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A b-METRIC APPROACH TO dq-K-METRIC SPACES AND APPLICATIONS

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Abstract. In this paper, we present a b-metric approach to dq-K-metric spaces and its application to the stability of functional equations.
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1. Introduction

In the paper [2], Brzdek *et al.* introduced the notion of the dq-K-metric and proposed a very general and uniform approach to many stability results of several recent papers. The dq-K-metric is a generalization of many distances, including the b-metric. The main result of [2] are stability results for the radical functional equation

$$f\big(p\big(\pi(x)+\pi(y)\big)\big) = f(x) + f(y),$$

where S is a non-empty set, $(P,+)$ and $(G,+)$ are groupoids, d is a dq-K-metric on G, $\pi : S \to P$ is surjective and $p : P \to S$ is a selection with respect to π, that is, $p(u) \in \pi^{-1}(u)$ for all $u \in P$. From this result, the authors deduced many stability results in concrete cases.

In this paper, from a given dq-K-metric, we construct a new b-metric and state some relationships between them. We also prove some stability results in b-metric spaces. Then, by using the b-metric induced by the given dq-K-metric, we deduce the stability result of [2] in dq-K-metric spaces.

2. Preliminaries

In this section, we recall the notions and properties helpful in the latter. The first is the notion of dq-K-metric space.

Definition 1 ([2], page 2 & page 5)**.** Let X be a non-empty set, $\kappa \geq 1$ and $d : X \times X \to [0, \infty)$ be a function such that for all $x, y, z \in X$, the following conditions hold:

(1) if $d(x, y) = d(y, x) = 0$ then $x = y$;
(2) $d(x, z) \leq \kappa[d(x, y) + d(y, z)]$.

Then we have the following definitions.

(1) d is called a *dq-K-metric* with the coefficient κ, and (X, d, κ) is called a *dq-K-metric space*.
(2) The sequence $\{x_n\}$ in X is called *convergent* to x, denoted by $\lim_{n\to\infty} x_n = x$, if
$$\lim_{n\to\infty} \max\{d(x_n, x), d(x, x_n)\} = 0.$$
(3) The sequence $\{x_n\}$ in X is called *Cauchy* if $\lim_{k\to\infty} \sup_{n,m\geq k} d(x_n, x_m) = 0$.
(4) The *dq-K*-metric space (X, d, κ) is called *complete* if each Cauchy sequence in X is a convergent sequence in X.

The next is the notion of a *b*-metric space.

Definition 2 ([3], page 263)**.** Let X be a non-empty set, $\kappa \geq 1$ and $\delta : X \times X \to [0, \infty)$ be a function such that for all $x, y, z \in X$, the following conditions hold:

(1) $\delta(x, y) = 0$ if and only if $x = y$;
(2) $\delta(x, y) = \delta(y, x)$;
(3) $\delta(x, z) \leq \kappa[\delta(x, y) + \delta(y, z)]$.

Then we have the following definitions.

(1) δ is called a *b-metric* on X and (X, δ, κ) is called a *b-metric space*.
(2) The sequence $\{x_n\}$ in X is called *convergent* to x if $\lim_{n\to\infty} \delta(x_n, x) = 0$, written by $\lim_{n\to\infty} x_n = x$.
(3) The sequence $\{x_n\}$ in X is called *Cauchy* if $\lim_{n,m\to\infty} \delta(x_n, x_m) = 0$.
(4) The *b*-metric space (X, δ, κ) is called *compete* if every Cauchy sequence is a convergent sequence.

The next is a metrizable result of a *b*-metric space in the sense that we have an equivalent metric from a given *b*-metric. This result has been used to solve usefully certain fixed point problems in *b*-metric spaces; see, for example, [4], [5].

Theorem 3 ([8], Proposition on page 4308)**.** *Let (X, δ, κ) be a b-metric space. Put $\theta = \log_{2\kappa} 2$, and*
$$\rho(x, y) = \inf \left\{ \sum_{i=1}^{n} \delta^\theta(x_i, x_{i+1}) : x_1 = x, x_2, \ldots, x_n, x_{n+1} = y \in X, n \geq 1 \right\}$$
for all $x, y \in X$. Then we have the following assertions:

(1) ρ is a metric on X satisfying
$$\frac{1}{4}\delta^\theta(x, y) \leq \rho(x, y) \leq \delta^\theta(x, y) \tag{2.1}$$
for all $x, y \in X$;

(2) if δ is a metric, then $\theta = 1$ and $\rho = \delta$.

The next are relevant notions to groupoids that will be used in the latter.

Definition 4 ([2], page 7)**.** Let G be a non-empty set and $+ : G^2 \to G$ be an inner operation satisfying the following conditions:

(1) $a + (b + c) = (a + b) + c$ for all $a, b, c \in G$;
(2) there exists an identity element $0 \in G$ such that $0 + a = a + 0 = a$ for all $a \in G$;
(3) for each $a \in G$, there exits $-a \in G$ such that $a + (-a) = (-a) + a = 0$.

Then we have the following definitions.

(1) $(G, +)$ is called a *groupoid*.
(2) The groupoid $(G, +)$ is called *square symmetric* if $2a + 2b = 2(a + b)$ for all $a, b \in G$, where $2a = a + a$.
(3) The groupoid $(G, +)$ is called *uniquely divisible by 2* if for each $a \in G$, there exists a unique $b \in G$ such that $a = 2b$. We also write $b = 2^{-1}a$ if $a = 2b$, and write $2^{-(n+1)}a = 2^{-1}2^{-n}a$ for all $n \in \mathbb{N}$.

The following are some basic remarks that have been presented in [2].

