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EXISTENCE OF SOLUTIONS FOR DOUBLE PHASE PROBLEMS INVOLVING CONVECTION TERM VIA A FIXED POINT APPROACH

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Abstract. In this paper, we study the existence of a non-trivial solution for the double phase problem for the problem. The proof is based on Schaefer's fixed point theorem. Our assumptions are suitable and different from those studied previously.
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1. Introduction and main results

In the past decade, the existence and multiplicity of solutions for double phase problems with different boundary value conditions have been widely investigated by a number of authors. Related works dealing double phase problems can be found in the works of Perera-Squassina [27], Papageorgiou-Radulescu-Repovs [25, 26], Cencelj-Radulescu-Repovs [7], Zhang-Radulescu [33], Radulescu [28], Cao-Ge-Yuan [6], Wang-Hou-Ge [30], Ge-Lv-Lu [17], Ge-Pucci [18], Liu-Dai [19, 20], Liu-Dai-Papageorgiou-Winkert [21], Colasuonno-Squassina [8], Crespo Blanco-Gasinski-Harjulehto-Winkert [11], Zeng-Radulescu-Winkert [32] and Liu-Papageorgiou [22]. For other regularity results on double phase equations, we can refer to the papers of Baroni-Colombo-Mingione [2, 3], Colombo-Mingione [9, 10], Byun-Lim [5], De Filippis-Mingione [12, 13, 14], Baasandorj-Byun-Oh [1] and Ok [24].

In this paper, we intend to show how Schaefer's fixed point theorem are able to solve a double phase problem involving nonlinearities with gradient. This kind of research is rare and leaves much space for us to discuss. The tools we use allows us to make simple assumptions, is also in contrast to other articles in this field.

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We consider the existence of nontrivial solution for the following problem in the bounded smooth domain $\Omega \subset \mathbb{R}^N$, $N \geq 2$:

$$\begin{cases} -\operatorname{div}(|\nabla u|^{p-2}\nabla u + \mu(x)|\nabla u|^{q-2}\nabla u) = f(x, u, \nabla u), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{P}$$

where $1 < p < q < N$ and

$$\frac{q}{p} < 1 + \frac{1}{N}, \ \mu : \overline{\Omega} \mapsto [0, +\infty) \text{ is Lipschitz continuous,} \tag{1.1}$$

and $f : \Omega \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^N \mapsto \mathbb{R}$ is a Carathéodory function satisfying suitable growth conditions.

In Gasinski-Winkert [16] the authors proved the existence of at least one weak solution for problem (P) under the following hypotheses on the nonlinearity $f(x, t)$:

(f_1) There exists $C > 0$ and a function $\alpha \in L^{\frac{r}{r-1}}(\Omega)$ such that

$$|f(x, t, \xi)| \leq C(\alpha(x) + |t|^{r-1} + |\xi|^{p\frac{r-1}{r}})$$

for a.a. $x \in \Omega$, for all $t \in \mathbb{R}$ and for all $\xi \in \mathbb{R}^N$, where $1 < r < p^*$ with the critical exponent $p^* = \frac{Np}{N-p}$;

(f_2) There are constants $b_1, b_2 \geq 0$ with $b_1 + b_2\lambda_{1,p}^{-1} < 1$, and a function $w \in L^1(\Omega)$ such that

$$f(x, t, \xi)s \leq b_1|\xi|^p + b_2|t|^p + w(x)$$

for a.a. $x \in \Omega$, for all $t \in \mathbb{R}$ and for all $\xi \in \mathbb{R}^N$, where $\lambda_{1,p}$ is the first eigenvalue of the following the p-Laplacian eigenvalue problem

$$\begin{cases} -\operatorname{div}(|\nabla u|^{p-2}\nabla u) = \lambda|u|^{p-2}u, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases}$$

Later in [30], by using the topological degree theory, Wang, Hou and Ge established the existence of at least one weak solution for (P) under the following assumptions:

(f_3) There are constants $r \in (1, p)$, $C > 0$ and a function $\alpha \in L^{\frac{r}{r-1}}(\Omega)$ such that

$$|f(x, t, \xi)| \leq C(\alpha(x) + |t|^{r-1} + |\xi|^{p\frac{r-1}{r}})$$

for a.a. $x \in \Omega$, for all $t \in \mathbb{R}$ and for all $\xi \in \mathbb{R}^N$.

