Fixed Point Theory, 26(2025), No. 1, 135-146
DOI: 10.24193/fpt-ro.2025.1.07
http://www.math.ubbcluj.ro/~nodeacj/sfptcj.html

RIGHT FIXED POINT PROPERTY IN BANACH LATTICES

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Abstract. In this paper, using the Right topology, we introduce three new properties in Banach lattices: the so-called Right orthogonality, the Right WORTH property, and the non-strictly Right Opial condition (and also positive versions of them). Moreover, Banach lattices in which these three properties coincide with order continuity of the norm are characterized. As an application, we give some sufficient conditions under which a Banach lattice has the Right fixed point property (or, positive Right fixed point property). In particular, it is established that for a Banach space X and a suitable Banach lattice F, a Banach lattice $\mathcal{M} \subset K(X, F)$ has the Right fixed point property (resp. positive Right fixed point property) if each evaluation operator ψ_{y^*} on $\mathcal{M}$ is a pseudo weakly compact (resp. positive pseudo weakly compact) operator, where $\psi_{y^*} : \mathcal{M} \to X^*$ is defined by $\psi_{y^*}(T) = T^*y^*$ for $y^* \in F^*$ and $T \in \mathcal{M}$.
Key Words and Phrases: Right topology, weak fixed point property, WORTH property, non-strictly Opial condition, weak orthogonality.
2020 Mathematics Subject Classification: 47H10, 46A40, 46B05, 46B30.

1. Introduction and Preliminaries

A Banach space X has the *fixed point property (fpp)* for nonexpansive mappings if every non-expansive self-map $T : C \to C$ of each nonempty, bounded, closed and convex subset C of X has a fixed point. When the same holds for every nonempty weakly compact convex subset of X, we say that X has the *weak fixed point property (w-fpp)*. Each Banach space with the fpp has the w-fpp, but the converse is false in general. In fact, c_0 and ℓ_1 have the w-fpp but neither of these two spaces have the fpp. A closed subspace $\mathcal{M}$ of $L^1[0,1]$ has the fpp if and only if $\mathcal{M}$ is reflexive [12]. A Banach space X has weak normal structure if for each weakly compact convex subset D there is an element $u \in D$ such that $\sup_{v\in D} \|u - v\|$ <diam (D). Every Banach space with the Schur property and every Banach space with the weak normal structure have the w-fpp [2, 16, 22].

For each Hilbert space H, a closed subspace $\mathcal{M} \subset K(H)$ has the fpp if and only if $\mathcal{M}$ is reflexive [7]. Also, for some closed subspace $\mathcal{M} \subset K(X, Y)$ (the Banach space of all compact operators between two Banach spaces X and Y), $\mathcal{M}^*$ has the Schur property if and only if all of the evaluation operators ϕ_x and ψ_{y^*} on $\mathcal{M}$ are compact, for $x \in X$, $y^* \in Y^*$ and $T \in \mathcal{M}$ [19, 23] and these results are improved in the Banach lattice setting in [5].

A Banach lattice E is called *weak orthogonal* if for each weakly null sequence (x_n) in E, $\||x_n| \wedge |x|\| \to 0$ for all $x \in E$. A Banach lattice E has the *WORTH property* (respectively, *non-strictly Opial condition*) if for each weakly null sequence (x_n) in E and $x \in E$, $\limsup_n \|x_n - x\| = \limsup_n \|x_n + x\|$ (respectively, $\limsup_n \|x_n\| \leq \limsup_n \|x_n + x\|$) [10, 21, 9]. The disjoint (positive) versions of these three concepts also is introduced in [14] and so the disjoint (positive) fpp is studied.

The norm of a Banach lattice is said to be uniformly monotone if for given $\epsilon > 0$ there is a $\delta > 0$ such that if $x, y \geq 0$ with $\|y\| = 1$ and $\|x + y\| \leq 1 + \delta$ then $\|x\| \leq \epsilon$. Each Banach lattice with uniformly monotone norm has order continuous norm (since it does not contain any copy of c_0).

In this article, we use the Right topology instead of weak topology to study the Right fpp. Right topology on a Banach space X is a locally convex topology on X which is obtained by restriction of the Mackey topology $\tau(X^{**}, X^*)$ to X (which is the topology of uniform convergence on relatively $\sigma(X^*, X^{**})$ compact subsets of X^*).