Remark 5 ([2], pages 7-8)**.** (1) If a groupoid $(G, +)$ is square symmetric, then for all $a, b \in G$ and all $n \in \mathbb{N}$,

$$2^n(a + b) = 2^n a + 2^n b.$$

Moreover, if $(G, +)$ is uniquely divisible by 2, then for all $a, b \in G$ and all $n \in \mathbb{N}$,

$$2^{-n}(a + b) = 2^{-n}a + 2^{-n}b.$$

(2) Every abelian semigroup is square symmetric.
(3) If $\mathbb{F}$ is a field, $A, B \in \mathbb{F}$, X is a linear space over $\mathbb{F}$, $x_0 \in X$ and $\oplus : X^2 \to X$ is defined by

$$x \oplus y = Ax + By + x_0$$

for all $x, y \in X$, then $(X, \oplus)$ is a square symmetric groupoid.
(4) If $(X, \oplus)$ is a square symmetric groupoid, D is a non-empty set, $h : D \to X$ is a bijection and $* : D^2 \to D$ is defined by

$$a * b = h^{-1}(h(a) \oplus h(b))$$

for all $a, b \in D$, then $(D, *)$ is a square symmetric groupoid.

One of main results of [2] is as follows.

Theorem 6 ([2], Theorem 2)**.** *Suppose that the following conditions hold.*

(1) $(G, +)$ and $(X, +)$ are two square symmetric groupoids, where $(X, +)$ is uniquely divisible by 2.
(2) (X, d, κ) is a complete dq-K-metric space such that for some $\xi > 0$ and all $x, y \in X$,

$$d(2^{-1}x, 2^{-1}y) \leq \xi d(x, y). \tag{2.2}$$

(3) The operation $+$ is continuous with respect to the dq-K-metric d.

(4) *There exist* $\varphi_i : G \times G \to [0, \infty)$, *where* $i = 1, 2$, *satisfying for each* $i = 1, 2$ *and for all* $x, y \in G$,

$$\Phi_i(x) := \kappa^2 \xi \sum_{j=0}^{\infty} (\kappa\xi)^j \varphi_i(2^j x, 2^j x) \quad < \quad \infty, \tag{2.3}$$

$$\lim_{j\to\infty} \xi^i \varphi_i(2^j x, 2^j y) \quad = \quad 0. \tag{2.4}$$

(5) $f : G \to X$ *is a map such that for all* $x, y \in G$,

$$d\big(f(x+y), f(x)+f(y)\big) \quad \leq \quad \varphi_1(x,y), \tag{2.5}$$

$$d\big(f(x)+f(y), f(x+y)\big) \quad \leq \quad \varphi_2(x,y). \tag{2.6}$$

Then there exists a unique map $\alpha : G \to X$ *such that for all* $x, y \in G$,

$$\begin{aligned} \alpha(x+y) &= \alpha(x)+\alpha(y), \\ d\big(\alpha(x), f(x)\big) &\leq \Phi_1(x), \\ d\big(f(x), \alpha(x)\big) &\leq \Phi_2(x). \end{aligned}$$

Moreover, for all $x \in G$,

$$\alpha(x) = \lim_{n\to\infty} f_n(x).$$

3. Stability of functional equations in b-metric spaces

First, we give the following example that shows the role of the convergence of the series $\kappa^2\xi \sum_{j=0}^{\infty} (\kappa\xi)^j \varphi_i(2^j x, 2^j x)$ in the assumption (2.3) of Theorem 6.

Example 7. Suppose that the following conditions hold.

(1)

$$G = L^{\frac{1}{4}}[0,1] = \big\{x : [0,1] \to \mathbb{R} : |x|^{\frac{1}{4}} \text{ is Lebesgue integrable } \big\},$$

$$X = L^{\frac{1}{2}}[0,1] = \big\{x : [0,1] \to \mathbb{R} : |x|^{\frac{1}{2}} \text{ is Lebesgue integrable } \big\}$$

with the usual addition of functions $(x+y)(t) = x(t)+y(t)$ and a dq-K-metric d with $\kappa = 2$ is defined by

$$d(x,y) = \Big(\int_0^1 |x(t) - y(t)|^{\frac{1}{2}} dt \Big)^2$$

for all $x, y \in X$ and all $t \in [0,1]$.

(2) $f : G \to X$ is defined by $f(x) = x + \sqrt{|x|}$ for all $x \in G$, where

$$(x + \sqrt{|x|})(t) = x(t) + \sqrt{|x(t)|}$$

for all $t \in [0,1]$.

(3) $\varphi_1 : G \times G \to [0,\infty)$ and $\varphi_2 : G \times G \to [0,\infty)$ are defined by

$$\varphi_1(x,y) = \varphi_2(x,y) = \Big(\int_0^1 \big| \sqrt{|x(t)+y(t)|} - \sqrt{|x(t)|} - \sqrt{|y(t)|} \big|^{\frac{1}{2}} dt \Big)^2$$

for all $x, y \in G$.

Then we have the following assertions.

(1) The assumptions (2.3) and (2.4) of Theorem 6 are not satisfied.
(2) All other assumptions of Theorem 6 are satisfied.

Proof. (1) We find that for $i = 1, 2$ and $\xi = \frac{1}{2}$ and $x, y \in G$,

$$\kappa^2 \xi \sum_{j=0}^{\infty} (\kappa\xi)^j \varphi_i(2^j x, 2^j y) = 2 \sum_{j=0}^{\infty} (\sqrt{2})^j \Big(\int_0^1 \big| \sqrt{|x(t) + y(t)|} - \sqrt{|x(t)} - \sqrt{|y(t)|} \big|^{\frac{1}{2}} dt \Big)^2.$$

Hence, for $\int_0^1 \big| \sqrt{|x(t) + y(t)|} - \sqrt{|x(t)|} - \sqrt{|y(t)|} \big|^{\frac{1}{2}} dt \neq 0$, we find that the assumptions (2.3) and (2.4) are not satisfied.

(2) By [7, Example 1], we find that (X, d, κ) is a complete dq-K-metric space with $\kappa = 2$. For $\xi = \frac{1}{2}$ and all $x, y \in X$, we have $d(2^{-1}x, 2^{-1}y) = \xi d(x, y)$. Note that for $x \in G$, we have $x + \sqrt{|x|} \in X$. Then the map $f : G \to X$ is well-defined. We also have for all $x, y \in G$,

$$\begin{aligned} d\big(f(x+y), f(x) + f(y)\big) &= \varphi_1(x, y), \\ d\big(f(x) + f(y), f(x+y)\big) &= \varphi_2(x, y). \end{aligned}$$

We find that the given addition is the usual addition of functions. Then $(G, +)$ and $(X, +)$ are two square symmetric groupoids, $(X, +)$ is uniquely divisible by 2. Moreover, the operation $+$ is continuous with respect to the dq-K-metric d. □

Now we prove a stability result for the generalized radical functional equation in b-metric spaces.

