(P+R)d(x_n,x^*). \end{aligned}$$

We have

$$\begin{aligned} d(x_{n+1},x^*) &\leq (I-P)^{-1}(P+R)d(x_n,x^*) \\ &\leq (I-P)^{-1}(P+R)\left[d(x_n,f(x_{n-1})) + d(f(x_{n-1},x^*))\right] \\ &\leq \left[(I-P)^{-1}(P+R)\right]^2 d(x_{n-1},x^*) \\ &\leq \dots \\ &\leq \left[(I-P)^{-1}(P+R)\right]^{n+1} d(x_0,x^*), \end{aligned}$$

so, this implies that $x_{n+1} \to x^*$ as $n \to \infty$. □

By similar proofs as above we have the following results.

Theorem 2.9. *Let (X,d) be an $\mathbb{R}^m_+$-metric space and $Y \subset X$ a nonempty subset. We suppose that $f : Y \to X$ is a ψ-PO with $F_f = \{x^*\}$. Then we have that:*

(i) *if $g : Y \to X$ is such that $d(f(x),g(x)) \leq \eta, \forall x \in Y$, for some $\eta \in \left(\mathbb{R}^*_+\right)^m$, then*

$$d(y^*,x^*) \leq \psi(\eta),\ \forall y^* \in F_g \cap (MI)_f;$$

(ii) *if $y \in Y \cap (MI)_f$ is such that*

$$d(y,f(y)) \leq \varepsilon,$$

for some $\varepsilon \in \left(\mathbb{R}^*_+\right)^m$, then

$$d(y,x^*) \leq \psi(\varepsilon),$$

i.e., the equation $x = f(x)$ is Ulam-Hyers stable;

(iii) *$x_n \in (MI)_f,\ n \in \mathbb{N},\ d(x_n,f(x_n)) \to 0$ as $n \to \infty$ implies that $x_n \to x^*$ as $n \to \infty$, i.e. the fixed point problem for f is well posed.*

Proof. (i)

$$\begin{aligned} d(y^*,x^*) &\leq \psi(d(y^*,f(y^*))) = \psi(d(f(y^*),g(y^*))) \\ &\leq \psi(\eta). \end{aligned}$$

(ii)

$$d(y,x^*) \leq \psi(d(y,f(y))) \leq \psi(\varepsilon).$$

(iii)

$$d(x_n,x^*) \leq \psi(d(x_n,f(x_n))) \underset{n\to\infty}{\to} \psi(0) = 0.$$

So $x_n \to x^*$as $n \to \infty$. □

Theorem 2.10. *Let (X, d) be an $\mathbb{R}_+^m$-metric space and $Y \subset X$ a nonempty subset. We suppose that $f : Y \to X$ is a quasi S-contraction. Then the following implication holds:*

$$x_n \in Y,\ n \in \mathbb{N},\ d(x_{n+1}, f(x_n)) \to 0 \text{ as } n \to \infty \text{ implies that } x_n \to x^* \text{ as } n \to \infty.$$

Proof. The conclusion follows from Theorem 2.8. □

Remark 2.11. The Theorem 2.7 and Theorem 2.8, in the case $P := \alpha \in \mathbb{R}_+,\ Q := \beta \in \mathbb{R}_+, R := \gamma \in \mathbb{R}_+$, take the following form:

Theorem 2.12. *Let (X, d) be an $\mathbb{R}_+$-metric space, $Y \subset X$ a nonempty subset and $f : Y \to X$ be an operator such that,*

$$d(f(x), f(y)) \leq \alpha d(x, f(x)) + \beta d(y, f(y)) + \gamma d(x, y), \tag{2.2}$$

for all $x, y \in Y$, where $\alpha, \beta, \gamma \in \mathbb{R}_+$. We suppose that, $F_f = \{x_f^*\}$ and $\gamma < 1$. Then we have that:

(i) *$d(x, x^*) \leq C d(x, f(x)),\ \forall x \in Y$, where $C := \dfrac{1+\alpha}{1-\gamma}$;*