It is important to note that the condition (f_1) means that the nonlinearity f has subcritical growth condition, and the condition (f_3) means sub-p linear growth. In this paper, we consider the problem (P) in the case when the nonlinearity f also satisfies sub-p linear growth. To this end, we assume that $f : \Omega \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^N \mapsto \mathbb{R}$ is a Carathéodory function satisfies the following assumptions:

(H_f) There are constants $r \in (1, p)$, $c > 0$ and a function $\alpha \in L^{\frac{p}{p-1}}(\Omega)$ such that

$$|f(x, t, \xi)| \leq \alpha(x) + c(|t|^{r-1} + |\xi|^{r-1})$$

for a.a. $x \in \Omega$, for all $t \in \mathbb{R}$ and all $\xi \in \mathbb{R}^N$;

(H_c) There exists some $\lambda \in [0, 1]$ such that

$$0 < c < \frac{1}{\lambda^{p-1}(C_r^r + C_r|\Omega|^{\frac{1}{\nu}})},$$

where $\nu = \frac{pr}{(p-r)(r-1)}$, C_H is defined in Proposition 2.2(2) and C_r is the best constants for the continuous embedding $L^p(\Omega) \hookrightarrow L^r(\Omega)$.

The aim of this paper is to prove the existence of at least nontrivial solution for (P) extending and refining the results in [30, 16] by using Schaefer's fixed point theorem. Now, we state our main results.

Theorem 1.1. *Under assumptions* (H_f) *and* (H_c)*, then problem* (P) *admits a nontrivial solution* $u \in W_0^{1,H}(\Omega)$.

This paper is divided into three sections. In Section 2, we will introduce some tools which we need to prove our main result in the last section. In Section 3, we give the proof of Theorem 1.1.

2. Preliminaries

Firstly, let us summary the most important results on the Musielak-Orlicz-Sobolev space $W_0^{1,H}(\Omega)$ and the basic properties of the double-phase operator. For more details, we orient the reader to [8, 23, 4, 15] and references therein. We also recall Schaefer's fixed point theorems [29].

In the entire paper, we always assume that $\Omega \subset \mathbb{R}^N$ is an open bounded subset with smooth boundary $\partial\Omega$, and assumption (1.1) holds.

Now, we consider the function $H : \Omega \times [0, +\infty) \to [0, +\infty)$ defined by

$$H(x,t) = t^p + \mu(x)t^q, \ \forall (x,t) \in \Omega \times [0, +\infty).$$

We define the following Musielak-Orlicz space

$$L^H(\Omega) = \Big\{ u | u : \Omega \to \mathbb{R} \text{ is measurable and } \int_\Omega H(x, |u|)dx < +\infty \Big\},$$

endowed with the Luxemburg norm

$$|u|_H = \inf\Big\{ \lambda > 0 : \int_\Omega H(x, |\frac{u}{\lambda}|)dx \leq 1 \Big\}.$$

We also defined the the corresponding Musielak-Orlicz-Sobolev space

$$W^{1,H}(\Omega) = \{ u \in L^H(\Omega) : |\nabla u| \in L^H(\Omega) \},$$

which is endowed with the norm $\|u\| = |u|_H + |\nabla u|_H$.

We also define $W_0^{1,H}(\Omega)$ as the subspace of $W^{1,H}(\Omega)$ which is the closure of $C_0^\infty(\Omega)$ with respect to the norm $\| \cdot \|$. According to the previous definitions, we known that $L^H(\Omega)$, $W_0^{1,H}(\Omega)$ and $W^{1,H}(\Omega)$ are separable reflexive Banach spaces (see [8]).

Later we need to use the modular and its properties, which is a mapping $\rho_H : L^H(\Omega) \to \mathbb{R}$ defined by

$$\rho_H(u) = \int_\Omega (|u|^p + \mu(x)|u|^q)dx.$$

Proposition 2.1. ([19, Proposition 2.1]) *For all* $u, v \in L^H(\Omega)$*, we have*

(i) $|u|_H < 1(\text{resp} = 1; > 1) \Leftrightarrow \rho_H(u) < 1(\text{resp} = 1; > 1)$.

(ii) $|u|_H \geq 1 \Rightarrow |u|_H^p \leq \rho_H(u) \leq |u|_H^q$.

