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NONTRIVIAL SOLUTIONS FOR A SEMIPOSITONE SUPERLINEAR EULER-BERNOULLI BEAM EQUATIONS WITH NEUMANN BOUNDARY CONDITIONS

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Abstract. Under some conditions concerning the first eigenvalue corresponding to the relevant linear operator, the existence of nontrivial solutions and positive solutions for nonlinear fourth-order equation with Neumann boundary conditions

$$\begin{cases} y^{(4)}(x) + (k_1 + k_2)y''(x) + k_1k_2y(x) = \lambda f(x, y(x)), & x \in [0, 1], \\ y'(0) = y'(1) = y'''(0) = y'''(1) = 0 \end{cases}$$

is obtained, where k_1 and k_2 are constants, $f \in C([0, 1] \times \mathbb{R}, \mathbb{R})$. And we discuss the properties of Green's function in detail according to the different classification of k_1 and k_2, an example is presented to illustrate the application of our main results. The main results are obtained by using the topological method and the fixed point theory of superlinear operators.
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1. Introduction

In this paper, we aim to investigate the existence of nontrivial solutions and positive solutions to the following nonlinear Euler-Bernoulli beam equation with Neumann boundary conditions (for short NBVP)

$$\begin{cases} y^{(4)}(x) + (k_1 + k_2)y''(x) + k_1k_2y(x) = \lambda f(x, y(x)), & x \in [0, 1], \\ y'(0) = y'(1) = y'''(0) = y'''(1) = 0, \end{cases} \tag{1.1}$$

where k_1 and k_2 are constants, λ is a positive parameter, $f \in C([0, 1] \times \mathbb{R}, \mathbb{R})$. One function $y \in C[0, 1]$ is called a positive solution of NBVP (1.1) if y is a solution of NBVP (1.1) and $y(x) > 0,\ x \in (0, 1)$. This problem is always used to describe the sliding braces at both ends of an elastic beam.

Boundary value problems of ordinary differential equations are of great importance in both theory and application, many of which come from classical mechanics and electricity. For example, the equation of the elastic beam studied in this paper is derived from the description of the deformation of the elastic beam in material mechanics. In material mechanics, the boundary value problem of the fourth order differential equation is used to describe the deformation of the elastic beam in the equilibrium state. In particular, the elastic beam equation is also called the Euler-Bernoulli beam equation. In the last decades, the equation in (1.1) with clamped beam boundary condition

$$y(0) = y(1) = y'(0) = y'(1) = 0 \tag{1.2}$$

has attracted the attention of many scholars, it describes the deformations of elastic beams with both fixed end-point, see [4, 11, 15, 26, 31, 33] as well as the references therein. In addition, the equation in (1.1) with Lidstone boundary condition

$$y(0) = y(1) = y''(0) = y''(1) = 0 \tag{1.3}$$

also has received a lot of attention, since it models the stationary states of the deflection of an elastic beam with both hinged ends, see [5, 12, 17, 31] and the references therein. In addition to the above mentioned common types of boundary conditions for elastic beams, another type of boundary conditions have also been considered due to the difference in the support forms of the elastic beams, namely, a simple support and a sliding support boundary condition

$$y(0) = y'(1) = y''(0) = y'''(1) = 0. \tag{1.4}$$

Besides, the elastic beams boundary value problem with local and nonlocal boundary conditions have been obtained in [13, 25].

Some of nonlinear analysis tools have been used to investigate the existence of solutions for the fourth-order elastic beam equation with boundary conditions (1.2) and (1.3), such as, lower and upper solutions [4, 20, 21, 23], monotone iterative technique [2, 5, 10, 22, 28], Krasnosel'skii fixed point theorem [8, 29, 31, 30], fixed-point index [5, 13, 33], Leray-Schauder degree [1, 9] and bifurcation theory [6, 17, 18, 19, 24]. In particular, by using the bifurcation techniques, Ma [17] considered the existence and multiplicity results of differential equation

$$y^{(4)}(x) + \eta y''(x) - \zeta y(x) = \lambda h(x) f(y(x)),\ x \in (0, 1) \tag{1.5}$$

with boundary condition (1.3), where $f \in C(\mathbb{R}, \mathbb{R})$ satisfies $yf(y) > 0$ for all $y \neq 0$, $\eta \in (-\infty, +\infty), \zeta \in [0, +\infty)$ are constants and satisfy the key condition $\frac{\eta}{\pi^2} + \frac{\zeta}{\pi^4} < 1$. Now, the interesting question is whether we could give a more general condition on η and ζ ?

In 2014, by using lower and upper solutions methods, Vrabel [23] discussed the existence of solution the following beam equation

$$y^{(4)}(x) + (k_1 + k_2)y''(x) + k_1k_2y(x) = f(x, y(x)),\ x \in (0, 1) \tag{1.6}$$

with hinged ends condition (1.3). Here, constants k_1 and k_2 satisfy the condition:

$$k_1 < k_2 < 0. \tag{1.7}$$

Now, if we take $\eta = k_1 + k_2$, $\zeta = -k_1 k_2$, then the left sides of equation (1.5) and (1.6) are the same. However, it is easy to see that the condition (1.7) is more general than the condition $\frac{\eta}{\pi^2} + \frac{\zeta}{\pi^4} < 1$. Later, Ma et al. [20, 21] discussed the same problem (1.6) with Lidstone boundary condition (1.3) under the restrictive condition

$$0 < k_1 < k_2 < \pi^2 \text{ and } k_1 < 0 < k_2 < \pi^2, \tag{1.8}$$

and obtained the existence of solution by using lower and upper methods. It is noted that Vrabel [23], Ma et al. [20, 21] only obtained the positivity of Green's function under the condition (1.7) and (1.8). Naturally, the question is: could we obtain sign properties of Green's function when k_1 and k_2 change and the positive solution to these kind of problem under the similar condition?

In particular, the existence and multiplicity of positive solutions for the elastic beam equations have been studied extensively when the nonlinear term satisfies assuming condition $(\mathbf{F}) : f(x, y) \geq 0, \forall y \geq 0$. However there are only a few papers concerned with the non-positone or semipositone elastic beam equations. For the case of non-positone, in 2022, Wang et al. [24] discussed the global structure of positive solutions for NBVP (1.1) by global bifurcation theory under assumption condition $f(x, 0) < 0, \forall x \in [0, 1]$. Similar conclusion have been studied on Yan et al. [27], and so on.

Yao [30] considered the existence of positive solutions of semipositone nonlinear elastic beam equation $y^{(4)}(x) = f(x, y(x), y''(x)), x \in [0, 1]$ with boundary condition (1.3) by using a special cone and the fixed point theorem of cone expansion-compression type, where nonlinear term $f(x, y)$ satisfies

$$f : [0, 1] \times \left[-\frac{5}{384}X, +\infty\right) \times \left(-\infty, \frac{1}{8}X\right] \to [-X, +\infty)$$

is continuous, where $X \geq 0$ is a constant. Ma [16] considered the existence and multiplicity of positive solutions for nonlinear fourth order boundary value problem

$$y^{(4)}(x) = \lambda f(x, y(x), y'(x)),\ x \in [0, 1] \tag{1.9}$$

with boundary condition (1.4) by using Guo-Krasnosel'skii fixed point theorem in cones. They make the following assumptions:

(A1)$f : [0, 1] \times \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ is continuous and there exists constant $X > 0$ such that $f(x, y, p) \geq -X$, for $(x, y, p) \in [0, 1] \times \mathbb{R}^+ \times \mathbb{R}^+$;

(A2) There exists a subinterval $[a, b] \subset (0, 1)$ with $a < b$ such that

$$\lim_{p \to +\infty} \frac{f(x, y, p)}{p} = \infty$$

holds uniformly for $(x, y) \in [a, b] \times \mathbb{R}^+$;

(A3)$f(x, y, 0) > 0, \quad (x, y) \in [0, 1] \times \mathbb{R}^+$;

and main results of paper [16] as follows:

Theorem A. Assume (A1) and (A2) hold. Then the problem (1.9) has at least one positive solution if $\lambda > 0$ is small enough;

Theorem B. Assume (A1), (A2) and (A3) hold. Then the problem (1.9) has at least two positive solutions if $\lambda > 0$ is small enough.