(ii) *if $g : Y \to X$ is such that $d(f(x), g(x)) \leq \eta, \forall x \in Y$, for some $\eta \in \mathbb{R}_+^*$, then*

$$d(y^*, x^*) \leq C\eta,\ \forall y^* \in F_g;$$

(iii) *if $y \in Y$ is such that*

$$d(y, f(y)) \leq \varepsilon,$$

for some $\varepsilon \in \mathbb{R}_+^*$, then

$$d(y, x^*) \leq C\varepsilon,$$

i.e., the equation $x = f(x)$ is Ulam-Hyers stable;

(iv) *$x_n \in Y,\ n \in \mathbb{N},\ d(x_n, f(x_n)) \to 0$ as $n \to \infty$ implies that $x_n \to x^*$as $n \to \infty$, i.e. the fixed point problem for f is well posed.*

Theorem 2.13. *Let (X, d) be an $\mathbb{R}_+$-metric space, $Y \subset X$ a nonempty subset and $f : Y \to X$ be an operator such that,*

$$d(f(x), f(y)) \leq \alpha d(x, f(x)) + \beta d(y, f(y)) + \gamma d(x, y), \tag{2.3}$$

for all $x, y \in Y$, where $\alpha, \beta, \gamma \in \mathbb{R}_+$. We suppose that $\alpha < 1$. Then we have that:

(i) *$d(f(x), x^*) \leq \dfrac{\alpha+\gamma}{1-\alpha} d(x, x^*),\ \forall x \in Y$, i.e., f is a quasicontraction;*

(ii) *$x_n \in Y,\ n \in \mathbb{N},\ d(x_{n+1}, f(x_n)) \to 0$ as $n \to \infty$ implies that $x_n \to x^*$as $n \to \infty$.*

3. Fibre non-self contraction principle

In this section we obtain the fibre contraction principle for non-self operators in $\mathbb{R}^m_+$-metric space.

Theorem 3.1. *Let (X,d) be an $\mathbb{R}^m_+$-metric space, $Y \subset X$ a nonempty set and (Y_1,d) a complete metric space. Let $g : Y \to X$, $h(x,\cdot) : Y_1 \to Y_1$ and $f : Y \times Y_1 \to X \times Y_1$, $f(x,y) = (g(x), h(x,y))$. We suppose that:*
(i) *g is a PO;*
(ii) *there exists S a convergent to zero matrix such that*

$$d(h(x,y), h(x,z)) \leq Sd(y,z),$$

for all $x \in (AB)_g$ and $y,z \in Y_1$;
(iii) *f is continuous.*
Then f is a PO.

Proof. First of all we remark that $(MI)_f = (MI)_g \times Y_1$ and $(MI)_g = (AB)_g$. Let $x_0 \in (AB)_g$ and $y_0 \in Y_1$. Define $x_{n+1} = g(x_n)$, $y_{n+1} = h(x_n, y_n)$ for $n \in \mathbb{N}$. It is clear that $x_n \to x^* \in F_g$ as $n \to \infty$. Since $h(x^*,\cdot)$ is an S-contraction, $F_{h(x^*,\cdot)} = \{y^*\}$. Let us prove that $y_n \to y^*$. We have

$$\begin{aligned} d(y_{n+1}, y^*) &= d(h(x_n,y_n), y^*) \\ &\leq d(h(x_n,y_n), h(x_n,y^*)) + d(h(x_n,y^*), y^*) \\ &\leq Sd(y_n,y^*) + d(h(x_n,y^*), y^*) \\ &... \\ &\leq S^{n+1}d(y_0,y^*) + S^n d(h(x_0,y^*), y^*) + \\ &... + Sd(h(x_{n-1},y^*), y^*) + d(h(x_n,y^*), y^*). \end{aligned}$$

Then $d(y_{n+1}, y^*) \to 0$, by a Cauchy-Toeplitz lemma (see [14]), so f is a PO. □