Theorem 8. *Suppose that the following conditions hold.*

(1) $(G, +)$ and $(X, +)$ are two square symmetric groupoids, where $(X, +)$ is uniquely divisible by 2.
(2) (X, δ, κ) is a complete b-metric space such that for some $\xi > 0$ and all $x, y \in X$,

$$\delta(2^{-1}x, 2^{-1}y) \leq \xi \delta(x, y). \tag{3.1}$$

(3) The operation $+$ is continuous with respect to the b-metric δ.
(4) There exists $\varphi : G \times G \to [0, \infty)$ such that for all $x, y \in G$,

$$\Phi(x) := \kappa^2 \xi \sum_{j=0}^{\infty} (\kappa\xi)^j \varphi(2^j x, 2^j x) \quad < \quad \infty \tag{3.2}$$

and

$$\lim_{n\to\infty} \xi^n \varphi(2^n x, 2^n y) \quad = \quad 0. \tag{3.3}$$

(5) $f : G \to X$ is a map such that for all $x, y \in G$,

$$\delta\big(f(x+y), f(x) + f(y)\big) \leq \varphi(x, y). \tag{3.4}$$

Then there exists a unique map $\alpha : G \to X$ such that for all $x, y \in G$,

$$\alpha(x+y) \quad = \quad \alpha(x) + \alpha(y), \tag{3.5}$$

$$\delta\big(\alpha(x), f(x)\big) \quad \leq \quad \Phi(x). \tag{3.6}$$

Moreover, for all $x \in G$,

$$\alpha(x) = \lim_{n\to\infty} f_n(x). \tag{3.7}$$

Proof. Put $f_0 = f$, and for each $n \in \mathbb{N}$ and all $x \in G$, put $f_n(x) = 2^{-n} f(2^n x)$. It follows from (3.1) and (3.4) that for each $n \in \mathbb{N}$,

$$\begin{aligned}
\delta\big(f_{n+1}(x), f_n(x)\big) &= \delta\big(2^{-(n+1)} f(2\cdot 2^n x), 2^{-(n+1)} 2f(2^n x)\big)\\
&\le \xi^{n+1}\delta\big(f(2\cdot 2^n x), 2f(2^n x)\big)\\
&= \xi^{n+1}\delta\big(f(2^n x + 2^n x), f(2^n x) + f(2^n x)\big)\\
&\le \xi^{n+1}\varphi(2^n x, 2^n x)
\end{aligned} \tag{3.8}$$

for all $x \in G$. For all $n, m \in \mathbb{N}$, by using (3.8), we have

$$\begin{aligned}
\delta\big(f_{n+m}(x), f_n(x)\big) &\le \sum_{j=1}^{m} \kappa^j \delta\big(f_{n+j}(x), f_{n+j-1}(x)\big)\\
&\le \sum_{j=1}^{m} \kappa^j \xi^{n+j}\varphi(2^{n+j-1}x, 2^{n+j-1}x)\\
&\le \sum_{j=1}^{m} \kappa^{n+j} \xi^{n+j}\varphi(2^{n+j-1}x, 2^{n+j-1}x)\\
&= \kappa\xi \sum_{j=n}^{m+n} (\kappa\xi)^j \varphi(2^j x, 2^j x)
\end{aligned} \tag{3.9}$$

for all $x \in G$. Letting $n, m \to \infty$ in (3.9) and using (3.2) we get

$$\lim_{n,m\to\infty} \delta\big(f_{n+m}(x), f_n(x)\big) = 0$$

for all $x \in G$. This proves that for each $x \in G$, we have $\{f_n(x)\}$ is a Cauchy sequence in (X, δ, κ). Since (X, δ, κ) is complete, there exists the map $\alpha : G \to X$ defined by

$$\alpha(x) = \lim_{n\to\infty} f_n(x) \tag{3.10}$$

for all $x \in G$. This proves that (3.7) holds.

For $n = 0$ and $m \in \mathbb{N}$ in (3.9), we have

$$\delta(f_m(x), f(x)) \le \kappa\xi \sum_{j=0}^{m} (\kappa\xi)^j \varphi(2^j x, 2^j x)$$

for all $x \in G$. Therefore, we have

$$\begin{aligned}
\delta\big(\alpha(x), f(x)\big) &\le \kappa\big[\delta\big(\alpha(x), f_m(x)\big) + \delta\big(f_m(x), f(x)\big)\big]\\
&\le \kappa\Big[\delta\big(\alpha(x), f_m(x)\big) + \kappa\xi \sum_{j=0}^{m} (\kappa\xi)^j \varphi(2^j x, 2^j x)\Big]
\end{aligned} \tag{3.11}$$

for all $x \in G$. Letting $m \to \infty$ in (3.11) and using (3.10), we get

$$\delta\big(\alpha(x), f(x)\big) \le \Phi(x)$$

for all $x \in G$. This proves that (3.6) holds.

Now, since G and X are square symmetric, by using Remark 5.(1) and (3.4), we get

$$\begin{aligned}\delta\big(f_n(x+y), f_n(x)+f_n(y)\big) &= \delta\big(2^{-n}f(2^nx+2^ny), 2^{-n}\big(f(2^nx)+f(2^ny)\big)\big)\\ &\leq \xi^n\varphi(2^nx, 2^ny) \qquad (3.12)\end{aligned}$$

for all $x, y \in G$. It follows from (3.2) that

$$\lim_{n\to\infty}\xi^n\varphi(2^nx, 2^ny) = 0 \qquad (3.13)$$

for all $x, y \in G$. Combining (3.12) and (3.13), we obtain

$$\lim_{n\to\infty}\delta\big(f_n(x+y), f_n(x)+f_n(y)\big) = 0 \qquad (3.14)$$

for all $x, y \in G$. We also find that

$$\begin{aligned}0 &\leq \delta\big(\alpha(x+y), \alpha(x)+\alpha(y)\big)\\ &\leq \kappa\big[\delta\big(\alpha(x+y), f_n(x+y)\big) + \delta\big(f_n(x+y), \alpha(x)+\alpha(y)\big)\big]\\ &\leq \kappa\big[\delta\big(\alpha(x+y), f_n(x+y)\big) + \kappa\delta\big(f_n(x+y), f_n(x)+f_n(y)\big)\\ &\quad +\kappa\delta\big(f_n(x)+f_n(y), \alpha(x)+\alpha(y)\big)\big] \qquad (3.15)\end{aligned}$$

for all $x, y \in G$. Now, letting $n \to \infty$ in (3.15) and using (3.10), (3.14) and the continuity of the operation $+$ with respect to δ, we have $\delta\big(\alpha(x+y), \alpha(x)+\alpha(y)\big) = 0$. This proves that (3.5) holds.