(iii) $|u|_H \leq 1 \Rightarrow |u|_H^q \leq \rho_H(u) \leq |u|_H^p$.

Based on the proof of Liu-Dai [8, Proposition 2.15, Proposition 2.18] we have the following the embedding result for the space $W_0^{1,H}(\Omega)$.

Proposition 2.2. *(1) Assume that $\vartheta \in [1, p^*)$. Then the embedding from $W_0^{1,H}(\Omega)$ to $L^{\vartheta}(\Omega)$ is continuous and compact.*

(2) Assume that hypotheses (1.1) is true. Then there is a constant $C_H > 0$ such that

$$|u|_H \leq C_H |\nabla u|_H, \; \forall u \in W_0^{1,H}(\Omega).$$

According to Proposition 2.2(1), there exists $c_\vartheta > 0$ such that

$$|u|_\vartheta \leq c_\vartheta \|u\|, \; \forall u \in W_0^{1,H}(\Omega),$$

where $|u|_s$ denotes the usual norm in $L^{\vartheta}(\Omega)$ for all $1 \leq \vartheta < p^*$. Moreover, the space $W_0^{1,H}(\Omega)$ has a norm $|\cdot|$ given by $\|u\| = |\nabla u|_H$ for all $u \in W_0^{1,H}(\Omega)$, which is equivalent to $\|\cdot\|$.

In addition, we consider the double-phase operator $\mathcal{L} : W_0^{1,H}(\Omega) \to (W_0^{1,H}(\Omega))^*$ defined by

$$\langle \mathcal{L}(u), v \rangle = \int_\Omega (|\nabla u|^{p-2} \nabla u \cdot \nabla v + \mu(x) |\nabla u|^{q-2} \nabla u \cdot \nabla v) dx,$$

for all $u, v \in W_0^{1,H}(\Omega)$, where $\langle \cdot, \cdot \rangle$ is the duality pairing between $W_0^{1,H}(\Omega)$ and its dual space $(W_0^{1,H}(\Omega))^*$. The next result summarizes the properties of the operator $\mathcal{L}$. It can be found, for example, in Liu-Dai [19, Proposition 3.1].

Proposition 2.3. *Let $E = W_0^{1,H}(\Omega)$ and $\mathcal{L} : E \to E^*$ be as above. Then $\mathcal{L}$ is bounded, continuous, monotone (hence maximal monotone), and of type (S_+).*

Let us recall the following classical Schaefer's fixed point theorem we will use later.

Theorem 2.4. [29] *Let X be a normed space, and $T : X \to X$ be a continuous mapping which maps bounded sets into relatively compact sets. Then one of the following statements holds:*

(i) the equation $u = T(u)$ has a solution, or

(ii) the set $B = \{x \in X : u = \lambda T(u) : 0 < \lambda < 1\}$ is unbounded.

3. Proof of the main results

Firstly, we will prove the key lemma of this paper.

Lemma 3.1. *Let assumption (H_f) be satisfied. Then we define the operator $S : E \to E^*$ by*

$$\langle Su, v \rangle = \int_\Omega f(x, u, \nabla u) v dx, \; \forall u, v \in E, \tag{3.1}$$

which is compact.

Proof. First, we define the operator $\phi : E \to L^{\frac{p}{p-1}}(\Omega)$ given by

$$\phi u = f(x, u, \nabla u), \; \forall u \in E.$$

Next, we prove that ϕ is bounded and continuous.

Let us verify that ϕ is bounded. By the virtue of assumptions of $H(f)$, one has

$$|\phi u|_{\frac{p}{p-1}}^{\frac{p}{p-1}} = \int_\Omega |f(x,u(x),\nabla u(x))|^{\frac{p}{p-1}} dx \le C(|\alpha|_{\frac{p}{p-1}}^{\frac{p}{p-1}} + |u|_\tau^\tau + |\nabla u|_\tau^\tau), \tag{3.2}$$

where $\tau = \frac{p(r-1)}{p-1} < p$. From the continuous embedding $E \hookrightarrow L^r(\Omega)$ $(1 < \tau < p < p^*)$ we have

$$|\phi u|_{\frac{p}{p-1}}^{\frac{p}{p-1}} \le C(|\alpha|_{\frac{p}{p-1}}^{\frac{p}{p-1}} + \|u\|^\tau + \|u\|^\tau). \tag{3.3}$$

This shows that ϕ is bounded on E.