The above result is very useful to study of the differentiability of solutions of operator equations with respect to a parameter. For example, let us consider the following equation

$$x(t,\lambda) = F(t, x(t,\lambda), \lambda), \;\; t \in [a,b], \;\; \lambda \in J \subset \mathbb{R} \tag{3.1}$$

and $F : [a,b] \times \mathbb{R}^m_+ \times J \to \mathbb{R}^m_+$. We suppose that:
(H1) $J \subset \mathbb{R}$ is a compact interval.
(H2) $F \in C([a,b] \times \mathbb{R}^m_+ \times J, \mathbb{R}^m_+)$.
(H3) $F(t,\cdot,\cdot) \in C^1(\mathbb{R}^m_+ \times J)$ for every $t \in [a,b]$.
(H4) $\left(\left|\dfrac{\partial F_j}{\partial u_i}(t,u,\lambda)\right|\right)_{i,j=1}^m \leq S$, S convergent to zero, for every $t \in [a,b]$, $u \in \mathbb{R}^m_+$, $\alpha_i \in \mathbb{R}$, $\lambda \in J$, $i = \overline{1,m}$.
(H5) equation (3.1) has at least one solution.
Then we have:

Theorem 3.2. *Under the conditions (H1)-(H5) the equation* (3.1) *has in $C([a,b] \times J, \mathbb{R}^m_+)$ a unique solution x^* and $x^*(t,\cdot) \in C^1(J)$ for every $t \in [a,b]$.*

Proof. Let $X = C([a,b] \times J, \mathbb{R}^m_+)$ with norm $\|.\|_C$ and let $B : C([a,b] \times J, \mathbb{R}^m_+) \to C([a,b] \times J, \mathbb{R}^m_+)$ be defined by $B(x)(t,\lambda) = F(t, x(t,\lambda), \lambda)$.

From conditions (H4) and (H5) it follows that $F_B = \{x^*\}$. Let $Y = \{x \in C([a,b] \times J, \mathbb{R}^m_+) : B(x)(t,\lambda) \in \mathbb{R}^m_+,\ \forall t \in [a,b],\ \lambda \in J\}$. It is clear that $x^* \in Y$, $B(Y) \subset Y$ and $B : Y \to Y$ is a PO. Let $x^0 \in Y$ be such that there exists $\dfrac{\partial x_i^0}{\partial \lambda}$ and $\dfrac{\partial x_i^0}{\partial \lambda} \in C([a,b] \times J)$. Let us suppose that there exists $\dfrac{\partial x_i^*}{\partial \lambda}$. Then we have that

$$\frac{\partial x_i^*(t,\lambda)}{\partial \lambda} = \frac{\partial F_i(t, x^*(t,\lambda), \lambda)}{\partial x_i} \cdot \frac{\partial x_i^*(t,\lambda)}{\partial \lambda} + \frac{\partial F_i(t, x^*(t,\lambda), \lambda)}{\partial \lambda},\ i = \overline{1,m}.$$

This relation suggests us to consider the following operators:

$$C_i : Y \times C([a,b] \times J) \to C([a,b] \times J)$$

defined by

$$C_i(x,y)(t,\lambda) = \frac{\partial F_i(t, x(t,\lambda), \lambda)}{\partial x_i} \cdot y(t,\lambda) + \frac{\partial F_i(t, x(t,\lambda), \lambda)}{\partial \lambda}$$

and

$$A : Y \times C([a,b] \times J) \to Y \times C([a,b] \times J)$$

with

$$A(x,y) = (B(x), C(x,y)).$$

From Theorem 3.1 we have that A is a PO. This implies that the sequences $x_{n+1} = B(x_n)$, $y_{n+1} = C(x_n, y_n)$ are convergent, $x_n \to x^*$, $y_n \to y^*$ and $x^* = B(x^*)$, $y^* = C(x^*, y^*)$.

Let us take $y_i^0 = \dfrac{\partial x_i^0}{\partial \lambda}$. Then $y_{i,n} = \dfrac{\partial x_{i,n}}{\partial \lambda}$. So

$$x_n \to x^* \text{ as } n \to \infty, \text{ with respect to the norm } \|\cdot\|_C$$

and

$$\frac{\partial x_{i,n}}{\partial \lambda} \to y_i^* \text{ as } n \to \infty.$$

These imply that $y^* \in C^1([a,b] \times J, \mathbb{R}^m_+)$ and $y_i^* = \dfrac{\partial x_i^*}{\partial \lambda}$, $i = \overline{1,m}$. □

For other results regarding fibre contractions, see [6], [15], [16].

