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EXISTENCE OF POSITIVE SOLUTIONS TO SINGULAR FRACTIONAL DIFFERENTIAL EQUATIONS WITH GENERALIZED LAPLACIAN

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Abstract. In this paper, we study the boundary value problem for a singular fractional differential equation with both generalized Laplacian and positive parameter. Based on the well-known results of fixed point index on cones due to Guo-Krasnoselskii and Leggett-Williams, we establish some sufficient conditions for the existence of at least one, two and three positive solutions, respectively. Meanwhile, some corresponding examples are also presented to illustrate the main results.
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1. Introduction

This paper deals with the existence of positive solutions for the boundary value problem of a fractional differential equation of the form

$$\begin{cases} D^{\beta}_{0+}(\varphi(D^{\alpha}_{0+}u(t))) = \lambda h(t)f(u(t)), \quad t \in (0,1), \\ u(0) = u'(0) = u'(1) = 0, \quad \varphi(D^{\alpha}_{0+}u(0)) = (\varphi(D^{\alpha}_{0+}u(1)))' = 0, \end{cases} \tag{1.1}$$

where $2 < \alpha \leq 3$, $1 < \beta \leq 2$, $\lambda > 0$, $h \not\equiv 0$ on any subinterval in $(0,1)$, D^{ς}_{0+} $(\varsigma = \alpha, \beta)$ denotes the standard Riemann-Liouville derivative of order ς throughout this paper. We also give some other assumptions as follows.

(*A*) $\varphi : \mathbb{R} \to \mathbb{R}$ is an odd increasing homeomorphism and there exist an increasing homeomorphism ψ from $(0, +\infty)$ onto $(0, +\infty)$ and a function γ from $(0, +\infty)$ into $(0, +\infty)$ such that

$$\psi(\sigma) \leq \frac{\varphi(\sigma x)}{\varphi(x)} \leq \gamma(\sigma), \quad \text{for all} \quad \sigma > 0, x \in \mathbb{R}/\{0\}.$$

(*H*) $h : (0, 1) \to [0, +\infty)$ is locally integrable and satisfies

$$\int_0^1 t^{\beta-1}(1-t)^{\beta-2}h(t)dt < +\infty.$$

(F_1) $f : [0, +\infty) \to [0, +\infty)$ is continuous.

(F_2) $f(u) > 0$, for all $u > 0$.

From condition (A), we see that φ covers two special cases $\varphi(x) = x$ and $\varphi(x) = |x|^{p-2}x$ $(p > 1)$. Here we call φ a generalized Laplacian operator.

The boundary value problems of nonlinear ordinary differential equations usually appear in various fields such as mathematics, physics and mechanics. In the past decades, the study of existence criteria for the boundary value problems of different kinds of integer-order differential equations (or systems) have attracted much attention (see [1, 2, 3, 5, 13, 15, 19, 20, 21, 22] and the references therein). For instance, Bai and Chen [3] applied the Leggett-Williams fixed point theorem to prove the existence of at least three solutions for the Dirichlet boundary value problem of the generalized Laplacian equation

$$\begin{cases} -\varphi(u'(t))' = \lambda h(t)f(u(t)), & t \in (0,1), \\ u(0) = u(1) = 0, \end{cases} \tag{1.2}$$

where $\lambda > 0$, $h \in C((0,1),[0,+\infty))$ with $0 < \int_0^1 h(t)dt < +\infty$, $f : [0,+\infty) \to [0,+\infty))$ is continuous and

$(A1)$ $\varphi : \mathbb{R} \to \mathbb{R}$ is an odd increasing homeomorphism and there exist two increasing homeomorphisms ψ_1 and ψ_2 of $(0, +\infty)$ onto $(0, +\infty)$ such that

$$\psi_1(\sigma)\varphi(x) \le \varphi(\sigma x) \le \psi_2(\sigma)\varphi(x), \text{ for all } \sigma \text{ and } x > 0.$$

We remark that the three solutions obtained in [3] may be all positive solutions, or there may be one nonnegative solution and two positive solutions.

Comparing with integer-order differential equations, fractional differential equations can be more accurate to describe phenomena in scientific areas such as fluid flows, biological chemical physics, electrical networks, control theory. Thus, more and more researchers have been interested in studying the boundary value problems for fractional differential equations. Specially, the topic about the existence of solutions for the boundary value problems of fractional differential equations has been widely investigated (see [4, 6, 8, 10, 11, 18, 26, 25, 24, 23, 27] and the references therein). For example, EI-Shahed [8] considered the boundary value problem of a fractional differential equation with positive parameter

$$\begin{cases} -D_{0+}^{\alpha} u(t) = \lambda h(t)f(u(t)), & t \in (0,1), \\ u(0) = u'(0) = u'(1) = 0, \end{cases} \tag{1.3}$$

where $2 < \alpha \le 3$, $h \in C((0,1),[0,+\infty))$ with $0 < \int_0^1 h(t)dt < +\infty$ and $f \in C([0,+\infty),[0,+\infty))$. By the Krasnoselskii fixed point theorem of cone, EI-Shahed [8] proved the existence and nonexistence of positive solutions for parameter λ belonging to some explicit intervals. Specially, when $h(t) \equiv 1$ and $f \in C([0,+\infty),(0,+\infty))$, Zhao *et al.* [26] derived the eigenvalue intervals that guarantee the existence of at least one or two positive solutions to (1.3). Recently, Zhang and Zhong [25] proved the existence of three positive solutions for a higher-order fractional differential equation with integral condition

$$\begin{cases} -D_{0+}^{\alpha}u(t)) = f(t,u(t)), \quad t\in(0,1),\\ u(0)=u'(0)=\cdots=u^{(n-2)}(0)=0,\\ D_{0+}^{\beta}u(1)=\lambda\int_0^{\eta}h(t)D_{0+}^{\beta}u(t)dt, \end{cases} \tag{1.4}$$

where $n-1<\alpha\le n$, $n\ge 3$, $\beta\ge 1$, $\alpha-\beta-1>0$, $0<\eta\le 1$, $0\le\lambda\int_0^{\eta}h(t)t^{\alpha-\beta-1}dt<1$. Note that the nonlinearity may have singularities at $t=0,1$ and $u=0$. Their proof was completed by the Leggett-Williams and Krasnoselskii fixed point theorems with aid of different height functions of the nonlinearity.

Since Leibenson [17] introduced the differential equations with p-Laplacian operator, the boundary value problems of fractional differential equations with p-Laplacian or generalized Laplacian have also been extensively studied because of their wild applications in various fields of engineering and science. There have been considerable works dealing with the existence of solutions (see [6, 10, 11, 18, 24, 23] and the references therein). For instance, by the fixed point theorem on cones, Chai [6] obtained existence and multiplicity results of positive solutions for the following boundary value problem of fractional differential equation with p-Laplacian

$$\begin{cases} -D_{0+}^{\beta}(\varphi_p(D_{0+}^{\alpha}u(t)))=f(t,u(t)), \quad t\in(0,1),\\ u(0)=0, \quad u(1)+\sigma D_{0+}^{\gamma}u(1)=0, \quad D_{0+}^{\alpha}u(0)=0, \end{cases} \tag{1.5}$$

where $1<\alpha\le 2$, $0<\beta\le 1$, $0<\gamma\le 1$, $\alpha-\gamma-1\ge 0$, $\sigma>0$, $\varphi_p(x)=|x|^{p-2}x, p>1$ and $f\in C([0,1]\times[0,+\infty),[0,+\infty))$. Later, Han *et al.* [10] considered the boundary value problem of a fractional differential equation with generalized Laplacian of the form

$$\begin{cases} D_{0+}^{\beta}(\varphi(D_{0+}^{\alpha}u(t)))=\lambda f(u(t)), \quad t\in(0,1),\\ u(0)=u'(0)=u'(1)=0, \quad \varphi(D_{0+}^{\alpha}u(0))=(\varphi(D_{0+}^{\alpha}u(1)))'=0, \end{cases} \tag{1.6}$$

where $2<\alpha\le 3$, $1<\beta\le 2$, $\lambda>0$, $f:(0,+\infty)\to(0,+\infty)$ is continuous and φ satisfies condition $(A1)$ that appeared in [3]. By combining the Guo-Krasnoselskii fixed point theorem on cones with the properties of Green function, Han *et al.* [10] showed that (1.6) has at least one or two positive solutions in terms of different eigenvalue intervals.

Inspired by the above works, we aim to study the boundary value problem of a class of fractional differential equations with both generalized Laplacian and singular weight like (1.1). To the best of our knowledge, there are few works dealing with the existence of positive solutions for this kind of fractional boundary value problems. It is worth noticing that condition (A) can be more general than condition $(A1)$, and the weight function h of model (1.1) may permit a singularity at $t=0$ and/or $t=1$ (see Example 3.4 in Section 3). Obviously, the model (1.6) considered in [10] is a special case of our model (1.1). Under different assumptions on nonlinearity f, we will find explicit eigenvalue intervals that guarantee the existence of at least one, two and three positive solutions to (1.1), respectively. Our main results not only extend the existence results in [10], but also seems to be initially discussing the existence of three positive solutions for fractional differential equations with parameter λ. For the proofs of main results, we will firstly establish the solution operator to (1.1), and then make use of the well-known results of fixed point index on cones due to Guo-Krasnoselskii and Leggett-Williams. But, the concurrent appearance of singularity and parameter λ will bring us difficulties to define a solution operator of (1.1) and make priori estimations on possible positive solutions under different assumptions on f. In particular, the process of solution operator set-up in [10] is not suitable for our model (1.1) (see Lemma 2.2 and Lemma 2.4 in [10]). For example, if $h(t) = t^{-\frac{4}{3}}$, then we can easily check that the Riemann-Liouville fractional integral $I_{0+}^{\beta} h(t)$ is not well-defined (see Definition 2.4 in Section 2). Thus, we will try other method to establish the solution operator of (1.1) (see Lemma 2.8 in Section 2 for details).

The outline of this paper is organized as follows. In Section 2, we give some preliminaries which will be useful in this paper. Then we present some sufficient conditions on the existence of positive solutions to (1.1), the detailed proofs and corresponding examples in Section 3.

2. Preliminaries

In this section, we mainly introduce the well-known fixed point theorem on cones, necessary definitions and preliminary lemmas from the fractional calculus theory. For the sake of convenience, we denote K as a cone of the Banach space $(E, \|\cdot\|)$ and make the following notations throughout this paper.

$$K_r = \{u \in K \mid \|u\| < r\},$$
$$\partial K_r = \{u \in K \mid \|u\| = r\},$$
$$K(\alpha, b, d) = \{u \in K \mid b \leq \alpha(u),\ \|u\| \leq d\},$$
$$\mathring{K}(\alpha, b, d) = \{u \in K \mid b < \alpha(u),\ \|u\| \leq d\},$$

where r, b, d are positive constants and α is a continuous functional.

Lemma 2.1(**Guo-Krasnoselskii** [7, 9, 14]) Let E be a Banach space and let K be a cone in E. Assume that $T : \overline{K_r} \to K$ is completely continuous such that $Tu \neq u$ for $u \in \partial K_r$.