Next, we will prove that ϕ is continuous. Let $u_n \to u$ in E. Then it is obvious that $u_n \to u$ in $L^p(\Omega)$ and $\nabla u_n \to \nabla u$ in $(L^p(\Omega))^N$. Hence there exists a subsequence of $\{u_{n_k}\}_{k=1}^\infty$ of $\{u_n\}_{n=1}^\infty$ and measurable functions $g \in L^p(\Omega)$ and $h \in (L^p(\Omega))^N$ such that

$$\begin{aligned} &u_{n_k}(x) \to u(x) \text{ and } \nabla u_{n_k}(x) \to \nabla u(x),\ \text{a.e. } x \in \Omega,\ \text{as } k \to +\infty; \\ &|u_{n_k}(x)| \le g(x) \text{ and } |\nabla u_{n_k}(x)| \le |h(x)|,\ \text{a.e. } x \in \Omega \text{ and all } k \in N. \end{aligned} \tag{3.4}$$

Since f satisfies the Carathéodory condition. So we obtain that

$$f(x, u_{n_k}(x), \nabla u_{n_k}(x)) \to f(x, u, \nabla u(x)),\ \text{a.e. } x \in \Omega,\ \text{as } k \to +\infty. \tag{3.5}$$

Moreover by (3.4), we know that

$$|f(x, u_{n_k}(x), \nabla u_{n_k}(x))| \le C\big(\alpha(x) + |u(x)|^{r-1} + |\nabla u(x)|^{r-1}\big), \tag{3.6}$$

for a.e. $x \in \Omega$ and all $k \in N$. Therefore, note that $\alpha + |u|^{r-1} + |\nabla u|^{r-1} \in L^{\frac{p}{p-1}}(\Omega)$, from (3.5), (3.6), and the Dominated Convergence Theorem, it follows that

$$\int_\Omega |f(x, u_{n_k}(x), \nabla u_{n_k}(x)) - f(x, u, \nabla u(x))|^{\frac{p}{p-1}} dx \to 0,\ \text{as } k \to +\infty,$$

from which it follows that

$$|\phi u_{n_k} - \phi u|_{\frac{p}{p-1}} \to 0,\ \text{as } k \to +\infty.$$

Then it follows that ϕu_n converges to ϕu in $L^{\frac{p}{p-1}}(\Omega)$.

Finally, we will prove that the $S : E \to E^*$ is compact. In fact, by compact imbedding of $i : E \to L^p(\Omega)$, the adjoint operator $i^* : L^{\frac{p}{p-1}}(\Omega) \to E^*$ is also compact. Therefore, it follows that $S = i^* \circ \phi : E \to E^*$ is compact. That finishes the proof. □

Now we will prove the main result of this paper.

Proof of Theorem 1.1. Because of the definition of the operator L and Lemma 3.1, we have that $u \in E$ is a weak solution of (P) if and only if

$$\mathcal{L}u = Su \Leftrightarrow u = \mathcal{L}^{-1}Su. \tag{3.7}$$

We define the operator $T : E \to E^*$ as follows:

$$T(u) = \mathcal{L}^{-1}Su.$$

Then it is not difficult to see that a solution of the problem (P) is a fixed point of the operator T. Hence, in view of Theorem 2.4, it suffices to show that,

(T_1) T is well defined;

(T_2) T is compact;

(T_3) The set $B = \{u \in E : u = \lambda T(u) : 0 < \lambda < 1\}$ is bounded.

Verification of (T_1). In view of Lemma 3.1 and Proposition 2.3, it is standard to check that the operator T can be considered as follows: $T : E \xrightarrow{S} E^* \xrightarrow{\mathcal{L}^{-1}} E$. So, T is well-defined.

Verification of (T_2). In fact, let $\{u_n\}$ be a bounded sequence in the reflexive space E. Then, there exist u_0 and a subsequence which we also denote $\{u_n\}$ such that u_n converges weakly to u_0 in E. By the compactness of S (see Lemma 3.1), we have $S(u_n) \to S(u_0)$ in E^*. Moreover, the operator $\mathcal{L}$ is bounded, continuous, and strictly monotone (see Proposition 2.3), so it is from Minty-Browder Theorem (see [31, Theorem 26A]) that the inverse operator $\mathcal{L}^{-1}$ is bounded and continuous. Hence we get $\mathcal{L}^{-1}S(u_n) \to \mathcal{L}^{-1}S(u_0)$, which implies that $T(u_n)$ converges strongly to $T(u_0)$.