In order to study the Right fpp, we need the following terminology and notation. If A is a norm bounded subset of X such that for each weakly null sequence (x_n^*) in X^*, $\lim_{n\to\infty} \sup_{a\in A} |\langle a, x_n^*\rangle| = 0$, then A is called a Dunford-Pettis set. Banach spaces in which Dunford-Pettis sets are relatively compact are said to have the Dunford-Pettis relatively compact property ($\mathrm{DP}_{rc}\mathrm{P}$) [13].

A Banach space X has the Dunford-Pettis property if each relatively weakly compact set in X is a Dunford-Pettis set. The reader is referred to [4, 3, 11] for the definitions and a discussion on the Dunford-Pettis property.

A subset A of X^* is a Right set if every Right null (i.e. weakly null and Dunford-Pettis) sequence (x_n) in X converges uniformly on A. An operator $T : X \to Y$ is called pseudo weakly compact (pwc) if it carries Right null sequences in X into norm null ones in Y [17].

The following general result will be introduced in our study: the Right orthogonality, the non-strictly Right Opial condition, the Right WORTH property (and also the disjoint version of them) in Banach lattices. As an application the connection between three properties and the Right fpp (or, positive Right fpp) is established. We will improve the results of [10, 25] and then we show that for each Banach space X and a suitable Banach lattice F, a Banach lattice $\mathcal{M} \subset K(X, F)$ has the Right fpp (resp. positive Right fpp) if each evaluation operator ψ_{y^*} on $\mathcal{M}$ is pseudo weakly compact (resp. positive pseudo weakly compact).

We recall some definitions and notations from Banach lattice theory. Throughout this article, X denotes the arbitrary Banach space, E denotes a Banach lattice and E^* refers to the dual of E, $E^+ = \{x \in E : x \geq 0\}$ refers to the positive cone of E and B_E is the closed unit ball of E. A Banach lattice E is said to be σ–Dedekind complete if every countable subset of E that is bounded above has a supremum. A Banach lattice E has order continuous norm if for each (x_α) with $x_\alpha \downarrow 0$ in E, we have $\|x_\alpha\| \to 0$, where the notation $x_\alpha \downarrow 0$ means that (x_α) is decreasing, its infimum exists and $\inf(x_\alpha) = 0$. An element $x \in E$ is discrete if $x > 0$ and $|y| \leq x$ implies $y = tx$ for some real number t. If every order interval $[0, y]$ in E contains a discrete element, then E is said to be a discrete Banach lattice. The spaces c_0, ℓ_p, where $1 \leq p < \infty$ and Schur spaces are discrete with order continuous norm. A norm bounded subset A of E is solid if $|x| \leq |y|$ for some $y \in A$ implies that $x \in A$. Each solid vector subspace of E is called an ideal. Also an ideal B of E is called a band if $\sup(A) \in B$ for every subset $A \subset B$ which has a supremum in E. A band B in a Banach lattice E is called a projection band if $E = B + B^{\perp}$, where $B^{\perp} = \{x \in E : |x| \wedge |y| = 0, \text{ for some } y \in B\}$. E is called an *AM-space* if $x \wedge y = 0$ in E implies $\|x \vee y\| = \max\{\|x\|, \|y\|\}$ and E is an *AL-space* if $\|x + y\| = \|x\| + \|y\|$ holds for all $x \wedge y = 0$ in E^+. A Banach lattice E has the *positive Schur* property if each positive weakly null sequence in E is norm null, or equivalently each disjoint positive weakly null sequence in E is norm null [24]. Note that all AL-spaces have the positive Schur property. Recall that a sequence (x_n) in a Banach lattice E is disjoint, if $|x_i| \wedge |x_j| = 0$ for each $i \neq j$. A Banach lattice E has the weakly sequentially continuous lattice operations if for every weakly null sequence (x_n) in E, $|x_n| \xrightarrow{w} 0$. We refer the reader for undefined terminologies to the classical references [3, 18].