Let $\beta : G \to X$ be also a map satisfying (3.5) and (3.6) for all $x, y \in G$, where β plays the role of α. Since α and β satisfy (3.5), for all $n \in \mathbb{N}$ and $x \in G$, we have

$$\alpha(2^nx) = 2^n\alpha(x) \text{ and } \beta(2^nx) = 2^n\beta(x). \qquad (3.16)$$

It follows from (3.1), (3.6) and (3.16) that for each $x \in G$ and $n \in \mathbb{N}$, we obtain

$$\begin{aligned}\delta(\alpha(x), \beta(x)) &= \delta\big(2^{-n}\alpha(2^nx), 2^{-n}\beta(2^nx)\big)\\ &\leq \xi^n\delta\big(\alpha(2^nx), \beta(2^nx)\big)\\ &\leq \xi^n\kappa\big[\delta\big(\alpha(2^nx), f(2^nx)\big) + \delta\big(f(2^nx), \beta(2^nx)\big)\big]\\ &\leq \xi^n\kappa\big[\Phi(2^nx) + \Phi(2^nx)\big]\\ &= 2\xi^n\kappa^3\xi\sum_{j=0}^{\infty}(\kappa\xi)^j\varphi(2^{j+n}x, 2^{j+n}x)\\ &= \frac{2\xi}{\kappa^{n-3}}\sum_{j=n}^{\infty}(\kappa\xi)^j\varphi(2^jx, 2^jx)\\ &\leq \frac{2\Phi(x)}{\kappa^{n-1}}. \qquad (3.17)\end{aligned}$$

Letting $n \to \infty$ in (3.17), we have $\delta(\alpha(x), \beta(x)) = 0$ for all $x \in G$. This proves that $\beta = \alpha$. Then α is the unique map satisfying (3.5) and (3.6) for all $x, y \in G$. □

In the next, we prove another stability result for the generalized radical functional equation in b-metric spaces.

Theorem 9. *Suppose that the following conditions hold.*

(1) $(G,+)$ *and* $(X,+)$ *are two square symmetric groupoids, where* $(X,+)$ *is uniquely divisible by 2.*
(2) (X,δ,κ) *is a complete b-metric space such that for some* $\xi>0$ *and all* $x,y\in X$,

$$\delta(2^{-1}x,2^{-1}y)\le\xi\delta(x,y). \tag{3.18}$$

(3) The operation + is continuous with respect to the b-metric δ.
(4) There exists $\varphi:G\times G\to[0,\infty)$ *satisfying for all* $x,y\in G$ *and* $\theta=\log_{2\kappa}2$,

$$\Phi(x):=\xi^{\theta}\sum_{j=0}^{\infty}\xi^{\theta j}\varphi^{\theta}(2^jx,2^jx)\quad<\quad\infty \tag{3.19}$$

and

$$\lim_{n\to\infty}\xi^n\varphi(2^nx,2^ny)\quad=\quad0. \tag{3.20}$$

(5) $f:G\to X$ *is a map such that for all* $x,y\in G$,

$$\delta\big(f(x+y),f(x)+f(y)\big)\le\varphi(x,y). \tag{3.21}$$

Then there exists a unique map $\alpha:G\to X$ *such that for all* $x,y\in G$,

$$\alpha(x+y)\quad=\quad\alpha(x)+\alpha(y), \tag{3.22}$$
$$\delta\big(\alpha(x),f(x)\big)\quad\le\quad(16)^{\frac{1}{\theta}}\Phi^{\frac{1}{\theta}}(x). \tag{3.23}$$

Moreover, for all $x\in G$,

$$\alpha(x)=\lim_{n\to\infty}f_n(x). \tag{3.24}$$

Proof. Put $f_0=f$, and for each $n\in\mathbb{N}$ and all $x\in G$, put $f_n(x)=2^{-n}f(2^nx)$. It follows from (3.18) and (3.21) that

$$\begin{aligned}\delta\big(f_{n+1}(x),f_n(x)\big)&=\delta\big(2^{-(n+1)}f(2.2^nx),2^{-(n+1)}2f(2^nx)\big)\\&\le\xi^{n+1}\delta\big(f(2.2^nx),2f(2^nx)\big)\\&=\xi^{n+1}\delta\big(f(2^nx+2^nx),f(2^nx)+f(2^nx)\big)\\&\le\xi^{n+1}\varphi(2^nx,2^nx)\end{aligned} \tag{3.25}$$

for all $x \in G$ and for all $n \in \mathbb{N}$. For all $n, m \in \mathbb{N}$, by using (2.1) and (3.25), we have

$$\begin{aligned} \frac{1}{4}\delta^\theta\big(f_{n+m}(x), f_n(x)\big) &\leq \rho\big(f_{n+m}(x), f_n(x)\big) \\ &\leq \sum_{j=1}^{m} \rho\big(f_{n+j}(x), f_{n+j-1}(x)\big) \\ &\leq \sum_{j=1}^{m} \delta^\theta\big(f_{n+j}(x), f_{n+j-1}(x)\big) \\ &\leq \sum_{j=1}^{m} \xi^{\theta(n+j)}\varphi^\theta(2^{n+j-1}x, 2^{n+j-1}x) \\ &= \xi^\theta \sum_{j=n}^{n+m-1} \xi^{\theta j}\varphi^\theta(2^j x, 2^j x) \qquad (3.26) \end{aligned}$$

for all $x \in G$. Letting $n, m \to \infty$ in (3.26) and using (3.19), we get

$$\lim_{n,m\to\infty} \delta\big(f_{n+m}(x), f_n(x)\big) = 0.$$

This proves that $\{f_n(x)\}$ is a Cauchy sequence in (X, δ, κ). Since (X, δ, κ) is complete, there exists the map $\alpha : G \to X$ defined by

$$\alpha(x) = \lim_{n\to\infty} f_n(x) \qquad (3.27)$$

for all $x \in G$. This proves that (3.24) holds.