(i) If $\|Tu\| \geq \|u\|$ for $u \in \partial K_r$, then
$$i(T, K_r, K) = 0.$$

(ii) If $\|Tu\| \leq \|u\|$ for $u \in \partial K_r$, then
$$i(T, K_r, K) = 1.$$

Definition 2.2([16]) A continuous functional $\alpha : K \to [0, +\infty)$ is called a concave positive functional on a cone K if α satisfies

$$\alpha(\kappa x + (1-\kappa)y) \geq \kappa\alpha(x) + (1-\kappa)\alpha(y), \text{ for all } x, y \in K, 0 \leq \kappa \leq 1.$$

Lemma 2.3(**Leggett-Williams** [16]) Let K be a cone in a real Banach space E and α be a concave positive functional on K such that $\alpha(x) \leq \|x\|$ for all $x \in \overline{K_c}$.

Suppose $T : \overline{K_c} \to \overline{K_c}$ is completely continuous and there exist numbers a, b and d, with $0 < d < a < b \leq c$, satisfying the following conditions:

(i) $\{u \in K(\alpha, a, b) : \alpha(u) > a\} \neq \emptyset$ and $\alpha(Tu) > a$ if $u \in K(\alpha, a, b)$;

(ii) $\|Tu\| < d$ if $u \in \overline{K_d}$;

(iii) $\alpha(Tu) > a$ for all $u \in K(\alpha, a, c)$ with $\|Tu\| > b$.

Then

$$i(T, K_d, \overline{K_c}) = 1,$$

$$i(T, \mathring{K}(\alpha, a, c), \overline{K_c}) = 1,$$

$$i(T, \overline{K_c} \backslash (\overline{K_d} \cup K(\alpha, a, c)), \overline{K_c}) = -1.$$

Furthermore, T has at least three fixed points u_1, u_2, u_3 in $\overline{K_c}$ such that $\|u_1\| < d$, $a < \alpha(u_2)$, $d < \|u_3\|$ with $\alpha(u_3) < a$.

Definition 2.4([12]) The Riemann-Liouville fractional integral of order $\alpha > 0$ of a function $y : (0, +\infty) \to \mathbb{R}$ is given by

$$I_{0+}^{\alpha} y(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s) ds,$$

provided the right side is pointwise defined on $(0, +\infty)$.

Definition 2.5([12]) The Riemann-Liouville fractional derivative of order $\alpha > 0$ of a continuous function $y : (0, +\infty) \to \mathbb{R}$ is given by

$$D_{0+}^{\alpha} y(t) = \frac{1}{\Gamma(n-\alpha)} (\frac{d}{dt})^n \int_0^t \frac{y(s)}{(t-s)^{\alpha-n+1}} ds,$$

where n is the smallest integer greater than or equal to α, provided that the right side is pointwise defined on $(0, +\infty)$.

Remark 2.6([4]) Note for $\lambda > -1$,

$$D_{0+}^{\alpha} t^{\lambda} = \frac{\Gamma(\lambda+1)}{\Gamma(\lambda-\alpha+1)} t^{\lambda-\alpha},$$

In particular, $D_{0+}^{\alpha} t^{\alpha-m} = 0$, $m = 1, 2, \cdots, N$, where N is the smallest integer greater than or equal to α.

Lemma 2.7([8]) Let $2 < \alpha \leq 3$ and $g \in C[0, 1]$. Then the following boundary value problem of fractional differential equation

$$\begin{cases} D_{0+}^{\alpha} w(t) + g(t) = 0, \quad 0 < t < 1, \\ w(0) = w'(0) = w'(1) = 0, \end{cases}$$

has a unique solution

$$w(t) = \int_0^1 G(t, s) g(s) ds,$$

where

$$G(t,s)=\begin{cases}\frac{t^{\alpha-1}(1-s)^{\alpha-2}}{\Gamma(\alpha)}, & 0\le t\le s\le 1,\\ \frac{t^{\alpha-1}(1-s)^{\alpha-2}-(t-s)^{\alpha-1}}{\Gamma(\alpha)}, & 0\le s\le t\le 1.\end{cases}\tag{2.1}$$

Lemma 2.8 Let $2<\alpha\le 3$, $1<\beta\le 2$ and g satisfies (H). Then the solution of the following boundary value problem

$$\begin{cases}D_{0^+}^{\beta}(\varphi(D_{0^+}^{\alpha}w(t)))=g(t), \quad t\in(0,1),\\ w(0)=w'(0)=w'(1)=0, \quad \varphi(D_{0^+}^{\alpha}w(0))=(\varphi(D_{0^+}^{\alpha}w(1)))'=0,\end{cases}\tag{2.2}$$

can be uniquely represented by

$$w(t)=\int_0^1 G(t,s)\varphi^{-1}\left(\int_0^1 H(s,\tau)g(\tau)d\tau\right)ds,$$

where $G(t,s)$ is the same as (2.1) and $H(s,\tau)$ is defined by

$$H(s,\tau)=\begin{cases}\frac{s^{\beta-1}(1-\tau)^{\beta-2}}{\Gamma(\beta)}, & 0\le s\le \tau\le 1,\\ \frac{s^{\beta-1}(1-\tau)^{\beta-2}-(s-\tau)^{\beta-1}}{\Gamma(\beta)}, & 0\le \tau\le s\le 1.\end{cases}\tag{2.3}$$

Proof. From condition (H), we see that g may be singular at $t=0$ and/or $t=1$. In order to deduce the solution operator of (2.2), we will divide the process into two cases as follows.

Case 1: By integrating both sides of the first equation in (2.2) from s to $\frac{1}{2}$ for $s\in(0,\frac{1}{2}]$, we have

$$\int_s^{\frac{1}{2}} D_{0^+}^{\beta}(\varphi(D_{0^+}^{\alpha}w(\tau)))d\tau=\int_s^{\frac{1}{2}} g(\tau)d\tau.\tag{2.4}$$

It follows from Definition 2.4 and Definition 2.5 that

$$D_{0^+}^{\beta}y(s)=(\frac{d}{ds})^2 I_{0^+}^{2-\beta}y(s).$$

By (2.4), we deduce

$$c_1-\frac{d}{ds}I_{0^+}^{2-\beta}\varphi(D_{0^+}^{\alpha}w(s))=\int_s^{\frac{1}{2}} g(\tau)d\tau,\tag{2.5}$$

where $c_1=\frac{d}{ds}I_{0^+}^{2-\beta}\varphi(D_{0^+}^{\alpha}w(\frac{1}{2}))$. We continue integrating both sides of (2.5) from t to $\frac{1}{2}$ for $t\in[0,\frac{1}{2}]$ and then obtain

$$\begin{aligned}c_1(\frac{1}{2}-t)-c_2+I_{0^+}^{2-\beta}\varphi(D_{0^+}^{\alpha}w(t))&=\int_t^{\frac{1}{2}}\int_s^{\frac{1}{2}} g(\tau)d\tau ds\\ &=\int_t^{\frac{1}{2}}\int_t^{\tau} g(\tau)dsd\tau\\ &=\int_t^{\frac{1}{2}}(\tau-t)g(\tau)d\tau.\end{aligned}$$

i.e.

$$c_1(\frac{1}{2}-t)-c_2+I_{0^+}^{2-\beta}\varphi(D_{0^+}^{\alpha}w(t))=\int_t^{\frac{1}{2}}(\tau-t)g(\tau)d\tau, \tag{2.6}$$

where $c_2=I_{0^+}^{2-\beta}\varphi(D_{0^+}^{\alpha}w(\frac{1}{2}))$. Taking $D_{0^+}^{2-\beta}$ on both sides of (2.6) and using Remark 2.6, we obtain

$$\varphi(D_{0^+}^{\alpha}w(t))-\frac{c_1}{\Gamma(\beta)}t^{\beta-1}-\frac{-\frac{1}{2}c_1+c_2}{\Gamma(\beta-1)}t^{\beta-2}=D_{0^+}^{2-\beta}\int_t^{\frac{1}{2}}(\tau-t)g(\tau)d\tau. \tag{2.7}$$

From Definition 2.5, we can calculate the right hand of (2.7). Exactly,

$$\begin{aligned}
&D_{0^+}^{2-\beta}\int_t^{\frac{1}{2}}(\tau-t)g(\tau)d\tau\\
=&\frac{1}{\Gamma(\beta-1)}\frac{d}{dt}\int_0^t(t-s)^{\beta-2}\int_s^{\frac{1}{2}}(\tau-s)g(\tau)d\tau ds\\
=&\frac{1}{\Gamma(\beta-1)}\frac{d}{dt}\int_0^t\int_s^{\frac{1}{2}}(t-s)^{\beta-2}(\tau-s)g(\tau)d\tau ds\\
=&\frac{1}{\Gamma(\beta-1)}\frac{d}{dt}\left[\int_0^t\int_0^{\tau}(t-s)^{\beta-2}(\tau-s)g(\tau)dsd\tau\right.\\
&\qquad\left.+\int_t^{\frac{1}{2}}\int_0^t(t-s)^{\beta-2}(\tau-s)g(\tau)dsd\tau\right]\\
=&\frac{1}{\Gamma(\beta-1)}\frac{d}{dt}\left[\int_0^t\left(\frac{\tau t^{\beta-1}}{\beta-1}+\frac{(t-\tau)^{\beta}}{(\beta-1)\beta}-\frac{t^{\beta}}{(\beta-1)\beta}\right)g(\tau)d\tau\right.\\
&\qquad\left.+\int_t^{\frac{1}{2}}\left(\frac{\tau t^{\beta-1}}{\beta-1}-\frac{t^{\beta}}{(\beta-1)\beta}\right)g(\tau)d\tau\right]\\
=&\int_0^t\frac{(t-\tau)^{\beta-1}-t^{\beta-1}}{\Gamma(\beta)}g(\tau)d\tau-\int_t^{\frac{1}{2}}\frac{t^{\beta-1}}{\Gamma(\beta)}g(\tau)d\tau+\int_0^{\frac{1}{2}}\frac{\tau t^{\beta-2}}{\Gamma(\beta-1)}g(\tau)d\tau.
\end{aligned}$$

Hence, for $t\in[0,\frac{1}{2}]$, we get

$$\begin{aligned}
&\varphi(D_{0^+}^{\alpha}w(t))-\frac{c_1}{\Gamma(\beta)}t^{\beta-1}-\frac{-\frac{1}{2}c_1+c_2}{\Gamma(\beta-1)}t^{\beta-2}\\
=&\int_0^t\frac{(t-\tau)^{\beta-1}-t^{\beta-1}}{\Gamma(\beta)}g(\tau)d\tau-\int_t^{\frac{1}{2}}\frac{t^{\beta-1}}{\Gamma(\beta)}g(\tau)d\tau+\int_0^{\frac{1}{2}}\frac{\tau t^{\beta-2}}{\Gamma(\beta-1)}g(\tau)d\tau.
\end{aligned} \tag{2.8}$$