It is worth pointing out that papers [16] and [30] only discussed the existence and multiplicity of positive solutions for fourth-order equations under the simply supported beam conditions (1.3) and (1.4), respectively. However, to the best of our knowledge, a fourth-order equation with parameters under Neumann boundary conditions in spite of its simple looking structure, is considered as a difficult problem in the literature. Therefore, relatively little is known about problem (1.1). Motivated by the above studies, the main purpose of this paper is to establish existence of nontrivial solutions and positive solutions for nonlinear problem (1.1) with Neumann boundary condition.

To sum up all the ideas mentioned in the introduction, we try to discuss the existence of solutions for the nonlinear NBVP (1.1) under a more general condition, like (1.7) and (1.8). Our method is also suitable for the problem (1.6),(1.7) and the problem (1.6),(1.8). Throughout this paper, we use the following assumptions:

(H1) There exists a constant $\zeta > 0$ satisfying

$$\liminf_{y\to+\infty} \frac{f(x,y)}{y} \geq \zeta, \quad \forall\, x \in [0,1];$$

(H2) There exists a constant $\eta > 0$ with $0 < \eta < \zeta$ satisfying

$$\limsup_{y\to 0} \left| \frac{f(x,y)}{y} \right| \leq \eta, \quad \forall\, x \in [0,1];$$

(H3) There exists a constant $X > 0$ such that

$$f(x,y) \geq -X, \quad \forall\, x \in [0,1], \quad y(x) \in \mathbb{R};$$

(H4) $f \in C([0,1] \times \mathbb{R}, \mathbb{R})$ satisfying superlinear condition

$$\lim_{y\to+\infty} \frac{f(x,y)}{y} = \infty.$$

This paper is divided into five sections; From the previous discussion, we find that the appearance of k_1 and k_2 lead to the absence of the positivity of Green's function in NBVP (1.1), which greatly increases the complexity of the calculation of Green's function. On this basis, in the second part of this paper, we discuss the properties of Green's function in detail according to the different classification of k_1 and k_2. Including the cases of $k_1 \leq k_2 < 0$, $k_1 < 0 < k_2 \leq \pi^2/4$ and $0 < k_1 < k_2 \leq \pi^2/4$, respectively. We introduce to the some preliminary results which we shall require. Section 3 and Section 4 are devoted to establishing the existence results of nontrivial solutions and positive solutions of NBVP (1.1) by the topological method and the fixed point theory of superlinear operators, and we give an example to illustrate our main results.

2. Green's function and its sign properties

Let $X = C[0,1]$ be a Banach space, with its usual normal $\|y\| = \max\{|y(x)|, x \in [0,1]\}$ for all $y \in X$. In this section, we discuss the Green's function for NBVP (1.1), the main results came from papers of Wang et al. [24].

2.1 Green's function and its sign properties in case $k_1 \le k_2 < 0$

From $k_1 \le k_2 < 0$, let $k_1 = -\alpha^2,\ k_2 = -\beta^2$, where α and β are constants greater than zero that satisfy $\alpha \ge \beta$. Then, the NBVP (1.1) is transformed into the following boundary value problem

$$\begin{cases} y^{(4)}(x) - (\alpha^2+\beta^2)y''(x) + \alpha^2\beta^2 y(x) = f(x, y(x)), \quad x \in [0,1], \\ y'(0) = y'(1) = y'''(0) = y'''(1) = 0. \end{cases} \tag{2.1}$$

Define linear operator $L : D(L) \to X$ as follows

$$Ly := y^{(4)} - (\alpha^2+\beta^2)y'' + \alpha^2\beta^2 y,\ y \in D(L),$$

where $D(L) := \{y \in C^4[0,1] : y'(0) = y'(1) = y'''(0) = y'''(1) = 0\}$.

To get the Green's function $\overline{G}(x,s)$ of $Ly = 0$, we define another linear operator

$$L_1 y := y'' - \alpha^2 y,\ D(L_1) := \{y \in C^2[0,1] : y'(0) = y'(1) = 0\}.$$

It's not difficult to calculate that the Green's function of $L_1 y = 0$ is

$$G_1(t,s) = -\begin{cases} \dfrac{\cosh[\alpha(1-t)]\cosh(\alpha s)}{\alpha \sinh \alpha}, & 0 \le s \le t \le 1, \\ \dfrac{\cosh[\alpha(1-s)]\cosh(\alpha t)}{\alpha \sinh \alpha}, & 0 \le t \le s \le 1. \end{cases}$$

Define linear operator

$$L_2 y := y'' - \beta^2 y,\ D(L_2) := \{y \in C^2[0,1] : y'(0) = y'(1) = 0\}.$$

Then the Green's function of $L_2 y = 0$ is

$$G_2(t,s) = -\begin{cases} \dfrac{\cosh[\beta(1-t)]\cosh(\beta s)}{\beta \sinh \beta}, & 0 \le s \le t \le 1, \\ \dfrac{\cosh[\beta(1-s)]\cosh(\beta t)}{\beta \sinh \beta}, & 0 \le t \le s \le 1. \end{cases}$$

It's easy to verify that the Green's function of $Ly = L_2 \circ (L_1 y)$, and $Ly = 0$ is

$$\overline{G}(x,s) := \int_0^1 G_2(x,t)G_1(t,s)dt,\ (x,s) \in [0,1] \times [0,1]. \tag{2.2}$$

Notice that if $\alpha = \beta$, then the characteristic equation $\mu^4 - 2\alpha^2\mu^2 + \alpha^4 = 0$ of (2.1) has double roots $\mu_1 = \alpha$, $\mu_2 = -\alpha$. In this case, the expression of $\overline{G}(x,s)$ can not be directly obtained from (2.2). Therefore, we divide two cases as follows:

Case 1. $\alpha = \beta > 0$

In this case,

$$y(x) = C_1\cosh(\alpha x) + C_2\sinh(\alpha x) + C_3 x\cosh(\alpha x) + C_4 x\sinh(\alpha x)$$

is the general solution of $y^{(4)}(x)-(\alpha^2+\beta^2)y''(x)+\alpha^2\beta^2 y(x)=0$. It's easy to compute that $\varphi(x)=(\alpha x\cosh(\alpha x)-\sinh(\alpha x))(2\alpha^3)^{-1}$ is the solution of initial value problem

$$\begin{cases} \varphi^{(4)}(x)-2\alpha^2\varphi''(x)+\alpha^4\varphi(x)=0, & x\in[0,1],\\ \varphi(0)=\varphi'(1)=\varphi''(0)=0, \varphi'''(1)=1. \end{cases}$$

From the theory of Green's function, we can obtain the explicit expression of Green's function of (2.1) as follows

$$\overline{G}(x,s)=\begin{cases} \dfrac{\sinh\alpha\cosh[\alpha(1-s)][\cosh(\alpha x)-\alpha x\sinh(\alpha x)]}{2\alpha^3\sinh^2\alpha} \\ \quad +\dfrac{\alpha\cosh(\alpha x)[\cosh(\alpha s+s\sinh\alpha\sinh[\alpha(1-s)]]}{2\alpha^3\sinh^2\alpha}, & s\le x,\\ \dfrac{\sinh\alpha\cosh[\alpha(1-x)][\cosh(\alpha s)-\alpha s\sinh(\alpha s)]}{2\alpha^3\sinh^2\alpha} \\ \quad +\dfrac{\alpha\cosh(\alpha s)[\cosh(\alpha x+x\sinh\alpha\sinh[\alpha(1-x)]]}{2\alpha^3\sinh^2\alpha}, & x\le s. \end{cases} \tag{2.3}$$

Case 2. $\alpha>\beta>0$

In this case, if $0\le x\le s\le 1$, then

$$\overline{G}(x,s)=\frac{1}{\alpha^2-\beta^2}\Big[\frac{\cosh(\beta x)\cosh[\beta(1-s)]}{\beta\sinh\beta}-\frac{\cosh(\alpha x)\cosh[\alpha(1-s)]}{\alpha\sinh\alpha}\Big].$$

By a similar calculation, when $0\le s\le x\le 1$,

$$\overline{G}(x,s)=\frac{1}{\alpha^2-\beta^2}\Big[\frac{\cosh[\beta(1-x)]\cosh(\beta s)}{\beta\sinh\beta}-\frac{\cosh[\alpha(1-x)]\cosh(\alpha s)}{\alpha\sinh\alpha}\Big].$$