For $n = 0$ and $m \in \mathbb{N}$ in (3.26), we have

$$\frac{1}{4}\delta^\theta(f_m(x), f(x)) \leq \xi^\theta \sum_{j=0}^{m-1} \xi^{\theta j}\varphi^\theta(2^j x, 2^j x)$$

for all $x \in G$. Therefore, for each $x \in G$, we obtain

$$\begin{aligned} \frac{1}{4}\delta^\theta\big(\alpha(x), f(x)\big) &\leq \rho\big(\alpha(x), f(x)\big) \\ &\leq \rho\big(\alpha(x), f_m(x)\big) + \rho\big(f_m(x), f(x)\big) \\ &\leq d^\theta\big(\alpha(x), f_n(x)\big) + d^\theta\big(f_m(x), f(x)\big) \\ &\leq d^\theta\big(\alpha(x), f_m(x)\big) + 4\xi^\theta \sum_{j=0}^{m-1} \xi^{\theta j}\varphi^\theta(2^j x, 2^j x) \qquad (3.28) \end{aligned}$$

for all $m \in \mathbb{N}$. Letting $m \to \infty$ in (3.28) and using (3.27), we get

$$\delta\big(\alpha(x), f(x)\big) \leq (16)^{\frac{1}{\theta}}\Phi^{\frac{1}{\theta}}(x)$$

for all $x \in G$. This proves that (3.23) holds.

Now, since G and X are square symmetric, by using Remark 5.(1) and (3.21), for each $x, y \in G$ and $n \in \mathbb{N}$, we get

$$\begin{aligned} \delta\big(f_n(x+y), f_n(x) + f_n(y)\big) &= \delta\big(2^{-n}f(2^n x + 2^n y), 2^{-n}\big(f(2^n x) + f(2^n y)\big)\big) \\ &\leq \xi^n\varphi(2^n x, 2^n y). \qquad (3.29) \end{aligned}$$

Combining (3.29) and (3.20), we get

$$\lim_{n\to\infty} \delta\big(f_n(x+y), f_n(x)+f_n(y)\big) = 0 \tag{3.30}$$

for all $x, y \in G$. We find that for each $n \in \mathbb{N}$, we obtain

$$\begin{aligned} 0 &\leq \delta\big(\alpha(x+y), \alpha(x)+\alpha(y)\big) \\ &\leq \kappa\delta\big(\alpha(x+y), f_n(x+y)\big) + \kappa\delta\big(f_n(x+y), \alpha(x)+\alpha(y)\big) \\ &\leq \kappa\delta\big(\alpha(x+y), f_n(x+y)\big) + \kappa^2\delta\big(f_n(x+y), f_n(x)+f_n(y)\big) \\ &\quad +\kappa^2\delta\big(f_n(x)+f_n(y), \alpha(x)+\alpha(y)\big) \end{aligned} \tag{3.31}$$

for all $x, y \in G$. Now, letting $n \to \infty$ in (3.31) and using (3.27), (3.30) and the continuity of the operation $+$ with respect to d, we have

$$\delta\big(\alpha(x+y), \alpha(x)+\alpha(y)\big) = 0$$

for all $x, y \in G$. This proves that (3.22) holds.

Let $\beta : G \to X$ be also a map satisfying (3.22) and (3.23) for all $x, y \in G$, where β plays the role of α. Since α and β satisfy (3.22), for all $n \in \mathbb{N}$ and $x \in G$ we have

$$\alpha(2^n x) = 2^n\alpha(x), \beta(2^n x) = 2^n\beta(x). \tag{3.32}$$

It follows from (3.18), (3.23) and (3.32) that for each $x \in G$ and $n \in \mathbb{N}$, we get

$$\begin{aligned} \delta(\alpha(x), \beta(x)) &= \delta\big(2^{-n}\alpha(2^n x), 2^{-n}\beta(2^n x)\big) \\ &\leq \xi^n\delta\big(\alpha(2^n x), \beta(2^n x)\big) \\ &\leq \xi^n\kappa\big[\delta\big(\alpha(2^n x), f(2^n x)\big) + \delta\big(f(2^n x), \beta(2^n x)\big)\big] \\ &\leq \xi^n\kappa\Big[(16)^{\frac{1}{\theta}}\Phi^{\frac{1}{\theta}}(2^n x) + (16)^{\frac{1}{\theta}}\Phi^{\frac{1}{\theta}}(2^n x)\Big] \\ &= 2\xi^n\kappa(16)^{\frac{1}{\theta}}\left(\xi^\theta\sum_{j=0}^{\infty}\xi^{\theta j}\varphi^\theta(2^j 2^n x, 2^j 2^n x)\right)^{\frac{1}{\theta}} \\ &= 2\kappa(16)^{\frac{1}{\theta}}\xi\left(\sum_{j=0}^{\infty}\xi^{\theta(j+n)}\varphi^\theta(2^{j+n}x, 2^{j+n}x)\right)^{\frac{1}{\theta}} \\ &= 2\kappa(16)^{\frac{1}{\theta}}\xi\left(\sum_{j=n}^{\infty}\xi^{\theta j}\varphi^\theta(2^j x, 2^j x)\right)^{\frac{1}{\theta}}. \end{aligned} \tag{3.33}$$

Note that from (3.19), for each $x \in G$, we have

$$\lim_{n\to\infty}\sum_{j=n}^{\infty}\xi^{\theta j}\varphi^\theta(2^j x, 2^j x) = 0. \tag{3.34}$$

Letting $n \to \infty$ in (3.33) and using (3.34), we have

$$\delta(\alpha(x), \beta(x)) = 0$$

for all $x \in G$. This proves that $\beta = \alpha$. Then α is the unique map satisfying (3.22) and (3.23) for all $x, y \in G$. □

The following example shows that Theorem 9 is a proper improvement of Theorem 6.