Verification of (T_3). Indeed, let $v \in B$, then it only remains to show that there exists $M > 0$ such that $|\nabla u|_H \le M$. We distinguish between two cases:

(i) if $|\nabla u|_H \le 1$, then B is bounded.

(ii) if $|\nabla u|_H > 1$, then from modular's properties, we deduce that $\lambda \ne 0$ and $u = \lambda T(u) = \lambda \mathcal{L}^{-1}S(u)$. Thus, from (H_f), Proposition 2.1, the Hölder inequality and the Young inequality, we obtain

$$\begin{aligned}
\frac{1}{\lambda^p}|\nabla u|_H^p \le & \int_\Omega (|\nabla(\frac{u}{\lambda})|^p + \mu(x)|\nabla(\frac{u}{\lambda})|^q)dx \\
=&\langle \mathcal{L}(\frac{u}{\lambda}), \frac{u}{\lambda}\rangle = \langle S(u), \frac{u}{\lambda}\rangle \\
=&\frac{1}{\lambda}\int_\Omega f(x,u,\nabla u)udx \\
\le&\frac{1}{\lambda}\int_\Omega (|\alpha(x)u(x)| + |u(x)|^r + |\nabla u(x)|^{r-1}|u|)dx \\
\le&\frac{1}{\lambda}\Big(|\alpha|_{\frac{p}{p-1}}|u|_p + c|u|_r^r + c\big||\nabla u|^{r-1}\big|_{\frac{p}{r-1}}|u|_r|1|_\nu\Big) \\
=&\frac{1}{\lambda}\Big(|\alpha|_{\frac{p}{p-1}}|u|_p + c|u|_r^r + c|\Omega|^{\frac{1}{\nu}}|\nabla u|_p^{r-1}|u|_r\Big) \\
\le&\frac{1}{\lambda}\Big(|\alpha|_{\frac{p}{p-1}}|u|_p + c|u|_r^r + c|\Omega|^{\frac{1}{\nu}}|\nabla u|_H^{r-1}|u|_r\Big).
\end{aligned}$$

where $\nu = \frac{pr}{(p-r)(r-1)} > 1$ because $1 < r < p$. Then from the Poincare's inequality and the continuous embedding $E \hookrightarrow L^r(\Omega)$, it follows that

$$\begin{aligned}
|\nabla u|_H^p \le&\lambda^{p-1}\Big(C_p|\alpha|_{\frac{p}{p-1}}|\nabla u|_H + cC_r^r|\nabla u|_H^r + c|\Omega|^{\frac{1}{\nu}}C_r|\nabla u|_H^r\Big) \\
\le&\lambda^{p-1}\Big(C_p|\alpha|_{\frac{p}{p-1}}|\nabla u|_H + cC_r^r|\nabla u|_H^p + c|\Omega|^{\frac{1}{\nu}}C_r|\nabla u|_H^p\Big)
\end{aligned}$$

and consequently,

$$\Big(1 - \lambda^{p-1}\big(cC_r^r + c|\Omega|^{\frac{1}{\nu}}C_r\big)\Big)|\nabla u|_H^{p-1} \le \lambda^{p-1}C_p|\alpha|_{\frac{p}{p-1}}.$$

By virtue of (H_c),

$$1 - \lambda^{p-1}\big(cC_r^r + c|\Omega|^{\frac{1}{\nu}}C_r\big) > 0.$$

Furthermore, we deduce

$$|\nabla u|_H^{p-1} \leq \frac{\lambda^{p-1} C_p |\alpha|_{\frac{p}{p-1}}}{\left(1 - \lambda^{p-1}\left(cC_r^r + c|\Omega|^{\frac{1}{\nu}} C_r\right)\right)},$$

and so $|\nabla u|_H$ is bounded. This shows that $\{u|u \in B\}$ is bounded.

Using Schaefer's fixed point theorem (see Theorem 2.4), we conclude that the operator T has a fixed point u which is the solution of the given problem (P). The proof is complete. □

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