Thus the concrete expression of Green's function of problem (2.1) is

$$\overline{G}(x,s)=\begin{cases} \dfrac{1}{\alpha^2-\beta^2}\Big[\dfrac{\cosh[\beta(1-s)]\cosh(\beta x)}{\beta\sinh\beta} \\ \qquad -\dfrac{\cosh[\alpha(1-s)]\cosh(\alpha x)}{\alpha\sinh\alpha}\Big], & 0\le x\le s\le 1,\\ \dfrac{1}{\alpha^2-\beta^2}\Big[\dfrac{\cosh[\beta(1-x)]\cosh(\beta s)}{\beta\sinh\beta} \\ \qquad -\dfrac{\cosh[\alpha(1-x)]\cosh(\alpha s)}{\alpha\sinh\alpha}\Big], & 0\le s\le x\le 1. \end{cases} \tag{2.4}$$

Theorem 2.1 [24] *If* $\alpha,\ \beta\in(0,+\infty)$ *with* $\alpha\ge\beta$*, then the Green's function of problem (2.1) satisfies* $\overline{G}(x,s)>0,\ (x,s)\in[0,1]\times[0,1]$.

Proof. According to literature [14], we know $G_i(t,s)<0,\ i=1,\ 2,\ (t,s)\in[0,1]\times[0,1]$, and from (2.2), we can get $\overline{G}(x,s)>0$, $(x,s)\in[0,1]\times[0,1]$.

2.2 Green's function and its sign properties in case $k_1<0<k_2\le\pi^2/4$

From $k_1<0<k_2\le\pi^2/4$, let $k_1=-\alpha^2,\ k_2=\beta^2$ and $\alpha\in(0,+\infty)$, $\beta\in(0,\pi/2]$, Then the NBVP (1.1) can be written as the following boundary value problem

$$\begin{cases} y^{(4)}(x)+(\beta^2-\alpha^2)y''(x)-\alpha^2\beta^2y(x)=f(x,y(x)), & x\in[0,1],\\ y'(0)=y'(1)=y'''(0)=y'''(1)=0. \end{cases} \tag{2.5}$$

Define linear operator $L : D(L) \to X$ as follows

$$Ly := y^{(4)} + (\beta^2 - \alpha^2)y'' - \alpha^2\beta^2 y,\ y \in D(L),$$

where $D(L) := \{y \in C^4[0,1] : y'(0) = y'(1) = y'''(0) = y'''(1) = 0\}$.

To get the Green's function $\widetilde{G}(x,s)$ of the operator $Ly = 0$, define linear operator

$$L_1 y := y'' - \alpha^2 y,\ D(L_1) := \{y \in C^2[0,1] : y'(0) = y'(1) = 0\}.$$

It's not difficult to calculate $G_1(t,s)$ is the Green's function of $L_1 y = 0$.

Define a linear operator

$$L_3 y := y'' + \beta^2 y,\ D(L_3) := \{y \in C^2[0,1] : y'(0) = y'(1) = 0\},$$

then the Green's function of $L_3 y = 0$ is

$$G_3(t,s) = \begin{cases} \dfrac{\cos[\beta(1-t)]\cos(\beta s)}{\beta\sin\beta}, & 0 \le s \le t \le 1, \\ \dfrac{\cos[\beta(1-s)]\cos(\beta t)}{\beta\sin\beta}, & 0 \le t \le s \le 1. \end{cases}$$

Obviously, if $\alpha > 0$, then $G_1(t,s) < 0$, $(t,s) \in [0,1]\times[0,1]$. If $0 < \beta < \pi/2$, then $G_3(t,s) > 0$; If $\beta = \pi/2$, then $G_3(t,s) \ge 0$. Especially, $G_3(t,s) = 0$ with $t = s = 0$ or $t = s = 1$.

Hence, $Ly = L_3 \circ (L_1 y)$, the Green's function of $Ly = 0$ is

$$\widetilde{G}(x,s) := \int_0^1 G_3(x,t)G_1(t,s)dt,\ (x,s) \in [0,1]\times[0,1]. \tag{2.6}$$

Moreover, if $0 \le x \le s \le 1$, then

$$-\widetilde{G}(x,s) = \frac{1}{\alpha^2+\beta^2}\Big[\frac{\cos(\beta x)\cos[\beta(1-s)]}{\beta\sin\beta} + \frac{\cosh(\alpha x)\cosh[\alpha(1-s)]}{\alpha\sinh\alpha}\Big].$$

By a similar calculation, if $0 \le s \le x \le 1$, then we can get

$$-\widetilde{G}(x,s) = \frac{1}{\alpha^2+\beta^2}\Big[\frac{\cos[\beta(1-x)]\cos(\beta s)}{\beta\sin\beta} + \frac{\cosh[\alpha(1-x)]\cosh(\alpha s)}{\alpha\sinh\alpha}\Big].$$

Thus, the concrete expression of Green's function of problem (2.5) is

$$-\widetilde{G}(x,s) = \begin{cases} \dfrac{1}{\alpha^2+\beta^2}\Big[\dfrac{\cos[\beta(1-s)]\cos(\beta x)}{\beta\sin\beta} \\ \qquad + \dfrac{\cosh[\alpha(1-s)]\cosh(\alpha x)}{\alpha\sinh\alpha}\Big], & 0 \le x \le s \le 1, \\ \dfrac{1}{\alpha^2+\beta^2}\Big[\dfrac{\cos[\beta(1-x)]\cos(\beta s)}{\beta\sin\beta} \\ \qquad + \dfrac{\cosh[\alpha(1-x)]\cosh(\alpha s)}{\alpha\sinh\alpha}\Big], & 0 \le s \le x \le 1. \end{cases} \tag{2.7}$$

The properties of Green's function $\widetilde{G}(x,s)$ are given as follows:

Theorem 2.2 [24] *If* $\alpha \in (0,+\infty)$, $\beta \in (0,\pi/2]$, *then* $\widetilde{G}(x,s) < 0$, $(x,s) \in [0,1]\times[0,1]$.

Proof. When $\alpha \in (0,+\infty)$, $\beta \in (0,\pi/2]$, it can be obtained directly from literature [14] that $G_1(t,s) < 0$, $G_3(t,s) > 0$, $(t,s) \in [0,1]\times[0,1]$. Combining (2.3), we know

$\widetilde{G}(x,s)<0,\ (x,s)\in[0,1]\times[0,1]$. When $\alpha\in(0,+\infty),\ \beta=\pi/2,$ we can get $G_1(t,s)<0$ and $G_3(x,t)\geq 0$.

In particular, $G_3(x,t)=0$ if and only if $x=t=0$ or $x=t=1$. Therefore, if $x=0$, combining this with (2.7), we can obtain

$$-\widetilde{G}(0,s)=\frac{1}{\alpha^2+\beta^2}\Big[\frac{\cos[\beta(1-s)]}{\beta\sin\beta}+\frac{\cosh[\alpha(1-s)]}{\alpha\sinh\alpha}\Big],\ 0\leq s\leq 1.$$

Because $x\sin x$ is increasing on $x\in(0,\pi/2)$, $\cos x$ is decreasing, so $\cos x/(x\sin x)^{-1}$ is a decreasing function, however, $\cos x/(x\sin x)^{-1}$ is positive on $x\in(0,\pi/2)$. Apparently, $\sinh x$ and $\cosh x$ are increasing and positive on $x\in(0,+\infty)$. Therefore, $-\widetilde{G}(0,s)>0,\ s\in[0,1]$, then we can get $\widetilde{G}(0,s)<0,\ s\in[0,1]$; If $x=1$,

$$-\widetilde{G}(1,s)=\frac{1}{\alpha^2+\beta^2}\Big[\frac{\cos(\beta s)}{\beta\sin\beta}+\frac{\cosh(\alpha s)}{\alpha\sinh\alpha}\Big],\ 0\leq s\leq 1.$$

Similar to reachable $\widetilde{G}(1,s)<0,\ s\in[0,1]$. To sum up, $\widetilde{G}(x,s)<0,\ (x,s)\in[0,1]\times[0,1]$.