Example 10. Consider the groupoids G, X, the functions φ_i, $i = 1, 2$, and the map f as in Example 7. Then we have the following assertions.

(1) All assumptions of Theorem 9 are satisfied. So Theorem 9 is applicable to given G, X, and f, and we have $\alpha(x) = x$ for all $x \in G$.

(2) The series $\kappa^2\xi \sum_{j=0}^{\infty} (\kappa\xi)^j \varphi_i(2^j x, 2^j x)$ is not convergent for some $x, y \in G$. So Theorem 6 is not applicable to given G, X, and f.

Proof. (1) Based on the proof of Example 7, we only need to show that for each $x, y \in G$,

$$\xi^\theta \sum_{j=0}^{\infty} \xi^{\theta j} \varphi^\theta(2^j x, 2^j y) < \infty$$

because it implies that (3.19) and (3.20) hold.

Note that $\theta = \log_{2\kappa} 2 = \log_4 2 = \frac{1}{2}$. Hence, for each $x, y \in G$, we have

$$\sum_{j=0}^{\infty} \xi^{\theta j} \varphi^\theta(2^j x, 2^j y) \quad = \quad \sum_{j=0}^{\infty} \frac{1}{2^{\frac{j}{2}}} \int_0^1 \left|\sqrt{|x(t) + y(t)|} - \sqrt{|x(t)|} - \sqrt{|y(t)|}\right|^{\frac{1}{2}} dt.$$

Since the series $\sum_{i=0}^{\infty} \frac{1}{2^{\frac{i}{2}}}$ is convergent, we find that the series $\xi^\theta \sum_{j=0}^{\infty} \xi^{\theta j} \varphi^\theta(2^j x, 2^j y)$ is also convergent.

Moreover, we find that

$$\alpha(x) = \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{f(2^n x)}{2^n} = x$$

for all $x \in G$.

(2) It follows from the proof of Example 7 that the series $\kappa^2\xi \sum_{j=0}^{\infty} (\kappa\xi)^j \varphi_i(2^j x, 2^j x)$ is not convergent for some $x, y \in G$. So Theorem 6 is not applicable to given G, X, and f. □

4. A b-metric approach to dq-K-metric spaces and applications to the stability of functional equations in dq-K-metric spaces

The following result shows that from a given dq-K-metric, we can construct a b-metric. The result also states some relationships of convergence and completeness between the given dq-K-metric and the new b-metric.

Theorem 11. *Let (X, d, κ) be a dq-K-metric space. Put*

$$\delta(x, y) = \begin{cases} 0 & \text{if } x = y \\ \max\{d(x, y), d(y, x)\} & \text{if } x \neq y. \end{cases} \tag{4.1}$$

Then we have the following assertions.

(1) δ is a b-metric on X with the coefficient κ.

(2) *Suppose that* $\{x_n\}$ *is a sequence in* X. *Then*

(a) *The sequence* $\{x_n\}$ *is convergent to* x *in the* b*-metric space* (X, δ, κ) *if and only if*

(i) *either it is convergent to* x *in the* dq*-*K*-metric space* (X, d, κ)

(ii) *or* $x_n = x$ *for* n *large enough.*

(b) *The sequence* $\{x_n\}$ *is Cauchy in the* b*-metric space* (X, δ, κ) *if and only if*

(i) *either it is Cauchy in the* dq*-*K*-metric space* (X, d, κ)

(ii) *or* $x_n = x_m$ *for* n, m *large enough.*

(3) *If the* dq*-*K*-metric space* (X, d, κ) *is complete, then the* b*-metric space* (X, δ, κ) *is also complete.*

Proof. (1) Let $x, y, z \in X$. We find that $\delta(x, y) \geq 0$, $\delta(x, y) = 0$ if and only if $x = y$, and $\delta(x, y) = \delta(y, x)$.

If x, y, z are not point-wise distinct, then

$$\delta(x, z) \leq \delta(x, y) + \delta(y, z) \leq \kappa[\delta(x, y) + \delta(y, z)].$$

If x, y, z are point-wise distinct, then we have

$$\begin{aligned} \delta(x, z) &= \max\{d(x, z), d(z, x)\} \\ &\leq \max\{\kappa[d(x, y) + d(y, z)], \kappa[d(z, y) + d(y, x)]\} \\ &\leq \kappa[\max\{d(x, y), d(y, x)\} + \max\{d(y, z), d(z, y)\}] \\ &= \kappa[\delta(x, y) + \delta(y, z)]. \end{aligned} \tag{4.2}$$

By the above relation, δ is a b-metric on X with the coefficient κ.

(2a) *Necessity.* Since $\lim_{n\to\infty} x_n = x$ in the b-metric space (X, δ, κ), we find that $\lim_{n\to\infty} \delta(x_n, x) = 0$. Therefore, we get the following assertion:

(i) either $x_n = x$ for n large enough

(ii) or there exists a subsequence $\{x_{k_n}\}$ such that all x_{k_n}, x_n and x are pointwise distinct. Then for all n we have

$$d(x_n, x) \leq \kappa[d(x_n, x_{k_n}) + d(x_{k_n}, x)] \leq \kappa[\delta(x_n, x_{k_n}) + \delta(x_{k_n}, x)]$$

and

$$d(x, x_n) \leq \kappa[d(x, x_n) + d(x_n, x_{k_n})] \leq \kappa[\delta(x, x_n) + \delta(x_n, x_{k_n})].$$

Then $\max\{d(x_n, x), d(x, x_n)\} \leq \kappa[\delta(x, x_n) + \delta(x_n, x_{k_n})]$ for all $n \in \mathbb{N}$. We find that

$$\delta(x_{k_n}, x) = \max\{d(x_{k_n}, x), d(x, x_{k_n})\}$$

for all $n \in \mathbb{N}$. This proves that $\lim_{n\to\infty} \delta(x_{k_n}, x) = 0$ in (X, δ, κ) if and only if

$$\lim_{n\to\infty} \max\{d(x_{k_n}, x), d(x, x_{k_n})\} = 0,$$

that is, $\lim_{n\to\infty} x_{k_n} = x$ in the dq-K-metric space (X, d, κ). Note that in this case we have $d(x, x) = 0$ since

$$d(x, x) \leq \kappa[d(x, x_{k_n}) + d(x_{k_n}, x)]$$

for all $n \in \mathbb{N}$. Therefore, $\lim_{n\to\infty} \max\{d(x_n, x), d(x, x_n)\} = 0$, that is, $\lim_{n\to\infty} x_n = x$ in the dq-K-metric space (X, d, κ).