Case 2: By integrating both sides of the first equation in (2.2) from $\frac{1}{2}$ to s for $s\in[\frac{1}{2},1)$, we get

$$\int_{\frac{1}{2}}^s D_{0^+}^{\beta}(\varphi(D_{0^+}^{\alpha}w(\tau)))d\tau=\int_{\frac{1}{2}}^s g(\tau)d\tau. \tag{2.9}$$

Based on the similar argument in Case 1 with aid of (2.9), we have

$$-c_1+\frac{d}{ds}I_{0^+}^{2-\beta}\varphi(D_{0^+}^{\alpha}w(s))=\int_{\frac{1}{2}}^s g(\tau)d\tau, \tag{2.10}$$

where $c_1 = \frac{d}{ds} I_{0^+}^{2-\beta} \varphi(D_{0^+}^{\alpha} w(\frac{1}{2}))$. Integrating on both sides of (2.10) from $\frac{1}{2}$ to t for $t \in [\frac{1}{2}, 1]$, we can deduce

$$\begin{aligned}
-c_1(t-\frac{1}{2}) - c_2 + I_{0^+}^{2-\beta}\varphi(D_{0^+}^{\alpha} w(t)) &= \int_{\frac{1}{2}}^{t}\int_{\frac{1}{2}}^{s} g(\tau)d\tau ds \\
&= \int_{\frac{1}{2}}^{t}\int_{\tau}^{t} g(\tau)ds d\tau \\
&= \int_{\frac{1}{2}}^{t} (t-\tau)g(\tau)d\tau.
\end{aligned}$$

i.e.

$$-c_1(t-\frac{1}{2}) - c_2 + I_{0^+}^{2-\beta}\varphi(D_{0^+}^{\alpha} w(t)) = \int_{\frac{1}{2}}^{t} (t-\tau)g(\tau)d\tau, \tag{2.11}$$

where $c_2 = I_{0^+}^{2-\beta}\varphi(D_{0^+}^{\alpha} w(\frac{1}{2}))$. Taking $D_{0^+}^{2-\beta}$ on both sides of (2.11) and using Remark 2.6, we obtain

$$\varphi(D_{0^+}^{\alpha} w(t)) - \frac{c_1}{\Gamma(\beta)} t^{\beta-1} - \frac{-\frac{1}{2}c_1 + c_2}{\Gamma(\beta-1)} t^{\beta-2} = D_{0^+}^{2-\beta}\int_{\frac{1}{2}}^{t} (t-\tau)g(\tau)d\tau. \tag{2.12}$$

By Definition 2.5, we can calculate the right hand of (2.12). Exactly,

$$\begin{aligned}
& D_{0^+}^{2-\beta}\int_{\frac{1}{2}}^{t} (t-\tau)g(\tau)d\tau \\
&= \frac{1}{\Gamma(\beta-1)}\frac{d}{dt}\int_0^t (t-s)^{\beta-2}\int_{\frac{1}{2}}^{s}(s-\tau)g(\tau)d\tau ds \\
&= \frac{1}{\Gamma(\beta-1)}\frac{d}{dt}\int_0^t \int_{\frac{1}{2}}^{s}(t-s)^{\beta-2}(s-\tau)g(\tau)d\tau ds \\
&= \frac{1}{\Gamma(\beta-1)}\frac{d}{dt}\Bigg[-\int_0^{\frac{1}{2}}\int_0^{\tau}(t-s)^{\beta-2}(s-\tau)g(\tau)ds d\tau \\
&\qquad\qquad + \int_{\frac{1}{2}}^{t}\int_{\tau}^{t}(t-s)^{\beta-2}(s-\tau)g(\tau)ds d\tau\Bigg] \\
&= \frac{1}{\Gamma(\beta-1)}\frac{d}{dt}\left[\int_0^{\frac{1}{2}}\left(\frac{\tau t^{\beta-1}}{\beta-1} + \frac{(t-\tau)^{\beta}}{(\beta-1)\beta} - \frac{t^{\beta}}{(\beta-1)\beta}\right)g(\tau)d\tau + \int_{\frac{1}{2}}^{t}\frac{(t-\tau)^{\beta}}{(\beta-1)\beta}g(\tau)d\tau\right] \\
&= \int_0^{\frac{1}{2}}\frac{(t-\tau)^{\beta-1} - t^{\beta-1}}{\Gamma(\beta)}g(\tau)d\tau + \int_{\frac{1}{2}}^{t}\frac{(t-\tau)^{\beta-1}}{\Gamma(\beta)}g(\tau)d\tau + \int_0^{\frac{1}{2}}\frac{\tau t^{\beta-2}}{\Gamma(\beta-1)}g(\tau)d\tau.
\end{aligned}$$

Hence, for $t \in [\frac{1}{2}, 1]$, we get

$$\begin{aligned}
& \varphi(D_{0^+}^{\alpha} w(t)) - \frac{c_1}{\Gamma(\beta)} t^{\beta-1} - \frac{-\frac{1}{2}c_1 + c_2}{\Gamma(\beta-1)} t^{\beta-2} \\
&= \int_0^{\frac{1}{2}}\frac{(t-\tau)^{\beta-1} - t^{\beta-1}}{\Gamma(\beta)}g(\tau)d\tau + \int_{\frac{1}{2}}^{t}\frac{(t-\tau)^{\beta-1}}{\Gamma(\beta)}g(\tau)d\tau + \int_0^{\frac{1}{2}}\frac{\tau t^{\beta-2}}{\Gamma(\beta-1)}g(\tau)d\tau. \quad (2.13)
\end{aligned}$$

Substituting boundary conditions $\varphi(D_{0+}^{\alpha}w(0))=0$ and $(\varphi(D_{0+}^{\alpha}w(1)))'=0$ into (2.8) and (2.13), respectively, we have

$$-\frac{-\frac{1}{2}c_1+c_2}{\Gamma(\beta-1)}=\int_0^{\frac{1}{2}}\frac{\tau}{\Gamma(\beta-1)}g(\tau)d\tau, \tag{2.14}$$

$$-\frac{c_1}{\Gamma(\beta)}=\int_0^{\frac{1}{2}}\frac{(1-\tau)^{\beta-2}-1}{\Gamma(\beta)}g(\tau)d\tau+\int_{\frac{1}{2}}^1\frac{(1-\tau)^{\beta-2}}{\Gamma(\beta)}g(\tau)d\tau. \tag{2.15}$$

Meanwhile, we can easily check that

$$\lim_{t\to\frac{1}{2}^+}\varphi(D_{0+}^{\alpha}w(t))=\lim_{t\to\frac{1}{2}^-}\varphi(D_{0+}^{\alpha}w(t)). \tag{2.16}$$

Thus, the solution w to (2.2) satisfy

$$\begin{aligned}&\varphi(D_{0+}^{\alpha}w(t))\\&=-\left[\int_0^t\frac{t^{\beta-1}(1-\tau)^{\beta-2}-(t-\tau)^{\beta-1}}{\Gamma(\beta)}g(\tau)d\tau+\int_t^1\frac{t^{\beta-1}(1-\tau)^{\beta-2}}{\Gamma(\beta)}g(\tau)d\tau\right]\\&=-\int_0^1 H(t,\tau)g(\tau)d\tau.\end{aligned}$$

Because $\varphi:\mathbb{R}\to\mathbb{R}$ is an odd increasing homeomorphism, we obtain

$$D_{0+}^{\alpha}w(t)=\varphi^{-1}\left(-\int_0^1 H(t,\tau)g(\tau)d\tau\right)=-\varphi^{-1}\left(\int_0^1 H(t,\tau)g(\tau)d\tau\right).$$

Thus, problem (2.2) can be represented by

$$D_{0+}^{\alpha}w(t)=-\varphi^{-1}\left(\int_0^1 H(t,\tau)g(\tau)d\tau\right),$$

$$w(0)=w'(0)=w'(1)=0.$$

By Lemma 2.7, we see that the solution to (2.2) can be uniquely rewritten as

$$w(t)=\int_0^1 G(t,s)\varphi^{-1}\left(\int_0^1 H(s,\tau)g(\tau)d\tau\right)ds.$$

Lemma 2.9 ([8]) Let $2<\alpha\le 3$, $1<\beta\le 2$. Functions $G(t,s)$ and $H(s,\tau)$ defined by (2.1) and (2.3), respectively, are continuous on $[0,1]\times[0,1]$ and satisfy

(i) $G(t,s)\ge 0$, $H(s,\tau)\ge 0$, for $t,s,\tau\in[0,1]$;

(ii) $G(t,s)\le G(1,s)$, $H(s,\tau)\le H(\tau,\tau)$, for $t,s,\tau\in[0,1]$;

(iii) $G(t,s)\ge t^{\alpha-1}G(1,s)$, $H(s,\tau)\ge s^{\beta-1}H(1,\tau)$, for $t,s,\tau\in[0,1]$.

Remark 2.10 ([21]) From condition (A), we get

$$\sigma x\le\varphi^{-1}[\gamma(\sigma)\varphi(x)],$$

and

$$\varphi^{-1}[\sigma\varphi(x)]\le\psi^{-1}(\sigma)x,\ \text{for } \sigma \text{ and } x>0.$$

In the following parts, we always take $E = C[0,1]$ as Banach space with norm $\|u\| = \max_{t\in[0,1]} |u(t)|$ and take a cone K defined by

$$K = \left\{ u \in E \mid u(t) \geq t^{\alpha-1}\|u\|,\ t \in [0,1] \right\}.$$

Let $\lambda > 0$, $u \in K$ and h satisfy (H), then $\lambda h f(u)$ also satisfies (H). By Lemma 2.8, we see that problem (1.1) can be equivalently rewritten as

$$u(t) = \int_0^1 G(t,s)\varphi^{-1}\left(\int_0^1 H(s,\tau)\lambda h(\tau)f(u(\tau))d\tau\right)ds.$$

Thus, we need continue to introduce Lemma 2.11 which will play an important role in Section 3.

Lemma 2.11 Let us give the solution operator $T_\lambda : K \to E$ defined by

$$T_\lambda(u)(t) = \int_0^1 G(t,s)\varphi^{-1}\left(\int_0^1 H(s,\tau)\lambda h(\tau)f(u(\tau))d\tau\right)ds.$$

Then $T_\lambda : K \to K$ is completely continuous.