2. Right fixed point property

Recently in [14] the authors introduced the disjoint (positive) version of the weak orthogonality, the non-strictly Opial condition and the WORTH property in Banach lattices and then considered Banach lattices in which these properties are equivalent. In this section replacing the weak topology by the Right topology, we will use the phrases "Right orthogonality, non-strictly Right Opial and Right WORTH property". We recall a practical characterization of the weak orthogonality of [14] as follows:

Lemma 2.1 *For a Banach lattice E, the following are equivalent:*

(a) *E is discrete with order continuous norm,*
(b) *E is a weak orthogonal Banach lattice.*

If we replace weakly null sequences by Right null sequences, instead of "weak orthogonality", we use the phrase "Right orthogonality". Recall that a sequence (x_n) in a Banach space X is Right null if and only if it is weakly null and Dunford-Pettis [15, Proposition 1]. It is important to note that a Banach space X has the DP$_{rc}$P if and only if for each Right null sequence (x_n) in E, $\|x_n\| \to 0$.

Definition 2.2 A Banach lattice E is called Right orthogonal if for each Right null sequence (x_n) in E, $\||x_n| \wedge |x|\| \to 0$ for all $x \in E$.

It is easily seen that if a Banach lattice E is weak orthogonal, then E is Right orthogonal, but the converse is false.

Proposition 2.3 *Each non-discrete reflexive Banach lattice E is Right orthogonal, but it is not weak orthogonal.*

Proof. From [20] each reflexive space has the DP$_{rc}$P; that is, each weakly null and Dunford-Pettis sequence in it is norm null. Then by [15, Proposition 1] each Right null sequence in E is norm null and so E is Right orthogonal. On the other hand, E is non-discrete and so it is not weak orthogonal.

For instance, each $L^p[0,1]$ $(1 < p < \infty)$ is Right orthogonal, but it is not weak orthogonal. It should be noted that $L^1[0,1]$ is not Right orthogonal. In fact, the *Rademacher sequences* (r_n) in $L^1[0,1]$ are weakly null and Dunford-Pettis (by the Dunford-Pettis property) and so they are Right null, but $\|r_n\| = 1$ for all n. Note that reflexive Banach lattices have order continuous norm.

We say the lattice operations in a Banach lattice E are Right weakly sequentially continuous if for every Right null sequence (x_n) in E, $(|x_n|)$ is weakly null. Clearly each discrete Banach lattice with order continuous norm has the Right weakly sequentially continuous lattice operations. All spaces $L^p[0,1]$ for all $1 < p < \infty$ have the Right weakly sequentially continuous lattice operations, but none of them have the weakly sequentially continuous lattice operations. We may then characterize Right orthogonal Banach lattices as follows:

Theorem 2.4 *Let E be a Banach lattice. Then the following are equivalent:*

(a) *E is Right orthogonal,*
(b) *E has order continuous norm and Right weakly sequentially continuous lattice operations.*

Proof. $(a) \Rightarrow (b)$. Let (x_n) be an order-bounded disjoint sequence in E^+. It is enough to show $\|x_n\| \to 0$. Note that there is an element $e \in E^+$ such that $(x_n) \subset [0,e]$. By [3, P. 185] and [4, Theorem 2.5], the sequence (x_n) is weakly null and Dunford-Pettis; that is, (x_n) is Right null. By the Right orthogonality, $\||x_n| \wedge |x|\| \to 0$ for all $x \in E$. Thus $\||x_n| \wedge e\| = \||x_n|\| = \|x_n\| \to 0$ and so E has order continuous norm. Next, we show that for each Right null sequence (x_n) in E, the sequence $(|x_n|)$ is weakly null in E. Since E is Right orthogonal, $\||x_n| \wedge |x|\| \to 0$ for all $x \in E$. From [3, Theorem 13.6], for each $f \in E^*_+$ there is an element $u \in E^+$ such that $f(|x_n| - u)^+ \to 0$ for all n. Hence

$$f(|x_n|) = f(|x_n| - u)^+ + f(|x_n| \wedge |x|) \to 0.$$

Thus E has the Right weakly sequentially continuous lattice operations.
$(b) \Rightarrow (a)$. Let (x_n) be a Right null sequence in E. We show that $\||x_n| \wedge |x|\| \to 0$ for all $x \in E$. Let $y_n := |x_n| \wedge |x|$. Since E has Right weakly sequentially continuous lattice operations, then positive sequences $(|x_n|)$ and so (y_n) are weakly null. If $\|y_n\| \nrightarrow 0$, then by [18, Corollary 2.3.5], there is a disjoint positive subsequence (y_{n_k}) with $\|y_{n_k}\| \nrightarrow 0$ which is a contradiction (since (y_{n_k}) is an order bounded disjoint sequence in E, by the order continuity of the norm on E it must be norm null).