Remark 2.3 It is worth noting that we get $\widetilde{G}(x,s)<0$ with the case of $k_1<0<k_2\leq\pi^2/4$. At this point, if the problem we are studying (1.1) is transformed into the following form

$$\begin{cases} y^{(4)}(x)+(k_1+k_2)y''(x)+k_1k_2y(x)+\lambda f(x,y(x))=0, & x\in[0,1],\\ y'(0)=y'(1)=y'''(0)=y'''(1)=0, \end{cases}\tag{2.8}$$

then the results obtained in this paper still held true for the above problems.

2.3 Green's function and its sign properties in case $0<k_1<k_2\leq\pi^2/4$

From $0<k_1<k_2\leq\pi^2/4$, let $k_1=\alpha^2,\ k_2=\beta^2$ and $0<\alpha<\beta\leq\pi/2$, then the NBVP (1.1) can be written as the following boundary value problem

$$\begin{cases} y^{(4)}(x)+(\alpha^2+\beta^2)y''(x)+\alpha^2\beta^2y(x)=f(x,y(x)), & x\in(0,1),\\ y'(0)=y'(1)=y'''(0)=y'''(1)=0. \end{cases}\tag{2.9}$$

Define linear operators $L:D(L)\to X$

$$Ly:=y^{(4)}+(\alpha^2+\beta^2)y''+\alpha^2\beta^2y,\ y\in D(L),$$

where

$$D(L):=\{y\in C^4[0,1]:y'(0)=y'(1)=y'''(0)=y'''(1)=0\}.$$

To get the Green's function $G(x,s)$ of the operator $Ly=0$, define another linear operator

$$L_4y:=y''+\alpha^2y,\ D(L_4):=\{y\in C^2[0,1]:y'(0)=y'(1)=0\}.$$

It's not difficult to calculate the Green's function of $L_4y=0$ is

$$G_4(t,s)=\begin{cases}\dfrac{\cos[\alpha(1-t)]\cos(\alpha s)}{\alpha\sin\alpha}, & 0\leq s\leq t\leq 1,\\[2mm] \dfrac{\cos[\alpha(1-s)]\cos(\alpha t)}{\alpha\sinh\alpha}, & 0\leq t\leq s\leq 1.\end{cases}$$

Define the linear operator

$$L_3y := y'' + \beta^2 y,\ D(L_3) := \{y \in C^2[0,1] : y'(0) = y'(1) = 0\}.$$

$G_3(t,s)$ is the Green's function of $L_3y = 0$.

It is easy to verify $Ly = L_3 \circ (L_4 y)$, then the Green's function of $Ly = 0$ is

$$G(x,s) := \int_0^1 G_3(x,t)G_4(t,s)dt,\ (x,s) \in [0,1] \times [0,1]. \tag{2.10}$$

Notice that if $\alpha = \beta$, then the characteristic equation $\mu^4 + 2\alpha^2\mu^2 + \alpha^4 = 0$ of (2.5) has double roots $\mu_1 = \alpha i$, $\mu_2 = -\alpha i$. In this case, the expression of $G(x,s)$ can not be directly obtained from (2.6).

Therefore, we divide two cases as follows:

Case 3. $\alpha = \beta < \pi/2$

In this case, $y(x) = C_1\cos(\alpha x) + C_2\sin(\alpha x) + C_3 x\cos(\alpha x) + C_4 x\sin(\alpha x)$ is the general solution of $y^{(4)}(x) + (\alpha^2+\beta^2)y''(x) + \alpha^2\beta^2 y(x) = 0$. It is easy to compute that $\varphi(x) = (\sin(\alpha x) - \alpha x\cos(\alpha x))/(2\alpha^3)^{-1}$ is the solution of initial value problem

$$\begin{cases} \varphi^{(4)}(x) + 2\alpha^2\varphi''(x) + \alpha^4\varphi(x) = 0, & x \in [0,1], \\ \varphi(0) = \varphi'(1) = \varphi''(0) = 0, \varphi'''(1) = 1. \end{cases}$$

Then, we can obtain the concrete expression of Green's function of problem (2.5) as follows

$$G(x,s) = \begin{cases} \dfrac{\sin\alpha\cos[\alpha(1-s)][\cos(\alpha x) + \alpha x\sin(\alpha x)]}{2\alpha^3\sin^2\alpha} \\ +\dfrac{\alpha\cos(\alpha x)[\cos(\alpha s - s\sin\alpha\sin[\alpha(1-s)]]}{2\alpha^3\sin^2\alpha}, & s \le x, \\ \dfrac{\sin\alpha\cos[\alpha(1-x)][\cos(\alpha s) + \alpha s\sin(\alpha s)]}{2\alpha^3\sin^2\alpha} \\ +\dfrac{\alpha\cos(\alpha s)[\cos(\alpha x - x\sin\alpha\sin[\alpha(1-x)]]}{2\alpha^3\sin^2\alpha}, & x \le s. \end{cases} \tag{2.11}$$

In particular, if $\alpha = \beta = \pi/2$, then $t = s = 0$ or $t = s = 1$, $G(x,s)$ contains zero.

Case 4. $0 < \alpha < \beta < \pi/2$

In this case, if $0 \le x \le s \le 1$, then

$$G(x,s) = \frac{1}{\beta^2 - \alpha^2}\Big[\frac{\cos(\alpha x)\cos[\alpha(1-s)]}{\alpha\sin\alpha} - \frac{\cos(\beta x)\cos[\beta(1-s)]}{\beta\sin\beta}\Big].$$

Similarly, if $0 \le s \le x \le 1$, then

$$G(x,s) = \frac{1}{\beta^2 - \alpha^2}\Big[\frac{\cos[\alpha(1-x)]\cos(\alpha s)}{\alpha\sin\alpha} - \frac{\cos[\beta(1-x)]\cos(\beta s)}{\beta\sin\beta}\Big].$$

So the concrete expression of Green's function of problem (2.5) is

$$G(x,s)=\begin{cases}\dfrac{1}{\beta^2-\alpha^2}\Big[\dfrac{\cos[\alpha(1-s)]\cos(\alpha x)}{\alpha\sin\alpha}\\ \qquad -\dfrac{\cos[\beta(1-s)]\cos(\beta x)}{\beta\sin\beta}\Big], & 0\le x\le s\le 1,\\ \dfrac{1}{\beta^2-\alpha^2}\Big[\dfrac{\cos[\alpha(1-x)]\cos(\alpha s)}{\alpha\sin\alpha}\\ \qquad -\dfrac{\cos[\beta(1-x)]\cos(\beta s)}{\beta\sin\beta}\Big], & 0\le s\le x\le 1.\end{cases} \tag{2.12}$$

The properties of Green's function $G(x,s)$ are given as follows:

Theorem 2.4 [24] *If* $0<\alpha<\beta\le\pi/2$, *then* $G(x,s)>0,\ (x,s)\in[0,1]\times[0,1]$.

Proof. According to literature [14], we know that $G_i(t,s)>0,\ i=3,4,\ (t,s)\in[0,1]\times[0,1]$. Combining (2.6), we can obtain $G(x,s)>0$, $(x,s)\in[0,1]\times[0,1]$.

If $0<\alpha<\beta=\pi/2$, then we can get $G_4(t,s)>0$ and $G_3(x,t)\ge 0$. Especially, $G_3(x,t)=0$ if and only if $x=t=0$ or $x=t=1$. Therefore, when $x=0$, by combining (2.12), we can obtain

$$G(0,s)=\frac{1}{\beta^2-\alpha^2}\Big[\frac{\cos[\alpha(1-s)]}{\alpha\sin\alpha}-\frac{\cos[\beta(1-s)]}{\beta\sin\beta}\Big],\ 0\le s\le 1.$$

Because $x\sin x$ is increasing on $x\in(0,\pi/2)$, $\cos x$ is decreasing, so $\cos x/(x\sin x)^{-1}$ is a decreasing function. Therefore $G(0,s)>0$, $s\in[0,1]$; When $x=1$,

$$G(1,s)=\frac{1}{\beta^2-\alpha^2}\Big[\frac{\cos(\alpha s)}{\alpha\sin\alpha}-\frac{\cos(\beta s)}{\beta\sin\beta}\Big],\ 0\le s\le 1.$$

Similarly, we can get $G(1,s)>0$, $s\in[0,1]$. To sum up, $G(x,s)>0$, $(x,s)\in[0,1]\times[0,1]$.