Proof. Firstly, we show that $T_\lambda(K) \subset K$. Using (ii)(iii) of Lemma 2.9, we can estimate that for $t \in [0,1]$

$$T_\lambda(u)(t) \leq \int_0^1 G(1,s)\varphi^{-1}\left(\int_0^1 H(s,\tau)\lambda h(\tau)f(u(\tau))d\tau\right)ds,$$

i.e.

$$\|T_\lambda(u)\| \leq \int_0^1 G(1,s)\varphi^{-1}\left(\int_0^1 H(s,\tau)\lambda h(\tau)f(u(\tau))d\tau\right)ds,$$

and

$$\begin{aligned} T_\lambda(u)(t) &\geq \int_0^1 t^{\alpha-1}G(1,s)\varphi^{-1}\left(\int_0^1 H(s,\tau)\lambda h(\tau)f(u(\tau))d\tau\right)ds \\ &= t^{\alpha-1}\int_0^1 G(1,s)\varphi^{-1}\left(\int_0^1 H(s,\tau)\lambda h(\tau)f(u(\tau))d\tau\right)ds. \end{aligned}$$

Consequently, we have for $t \in [0,1]$

$$T_\lambda(u)(t) \geq t^{\alpha-1}\|T_\lambda(u)\|.$$

Meanwhile, the continuity of T_λ can be derived by the Lebesgue Dominated Convergence Theorem and continuity of functions G, H and f.

Secondly, by the Arzela-Ascoli theorem, we show that $T_\lambda(B)$ is uniform bounded and equicontinuous for any bounded subset B in K. The proof is standard and can be completed by making minor changes to the proof of Lemma 3.1 in [10]. Here we omit it.

Remark 2.12 By the definition of cone K, the range of α and Lemma 2.11, we can easily get $\|T_\lambda(u)\| = T_\lambda(u)(1)$ for any $u \in K$.

3. Sufficient conditions on the existence of positive solutions

Based on the well-known results of fixed point index on cones due to Guo-Krasnoselskii and Leggett-Williams, we derive some sufficient conditions that guarantee the existence of at least one, two and three positive solutions to (1.1), respectively. As applications, we present the corresponding examples at the end of every subsection. Specially, by a nonnegative solution u to (1.1), we understand a function $u \in E$ with $u(t) \geq 0$ for all $t \in [0,1]$, which satisfies (1.1). If u is a nonnegative solution of (1.1) and satisfies $\|u\| > 0$, then u is called a positive solution of (1.1).

For the sake of illustration, we introduce the following notations.

$$f_0 = \liminf_{u \to 0^+} \frac{f(u)}{\varphi(u)}, \qquad f_\infty = \liminf_{u \to +\infty} \frac{f(u)}{\varphi(u)},$$

$$F_0 = \limsup_{u \to 0^+} \frac{f(u)}{\varphi(u)}, \qquad F_\infty = \limsup_{u \to +\infty} \frac{f(u)}{\varphi(u)}.$$

3.1. **Existence of at least one positive solution.**

Theorem 3.1 Suppose that $(A)(H)$ and (F_1) hold. If either

$$\frac{4\gamma\left(\frac{16}{\int_{\frac{1}{4}}^{\frac{3}{4}} G(1,s)ds}\right)}{\int_{\frac{1}{4}}^{\frac{3}{4}} H(1,\tau)h(\tau)d\tau f_\infty} (\triangleq \lambda_*) < \frac{\psi\left(\frac{1}{\psi^{-1}(\int_0^1 H(\tau,\tau)h(\tau)d\tau)\int_0^1 G(1,s)ds}\right)}{F_0} (\triangleq \lambda^*),$$

or

$$\frac{4\gamma\left(\frac{16}{\int_{\frac{1}{4}}^{\frac{3}{4}} G(1,s)ds}\right)}{\int_{\frac{1}{4}}^{\frac{3}{4}} H(1,\tau)h(\tau)d\tau f_0} (\triangleq \lambda_*) < \frac{\psi\left(\frac{1}{\psi^{-1}(\int_0^1 H(\tau,\tau)h(\tau)d\tau)\int_0^1 G(1,s)ds}\right)}{F_\infty} (\triangleq \lambda^*),$$

then the boundary value problem (1.1) has at least one positive solution for any $\lambda \in (\lambda_*, \lambda^*)$.

Proof. (1) We prove that the result is true for the first case

$$\frac{4\gamma\left(\frac{16}{\int_{\frac{1}{4}}^{\frac{3}{4}} G(1,s)ds}\right)}{\int_{\frac{1}{4}}^{\frac{3}{4}} H(1,\tau)h(\tau)d\tau f_\infty} (\triangleq \lambda_*) < \frac{\psi\left(\frac{1}{\psi^{-1}(\int_0^1 H(\tau,\tau)h(\tau)d\tau)\int_0^1 G(1,s)ds}\right)}{F_0} (\triangleq \lambda^*).$$

For any $\lambda \in (\lambda_*, \lambda^*)$, we can choose $\epsilon > 0$ sufficiently small so that

$$\frac{4\gamma\left(\frac{16}{\int_{\frac{1}{4}}^{\frac{3}{4}} G(1,s)ds}\right)}{\int_{\frac{1}{4}}^{\frac{3}{4}} H(1,\tau)h(\tau)d\tau(f_\infty - \epsilon)} \le \lambda \le \frac{\psi\left(\frac{1}{\psi^{-1}(\int_0^1 H(\tau,\tau)h(\tau)d\tau)\int_0^1 G(1,s)ds}\right)}{F_0+\epsilon}. \tag{3.1}$$

On the one hand, the definition of F_0 implies that, for ϵ given above, there must exist a positive constant r_1 satisfying

$$f(u) \le (F_0+\epsilon)\varphi(u), \quad \text{for} \quad 0 < u \le r_1. \tag{3.2}$$

Let $u \in \partial K_{r_1}$. By applying Lemma 2.9, Remark 2.10, Remark 2.12 and (3.1)(3.2), we have

$$\begin{aligned}
\|T_\lambda(u)\| = T_\lambda(u)(1) &= \int_0^1 G(1,s)\varphi^{-1}\left(\int_0^1 H(s,\tau)\lambda h(\tau)f(u(\tau))d\tau\right)ds\\
&\le \int_0^1 G(1,s)\varphi^{-1}\left(\int_0^1 H(\tau,\tau)\lambda h(\tau)f(u(\tau))d\tau\right)ds\\
&\le \int_0^1 G(1,s)\varphi^{-1}\left(\int_0^1 H(\tau,\tau)\lambda h(\tau)(F_0+\epsilon)\varphi(u(\tau))d\tau\right)ds\\
&\le \int_0^1 G(1,s)\varphi^{-1}\left(\int_0^1 H(\tau,\tau)\lambda h(\tau)(F_0+\epsilon)\varphi(r_1)d\tau\right)ds\\
&\le \psi^{-1}\left(\lambda(F_0+\epsilon)\right)\varphi^{-1}\left(\int_0^1 H(\tau,\tau)h(\tau)d\tau\varphi(r_1)\right)\int_0^1 G(1,s)ds\\
&\le \psi^{-1}\left(\lambda(F_0+\epsilon)\right)\psi^{-1}\left(\int_0^1 H(\tau,\tau)h(\tau)d\tau\right)\int_0^1 G(1,s)ds\cdot r_1 \le r_1.
\end{aligned}$$

i.e.

$$\|T_\lambda(u)\| \le \|u\|, \text{ for } u \in \partial K_{r_1}. \tag{3.3}$$

On the other hand, the definition of f_∞ implies that, for ϵ mentioned above, we can choose a positive constant r such that

$$f(u) \ge (f_\infty - \epsilon)\varphi(u), \text{ for } u \ge r.$$

Take $r_2 > \max\{r_1, 16r\}$ and let $u \in \partial K_{r_2}$. Then we can derive

$$\min_{t\in[\frac{1}{4},\frac{3}{4}]} u(t) \ge \min_{t\in[\frac{1}{4},\frac{3}{4}]} t^{\alpha-1}\|u\| \ge \frac{1}{16}\|u\| > r,$$

$$f(u(t)) \ge (f_\infty-\epsilon)\varphi(u(t)) \ge (f_\infty-\epsilon)\varphi\left(\frac{1}{16}\|u\|\right), \text{ for } t \in [\frac{1}{4},\frac{3}{4}]. \tag{3.4}$$

Using Lemma 2.9, Remark 2.10, Remark 2.12 and (3.1)(3.4), we can continue to obtain

$$\begin{aligned}
\|T_\lambda(u)\| = T_\lambda(u)(1) &= \int_0^1 G(1,s)\varphi^{-1}\left(\int_0^1 H(s,\tau)\lambda h(\tau)f(u(\tau))d\tau\right)ds\\
&\ge \int_0^1 G(1,s)\varphi^{-1}\left(\int_0^1 s^{\beta-1}H(1,\tau)\lambda h(\tau)f(u(\tau))d\tau\right)ds
\end{aligned}$$

$$= \int_0^1 G(1,s)\varphi^{-1}\left(s^{\beta-1}\int_0^1 H(1,\tau)\lambda h(\tau)f(u(\tau))d\tau\right)ds$$
$$\geq \int_{\frac{1}{4}}^{\frac{3}{4}} G(1,s)\varphi^{-1}\left(s^{\beta-1}\int_{\frac{1}{4}}^{\frac{3}{4}} H(1,\tau)\lambda h(\tau)f(u(\tau))d\tau\right)ds$$
$$\geq \int_{\frac{1}{4}}^{\frac{3}{4}} G(1,s)\varphi^{-1}\left((\frac{1}{4})^{\beta-1}\int_{\frac{1}{4}}^{\frac{3}{4}} H(1,\tau)\lambda h(\tau)(f_\infty-\epsilon)\varphi\left(\frac{1}{16}\|u\|\right)d\tau\right)ds$$
$$\geq \varphi^{-1}\left(\frac{1}{4}\lambda(f_\infty-\epsilon)\int_{\frac{1}{4}}^{\frac{3}{4}} H(1,\tau)h(\tau)d\tau\varphi\left(\frac{1}{16}\|u\|\right)\right)\int_{\frac{1}{4}}^{\frac{3}{4}} G(1,s)ds$$
$$\geq \varphi^{-1}\left(\gamma\left(\frac{16}{\int_{\frac{1}{4}}^{\frac{3}{4}} G(1,s)ds}\right)\varphi\left(\frac{1}{16}\|u\|\right)\right)\int_{\frac{1}{4}}^{\frac{3}{4}} G(1,s)ds \geq \|u\|.$$

i.e.

$$\|T_\lambda(u)\| \geq \|u\|, \text{ for } u \in \partial K_{r_2}. \tag{3.5}$$

Suppose that

$$T_\lambda(u) = u, \text{ for } u \in \partial K_{r_i},\ i = 1 \text{ or } 2.$$

Then T_λ must have a fixed point on ∂K_{r_i} for $i = 1$ or 2. i.e. Problem (1.1) has at least one positive solution for any (λ_*, λ^*). Otherwise, by Lemma 2.1 and (3.3)(3.5), we have

$$i(T_\lambda, K_{r_1}, K) = 1, \quad i(T_\lambda, K_{r_2}, K) = 0.$$

Applying the additivity of the fixed point index, we see that

$$i(T_\lambda, K_{r_2} \setminus \overline{K_{r_1}}, K) = -1.$$

Thus, we can conclude that T_λ has a fixed point $u \in K_{r_2} \setminus \overline{K_{r_1}}$. That is to say, u is a positive solution of the boundary value problem (1.1) with $r_1 < \|u\| < r_2$ for any $\lambda \in (\lambda_*, \lambda^*)$.