The following result is easily derived from the theorem:

Corollary 2.5 *For a Banach lattice E, the following are equivalent:*

(a) *E is a discrete Right orthogonal Banach lattice,*
(b) *E is a weak orthogonal Banach lattice.*

We are now ready to discuss the Right fixed point property for Banach lattices:

Definition 2.6 A Banach lattice E has the *Right fixed point property or R-fpp* if every non-expansive self-map $T : K \to K$ of each nonempty, convex and Right compact subset K of E has a fixed point.

It is easily verified that a norm bounded subset A of a Banach space X is Right compact if and only if A is relatively weakly compact and Dunford-Pettis. Hence each Banach space with the w-fpp has the R-fpp, but the converse is not true. All Banach

spaces with the $\mathrm{DP}_{rc}\mathrm{P}$, such as reflexive spaces, have the R-fpp.
Also, a Banach space X has the Dunford-Pettis property if and only if each relatively weakly compact set in X is Right compact [17, Proposition 3.15].

It follows easily from the definition that in each Banach space with the Dunford-Pettis property, w-fpp and R-fpp are the same.
Using the fact that a Banach space X has the Schur property if and only if X has the Dunford-Pettis property and contains no copy of ℓ_1, it follows that: no infinite dimensional reflexive Banach space can have the Dunford-Pettis property. In [21], it is proved that each weak orthogonal Banach lattice has the w-fpp. By the same techniques we can show that each Right orthogonal Banach lattice has the R-fpp.
Definition 2.7 Let E be a Banach lattice. Then:

(a) E has the Right WORTH property if for each Right null sequence (x_n) in E and $x \in E$, $\limsup_n \|x_n - x\| = \limsup_n \|x_n + x\|$.
(b) E has the non-strictly Right Opial condition if for each Right null sequence (x_n) in E and $x \in E$ we have $\limsup_n \|x_n\| \leq \limsup_n \|x_n + x\|$.

It is clear that each Banach lattice with the WORTH property (resp. non-strictly Opial condition) has the Right WORTH (resp. non-strictly Right Opial condition) property too, but the converse is not true. In fact, all reflexive spaces $L^p[0,1]$ $(1 < p < \infty)$ have the Right WORTH property (resp. non-strictly Right Opial condition) but only $L^2[0,1]$ has the WORTH property (resp. non-strictly Opial condition).

In the following theorem a connection between the Right orthogonality, Right WORTH property and non-strictly Right Opial condition is discussed:
Theorem 2.8 *Let E be a Banach lattice. Then for the following assertions:*

(a) *E is Right orthogonal,*
(b) *E has the Right WORTH property,*
(c) *E has the non-strictly Right Opial condition.*

the implications $(a) \Rightarrow (b) \Rightarrow (c)$ are valid.
Proof. $(a) \Rightarrow (b)$. Let (x_n) be a Right null sequence in E. Then,

$$||x_n + x| - |x_n - x|| = 2(|x_n| \wedge |x|)$$

for all $x \in E$ and so by the Right orthogonality, we have

$$\||x_n + x| - |x_n - x|\| = 2(\||x_n| \wedge |x|\|) \to 0.$$

Thus E has the Right WORTH property.
$(b) \Rightarrow (c)$. Let (x_n) be a Right null sequence in E. Then, by the Right WORTH property, we have

$$\limsup_n \|x_n - x\| = \limsup_n \|x_n + x\|$$

for all $x \in E$. Hence,

$$\limsup_n \|x_n\| \leq \frac{1}{2}(\limsup_n \|x_n - x\| + \limsup_n \|x_n + x\|) = \limsup_n \|x_n + x\|$$

and so E has the non-strictly Right Opial condition.

The following conditions on the underlying Banach lattices ensures that the three concepts in the Theorem 2.8 to be equivalent.

Theorem 2.9 *For each σ-Dedekind complete Banach lattice E with the Right weakly sequentially continuous lattice operations, the following are equivalent:*

(a) *E is Right orthogonal,*
(b) *E has Right WORTH property,*
(c) *E has the non-strictly Right Opial condition.*

Proof. $(a) \Rightarrow (b) \Rightarrow (c)$. It follows from Theorems 2.8.
$(c) \Rightarrow (a)$. If E has the non-strictly Right Opial condition, then E does not contain a copy of ℓ_∞. Indeed, Banach lattice ℓ_∞ does not have the non-strictly Right Opial condition. In fact, the standard unit basis (e_n) in ℓ_∞ is a weakly null and Dunford-Pettis (Right null) sequence and $\|e_n\| = 1$ for all n. Also, one can find an element $x \in \ell_\infty$ with $1 = \limsup_n \|e_n\| > \limsup_n \|e_n + x\|$. Also, E is σ-Dedekind complete and then E has order continuous norm. Since E has the Right weakly sequentially continuous lattice operations, then it is Right orthogonal.