Sufficiency. If $x_n = x$ for n large enough, then $\lim_{n\to\infty} x_n = x$ in the b-metric space (X, δ, κ). If $\lim_{n\to\infty} x_n = x$ in the dq-K-metric space (X, d, κ), then

$$\lim_{n\to\infty} \max\{d(x_n, x), d(x, x_n)\} = 0.$$

Note that for each $n \in \mathbb{N}$, we have

$$\delta(x_n, x) \le \max\{d(x_n, x), d(x, x_n)\}.$$

This proves that $\lim_{n\to\infty} \delta(x_n, x) = 0$, that is, $\lim_{n\to\infty} x_n = x$ in the b-metric space (X, δ, κ).

(2b) *Necessity.* Since $\{x_n\}$ is a Cauchy sequence in the b-metric space (X, δ, κ), we have $\lim_{n,m\to\infty} \delta(x_n, x_m) = 0$. So we get the following assertion:

(i) either $x_n = x_m$ for n, m large enough
(ii) or the exists a subsequence $\{x_{k_n}\}$ of $\{x_n\}$ such that all x_{k_n} are point-wise distinct and $k_n > n$ for all n. Then for all each n, m, we have

$$\begin{aligned} d(x_n, x_m) &\le \kappa[d(x_n, x_{k_n}) + d(x_{k_n}, x_m)] \\ &\le \kappa d(x_n, x_{k_n}) + \kappa^2[d(x_{k_n}, x_{k_m}) + d(x_{k_m}, x_m)] \\ &\le \kappa\delta(x_n, x_{k_n}) + \kappa^2[\delta(x_{k_n}, x_{k_m}) + \delta(x_{k_m}, x_m)]. \end{aligned} \tag{4.3}$$

Note that

$$\lim_{n\to\infty} \delta(x_n, x_{k_n}) = \lim_{n,m\to\infty} \delta(x_{k_n}, x_{k_m}) = 0. \tag{4.4}$$

It follows from (4.3) and (4.4) that $\lim_{n,m\to\infty} d(x_n, x_m) = 0$. Then $\{x_n\}$ is Cauchy in dq-K-metric space (X, d, κ).

Sufficiency. If $x_n = x_m$ for n, m large enough, then $\{x_n\}$ is a Cauchy sequence in the b-metric space (X, δ, κ). If $\{x_n\}$ is a Cauchy sequence in the dq-K-metric space (X, d, κ). Then $\lim_{n,m\to\infty} d(x_n, x_m) = 0$. Note that

$$\delta(x_n, x_m) \le \max\{d(x_n, x_m), d(x_m, x_n)\}$$

for all n, m. So $\lim_{n,m\to\infty} \delta(x_n, x_m) = 0$. This proves that $\{x_n\}$ is a Cauchy sequence in the b-metric space (X, δ, κ).

(3). Suppose that $\{x_n\}$ is a Cauchy sequence in the b-metric space (X, δ, κ). Then $\lim_{n,m\to\infty} \delta(x_n, x_m) = 0$.

If $\{x_{k_n}\}$ is a subsequence of $\{x_n\}$ and $x_{k_n} = x$ for all k_n, then $\delta(x_{k_n}, x) = 0$ for all n. This proves that $\lim_{n\to\infty} x_{k_n} = x$ in (X, δ, κ).

If $\{x_{k_n}\}$ is a subsequence of $\{x_n\}$ and all x_{k_n}'s are point-wise distinct, then

$$\delta(x_{k_n}, x_{k_m}) = \max\{d(x_{k_n}, x_{k_m}), d(x_{k_m}, x_{k_n})\}.$$

Since $\lim_{n,m\to\infty} \delta(x_{k_n}, x_{k_m}) = 0$, we have

$$\lim_{n,m\to\infty} \max\{d(x_{k_n}, x_{k_m}), d(x_{k_m}, x_{k_n})\} = 0.$$

This proves that $\lim_{k\to\infty} \sup_{k_n,k_m \geq k} d(x_{k_n}, x_{k_m}) = 0$. So $\{x_{k_n}\}$ is a Cauchy sequence in the dq-K-metric space (X, d, κ). Since (X, d, κ) is complete, there exists $\lim_{n\to\infty} x_{k_n} = y$ in (X, d, κ). It follows from (2a) that $\lim_{n\to\infty} x_{k_n} = y$ in (X, δ, κ).

Note that $\lim_{n,m\to\infty} \delta(x_n, x_m) = 0$. So if there exist x and y as the above, then $x = y$. This proves that $\lim_{n\to\infty} x_n = x$ in the b-metric space (X, δ, κ). So the b-metric space (X, δ, κ) is complete. □

In the case $\kappa = 1$, we have the following corollary which shows that for a given dq-metric, we can construct a metric.

Corollary 12. *Let (X, d) be a dq-metric space. Put*

$$\delta(x, y) = \begin{cases} 0 & \text{if } x = y \\ \max\{d(x, y), d(y, x)\} & \text{if } x \neq y. \end{cases} \tag{4.5}$$

Then we have the following assertions.