(2) We show that the result is also true for the second case

$$\frac{4\gamma\left(\frac{16}{\int_{\frac{1}{4}}^{\frac{3}{4}} G(1,s)ds}\right)}{\int_{\frac{1}{4}}^{\frac{3}{4}} H(1,\tau)h(\tau)d\tau f_0}(\triangleq \lambda_*) < \frac{\psi\left(\frac{1}{\psi^{-1}(\int_0^1 H(\tau,\tau)h(\tau)d\tau)\int_0^1 G(1,s)ds}\right)}{F_\infty}(\triangleq \lambda^*).$$

For any $\lambda \in (\lambda_*, \lambda^*)$, we can also choose $\epsilon > 0$ sufficiently small so that

$$\frac{4\gamma\left(\frac{16}{\int_{\frac{1}{4}}^{\frac{3}{4}} G(1,s)ds}\right)}{\int_{\frac{1}{4}}^{\frac{3}{4}} H(1,\tau)h(\tau)d\tau(f_0-\epsilon)}(\triangleq \lambda_*) \leq \lambda \leq \frac{\psi\left(\frac{1}{\psi^{-1}(\int_0^1 H(\tau,\tau)h(\tau)d\tau)\int_0^1 G(1,s)ds}\right)}{F_\infty+\epsilon}(\triangleq \lambda^*). \tag{3.6}$$

On the one hand, the definition of f_0 implies that, for ϵ given above, there must exist a positive constant r_1 satisfying

$$f(u) \geq (f_0 - \epsilon)\varphi(u), \quad \text{for } \ 0 < u \leq r_1. \tag{3.7}$$

Let $u \in \partial K_{r_1}$. By applying the similar processes of the first case with aid of (3.7), we can derive

$$\|T_\lambda(u)\| \geq \|u\|, \text{ for } u \in \partial K_{r_1}. \tag{3.8}$$

On the other hand, the definition of F_∞ implies that, for ϵ mentioned above, we can choose a positive constant R satisfying

$$f(u) \leq (F_\infty + \epsilon)\varphi(u), \text{ for } u \geq R. \tag{3.9}$$

We will consider two subcases to estimate $\|T_\lambda(u)\|$ in the following proof.

(i) If f is bounded, then there exists a positive constant M satisfying

$$f(u) \leq M, \text{ for } u \geq 0.$$

Take $r_3 > \max\{r_1, \psi^{-1}(\lambda M)\,\varphi^{-1}\left(\int_0^1 H(\tau,\tau)h(\tau)d\tau\right)\int_0^1 G(1,s)ds\}$ and then let $u \in K$ with $\|u\| = r_3$. It follows from Remark 2.10 and Remark 2.12 that

$$\begin{aligned}
\|T_\lambda(u)\| = T_\lambda(u)(1) &= \int_0^1 G(1,s)\varphi^{-1}\left(\int_0^1 H(s,\tau)\lambda h(\tau)f(u(\tau))d\tau\right)ds \\
&\leq \int_0^1 G(1,s)\varphi^{-1}\left(\lambda M\int_0^1 H(\tau,\tau)h(\tau)d\tau\right)ds \\
&\leq \psi^{-1}(\lambda M)\,\varphi^{-1}\left(\int_0^1 H(\tau,\tau)h(\tau)d\tau\right)\int_0^1 G(1,s)ds \leq r_3 = \|u\|.
\end{aligned}$$

(ii) If f is unbounded, then there must exist a positive constant $r_4 > \max\{r_1, R\}$ satisfying

$$f(u) \leq f(r_4), \text{ for } 0 < u \leq r_4. \tag{3.10}$$

Let $u \in K$ with $\|u\| = r_4$. By Remark 2.10, Remark 2.12 and (3.6)(3.9)(3.10), we get

$$\begin{aligned}
\|T_\lambda(u)\| = T_\lambda(u)(1) &= \int_0^1 G(1,s)\varphi^{-1}\left(\int_0^1 H(s,\tau)\lambda h(\tau)f(u(\tau))d\tau\right)ds \\
&\leq \int_0^1 G(1,s)\varphi^{-1}\left(\int_0^1 H(\tau,\tau)\lambda h(\tau)f(r_4)d\tau\right)ds \\
&\leq \int_0^1 G(1,s)\varphi^{-1}\left(\int_0^1 H(\tau,\tau)\lambda h(\tau)(F_\infty+\epsilon)\varphi(r_4)d\tau\right)ds \\
&\leq \psi^{-1}(\lambda(F_\infty+\epsilon))\,\varphi^{-1}\left(\int_0^1 H(\tau,\tau)h(\tau)d\tau\varphi(r_4)\right)\int_0^1 G(1,s)ds \\
&\leq \psi^{-1}(\lambda(F_\infty+\epsilon))\,\psi^{-1}\left(\int_0^1 H(\tau,\tau)h(\tau)d\tau\right)\int_0^1 G(1,s)ds\cdot r_4 \leq r_4 = \|u\|.
\end{aligned}$$

Based on the arguments of (i)(ii), we can take $r_2 = \max\{r_3, r_4\}$. Then, we have

$$\|T_\lambda(u)\| \leq \|u\|, \text{ for } u \in \partial K_{r_2}. \tag{3.11}$$

Suppose that

$$T_\lambda(u) = u, \text{ for } u \in \partial K_{r_i},\ i = 1 \text{ or } 2.$$

Then T_λ must have a fixed point on ∂K_{r_i} for $i = 1$ or 2. i.e. Problem (1.1) has at least one positive solution for any (λ_*, λ^*). Otherwise, by Lemma 2.1 and (3.8)(3.11), we have

$$i(T_\lambda, K_{r_1}, K) = 0, \quad i(T_\lambda, K_{r_2}, K) = 1.$$

Applying the additivity of the fixed point index, we see that

$$i(T_\lambda, K_{r_2} \setminus \overline{K_{r_1}}, K) = 1.$$

Hence, we can conclude that T_λ has a fixed point $u \in K_{r_2} \setminus \overline{K_{r_1}}$. That is to say, u is a positive solution of the boundary value problem (1.1) with $r_1 < \|u\| < r_2$ for any $\lambda \in (\lambda_*, \lambda^*)$.

Example 3.2 Consider the following boundary value problem

$$\begin{cases} D_{0^+}^{\frac{3}{2}}(\varphi(D_{0^+}^{\frac{5}{2}}u(t))) = \lambda t^{-1}(1000 - 999e^{-u})u^{\frac{1}{3}}, & t \in (0,1), \\ u(0) = u'(0) = u'(1) = 0, \quad \varphi(D_{0^+}^{\frac{5}{2}}u(0)) = (\varphi(D_{0^+}^{\frac{5}{2}}u(1)))' = 0. \end{cases} \tag{3.12}$$

Here we take $\varphi(x) = x^{\frac{1}{3}}$, $x \in \mathbb{R}$ and $\psi(x) = \gamma(x) \equiv \varphi(x)$. It is obvious that φ is an odd increasing homeomorphism and satisfies condition (A). Since $h(t) = t^{-1}$ and $\beta = \frac{3}{2}$, we see that h is singular at $t = 0$ and

$$\int_0^1 t^{\beta-1}(1-t)^{\beta-2}h(t)dt = \int_0^1 t^{\frac{1}{2}}(1-t)^{-\frac{1}{2}}t^{-1}dt = \int_0^1 t^{-\frac{1}{2}}(1-t)^{-\frac{1}{2}}dt = \pi,$$

which implies that condition (H) is true. While, $f(u) = (1000 - 999e^{-u})u^{\frac{1}{3}}$ satisfies condition (F_1). Moreover, by the definitions of f_∞, F_0, G and H, we can calculate that

$$f_\infty = \liminf_{u\to+\infty} \frac{(1000 - 999e^{-u})u^{\frac{1}{3}}}{u^{\frac{1}{3}}} = \liminf_{u\to+\infty}(1000 - 999e^{-u}) = 1000,$$

$$F_0 = \limsup_{u\to0^+} \frac{(1000 - 999e^{-u})u^{\frac{1}{3}}}{u^{\frac{1}{3}}} = \limsup_{u\to0^+}(1000 - 999e^{-u}) = 1,$$

$$\int_{\frac{1}{4}}^{\frac{3}{4}} G(1,s)ds = \int_{\frac{1}{4}}^{\frac{3}{4}} \frac{(1-s)^{\frac{1}{2}} - (1-s)^{\frac{3}{2}}}{\Gamma(\frac{5}{2})}ds \doteq 0.13, \tag{3.13}$$

$$\int_{\frac{1}{4}}^{\frac{3}{4}} H(1,\tau)h(\tau)d\tau = \int_{\frac{1}{4}}^{\frac{3}{4}} \frac{(1-\tau)^{-\frac{1}{2}} - (1-\tau)^{\frac{1}{2}}}{\Gamma(\frac{3}{2})}\tau^{-1}d\tau \doteq 0.83, \tag{3.14}$$

$$\int_0^1 G(1,s)ds = \int_0^1 \frac{(1-s)^{\frac{1}{2}} - (1-s)^{\frac{3}{2}}}{\Gamma(\frac{5}{2})}ds = \frac{16}{45\sqrt{\pi}}, \tag{3.15}$$

$$\int_0^1 H(\tau,\tau)h(\tau)d\tau = \int_0^1 \frac{\tau^{\frac{1}{2}}(1-\tau)^{-\frac{1}{2}}}{\Gamma(\frac{3}{2})}\tau^{-1}d\tau = 2\sqrt{\pi}. \tag{3.16}$$

Thus, we have

$$\frac{4\gamma\left(\frac{16}{\int_{\frac{1}{4}}^{\frac{3}{4}} G(1,s)ds}\right)}{\int_{\frac{1}{4}}^{\frac{3}{4}} H(1,\tau)h(\tau)d\tau f_\infty} \doteq 0.024,$$

$$\frac{\psi\left(\frac{1}{\psi^{-1}(\int_0^1 H(\tau,\tau)h(\tau)d\tau)\int_0^1 G(1,s)ds}\right)}{F_0} \doteq 0.48,$$

which belongs to the first case of Theorem 3.1. Therefore, by Theorem 3.1 , we can conclude that problem (3.12) has at least one positive solution for $\lambda \in (0.024, 0.48)$.