It is known that a Banach space X has the Dunford-Pettis property if and only if every relatively weakly compact set in X is Dunford-Pettis. By [17, Proposition 3.15] on a Banach lattice E with the Dunford-Pettis property two topologies weak and Right coincide sequentially and so each weakly null sequence (x_n) in E is Right null.

Note that all AL-spaces and AM-spaces have the Dunford-Pettis property and we may therefore conclude that:

Theorem 2.10 *For a Banach lattice E we have:*

(a) *If E is an AL-space, then E has the w-fpp if and only if E contains no isometric copy of $L^1[0,1]$.*
(b) *If E is a σ-Dedekind complete AM-space, then E has the w-fpp if and only if E is weak orthogonal.*

Proof. (a). It follows from [8, Corollary 6.3]. In fact, an abstract L_p-space, $1 \leq p \leq \infty$ (whenever its norm is p-additive) has the w-fpp if and only if E contains no isometric copy of $L^1[0,1]$. For $p = 1$, we use the fact that any abstract AL-space either (i) contains a copy of $L^1[0,1]$ or (ii) is discrete. In the later case, E has the w-fpp. Note that $L^1[0,1]$ is a non-discrete Banach lattice with order continuous norm.
(b). If E is weak orthogonal, then E has the w-fpp. For the converse, since E the the w-fpp, by [8, Theorem 6.1] it has no isometric copy of $L^1[0,1]$. Hence, E has no norm copy of ℓ_∞ and so it has order continuous norm. Therefore, E is order isometric to $c_0(\Gamma)$ for some index set Γ, and then it is weak orthogonal.

The following result is easily proved.

Corollary 2.11 *For a Banach lattice E with the Dunford-Pettis property, the following are valid:*

(b) *E is weak orthogonal if and only if E is Right orthogonal.*
(b) *E has the WORTH property if and only if E has the Right WORTH property.*
(c) *E has the non-strictly Right Opial condition if and only if E has the non-strictly Opial condition.*

The previous results provide some conditions for a Banach lattice under which the non-strictly Right Opial condition implies the R-fpp. In fact, each σ-Dedekind complete Banach lattice with the non-strictly Right Opial condition and Right weakly sequentially continuous lattice operations is Right orthogonal and so it has the R-fpp.

Definition 2.12 A Banach space X has Right normal structure if for each Right compact convex subset D of X there is an element $u \in D$ such that $\sup_{v\in D} \|u-v\| < diam(D)$.

It is clear that weak normal structure implies Right normal structure but the converse is not true. Also note that c_0 does not have Right normal structure. In fact, the unit vector basis (e_n) of c_0 is weakly null and Dunford-Pettis (Right null). Hence the set $\{e_n : n \in N\}$ is a relatively weakly compact and Dunford-Pettis (Right compact) set in c_0 and so, by the Krein-Smulian Theorem $K = \bar{co}(\{e_n : n \in N\})$, the closed convex hull of $\{e_n : n \in N\}$, is a Right compact convex subset of c_0 and $\lim_n \|x - e_n\| = diam(K) = 1$, for all $x \in K$.

Two properties Right normal structure and Right orthogonality are different, but we show that under some conditions there is a connection between them.

Theorem 2.13 *Let E be a Banach lattice. The following assertions hold:*

(a) *If E has the Right weakly sequentially continuous lattice operations and Right normal structure, then E is Right orthogonal.*
(b) *If E is Right orthogonal with uniformly monotone norm, then E has the Right normal structure.*

Proof. (a). Since c_0 does not have Right normal structure, then E does not contain any copy of c_0 and so it has order continuous norm. From Theorem 2.4 each Banach lattice with Right weakly sequentially continuous lattice operations and order continuous norm, is Right orthogonal.

(b). If E is Right orthogonal, then it has order continuous norm and the Right weakly sequentially continuous lattice operations. On the other hand E has a uniformly monotone norm and so it has the Right normal structure. There is a similar result in [10].