4. Data dependence in terms of ψ-PO

Let (X,d) be a $\mathbb{R}^m_+$-metric space, $Y \subset X$ a nonempty subset of X and $f, g : Y \to X$ two operators.

Theorem 4.1. *Assume that the following conditions are satisfied:*
(i) f *is* ψ*-PO with* $F_f = \{x^*\}$*;*
(ii) $F_g \subset (BA)_f$;
(iii) *there exists* $\eta \in \mathbb{R}^m_+$ *such that*

$$d(f(x), g(x)) \le \eta, \quad \textit{for all } x \in Y.$$

Then

$$d(x^*, y^*) \le \psi(\eta),\ \forall y^* \in F_g.$$

Proof. Let $y^* \in F_g,\ y^* \in (BA)(x^*)$. Then

$$d(y^*, x^*) \le \psi(d(y^*, f(y^*))) = \psi(d(g(y^*), f(y^*))) \le \psi(\eta). \qquad \square$$

Another result in the case of strict φ-contractions is the following.

Theorem 4.2. *Assume that the following conditions are satisfied:*
(i) f *is a strict* φ*-contraction with* $F_f = \{x_f^*\}$;
(ii) $F_g \neq \emptyset$;
(iii) *there exists* $\eta \in \mathbb{R}_+^m$ *such that*

$$d(f(x), g(x)) \le \eta, \quad \textit{for all } x \in Y.$$

Then

$$d(x_g^*, x_f^*) \le \psi_\varphi(\eta), \quad \textit{for all } x_g^* \in F_g.$$

Proof. Let $x_g^* \in F_g$. We have

$$\begin{aligned} d(x_g^*, x_f^*) &\le d(x_g^*, f(x_g^*)) + d(f(x_g^*), x_f^*) \\ &= d(g(x_g^*), f(x_g^*)) + d(f(x_g^*), f(x_f^*)) \\ &\le \eta + \varphi(d(x_g^*, x_f^*)). \end{aligned}$$

Hence

$$d(x_g^*, x_f^*) - \varphi(d(x_g^*, x_f^*)) \le \eta.$$

Then

$$d(x_g^*, x_f^*) \le \psi_\varphi(\eta). \qquad \square$$

We also have the following result:

Theorem 4.3. *Assume that the following conditions are satisfied:*
(i) *there exist* $P, Q \in \mathbb{R}_+^{m\times m}$, P *convergent to* 0 *matrix, such that*

$$d(f(x), f(y)) \le Pd(x, y) + Q[d(x, f(x)) + d(y, f(y))]$$

for all $x, y \in X$, *and let* $F_f = \{x_f^*\}$;
(ii) $F_g \neq \emptyset$;
(iii) *there exists* $\eta \in \mathbb{R}_+^m$ *such that*

$$d(f(x), g(x)) \le \eta, \quad \textit{for all } x \in Y.$$

Then

$$d(x_g^*, x_f^*) \le (I - P)^{-1}(I + Q)\eta, \quad \textit{for all } x_g^* \in F_g. \tag{4.1}$$

Proof. Let $x_g^* \in F_g$. We have

$$\begin{aligned} d(x_g^*, x_f^*) &\leq d(x_g^*, f(x_g^*)) + d(f(x_g^*), x_f^*) \\ &= d(g(x_g^*), f(x_g^*)) + d(f(x_g^*), f(x_f^*)) \\ &\leq \eta + Pd(x_g^*, x_f^*) + Q\left[d(x_g^*, f(x_g^*)) + d(x_f^*, f(x_f^*))\right] \\ &= \eta + Pd(x_g^*, x_f^*) + Qd(x_g^*, f(x_g^*)) \\ &= \eta + Pd(x_g^*, x_f^*) + Qd(g(x_g^*), f(x_g^*)) \\ &\leq \eta + Q\eta + Pd(x_g^*, x_f^*). \end{aligned}$$

Then

$$(I - P)\, d(x_g^*, x_f^*) \leq (I + Q)\, \eta,$$

so

$$d(x_g^*, x_f^*) \leq (I - P)^{-1} (I + Q)\, \eta, \text{ for all } x_g^* \in F_g. \qquad \square$$

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