3.2. **Existence of at least two positive solutions.**

Theorem 3.3 Suppose that $(A)(H)(F_1)$ and (F_2) hold.
(1) If $f_0 = f_\infty = +\infty$, then the boundary value problem (1.1) has at least two positive solutions for any $\lambda \in (0, \lambda^0)$, where

$$\lambda^0 = \sup_{r>0} \frac{\varphi(r)}{\int_0^1 H(\tau,\tau)h(\tau)d\tau \max_{u\in[0,r]} f(u)}.$$

(2) If $F_0 = F_\infty = 0$, then the boundary value problem (1.1) has at least two positive solutions for any $\lambda > \lambda_0$, where

$$\lambda_0 = \inf_{r>0} \frac{4\varphi\left(\frac{r}{\int_{\frac{1}{4}}^{\frac{3}{4}} G(1,s)ds}\right)}{\int_{\frac{1}{4}}^{\frac{3}{4}} H(1,\tau)h(\tau)d\tau \min_{u\in[\frac{r}{16},r]} f(u)}.$$

Proof. (1) By (F_2), we can easily get that $\max_{u\in[0,r]} f(u) > 0$ for any $r > 0$. Let $u \in K$ with $\|u\| = r$. Using Remark 2.10 and Remark 2.12 again, we can get

$$\begin{aligned}
\|T_\lambda(u)\| = T_\lambda(u)(1) &= \int_0^1 G(1,s)\varphi^{-1}\left(\int_0^1 H(s,\tau)\lambda h(\tau)f(u(\tau))d\tau\right)ds \\
&\leq \int_0^1 G(1,s)\varphi^{-1}\left(\lambda\int_0^1 H(\tau,\tau)h(\tau)d\tau \max_{u\in[0,r]} f(u)\right)ds \\
&= \varphi^{-1}\left(\lambda\int_0^1 H(\tau,\tau)h(\tau)d\tau \max_{u\in[0,r]} f(u)\right)\int_0^1 G(1,s)ds.
\end{aligned}$$

By the definition of function $G(t,s)$, we can easily check that $\int_0^1 G(1,s)ds < 1$. Thus, we have

$$\|T_\lambda(u)\| \leq \varphi^{-1}\left(\lambda\int_0^1 H(\tau,\tau)h(\tau)d\tau \max_{u\in[0,r]} f(u)\right). \tag{3.17}$$

To show the existence of λ^0, we need to give a new function $x : (0,+\infty) \to (0,+\infty)$ defined by

$$x(r) = \frac{\varphi(r)}{\int_0^1 H(\tau,\tau)h(\tau)d\tau \max_{u\in[0,r]} f(u)}.$$

Clearly, $x(r)$ is a continuous function. From $f_0 = f_\infty = +\infty$, we see that $\lim_{r\to 0} x(r) = \lim_{r\to+\infty} x(r) = 0$, and there exists a positive constant r^* such that $x(r^*) = \sup_{r>0} x(r)$. Take $\lambda^0 = x(r^*)$, and then for any $\lambda \in (0, \lambda^0)$, we can find two positive constants r_1, r_2 such that $x(r_1) = x(r_2) = \lambda$. Using (3.17), for $u \in K$ with $\|u\| = r_i$ $(i = 1, 2)$, we have

$$\begin{aligned}
&\|T_\lambda(u)\| \\
&\le \varphi^{-1}\left(\lambda \int_0^1 H(\tau,\tau)h(\tau)d\tau \max_{u\in[0,r]} f(u)\right) \\
&= \varphi^{-1}\left(\frac{\varphi(r_i)}{\int_0^1 H(\tau,\tau)h(\tau)d\tau \max_{u\in[0,r_i]} f(u)} \int_0^1 H(\tau,\tau)h(\tau)d\tau \max_{u\in[0,r_i]} f(u)\right) = r_i.
\end{aligned}$$

Thus, we get

$$\|T_\lambda(u)\| \le \|u\|, \text{ for } u \in \partial K_{r_i},\ i = 1, 2. \tag{3.18}$$

Moreover, applying the similar arguments in the proof of Theorem 3.1 with aids of $f_0 = f_\infty = +\infty$, for any $\lambda \in (0, \lambda^0)$ mentioned above, we can find two positive constants r_3, r_4 such that $r_3 < r_1 < r_2 < r_4$ and

$$\|T_\lambda(u)\| \ge \|u\|, \text{ for } u \in \partial K_{r_j},\ j = 3, 4. \tag{3.19}$$

Finally, we need consider three cases to complete the proof of this theorem.

Case 1: Suppose that

$$T_\lambda(u) \ne u, \text{ for } u \in \partial K_{r_i},\ i = 1, 2, 3, 4.$$

By Lemma 2.1 and (3.18)(3.19), we have

$$i(T_\lambda, K_{r_1}, K) = 1, \quad i(T_\lambda, K_{r_2}, K) = 1,$$

and

$$i(T_\lambda, K_{r_3}, K) = 0, \quad i(T_\lambda, K_{r_4}, K) = 0.$$

Applying the additivity of the fixed point index, we see that

$$i(T_\lambda, K_{r_1} \setminus \overline{K_{r_3}}, K) = 1, \quad i(T_\lambda, K_{r_4} \setminus \overline{K_{r_2}}, K) = -1.$$

Thus, we can conclude that T_λ has two fixed points u_1, u_2 in $K_{r_1} \setminus \overline{K_{r_3}}$ and $K_{r_4} \setminus \overline{K_{r_2}}$, respectively. That is to say, u_1, u_2 are two positive solutions of the boundary value problem (1.1) with $r_3 < \|u_1\| < r_1 < r_2 < \|u_2\| < r_4$ for any $\lambda \in (0, \lambda^0)$.

Case 2: Suppose that there exists only one component $r_0 \in \{r_1, r_2, r_3, r_4\}$ such that $T_\lambda(u) = u$ for $u \in \partial K_{r_0}$. Without loss of generality, we assume that

$$T_\lambda(u) = u, \text{ for } u \in \partial K_{r_1}, \tag{3.20}$$

and

$$T_\lambda(u) \ne u, \text{ for } u \in \partial K_{r_i},\ i = 2, 3, 4. \tag{3.21}$$

From (3.20), we see that T_λ must have a fixed point $u_1 \in \partial K_{r_1}$. Meanwhile, by Lemma 2.1 and (3.21), we have

$$i(T_\lambda, K_{r_2}, K) = 1, \quad i(T_\lambda, K_{r_4}, K) = 0.$$

Applying the additivity of the fixed point index, we obtain

$$i(T_\lambda, K_{r_4} \setminus \overline{K_{r_2}}, K) = -1.$$

i.e. T_λ has a fixed point $u_2 \in K_{r_4} \setminus \overline{K_{r_2}}$. That is to say, u_1, u_2 are two positive solutions of the boundary value problem (1.1) with $\|u_1\| = r_1 < r_2 < \|u_2\| < r_4$ for any $\lambda \in (0, \lambda^0)$.

Case 3: Suppose that there exist at least two components $r_0, r_* \in \{r_1, r_2, r_3, r_4\}$ such that $T_\lambda(u) = u$ for $u \in \partial K_{r_i}, r_i = r_0, r_*$. Without loss of generality, we assume that

$$T_\lambda(u) = u, \text{ for } u \in \partial K_{r_i}, \ i = 1, 2.$$

Then T_λ must have two fixed points $u_1, u_2 \in \partial K_{r_i}, \ i = 1, 2$. That is to say, u_1, u_2 are two positive solutions of the boundary value problem (1.1) with $\|u_1\| = r_1 < r_2 = \|u_2\|$ for any $\lambda \in (0, \lambda^0)$.

(2) By (F_2), we can easily get that $\min_{u\in[\frac{r}{16}, r]} f(u) > 0$ for any $r > 0$. If $u \in K$ with $\|u\| = r$, then we have

$$r \geq \min_{t\in[\frac{1}{4},\frac{3}{4}]} u(t) \geq \min_{t\in[\frac{1}{4},\frac{3}{4}]} t^{\alpha-1}\|u\| \geq \frac{1}{16}\|u\| = \frac{r}{16},$$

$$f(u(t)) \geq \min_{u\in[\frac{r}{16},r]} f(u), \text{ for } t \in [\frac{1}{4}, \frac{3}{4}]. \tag{3.22}$$

Applying Lemma 2.9, Remark 2.10, Remark 2.12 and (3.22) again, we can get

$$\begin{aligned}
\|T_\lambda(u)\| = T_\lambda(u)(1) &= \int_0^1 G(1,s)\varphi^{-1}\left(\int_0^1 H(s,\tau)\lambda h(\tau) f(u(\tau))d\tau\right) ds \\
&\geq \int_0^1 G(1,s)\varphi^{-1}\left(\int_0^1 s^{\beta-1}H(1,\tau)\lambda h(\tau) f(u(\tau))d\tau\right) ds \\
&= \int_0^1 G(1,s)\varphi^{-1}\left(s^{\beta-1}\int_0^1 H(1,\tau)\lambda h(\tau) f(u(\tau))d\tau\right) ds \\
&\geq \int_{\frac{1}{4}}^{\frac{3}{4}} G(1,s)\varphi^{-1}\left(s^{\beta-1}\int_{\frac{1}{4}}^{\frac{3}{4}} H(1,\tau)\lambda h(\tau) f(u(\tau))d\tau\right) ds \\
&\geq \int_{\frac{1}{4}}^{\frac{3}{4}} G(1,s)\varphi^{-1}\left((\frac{1}{4})^{\beta-1}\int_{\frac{1}{4}}^{\frac{3}{4}} H(1,\tau)\lambda h(\tau) \min_{u\in[\frac{r}{16},r]} f(u)d\tau\right) ds \\
&\geq \varphi^{-1}\left(\frac{1}{4}\lambda\int_{\frac{1}{4}}^{\frac{3}{4}} H(1,\tau)h(\tau)d\tau \min_{u\in[\frac{r}{16},r]} f(u)\right)\int_{\frac{1}{4}}^{\frac{3}{4}} G(1,s)ds.
\end{aligned}$$

Obviously, $\int_{\frac{1}{4}}^{\frac{3}{4}} G(1,s)ds < 1$. Thus, we have

$$\|T_\lambda(u)\| \geq \varphi^{-1}\left(\frac{1}{4}\lambda\int_{\frac{1}{4}}^{\frac{3}{4}} H(1,\tau)h(\tau)d\tau \min_{u\in[\frac{r}{16},r]} f(u)\right)\int_{\frac{1}{4}}^{\frac{3}{4}} G(1,s)ds. \tag{3.23}$$

To show the existence of λ_0, we need to give a new function $y : (0, +\infty) \to (0, +\infty)$ defined by

$$y(r) = \frac{4\varphi\left(\frac{r}{\int_{\frac{1}{4}}^{\frac{3}{4}} G(1,s)ds}\right)}{\int_{\frac{1}{4}}^{\frac{3}{4}} H(1,\tau)h(\tau)d\tau \min_{u\in[\frac{r}{16},r]} f(u)}.$$

Clearly, $y(r)$ is a continuous function. From $F_0 = F_\infty = 0$, we see that $\lim_{r\to 0} y(r) = \lim_{r\to+\infty} y(r) = +\infty$, and there exists a positive constant r_* such that $y(r_*) =$

$\inf_{r>0} y(r)$. Take $\lambda_0 = y(r_*)$, and then for any $\lambda > \lambda_0$, we can find two positive constants r_1, r_2 such that $y(r_1) = y(r_2) = \lambda$. Using (3.23), for $u \in K$ with $\|u\| = r_i$ $(i = 1, 2)$, we have

$$\begin{aligned}
&\|T_\lambda(u)\| \\
&\geq \varphi^{-1}\left(\frac{1}{4}\lambda \int_{\frac{1}{4}}^{\frac{3}{4}} H(1,\tau)h(\tau)d\tau \min_{u\in[\frac{r}{16},r]} f(u)\right) \int_{\frac{1}{4}}^{\frac{3}{4}} G(1,s)ds \\
&\geq \varphi^{-1}\left(\frac{1}{4}\cdot \frac{4\varphi\left(\frac{r_i}{\int_{\frac{1}{4}}^{\frac{3}{4}} G(1,s)ds}\right)}{\int_{\frac{1}{4}}^{\frac{3}{4}} H(1,\tau)h(\tau)d\tau \min_{u\in[\frac{r_i}{16},r_i]} f(u)} \int_{\frac{1}{4}}^{\frac{3}{4}} H(1,\tau)h(\tau)d\tau \min_{u\in[\frac{r_i}{16},r_i]} f(u)\right) \\
&\qquad \times \int_{\frac{1}{4}}^{\frac{3}{4}} G(1,s)ds \\
&= r_i.
\end{aligned}$$

Thus, we get

$$\|T_\lambda(u)\| \geq \|u\|, \text{ for } u \in \partial K_{r_i},\ i = 1, 2. \tag{3.24}$$

Moreover, applying the similar arguments in the proof of Theorem 3.1 with aids of $F_0 = F_\infty = 0$, for any $\lambda > \lambda_0$ mentioned above, we can find two positive constants r_3, r_4 such that $r_3 < r_1 < r_2 < r_4$ and

$$\|T_\lambda(u)\| \leq \|u\|, \text{ for } u \in \partial K_{r_j},\ j = 3, 4. \tag{3.25}$$

Finally, we need consider three cases to complete the proof of this theorem.