Remark 2.5 [24] It should be noted that in the three cases discussed in this section, if the parameter $k_1=0$ or $k_2=0$, the operator Ly has eigenvalue $\lambda_0=0$ and $Ly=0$ has nontrivial solution $y\equiv C\,(C\neq 0)$. Therefore, according to the Fredholm alternative theorem, there is no solution to the problem (1.1), so the parameters in this paper meet the requirement that $k_1k_2\neq 0$ are always valid. In particular, the parameter $k_1=k_2=\pi^2/4$, if $t=s=0$ or $t=s=1$, then $G(x,s)$ contains zero.

2.4 Preliminaries

Based on the sign of Green's function of NBVP (1.1), without loss of generality, we discuss the case of $0<k_1<k_2\le\pi^2/4$.

Obviously, $y(x)$ is a solution of the problem

$$\begin{cases} y^{(4)}(x)+(k_1+k_2)y''(x)+k_1k_2y(x)=h(x), & x\in[0,1],\\ y'(0)=y'(1)=y'''(0)=y'''(1)=0,\end{cases}$$

then

$$y(x)=\int_0^1 G(x,s)h(s)ds,\ x\in[0,1],$$

where $G(x,s)$ is given by (2.12).

From Theorem 2.4, there exist $0 < m < M$ such that

$$m = \min_{x,s\in[0,1]} G(x,s),\ M = \max_{x,s\in[0,1]} \overline{G}(x,s).$$

Besides, by (2.12) we have
($\mathbf{G}_1$) $G(x,s) = G(s,x),\ 0 \le x,s \le 1$;
($\mathbf{G}_2$) There exist nonnegative function $g(x) \in C[0,1]$ such that

$$G(x,s) \ge g(x)G(\tau,s),\ 0 \le x,s,\tau \le 1.$$

Consider 4-dimensional Banach space

$$E = \{y \in C^4[0,1] : y'(0) = y'(1) = y'''(0) = y'''(1) = 0\}$$

with the norm $\|y\| = \max_{0\le x\le 1} |y(x)|$ for all $y \in E$ and the cone P in E given by

$$P = \left\{y \in E, y(x) \ge 0, y(x) \ge \sigma\|y\|,\ \sigma = mM^{-1}\right\}.$$

For $u,v \in E$, we write $u \le v$ if $u(x) \le v(x)$ for any $x \in [0,1]$. For any $r > 0$, let $B_r = \{y \in E : \|y\| < r\}$ and $\partial B_r = \{y \in E : \|y\| = r\}$. We denote θ is the zero element in E.
Lemma 2.6 *Define operators $A, K : E \to E$, by*

$$(Ay)(x) = \int_0^1 G(x,s)f(s,y(s))ds,\ y \in E,\ x \in [0,1]; \tag{2.13}$$

$$(Ky)(x) = \int_0^1 G(x,s)y(s)ds,\ y \in E,\ x \in [0,1]; \tag{2.14}$$

Then $K(P) \subset P$, $A(P) \subset P$ and $K : E \to E,\ A : E \to E$ are completely continuous.
Proof. By the definitions of m and M, it follows that

$$Ay(x) = \int_0^1 G(x,s)f(s,y(s))ds \ge m\int_0^1 f(s,y(s))ds, x \in [0,1],$$

$$Ay(x) = \int_0^1 G(x,s)f(s,y(s))ds \le M\int_0^1 f(s,y(s))ds, x \in [0,1].$$

Accordingly,

$$Ay(x) \ge \sigma \max_{x\in[0,1]} Ay(x) = \sigma\|Ay\|,$$

where $\sigma = mM^{-1}$. So $A(P) \subset P$, by using the Arzelà-Ascoli theorem, A is a completely continuous operator. By using similar method it yields that $K(P) \subset P$ and $K : E \to E$ is completely continuous.

It is evident that $y \in P$ is a fixed point of the operator λA if and only if y is a solution of NBVP (1.1). K defined by (2.14) is an important operator in our later discussion. We present some properties of it as follows.
Lemma 2.7 *Suppose that the linear operator K is defined by (2.14). Then the spectral radius $r(K) > 0$, and there exist $\xi \in E$ with $\xi > 0$ on [0,1] such that*

$$K\xi = r(K)\xi \quad \textit{and} \quad \int_0^1 \xi(s)ds = r^{-1}(K).$$

Moreover, $\lambda_1 = r^{-1}(K)$ *is the first positive eigenvalue of the linear NBVP (1.1) and*

$$\int_0^1 (Ky)(s)\xi(s)ds = \frac{1}{\lambda_1}\int_0^1 y(s)\xi(s)ds,\ \forall\ y \in E. \tag{2.15}$$

Proof. Define the cone

$$P_0 = \{y \in E : y(x) \geq 0, \forall\ x \in [0,1]\}.$$

Then the cone P_0 is normal and has nonempty interiors int P_0. It is clear that P_0 is also a total cone of E, that is, $E = \overline{P_0 - P_0}$, which means that the set $P_0 - P_0 = \{u - v : u, v \in P_0\}$ is dense in E. It follows from Lemma 2.6 that K is strongly positive, that is,

$$K(y) \in intP_0 \quad \text{for} \quad y \in P_0\backslash\{\theta\}.$$

Obviously, $K(P_0) \subseteq P_0$. By the Krein-Rutman theorem ([7], Theorem 19.3; [32], Theorem 7.C), the spectral radius $r(K) > 0$ and there exists $\xi_0 \in E$ with $\xi_0 > 0$ on [0,1] such that $K\xi_0 = r(K)\xi_0$. Let

$$\xi = \frac{\xi_0}{r(K)\int_0^1 \xi_0(s)ds}.$$

Obviously, $\xi > 0$ on [0,1], $K\xi = r(K)\xi$ and $\int_0^1 \xi(s)ds = r^{-1}(K)$.

Notice that $K\xi = r(K)\xi$ is equivalent to the following NBVP

$$\begin{cases} \xi^{(4)}(x) + (k_1 + k_2)\xi''(x) + k_1k_2\xi(x) = \dfrac{1}{r(K)}\xi(x), \quad x \in [0,1], \\ \xi'(0) = \xi'(1) = \xi'''(0) = \xi'''(1) = 0, \end{cases}$$

we can obtain that $\lambda_1 = r^{-1}(K)$ is an eigenvalue of the linear NBVP (1.1). From the strong positivity of K, we know that there exist $\eta \in P_0$ and a constant $c > 0$ such that $cK\eta \geq \eta$ on [0,1]. Then λ_1 is the first positive eigenvalue of the linear problem NBVP (1.1). Since $\xi'(0) = \xi'(1) = \xi'''(0) = \xi'''(1) = 0$, we have

$$\begin{aligned} &\lambda_1 \int_0^1 (Ky)(s)\xi(s)ds = \int_0^1 (Ky)(s)\{\xi^{(4)}(s) + (k_1 + k_2)\xi''(s) + k_1k_2\xi(s)\}ds \\ &= \int_0^1 \xi(s)(Ky)^{(4)}(s)ds + (k_1 + k_2)\int_0^1 \xi(s)(Ky)''(s)ds + k_1k_2\int_0^1 \xi(s)(Ky)(s)ds \\ &= \int_0^1 y(s)\xi(s)ds. \end{aligned}$$

Then (2.15) holds.

The proof of the main theorems are based on the Leray-Schauder theorem. The following three well-known theorem in [3, 9] are needed in our argument.

Lemma 2.8 [3] *Let* $\Omega \subset E$ *be a bounded open set,* $\theta \in \Omega$*, and* $A : \overline{\Omega} \to E$ *a completely continuous operator. If* $Ax \neq \mu x$, $x \in \partial\Omega$, $\mu \geq 1$, *then* $\deg(I - A, \Omega, \theta) = 1$.

Lemma 2.9 [3] *Let* $\Omega \subset E$ *be a bounded open set and* $A : \overline{\Omega} \to E$ *a completely continuous operator. If there exists* $y_0 \in E\backslash\{\theta\}$ *such that* $y - Ay \neq \mu y_0$, $y \in \partial\Omega$, $\mu \geq 0$, *then* $\deg(I - A, \Omega, \theta) = 0$.