Corollary 2.14 *Let E be a discrete Banach lattice with uniformly monotone norm. The following assertions hold:*

(a) *E is weak orthogonal.*
(b) *E has a weak normal structure.*
(c) *E has a Right normal structure.*
(d) *E is Right orthogonal.*

There is another version of Opial conditions, the so-called uniformly Opial condition. All Opials condition, non-strict Opial condition and uniformly Opial condition have an important role for the fpp in Banach spaces. A Banach space E has uniformly Opial's condition if for each $c > 0$ there is an $r > 0$ such that $1+r \leq \liminf_n \|x_n + x\|$ for each $x \in X$ with $\|x\| \geq c$ and each weakly null sequence (x_n) in X such that $\liminf_n \|x_n\| \geq 1$. In general, uniformly Opial's condition implies the Opial's condition and so the non-strictly Opial's condition.

Note that each Banach lattice with uniformly monotone norm has order continuous norm. So we can improve [10, Theorem 3.5]: each discrete Banach lattice with uniformly monotone norm satisfies uniform Opial's condition.

If we use the Right null sequence instead of weakly null ones in the definition of uniformly Opial's condition, then we have the uniformly Right Opial's condition. Moreover, we can prove that each Banach lattice with uniformly monotone norm and Right weakly sequentially continuous lattice operations, has the uniformly Right Opial's condition.

3. positive Right fixed point property

Using the positive weakly null sequences instead of weakly null ones, the positive versions of weak orthogonality, WORTH property and non-strictly Opial condition are studied in [14]. Here we also discuss the positive versions of three concepts that were previously introduced and consider their applications to study of the positive R-fpp.

We consider our results using the positive Right null sequences:

Definition 3.1 Let E be a Banach lattice. Then:

(a) E is positive Right orthogonal if for each positive Right null sequence (x_n) in E and $x \in E$, $\||x_n| \wedge |x|\| \to 0$,

(b) E is positive Right WORTH property if for each positive Right null sequence (x_n) in E and $x \in E$, we have $\limsup_n \|x_n - x\| = \limsup_n \|x_n + x\|$,

(c) E is positive non-strictly Right Opial condition if for each positive Right null sequence (x_n) in E and $x \in E$, we have $\limsup_n \|x_n\| \leq \limsup_n \|x_n + x\|$.

If we replace positive Right null sequences in Definition 3.1 by disjoint Right null sequences, instead of "positive Right orthogonal, positive Right WORTH property and positive Right Opial condition", we use the phrase "disjoint Right orthogonal, disjoint Right WORTH property and disjoint Right Opial condition".

Recall that a Banach lattice E has the positive DP$_{rc}$P if each Right null sequence in E with the positive terms in E is norm null or equivalently, each disjoint Right null sequence in E is norm null. It is easily verified that each Banach lattice with the positive DP$_{rc}$P is positive Right orthogonal. For instance, $L^1[0,1] \oplus L^2[0,1]$ has the positive DP$_{rc}$P, but it does not have the DP$_{rc}$P and so it is positive Right orthogonal but not Right orthogonal. Note that in discrete Banach lattices, DP$_{rc}$P and positive DP$_{rc}$P are the same. The reader is referred to [6] for information on the positive DP$_{rc}$P.

The following characterization is proved similar to [14, Theorem 2.2] and it shows that a Banach lattice E is positive Right orthogonal if and only if E is disjoint Right orthogonal.

Proposition 3.2 *For a Banach lattice E, the following are equivalent:*

(a) *E has order continuous norm,*

(b) *E is a positive Right orthogonal Banach lattice,*

(c) *E is a disjoint Right orthogonal Banach lattice.*

In the rest of this article we can replace "positive" with "disjoint". Each Right orthogonal Banach lattice is positive Right orthogonal, but the converse is false. The converse holds for each Banach lattice E with Right weakly sequentially continuous lattice operations.

It is easily verified that each Banach lattice with the Right WORTH property has the positive Right WORTH property too, but the converse is not true. In fact, $L^1[0,1]$ has the positive Right WORTH property but not the Right WORTH property. Also, $L^1[0,1]$ has the positive non-strictly Right Opial condition but not the non-strictly Right Opial condition.