(1) *δ is a metric on X.*
(2) *Suppose that $\{x_n\}$ is a sequence in X. Then*
   (a) *The sequence $\{x_n\}$ is convergent to x in the metric space (X, δ) if and only if*
      (i) *either it is convergent to x in the dq-metric space (X, d)*
      (ii) *or $x_n = x$ for n large enough.*
   (b) *The sequence $\{x_n\}$ is Cauchy in the metric space (X, δ) if and only if*
      (i) *either it is Cauchy in the dq-metric space (X, d)*
      (ii) *or $x_n = x_m$ for n, m large enough.*
(3) *If the dq-metric space (X, d) is complete, then the metric space (X, δ) is complete.*

For the stability results of functional equations in dq-metric spaces, see for example [1], [6].

Next, we shall apply Theorem 11 to prove stability results of the generalized radical functional equation in dq-K-metric spaces. From Theorem 11 and Theorem 8, we get Corollary 13, which is a very similar result to Theorem 6. The differences between them are only as follows.

(1) We need not to assume the condition (2.4) in Theorem 6.
(2) The approximation (4.12) is a combination of the approximations (2.5) and (2.6) in Theorem 6.

Corollary 13. *Suppose that the following conditions hold.*

(1) *$(G, +)$ and $(X, +)$ are two square symmetric groupoids, where $(X, +)$ is uniquely divisible by 2.*
(2) *(X, d, κ) is a complete dq-K-metric space such that for some $\xi > 0$ and all $x, y \in X$,*
$$d(2^{-1}x, 2^{-1}y) \leq \xi d(x, y). \tag{4.6}$$
(3) *The operation $+$ is continuous with respect to the dq-K-metric d.*

(4) There exist $\varphi_i : G \times G \to [0,\infty)$, $i = 1, 2$, satisfying for all $x, y \in G$,

$$\Phi_i(x) := \kappa^2\xi\sum_{j=0}^{\infty}(\kappa\xi)^j\varphi_i(2^jx, 2^jx) \quad < \quad \infty \tag{4.7}$$

and

$$\lim_{n\to\infty}\xi^n\varphi(2^nx, 2^ny) \quad = \quad 0. \tag{4.8}$$

(5) $f : G \to X$ is a map such that for all $x, y \in G$,

$$d\big(f(x+y), f(x)+f(y)\big) \quad \le \quad \varphi_1(x,y), \tag{4.9}$$

$$d\big(f(x)+f(y), f(x+y)\big) \quad \le \quad \varphi_2(x,y). \tag{4.10}$$

Then there exists a unique map $\alpha : G \to X$ such that for all $x, y \in G$,

$$\alpha(x+y) \quad = \quad \alpha(x)+\alpha(y) \tag{4.11}$$

and for all $x \in G$ with $f(x) \neq \alpha(x)$,

$$\max\{d\big(\alpha(x), f(x)\big), d\big(f(x), \alpha(x)\big)\} \quad \le \quad \Phi_1(x)+\Phi_2(x). \tag{4.12}$$

Moreover, for all $x \in G$,

$$\alpha(x) = \lim_{n\to\infty} f_n(x). \tag{4.13}$$

Proof. Let δ be the b-metric defined in Theorem 11. If $x = y$, then $\delta(x, y) = 0$. If $x \neq y$, then from (4.6), we have

$$\begin{aligned}\delta(2^{-1}x, 2^{-1}y) &= \max\{d(2^{-1}x, 2^{-1}y), d(2^{-1}y, 2^{-1}x)\}\\ &\le \max\{\xi d(x,y), \xi d(y,x)\}\\ &= \xi\delta(x,y)\end{aligned}$$

for all $x, y \in G$. This proves that (3.1) holds.

It follows from Theorem 11 in (3) that (X, δ, κ) is a complete b-metric space. For all $x, y \in X$, put $\varphi(x, y) = \varphi_1(x, y) + \varphi_2(x, y)$. From (4.7), for each $x \in G$, we get

$$\Phi(x) := \kappa^2\xi\sum_{j=0}^{\infty}(\kappa\xi)^j\varphi(2^jx, 2^jx) = \Phi_1(x)+\Phi_2(x) < \infty.$$

This proves that (3.2) holds.

It follows from (4.9) and (4.10) that

$$\begin{aligned}\delta(f(x+y), f(x)+f(y)) &\le \max\{d(f(x+y), f(x)+f(y)), d(f(x)+f(y), f(x+y))\}\\ &\le \max\{\varphi_1(x,y), \varphi_2(x,y)\}\\ &\le \varphi(x,y)\end{aligned}$$

for all $x, y \in G$. This proves that (3.4) holds.

So all assumptions of Theorem 8 hold. Then the exists a unique $\alpha : G \to X$ such that (3.5), (3.6) and (3.7) hold. Then (4.11) holds. It follows from (3.6) and (4.1) that (4.12) also hold. □

Remark 14. We can apply the mentioned approach with Corollary 12 to certain results in [1], [6].

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References

[1] J. Brzdek, E.-s. El-hady, Z. Leśniak, *Fixed-point theorem in classes of function with values in a dq-metric space*, J. Fixed Point Theory Appl., **20**(2018), no. 4, 1–16.

[2] J. Brzdek, E.-s. El-hady, J. Schwaiger, *Investigations on the Hyers–Ulam stability of generalized radical functional equations*, Aequat. Math., **94**(2020), 575–593.

[3] S. Czerwik, *Nonlinear set-valued contraction mappings in b-metric spaces*, Atti Sem. Math. Fis. Univ. Modena, **46**(1998), 263–276.

[4] N. V. Dung, T. V. An, V. T. L. Hang, *Remarks on Frink's metrization technique and applications*, Fixed Point Theory, **20**(2019), no. 1, 157–176.

[5] N. V. Dung, V. T. L. Hang, *On two questions of A. Petruşel and G. Petruşel in b-metric fixed point theory, J. Fixed Point Theory Appl.*, **20**(2018), no. 3, 1–16.

[6] B. Hejmej, *Stability of functional equations in dislocated quasi-metric spaces*, Ann. Math. Sil., **32**(2018), 215–225.

[7] L. Maligranda, *Tosio Aoki (1910-1989)*, In: International symposium on Banach and function spaces: 14/09/2006-17/09/2006, pages 1–23, Yokohama, 2008. Yokohama Publishers.

[8] M. Paluszyński, K. Stempak, *On quasi-metric and metric spaces*, Proc. Amer. Math. Soc., **137**(2009), no. 12, 4307–4312.

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