Lemma 2.10 [9] *Let* $\Omega \subset E$ *be a bounded open set,* $\theta \in \Omega$*, and* $A : \overline{\Omega} \to E$ *a* *completely continuous operator with $A\theta = \theta$. Suppose that the Frechet derivative A'_θ of A at θ exists and I is not an eigenvalue of A'_θ. Then there exists $r_0 > 0$ such that for any $0 < r < r_0$, $\deg(I - A, T_r, \theta) = \deg(I - A'_\theta, T_r, \theta) = (-1)^\kappa$, where κ is the sum of algebraic multiplicities for all eigenvalues of eigenvalues of A'_θ lying in the interval (0,1) and $T_r = \{x \in E | \|x\| < r\}$.*

3. Nontrivial solutions

In this section, we give the existence of nontrivial solutions for NBVP (1.1), and the main results as follows:

Theorem 3.1 *Suppose that (H1)-(H3) hold. Then for any*

$$\lambda \in \left(\frac{\lambda_1}{\zeta}, \frac{\lambda_1}{\eta}\right),$$

NBVP (1.1) has at least one nontrivial solution, where $\lambda_1 = r^{-1}(K)$ and K is given by (2.14).

Proof. Let $(\mathcal{F}y)(x) = f(x, y(x))$ for all $y \in E$, then by (2.14), $A = K\mathcal{F}$. Applying the Arzelà-Ascoli theorem and a standard argument, we can prove that $(\lambda A) : E \to E$ is a completely continuous operator. It is known to all that the nonzero fixed points of the operator λA are the nontrivial solutions of the NBVP (1.1).

Now we show that there exists $R_0 > 0$, such that for any $R > R_0$,

$$y - \lambda Ay \neq \mu y^*, \quad \forall\ \mu \geq 0, \quad y \in \partial T_R, \tag{3.1}$$

where y^* is the positive eigenfunction of K corresponding to its first eigenvalue $\lambda_1 = r^{-1}(K)$, $T_R = \{y \in C[0,1] | \|y\| < R\}$ is a bounded open subset of E.

If (3.1) is not true, then there exists $\mu_0 > 0$ (if $\mu_0 = 0$, then Theorem 3.1 holds) and $y_0 \in \partial T_R$ such that

$$y_0 - \lambda Ay_0 = \mu_0 y^*. \tag{3.2}$$

By conditions (H1) and (H2), there exists constant $a > 0$, $r_0 > 0$ and

$$0 < \varepsilon < \min\left\{\frac{\zeta\lambda - \lambda_1}{\lambda}, \frac{\eta\lambda - \lambda_1}{\lambda}\right\}$$

such that

$$f(x, y) \geq (\zeta - \varepsilon)y - a, \quad x \in [0,1], \quad y \geq 0, \tag{3.3}$$

$$|f(x, y)| \leq (\eta + \varepsilon)|y|, \quad x \in [0,1], \quad y \leq r_0. \tag{3.4}$$

It follows from (H3) that

$$f(x, y) \geq -X \geq (\zeta - \varepsilon)y - X, \quad x \in [0,1], \quad y \leq 0. \tag{3.5}$$

Take $w = \max\{X, a\}$, then by (3.3) and (3.5), we have

$$f(x, y) \geq (\zeta - \varepsilon)y - w, \quad x \in [0,1], \quad y \in \mathbb{R}. \tag{3.6}$$

Let $\zeta_1 = \zeta - \varepsilon$, then

$$\lambda Ay = \lambda K\mathcal{F}y \geq \lambda\zeta_1 Ky - y_1, \quad \forall y \in E, \tag{3.7}$$

where $y_1 = \int_0^1 G(x,s)\lambda w ds$. Take

$$\delta = m\lambda_1 \int_0^1 y^*(s)ds > 0.$$

By (2.14) and the definition of m we have

$$\begin{aligned} y^*(x) = \lambda_1 K y^*(x) &= \lambda_1 \int_0^1 G(x,s) y^*(s) ds \\ &\geq m\lambda_1 \int_0^1 y^*(s) ds = \delta. \end{aligned}$$

Let

$$P_1 = \left\{ y \in P \,\middle|\, \int_0^1 y^*(x) y(x) dx \geq \sigma \lambda_1^{-1} \delta \|y\| \right\}.$$

For any $y \in P$, we have

$$\begin{aligned} \int_0^1 y^*(x)(\lambda K y)(x) dx &= \int_0^1 y^*(x) \left\{ \int_0^1 G(x,s) \lambda y(s) ds \right\} dx \\ &= \int_0^1 \lambda y(s) \left\{ \int_0^1 G(s,x) y^*(x) dx \right\} ds \\ &= \int_0^1 \lambda y(s) K y^*(s) ds \\ &\geq \sigma \lambda_1^{-1} \delta \int_0^1 G(x,s) \lambda y(s) ds \\ &= \sigma \lambda_1^{-1} \delta (\lambda K y)(x). \end{aligned}$$

And so

$$\int_0^1 y^*(x)(\lambda K y)(x) dx \geq \sigma \lambda_1^{-1} \delta \|\lambda K y\|,$$

i.e. $(\lambda K)(P) \subset P_1$. Since $\mathcal{F} y_0 + w \in P$,

$$(\lambda K)(\mathcal{F} y_0 + w) \in P_1 \text{ and } \mu_0 y^* = \mu_0 \lambda_1 K y^* \in P_1.$$

By (3.2) we have

$$y_0 + \lambda K w = \lambda A y_0 + \mu_0 y^* + \lambda K w = \lambda K(\mathcal{F} y_0 + w) + \mu_0 y^* \in P_1.$$

Thus

$$\begin{aligned} \int_0^1 y * (x)(y_0 + \lambda K w)(x) dx &\geq \sigma \lambda_1^{-1} \delta \|y_0 + \lambda K w\| \\ &\geq \sigma \lambda_1^{-1} \delta \|y_0\| - \sigma \lambda_1^{-1} \delta \|\lambda K w\|. \end{aligned} \tag{3.8}$$

Take $\varepsilon_0 = \zeta_1 \lambda r(K) - 1 > 0$. By (3.7) we have

$$\begin{aligned}\int_0^1 y^*(x)(\lambda A y_0)(x)dx &\geq \int_0^1 y^*(x)\zeta_1(\lambda K y_0)(x)dx - \int_0^1 y^*(x)y_1(x)dx \\ &= \zeta_1 \lambda r(K) \int_0^1 y^*(x)y_0(x)dx - \int_0^1 y^*(x)y_1(x)dx \\ &= \int_0^1 y^*(x)y_0(x)dx + \varepsilon_0 \int_0^1 y^*(x)(y_0 + \lambda K w)(x)dx \\ &- \varepsilon_0 \int_0^1 y^*(x)(\lambda K w)(x)dx - \int_0^1 y^*(x)y_1(x)dx.\end{aligned}$$

Take

$$\begin{aligned}R_0 =& \frac{\lambda_1}{\varepsilon_0 \sigma \delta} \left\{ \varepsilon_0 \lambda_1^{-1} \delta \|\lambda K w\| + \varepsilon_0 \int_0^1 y * (x)(\lambda K w)(x)dx \right\} \\ &+ \frac{\lambda_1}{\varepsilon_0 \delta} \int_0^1 y^*(x)y_1(x)dx.\end{aligned}$$

For any $\|y_0\| = R > R_0$, by (3.8) we have

$$\begin{aligned}\int_0^1 y^*(x)(\lambda A y_0 - y_0)(x)dx \geq& \varepsilon_0[\sigma \lambda_1^{-1} \delta \|y_0\| - \sigma \lambda_1^{-1} \delta \|\lambda K w\|] \\ &- \varepsilon_0 \int_0^1 y^*(x)(\lambda K w)(x)dx - \int_0^1 y^*(x)y_1(x)dx \\ &> \varepsilon_0 \sigma \lambda_1^{-1} \delta R_0 - \varepsilon_0 \sigma \lambda_1^{-1} \delta \|\lambda K w\| \\ &- \varepsilon_0 \int_0^1 y^*(x)(\lambda K w)(x)dx - \int_0^1 y^*(x)y_1(x)dx = 0.\end{aligned}$$