Case 1: Suppose that

$$T_\lambda(u) \neq u, \text{ for } u \in \partial K_{r_i},\ i = 1, 2, 3, 4.$$

By Lemma 2.1 and (3.24)(3.25) that

$$i(T_\lambda, K_{r_1}, K) = 0, \quad i(T_\lambda, K_{r_2}, K) = 0,$$

and

$$i(T_\lambda, K_{r_3}, K) = 1, \quad i(T_\lambda, K_{r_4}, K) = 1.$$

Applying the additivity of the fixed point index, we see that

$$i(T_\lambda, K_{r_1} \setminus \overline{K_{r_3}}, K) = -1, \quad i(T_\lambda, K_{r_4} \setminus \overline{K_{r_2}}, K) = 1.$$

Thus, we can conclude that T_λ has two fixed points u_1, u_2 in $K_{r_1} \setminus \overline{K_{r_3}}$ and $K_{r_4} \setminus \overline{K_{r_2}}$, respectively. That is to say, u_1, u_2 are two positive solutions of the boundary value problem (1.1) with $r_3 < \|u_1\| < r_1 < r_2 < \|u_2\| < r_4$ for any $\lambda > \lambda_0$.

Case 2: Suppose that there exists only one component $r_0 \in \{r_1, r_2, r_3, r_4\}$ such that $T_\lambda(u) = u$ for $u \in \partial K_{r_0}$. Without loss of generality, we assume that

$$T_\lambda(u) = u, \text{ for } u \in \partial K_{r_1}, \tag{3.26}$$

and

$$T_\lambda(u) \neq u, \text{ for } u \in \partial K_{r_i},\ i = 2, 3, 4. \tag{3.27}$$

From (3.26), we see that T_λ must have a fixed point $u_1 \in \partial K_{r_1}$. Meanwhile, by Lemma 2.1 and (3.27), we have

$$i(T_\lambda, K_{r_2}, K) = 0, \quad i(T_\lambda, K_{r_4}, K) = 1.$$

Applying the additivity of the fixed point index, we obtain

$$i(T_\lambda, K_{r_4} \setminus \overline{K_{r_2}}, K) = 1.$$

i.e. T_λ has a fixed point $u_2 \in K_{r_4} \setminus \overline{K_{r_2}}$. That is to say, u_1, u_2 are two positive solutions of the boundary value problem (1.1) with $\|u_1\| = r_1 < r_2 < \|u_2\| < r_4$ for any $\lambda > \lambda_0$.

Case 3: Suppose that there exist at least two components $r_0, r_* \in \{r_1, r_2, r_3, r_4\}$ such that $T_\lambda(u) = u$ for $u \in \partial K_{r_i}, r_i = r_0, r_*$. Without loss of generality, we assume that

$$T_\lambda(u) = u, \text{ for } u \in \partial K_{r_i}, \ i = 1, 2.$$

Then T_λ must have two fixed points $u_1, u_2 \in \partial K_{r_i}, \ i = 1, 2$. That is to say, u_1, u_2 are two positive solutions of the boundary value problem (1.1) with $\|u_1\| = r_1 < r_2 = \|u_2\|$ for any $\lambda > \lambda_0$.

Example 3.4 Consider the following boundary value problem

$$\begin{cases} D_{0^+}^{\frac{3}{2}}(\varphi(D_{0^+}^{\frac{5}{2}} u(t))) = \lambda t^{-1} f(u), \quad t \in (0,1), \\ u(0) = u'(0) = u'(1) = 0, \quad \varphi(D_{0^+}^{\frac{5}{2}} u(0)) = (\varphi(D_{0^+}^{\frac{5}{2}} u(1)))' = 0. \end{cases} \tag{3.28}$$

Here we take $\varphi(x) = |x|x + x$, $x \in \mathbb{R}$, and

$$f(u) = \begin{cases} u^3, & 0 \le u < 1, \\ u, & u \ge 1. \end{cases}$$

Combining Example 4.1 of [15] and Example 3.2 mentioned above, we can prove that problem (3.28) satisfies conditions $(A)(H)$ (F_1) and (F_2). We can also calculate that

$$F_0 = \limsup_{u \to 0^+} \frac{u^3}{u^2 + u} = 0,$$

$$F_\infty = \limsup_{u \to +\infty} \frac{u}{u^2 + u} = 0,$$

which belongs to the second case of Theorem 3.3. Additionally, for any $r > 0$, we have

$$\min_{u \in [\frac{r}{16}, r]} f(u) = f(\frac{r}{16}) = \begin{cases} \frac{r^3}{4096}, & 0 < r < 16, \\ \frac{r}{16}, & r \ge 16. \end{cases}$$

Using (3.13) and (3.14), for $0 < r < 16$, we have

$$y(r) = \frac{4\varphi\left(\frac{r}{\int_{\frac{1}{4}}^{\frac{3}{4}} G(1,s)ds}\right)}{\int_{\frac{1}{4}}^{\frac{3}{4}} H(1,\tau)h(\tau)d\tau \cdot \frac{r^3}{4096}} \doteq \frac{1167409.41r + 151798.75}{r^2},$$

and $y'(r) < 0$. While, for $r \geq 16$, we have

$$y(r) = \frac{4\varphi\left(\frac{r}{\int_{\frac{1}{4}}^{\frac{3}{4}} G(1,s)ds}\right)}{\int_{\frac{1}{4}}^{\frac{3}{4}} H(1,\tau)h(\tau)d\tau \cdot \frac{r}{16}} \doteq 4560.29r + 592.98,$$

and $y'(r) > 0$. Thus, we can get

$$\lambda_0 = \inf_{r>0} y(r) = y(16) \doteq 73557.62.$$

Therefore, by Theorem 3.3, we can conclude that problem (3.28) has at least two positive solutions for $\lambda > 73557.62$.

3.3. **Existence of at least three positive solutions.**

Theorem 3.5 Suppose that $(A)(H)$ (F_1) hold and $F_\infty < 1$. If there exist two constants $0 < d < k$ such that

(i) $\max_{u\in[0,d]} f(u) \leq \varphi(d)$;

(ii) $f(u) \geq \varphi(\theta u)$ for all $k \leq u \leq 16k$, where $\theta > 0$ is a constant satisfying

$$\frac{4\gamma\left(\frac{16}{\theta\int_{\frac{1}{4}}^{\frac{3}{4}} G(1,s)ds}\right)}{\int_{\frac{1}{4}}^{\frac{3}{4}} H(1,\tau)h(\tau)d\tau}(\triangleq \underline{\lambda}) < \psi\left(\frac{1}{\psi^{-1}(\int_0^1 H(\tau,\tau)h(\tau)d\tau)\int_0^1 G(1,s)ds}\right)(\triangleq \overline{\lambda}).$$

Then the boundary value problem (1.1) has at least one nonnegative solution and two positive solutions for any $\lambda \in (\underline{\lambda}, \overline{\lambda})$.

Proof. The condition on θ implies that the interval $(\underline{\lambda}, \overline{\lambda})$ is not empty. Exactly, we will divide the proof into four steps.

Step 1: Show that $T_\lambda : \overline{K_c} \to \overline{K_c}$ is completely continuous for some positive constant $c > 0$.

From $F_\infty < 1$, we can find two constants ϱ, δ satisfying $0 < \varrho < 1$, $\delta > 0$ and

$$f(u) \leq \varrho\varphi(u), \text{ for } u \geq \delta.$$

By (F_1), we can denote $\eta = \max_{0\leq u\leq\delta} f(u)$. Thus, we get

$$f(u) \leq \varrho\varphi(u) + \eta, \text{ for } u \geq 0. \tag{3.29}$$

Take $c > \max\left\{16k, \varphi^{-1}(\frac{\eta}{1-\varrho})\right\}$ and let $u \in K_c$. By Lemma 2.9 and Remark 2.12, we can obtain

$$\begin{aligned}\|T_\lambda(u)\| = T_\lambda(u)(1) &= \int_0^1 G(1,s)\varphi^{-1}\left(\int_0^1 H(s,\tau)\lambda h(\tau)f(u(\tau))d\tau\right)ds \\ &\leq \int_0^1 G(1,s)\varphi^{-1}\left(\int_0^1 H(\tau,\tau)\lambda h(\tau)(\varrho\varphi(u(\tau)) + \eta)d\tau\right)ds\end{aligned}$$

$$\leq \int_0^1 G(1,s)\varphi^{-1}\left(\int_0^1 H(\tau,\tau)\lambda h(\tau)(\varrho\varphi(c)+\eta)d\tau\right)ds.$$

By the choice of c, we can derive that $\varrho\varphi(c)+\eta < \varphi(c)$. Thus, we have

$$\|T_\lambda(u)\| \leq \int_0^1 G(1,s)\varphi^{-1}\left(\int_0^1 H(\tau,\tau)\lambda h(\tau)\varphi(c)d\tau\right)ds.$$

Combining Remark 2.10 with the range of λ, we see that

$$\|T_\lambda(u)\| \leq \psi^{-1}(\lambda)\varphi^{-1}\left(\int_0^1 H(\tau,\tau)\lambda h(\tau)\varphi(c)d\tau\right)\int_0^1 G(1,s)ds$$

$$\leq \psi^{-1}(\lambda)\psi^{-1}\left(\int_0^1 H(\tau,\tau)h(\tau)d\tau\right)\int_0^1 G(1,s)ds\cdot c < c,$$

which implies that $T_\lambda(\overline{K_c}) \subset \overline{K_c}$. By Lemma 2.11, the proof of Step 1 is done.