Theorem 3.3 *For each σ-Dedekind complete Banach lattice E, the following are equivalent:*

(a) *E is positive Right orthogonal,*
(b) *E has the positive Right WORTH property,*
(c) *E has the positive non-strictly Right Opial condition.*

Proof. $(a) \Rightarrow (b) \Rightarrow (c)$. It is proved similar to Theorem 2.8.
$(c) \Rightarrow (a)$. If E has the positive non-strictly Right Opial condition, then E contains no copy of ℓ_∞. Since E is σ-Dedekind complete, E has order continuous norm. Hence E is positive Right orthogonal.

Using the above results the following can be derived:

Corollary 3.4 *For a Banach lattice E, the following are valid:*

(a) *If E has the Right weakly sequentially continuous lattice operations, then E is positive Right orthogonal if and only if E is Right orthogonal.*
(b) *If E is discrete σ-Dedekind complete, then E has the positive Right WORTH property if and only if E has the Right WORTH property.*
(c) *If E is discrete σ-Dedekind complete, then E has the positive non-strictly Right Opial condition if and only if E has the non-strictly Right Opial condition.*

Following the discussion in section 2 in connection with so called R-fpp, we now introduce the notion of positive R-fpp:

Definition 3.5 A Banach lattice E has the *positive R-fpp* if every non-expansive self-map $T : K \to K$ of each nonempty, convex and Right compact with the positive terms subset K of E has a fixed point.

Each Right orthogonal Banach lattice has the R-fpp and by the same techniques we can show that each positive Right orthogonal Banach lattice has the positive R-fpp. On the other hand, each Banach lattice with the R-fpp has the positive R-fpp, but the converse is false. In fact, $L^1[0,1]$ has the positive Schur property and so it is a positive Right orthogonal Banach lattice. Then $L^1[0,1]$ has the positive R-fpp, but only reflexive subspaces of $L^1[0,1]$ have the R-fpp. Note that all AL-spaces have the positive Schur property and so they have the positive R-fpp.

4. Right fpp for some operator spaces

This section leads to the extension and improvement of results in [25] for the R-fpp and the positive R-fpp of a Banach lattice $\mathcal{M}$ of compact operators between suitable Banach lattices.

From [25] if X is a Banach space with the shrinking finite dimensional Schauder decomposition and Y is a c_0 or ℓ_p-direct sum of finite dimensional Banach spaces, it is proved that the complete continuity of all evaluation operators on a closed subspace $\mathcal{M} \subset K(X,Y)$ is a sufficient condition for the w-fpp of $\mathcal{M}$. The reader should note that two lemmas 2.3 and 2.5 of [25] also hold without any conditions on X (for our detailed discussion the reader is referred to [19, Lemma 3.2]).

Based on our discussion and by the same arguments in [5, 25, 14], we can assume that X is an arbitrary Banach space and then improve [25, Theorem 2.6] for R-fpp (or the positive R-fpp) for a suitable Banach lattice of some compact operator spaces from a Banach space into a Banach lattice as follows: First observe that:

Definition 4.1 An operator $T : E \to X$ is called pseudo weakly compact (briefly, pwc) (resp. positive pwc) if for every Right null (resp. positive Right null) sequence (x_n) in E, $\|Tx_n\| \to 0$.

It is clear that each pwc operator is positive pwc, but the converse is not true. In fact, the identity operator on each Banach lattice with the positive $\mathrm{DP}_{rc}\mathrm{P}$ and without the $\mathrm{DP}_{rc}\mathrm{P}$ such as $L^1[0,1]$ is positive pwc, but it is not pwc. To continue, we need two following lemmas. For the first lemma, we may use Remark 2.3 in [1] in a similar fashion to prove that:

Lemma 4.2 *Let X and Y be two Banach spaces and $\mathcal{M} \subset L(X,Y)$ be a closed subspace. If all evaluation operators ψ_{y^*} are pwc (resp. positive pwc), then, the operator $K \to TK$ from $\mathcal{M}$ into $L(X,Y)$ is pwc (resp. positive pwc) for all compact operators $T \in K(Y)$.*

For the second lemma (which can be proved by the same arguments in [5, 25]) we recall some notations. Let F be a discrete Banach lattice with complete disjoint systems consisting of discrete elements $\{u_i\}_{i\in I}$. Then $W = \sum_{i\in I} I_{u_i}$ is a projection band and $F = W + W^{\perp}$. Also, every $x \in E$ can be written as $x = x_1 + x_2$; where $x_1 \in W$ and $x_2 \in W^{\perp}$. The projection $P_W : E \to E$ defined by $P_W(x) = x_1$ is the band projection onto W. It is clear that $\|P_W\| = 1$.