But we see from (3.2) that

$$\int_0^1 y^*(x)(y_0 - \lambda A y_0)(x)dx = \int_0^1 y^*(x)\mu_0 y^*(x)dx \geq 0,$$

which is a contradiction. So (3.1) is true. By Lemma 2.9 we have

$$\deg(I - \lambda A, T_R, \theta) = 0. \tag{3.9}$$

Next we show that

$$(\lambda A)y \neq \mu y, \quad y \in \partial T_r, \quad \mu \geq 1, \tag{3.10}$$

where $0 < r < \min\{r_0, R_0\}$. Assume on the contrary that exist $y_0 \in \partial T_r$ and $\mu_0 \geq 1$ such that $(\lambda A)y_0 = \mu_0 y_0$. Since λA has no fixed point on ∂T_r, we have $\mu_0 > 1$. Let $K_1 = \lambda(\eta + \varepsilon)K$, then $r(K_1) < 1$. By (3.4), we have

$$|\lambda A y_0| \leq (\eta + \varepsilon)\lambda K_1 |y_0| = K_1 |y_0|,$$

then $\mu_0 |y_0| \leq K_1 |y_0|$, and therefore

$$\mu_0^n |y_0| \leq K_1^n |y_0|. \tag{3.11}$$

Let $D = \{\xi | \xi \geq |y_0|\}$. It follows from (3.11) that $\{\mu_0^{-n} K_1^n |y_0| \; | n = 1, 2, \cdots\} \subset D$. And $\theta \in T_r$ implies that $d = d(\theta, D) > 0$. Then one can have that

$$\|K_1^n\| \geq \frac{1}{\|y_0\|} \|K_1^n y_0\| \geq \frac{d}{\|y_0\|} \mu_0^n, \quad n = 1, 2, \cdots,$$

which shows

$$r(K_1) = \lim_{n \to \infty} (\|K_1^n\|)^{1/n} \geq \lim_{n \to \infty} \left(\frac{d}{\|y_0\|} \mu_0^n \right)^{1/n} = \mu_0 > 1.$$

This contradicts $r(K_1) < 1$. So (3.10) holds. By Lemma 2.8, we have

$$\deg(I - \lambda A, T_r, \theta) = 1. \tag{3.12}$$

By (3.9) and (3.12), we have

$$\deg(I - \lambda A, T_R \backslash \overline{T_r}, \theta) = \deg(I - \lambda A, T_R, \theta) - \deg(I - \lambda A, T_r, \theta) = -1.$$

Then λA has at least one fixed point in $T_R \backslash \overline{T_r}$. This means that NBVP (1.1) has at least one nontrivial solution.

Remark 3.2 For any $\zeta > 0$ and $\eta = 0$, the proof of Theorem 3.1 is valid. Then for any $\lambda > 0$, NBVP (1.1) has at least one nontrivial solution.

Theorem 3.3 *Suppose that (H1) and (H2) hold. Assume there exists a constant $X^* > 0$ such that*

$$f(x, y) \geq -\frac{X^*}{\mathcal{C}}, \quad \forall \; x \in [0, 1], \quad y \geq -\frac{\lambda_1 X^*}{\eta},$$

then for any

$$\lambda \in \left(\frac{\lambda_1}{\zeta}, \frac{\lambda_1}{\eta} \right)$$

NBVP (1.1) has at least one nontrivial solution, where $\mathcal{C} = \max_{x \in [0,1]} \int_0^1 G(x, s) ds$.

Proof. Let

$$f_1(x, y) = \begin{cases} f(x, y(x)), & \forall \; y \geq -\dfrac{\lambda_1 X^*}{\eta}, \quad x \in [0, 1], \\ f\left(x, -\dfrac{\lambda_1 X^*}{\eta}\right), & \forall \; y < -\dfrac{\lambda_1 X^*}{\eta}, \quad x \in [0, 1], \end{cases}$$

and

$$(A_1 y)(x) = \int_0^1 G(x, s) f_1(s, y(s)) ds.$$

Then all conditions of Theorem 3.1 hold for f_1. By Theorem 3.1, λA_1 has at least one nonzero fixed point $\xi^*(x)$, and

$$\xi^*(x) = \lambda \int_0^1 G(x, s) f_1(s, \xi^*(s)) ds \geq -\lambda \frac{X^*}{\mathcal{C}} \int_0^1 G(x, s) ds \geq -\frac{\lambda_1 X^*}{\eta}.$$

Thus

$$\xi^*(x) = \lambda \int_0^1 G(x, s) f_1(s, \xi^*(s)) ds = \lambda \int_0^1 G(x, s) f(s, \xi^*(s)) ds = \lambda A \xi^*(x).$$

This indicates $\xi^*(x)$ is a nontrivial solution of NBVP (1.1).

Theorem 3.4 *Suppose that (H1) and (H3) hold. Let* $f(x,0) \equiv 0,\ \forall\ x \in [0,1]$ *and*

$$\lim_{y\to\infty} \frac{f(x,y)}{y} = \phi. \tag{3.13}$$

Then for any

$$\lambda \in \left(\frac{\lambda_1}{\zeta}, +\infty\right) \quad \text{and} \quad \lambda \neq \frac{\lambda_1}{\phi},$$

NBVP (1.1) has at least one nontrivial solution.

Proof. From the proof of Theorem 3.1, if (H1) and (H3) hold, then there exists $R_0 > 0$ such that for any $R > R_0$ and $\lambda > \lambda_1\zeta^{-1}$ (3.9) holds.

Since $f(x,0) \equiv 0,\ \forall\ x \in [0,1],\ A\theta = \theta$. By (3.13) we have that the Frechet derivative A'_θ of A at θ exists and

$$(A'_\theta y)(x) = \int_0^1 G(x,s)\phi y(s)ds.$$

Notice that $\lambda \neq \lambda_1\phi^{-1}$, then 1 is not an eigenvalue of $\lambda A'_\theta$. By Lemma 2.10 there exists $r_0 > 0$, for any $0 < r < \min\{r_0, R_0\}$,

$$\deg(I - \lambda A, T_r, \theta) = \deg(I - \lambda A'_\theta, T_r, \theta) = (-1)^\kappa \neq 0, \tag{3.14}$$

where κ is the sum of algebraic multiplicities for all eigenvalues of $\lambda A'_\theta$ lying in the interval $(0,1)$.

By (3.9) and (3.14) λA has at least one nonzero fixed point. Thus NBVP (1.1) has at least one nontrivial solution.

4. Positive solutions

In many realistic problems, the positive solution is more significant. In this section we will study this question.

Theorem 4.1 *Suppose that (H4) holds. Then there exists* $\lambda^* > 0$ *such that for any* $0 < \lambda < \lambda^*$ *NBVP (1.1) has at least one positive solution.*

Proof. Let $D = [0,1]$, $D_0 = [x_1, x_2] \subset (0,1) \subset D$, and constant

$$\gamma = \min_{x_1 \le x \le x_2} g(x) > 0.$$

By (H4), there exist $X_1 > 0$ and $R_1 > 0$ such that

$$\begin{aligned} f(x,y) &\geq -X_1, \quad \forall\ x \in [0,1], \quad y \geq 0, \\ f(x,y) &\geq -\gamma^{-1}NX_1, \quad \forall\ x \in [x_1,x_2], \quad y \geq R_1, \end{aligned} \tag{4.1}$$

where $N > [1-(x_2-x_1)](x_2-x_1)^{-1}$ is a natural number. Let

$$f_2(x,y) = \begin{cases} f(x,y(x)), & y \geq 0, \\ f(x,-y(x)), & y < 0. \end{cases}$$

Then

$$f_2(x,y) \geq -X_1, \quad \forall\ x \in [0,1], \quad y \in \mathbb{R}. \tag{4.2}$$

Let

$$(A_2 y)(x) = \int_0^1 G(x,s) f_2(s, y(s))ds.$$

Obviously, $A_2 : E \to E$ is a completely continuous operator.

From Remark 3.2 and the proof of Theorem 3.1, there exists $R_0 > 0$, for any $R > R_0$,

$$\deg(I - \lambda A_2, T_R, \theta) = 0, \quad \forall\ \lambda > 0. \tag{4.3}$$

Take $0 < r < R_0$. Let

$$m_1 = \max_{x\in[0,1],|y|<r} |f_2(x,y)|, \quad M = \max_{0\le x,s\le 1} G(x,s), \quad \overline{\lambda} = r(m_1 M)^{-1}.$$

For any $0 < \lambda < \overline{\lambda},\ y \in \partial T_r$, we have

$$\|\lambda A_2 y\| = \max_{x\in[0,1]} \left| \int_0^1 \lambda G(x,s) f_2(s,y(s))ds \right| < \overline{\lambda} M m_1 = r = \|y\|.$$

Thus

$$\deg(I - \lambda A_2, T_r, \theta) = 1, \quad \forall\ 0 < \lambda < \overline{\lambda}. \tag{4.4}$$

From (4.3) and (4.4) we have that for any $0 < \lambda < \overline{\lambda}$, there exists $y_\lambda \in C[0,1]$ with $\|y_\lambda\| > r$ such that $y_\lambda = \lambda A_2 y_\lambda$. Now we show

$$\lim_{\lambda\to 0^+, y_\lambda=\lambda A_2 y_\lambda, \|y_\lambda\|>r} \|y_\lambda\| = +\infty. \tag{4.5}$$

In fact, if (4.5) doesn't hold, then there exist $\lambda_n > 0$, $y_{\lambda_n} \in C[0,1]$ such that $\lambda_n \to 0$, $r < \|y_{\lambda_n}\| < c$ ($c > 0$ is a constant), and

$$y_{\lambda_n} = \lambda_n A_2 y_{\lambda_n}. \tag{4.6}$$

Since A_2 is completely continuous, then $\{y_{\lambda_n}\}$ has a subsequence (assume without loss of generality that it is $\{y_{\lambda_n}\}$) converging to $y_* \in C[0,1]$. Let $n \to \infty$ in (4.6), we have $y_* = \theta$, which is a contradiction of $\|y_{\lambda_n}\| > r > 0$. Then (4.5) holds.

Next we show that there exists $R = R(\overline{\lambda}) > 0$ such that if $0 < \lambda_0 < \overline{\lambda}$, $\|y_0\| \ge R$ and $y_0 = \lambda_0 A y_0$, then $y_0(x) \ge 0$. Take

$$R = R(\overline{\lambda}) = \max\{2\gamma^{-1}R_1, 2\gamma^{-1}\overline{\lambda}X_1 M, 2\overline{\lambda}X_1 M\}. \tag{4.7}$$

Assume that $0 < \lambda_0 \le \overline{\lambda}$, $\|y_0\| \ge R$ and $y_0 = \lambda_0 A y_0$. Take any $\overline{x} \in [x_1, x_2]$, by ($\mathbf{G}_2$) we have that for any $\tau \in [0,1]$,

$$\begin{aligned} y_0(\overline{x}) &= \lambda_0 \int_0^1 G(\overline{x},s)[f_2(s,y_0(s)) + X_1]ds - \lambda_0 \int_0^1 X_1 G(\overline{x},s)ds \\ &\ge \lambda_0\gamma \int_0^1 G(\tau,s) f_2(s,y_0(s))ds - \overline{\lambda}X_1 M \\ &= \gamma y_0(\tau) - \overline{\lambda}X_1 M. \end{aligned} \tag{4.8}$$

On account of the continuity of y_0, there exists $x^* \in [0,1]$ such that $y_0(x^*) = \|y_0\|$. Take $\tau = x^*$ in (4.8), by (4.7) we have

$$y_0(\overline{x}) \ge \gamma\|y_0\| - \overline{\lambda}X_1 M \ge \gamma R - \overline{\lambda}X_1 M = \frac{1}{2}\gamma R + \frac{1}{2}\gamma R - \overline{\lambda}X_1 M \ge \frac{1}{2}\gamma R \ge R_1.$$

Thus $y_0(\overline{x}) \ge R_1$, for any $\overline{x} \in [x_1, x_2]$. By (4.1) we have

$$f_2(s, y_0(s)) \ge \gamma^{-1} N X_1, \quad \forall\ s \in D_0 = [x_1, x_2]. \tag{4.9}$$

It follows from $(\mathbf{G}_1)$ and $(\mathbf{G}_2)$ that for any $s \in [x_1, x_2]$ and $x, \tau \in [0,1]$

$$G(x,s) = G(s,x) \geq \gamma G(\tau, x) = \gamma G(x,\tau). \tag{4.10}$$

Take $D_i \subset D(i = 1, 2, \cdots, N)$ such that $\text{mes}D_i = \text{mes}D_0$, $\bigcup_{i=1}^{N} D_i \supset D \backslash D_0$. By (4.9) and (4.10) we have that for any $x \in D$, $s \in D_0$, $\tau \in D_i(i = 1, 2, \cdots, N)$,

$$\frac{1}{N} G(x,s) f_2(s, y_0(s)) \geq X_1 G(x,\tau). \tag{4.11}$$

Notice that $\text{mes}D_i = \text{mes}D_0(i = 1, 2, \cdots, N)$, then

$$\frac{1}{N} \int_{D_0} G(x,s) f_2(d, y_0(s)) ds \geq \int_{D_i} X_1 G(x,\tau) d\tau, \ i = 1, 2, \cdots, N.$$

Thus

$$\begin{aligned} \int_{D_0} G(x,s) f_2(s, y_0(s)) ds &\geq \sum_{i=1}^{N} \int_{D_i} X_1 G(x,\tau) d\tau \\ \geq \int_{D \backslash D_0} X_1 G(x,\tau) d\tau &= \int_{D \backslash D_0} X_1 G(x,s) ds. \end{aligned} \tag{4.12}$$

By (4.12) and (4.1) we have that for any $x \in [0,1]$,

$$y_0(x) = \lambda_0 \int_{D_0} G(x,s) f_2(s, y_0(s)) ds + \lambda_0 \int_{D \backslash D_0} G(x,s) f_2(s, y_0(s)) ds \geq 0.$$

For R in (4.7), by (4.5), there exists $\lambda^* > \overline{\lambda}$ such that if $0 < \lambda \leq \lambda^*$, $\|y_\lambda\| \geq r$ and $y_\lambda = \lambda A_2 y_\lambda$, then $\|y_\lambda\| \geq R$, thus $y_\lambda(x) \geq 0$. By the definitions of A_2 and f_2 we have

$$y_\lambda(x) = \lambda \int_0^1 G(x,s) f_2(s, y_\lambda(s)) ds + \lambda \int_0^1 G(x,s) f(s, y_\lambda(s)) ds = \lambda A_2 y_\lambda(x).$$

So $y_\lambda(x)$ is a positive solution of NBVP (1.1).

Remark 4.2 In Theorem 4.1 we obtain the existence of positive solutions for the semipositone boundary value problem (1.1) without that assuming $(\mathbf{F})$ holds.

Remark 4.3 Since we only study the existence of positive solutions for the boundary value problem (1.1), which is irrelevant to the value of $f(x,y)$ when $y \leq 0$, we only suppose that $f(x,y)$ is bounded below when $y \geq 0$. The nonlinear term $f(x,y)$ may be unbounded from below when $y \leq 0$.

Example 4.4 Consider the fourth-order Neumann boundary value problem

$$\begin{cases} y^{(4)}(x) + \frac{5}{8} y''(x) + \frac{3}{16} y(x) = \lambda[(x^{1/2} + 1) y^3 - y^{1/3}], \quad x \in [0,1], \\ y'(0) = y'(1) = y'''(0) = y'''(1) = 0. \end{cases} \tag{4.13}$$

In this example, $f(x,y) = (x^{1/2} + 1) y^3 - y^{1/3}$, then

$$\lim_{y \to +\infty} \frac{f(x,y)}{y} = \lim_{y \to +\infty} \left(\sqrt{x} + 1) y^2 - \frac{1}{y^{2/3}} \right) = +\infty,$$

which means that (H4) holds. By Theorem 4.1 there exists $\lambda^* > 0$ such that for any $0 < \lambda < \lambda^*$ NBVP (4.13) has at least one positive solution. In particular, the nonlinear term f doesn't satisfy condition $(\mathbf{F})$ and (H3), but the existence of positive solutions of NBVP (4.13) is obtained by using our result.

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