Lemma 4.3 *Let X be a Banach space, F be a discrete Banach lattice with order continuous norm and $\mathcal{M} \subset K(X,F)$ be a Banach lattice. Then the following assertions hold:*

(a) *If $K_1, K_2, ...K_n \in K(X,F)$ and $\epsilon > 0$, then there is a finite dimensional projection band $W \subset F$ such that $\|P_{W^{\perp}} K_i\| \leq \epsilon$ for all $i = 1, 2, ...n$.*
(b) *If all of the evaluation operators ψ_{y^*} are pwc (resp. positive pwc), (K_n) is a Right null (resp. positive Right null) sequence in $\mathcal{M}$, then there is a subsequence (K_{n_i}) of (K_n) and a sequence (U_i) of $K(X,F)$ such that $\lim_{i\to\infty} \|U_i - K_{n_i}\| = 0$*

Now, we give some sufficient conditions for the R-fpp (respectively positive R-fpp) of a Banach lattice $\mathcal{M}$ of some compact operators from a Banach space X into a Banach lattice F with respect to pwc-ness (respectively positive pwc-ness) of all evaluation operators. The reader should note that following the same arguments as in the proof of Theorem 3.5 of [14] we can conclude the following theorem.

Theorem 4.4 *Let X be a Banach space, F be an AM-space with order continuous norm and $\mathcal{M} \subset K(X,F)$ be a Banach lattice. If all of the evaluation operators ψ_{y^*} are pwc (resp. positive pwc), then $\mathcal{M}$ has the R-fpp (resp. positive R-fpp).*

The reader should note that if $T : X \to Y$ is an operator between two Banach spaces such that one of them has the $\mathrm{DP}_{rc}\mathrm{P}$ (such as reflexive spaces or discrete KB-spaces), then T is pwc. Recall that E is called a KB-space if every increasing norm bounded sequence of E^+ is norm convergent. Similarly, if $T : E \to F$ is an operator between two Banach lattices such that one of them has the positive $\mathrm{DP}_{rc}\mathrm{P}$ (such as $L^1[0,1]$), then each the operator T is positive pwc.

We conclude our paper review with some examples. At first, using the fact that a Banach lattice E is an AM-space with order continuous norm if and only if E is lattice isometric to $c_0(\Omega)$, where Ω is a nonempty set, it follows that:

Example 4.5 Let X be a reflexive Banach space and F be an AM-space with order continuous norm. Then each Banach lattice $\mathcal{M} \subset K(X,F)$ has the R-fpp. For instance, each Banach lattice $\mathcal{M} \subset K(\ell_2, c_0)$ has the R-fpp.

In fact, since ℓ_2 has the $\mathrm{DP}_{rc}\mathrm{P}$ then for the Banach lattice $\mathcal{M} \subset K(\ell_2, c_0)$ all evaluation operators $\psi_{y^*} : \mathcal{M} \to \ell_2$ are pwc. Also, c_0 is an AM-space with order continuous norm and then by Theorem 4.4, $\mathcal{M}$ has the R-fpp.

The following example shows that order continuity of the norm on an AM-space F in Theorem 4.4 cannot be removed.

Example 4.6 Banach lattice ℓ_∞ is an AM-space without order continuous norm and ℓ_∞ can be embedded isometrically into $K(\ell_2, \ell_\infty)$. It is easy to see that all evaluation operators $\psi_{y^*} : \mathcal{M} \to \ell_2$ are pwc, but ℓ_∞ (and so $K(\ell_2, \ell_\infty)$) does not have the R-fpp (note that, ℓ_∞ does not have the w-fpp [8]).

Similar to [14, Example 3.9] if X is a Banach space and H is a Hilbert space such that all of the evaluation operators ψ_y with $y \in H$ are pwc, then each Banach lattice $\mathcal{M} \subset K(X,H)$ has the R-fpp.

It should be noted that AL-spaces do not have the w-fpp, in general. For instance, only reflexive subspaces of $L^1[0,1]$ have the w-fpp (fpp). Also, ℓ_∞ is a σ-Dedekind complete AM-space without w-fpp.

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Received: March 6, 2023; Accepted: February 25, 2024.