Step 2: Show that $\|T_\lambda(u)\| < d$ for $u \in \overline{K_d}$.

From condition (i) in this theorem, we see that

$$f(u) \leq \max_{u\in[0,d]} f(u) \leq \varphi(d), \text{ for } 0 \leq u \leq d. \tag{3.30}$$

Applying the similar process of Step 1 with the aid of (3.30), we can derive that for $u \in \overline{K_d}$

$$\|T_\lambda(u)\| \leq \psi^{-1}(\lambda)\psi^{-1}\left(\int_0^1 H(\tau,\tau)h(\tau)d\tau\right)\int_0^1 G(1,s)ds\cdot d < d,$$

which completes the proof of Step 2.

Step 3: Show that there exist two positive constants a, b with $a < b$ and a nonnegative continuous concave functional α on K satisfying $\{u \in K(\alpha,a,b) : \alpha(x) > a\} \neq \emptyset$ and $\alpha(T_\lambda(u)) > a$ if $u \in K(\alpha,a,b)$.

By the definition of cone K, we give a nonnegative continuous concave functional α defined by

$$\alpha(u) = \min_{t\in[\frac{1}{4},\frac{3}{4}]} u(t), \text{ on } K.$$

Choosing $a = k$, $b = 16k$ and $u(t) \equiv 2k$ for $t \in [0,1]$, we can easily see that $k < u(t) \equiv 2k < 16k$. That is to say, $\{u \in K(\alpha,a,b) : \alpha(x) > a\} \neq \emptyset$.

Next, we need continue to show $\alpha(T_\lambda(u)) > a$ for $u \in K(\alpha,a,b)$. By the definition of α and $T_\lambda(u) \in K$ for $u \in K(\alpha,a,b)$, we obtain

$$\alpha(T_\lambda(u)) = \min_{t\in[\frac{1}{4},\frac{3}{4}]} T_\lambda(u)(t) \geq \min_{t\in[\frac{1}{4},\frac{3}{4}]} t^{\alpha-1}\|T_\lambda(u)\| \geq \frac{1}{16}\|T_\lambda(u)\|.$$

Then, the problem can be transferred to estimate $\|T_\lambda(u)\|$ for $u \in K(\alpha,a,b)$. From the definition of $K(\alpha,a,b)$, we see that $\min_{t\in[\frac{1}{4},\frac{3}{4}]} u(t) = \alpha(u) \geq a = k$ and $\|u\| \leq b = 16k$. Applying condition (ii) in this theorem, we can obtain

$$f(u(t)) \geq \varphi(\theta u(t)) \geq \varphi(\theta k), \text{ for } t \in [\frac{1}{4},\frac{3}{4}]. \tag{3.31}$$

By Lemma 2.9, Remark 2.12 and (3.31), we have

$$\begin{aligned}
\|T_\lambda(u)\| = T_\lambda(u)(1) &= \int_0^1 G(1,s)\varphi^{-1}\left(\int_0^1 H(s,\tau)\lambda h(\tau)f(u(\tau))d\tau\right)ds \\
&\geq \int_0^1 G(1,s)\varphi^{-1}\left(\int_0^1 s^{\beta-1}H(1,\tau)\lambda h(\tau)f(u(\tau))d\tau\right)ds \\
&= \int_0^1 G(1,s)\varphi^{-1}\left(s^{\beta-1}\int_0^1 H(1,\tau)\lambda h(\tau)f(u(\tau))d\tau\right)ds \\
&\geq \int_{\frac{1}{4}}^{\frac{3}{4}} G(1,s)\varphi^{-1}\left(s^{\beta-1}\int_{\frac{1}{4}}^{\frac{3}{4}} H(1,\tau)\lambda h(\tau)f(u(\tau))d\tau\right)ds \\
&\geq \int_{\frac{1}{4}}^{\frac{3}{4}} G(1,s)\varphi^{-1}\left((\frac{1}{4})^{\beta-1}\int_{\frac{1}{4}}^{\frac{3}{4}} H(1,\tau)\lambda h(\tau)\varphi(\theta k)d\tau\right)ds \\
&\geq \varphi^{-1}\left(\frac{1}{4}\lambda\int_{\frac{1}{4}}^{\frac{3}{4}} H(1,\tau)h(\tau)d\tau\varphi(\theta k)\right)\int_{\frac{1}{4}}^{\frac{3}{4}} G(1,s)ds.
\end{aligned}$$

From condition (A) on φ, h is nonnegative and nontrivial on any subinterval of $(0,1)$, and Remark 2.10, we can derive that for any $\lambda \in (\underline{\lambda}, \overline{\lambda})$

$$\begin{aligned}
\|T_\lambda(u)\| &> \varphi^{-1}\left(\frac{1}{4}\underline{\lambda}\int_{\frac{1}{4}}^{\frac{3}{4}} H(1,\tau)h(\tau)d\tau\varphi(\theta k)\right)\int_{\frac{1}{4}}^{\frac{3}{4}} G(1,s)ds \\
&= \varphi^{-1}\left(\gamma\left(\frac{16}{\theta\int_{\frac{1}{4}}^{\frac{3}{4}} G(1,s)ds}\right)\varphi(\theta k)\right)\int_{\frac{1}{4}}^{\frac{3}{4}} G(1,s)ds \\
&\geq \frac{16}{\theta\int_{\frac{1}{4}}^{\frac{3}{4}} G(1,s)ds}\cdot\theta k\cdot\int_{\frac{1}{4}}^{\frac{3}{4}} G(1,s)ds = 16k = 16a.
\end{aligned}$$

Thus, we can show that

$$\alpha(T_\lambda(u)) \geq \frac{1}{16}\|T_\lambda(u)\| > \frac{1}{16}\cdot 16a = a, \text{ for } u \in K(\alpha,a,b),$$

which completes the proof of Step 3.

Step 4: For all $u \in K(\alpha,a,c)$ with $\|T_\lambda(u)\| > b$, we can easily check that

$$\alpha(T_\lambda(u)) = \min_{t\in[\frac{1}{4},\frac{3}{4}]} T_\lambda(u)(t) \geq \min_{t\in[\frac{1}{4},\frac{3}{4}]} t^{\alpha-1}\|T_\lambda(u)\| \geq \frac{1}{16}\|T_\lambda(u)\| > \frac{b}{16} = a.$$

Based on the four steps above and Lemma 2.3, we can obtain that T_λ has at least three fixed points u_1, u_2, u_3 in $\overline{K_c}$ such that $\|u_1\| < d$, $a < \alpha(u_2)$, $d < \|u_3\|$ with $\alpha(u_3) < a$. That is to say, u_1 is a nonnegative solution and u_2, u_3 are two positive solutions of the boundary value problem (1.1) with $\|u_1\| < d$, $\min_{t\in[\frac{1}{4},\frac{3}{4}]} u_2(t) > a = k$, $\|u_3\| > d$ and $\min_{t\in[\frac{1}{4},\frac{3}{4}]} u_3(t) < a = k$ for any $\lambda \in (\underline{\lambda}, \overline{\lambda})$.

Theorem 3.6 Suppose that $(A)(H)$ (F_1) hold and $F_\infty < 1$. If there exist three constants $0 < e < d < k$ such that

(i) $\max_{u\in[0,d]} f(u) \le \varphi(d)$;

(ii) $f(u) \ge \varphi(\theta_1 u)$ for all $\frac{e}{16} \le u \le e$, $f(u) \ge \varphi(\theta_2 u)$ for all $k \le u \le 16k$, where $\theta_1, \theta_2 > 0$ are two constants satisfying

$$\frac{4\gamma\left(\frac{16}{\min\{\theta_1,\theta_2\}\int_{\frac{1}{4}}^{\frac{3}{4}} G(1,s)ds}\right)}{\int_{\frac{1}{4}}^{\frac{3}{4}} H(1,\tau)h(\tau)d\tau}(\triangleq \underline{\lambda}) < \psi\left(\frac{1}{\psi^{-1}(\int_0^1 H(\tau,\tau)h(\tau)d\tau)\int_0^1 G(1,s)ds}\right)(\triangleq \overline{\lambda}).$$

Then the boundary value problem (1.1) has at least three positive solutions for any $\lambda \in (\underline{\lambda}, \overline{\lambda})$.

Proof. The condition on θ_1, θ_2 implies that the interval $(\underline{\lambda}, \overline{\lambda})$ is not empty. Based upon the arguments of Theorem 3.5 with aid of conditions in this theorem, we can derive that

$$i(T_\lambda, K_d, \overline{K_c}) = 1, \tag{3.32}$$

$$i(T_\lambda, \mathring{K}(\alpha,a,c), \overline{K_c}) = 1, \tag{3.33}$$

$$i(T_\lambda, \overline{K_c}\backslash(\overline{K_d} \cup K(\alpha,a,c)), \overline{K_c}) = -1. \tag{3.34}$$

Moreover, if $u \in K$ with $\|u\| = e$, then for $t \in [\frac{1}{4}, \frac{3}{4}]$

$$e \ge u(t) \ge t^{\alpha-1}\|u\| \ge \frac{1}{16}\|u\| = \frac{e}{16},$$

$$f(u(t)) \ge \varphi(\theta_1 u(t)) \ge \varphi(\theta_1 \cdot \frac{e}{16}). \tag{3.35}$$

Let $u \in \partial K_e$. From Lemma 2.9, condition (A) on φ, h is nonnegative and nontrivial on any subinterval of $(0,1)$, Remark 2.10 , Remark 2.12 and (3.35), we can obtain that for any $\lambda \in (\underline{\lambda}, \overline{\lambda})$

$$\begin{aligned}
\|T_\lambda(u)\| = T_\lambda(u)(1) &= \int_0^1 G(1,s)\varphi^{-1}\left(\int_0^1 H(s,\tau)\lambda h(\tau)f(u(\tau))d\tau\right)ds \\
&\ge \int_0^1 G(1,s)\varphi^{-1}\left(\int_0^1 s^{\beta-1}H(1,\tau)\lambda h(\tau)f(u(\tau))d\tau\right)ds \\
&= \int_0^1 G(1,s)\varphi^{-1}\left(s^{\beta-1}\int_0^1 H(1,\tau)\lambda h(\tau)f(u(\tau))d\tau\right)ds \\
&\ge \int_{\frac{1}{4}}^{\frac{3}{4}} G(1,s)\varphi^{-1}\left(s^{\beta-1}\int_{\frac{1}{4}}^{\frac{3}{4}} H(1,\tau)\lambda h(\tau)f(u(\tau))d\tau\right)ds \\
&\ge \int_{\frac{1}{4}}^{\frac{3}{4}} G(1,s)\varphi^{-1}\left((\frac{1}{4})^{\beta-1}\int_{\frac{1}{4}}^{\frac{3}{4}} H(1,\tau)\lambda h(\tau)\varphi(\theta_1 \cdot \frac{e}{16})d\tau\right)ds \\
&\ge \varphi^{-1}\left(\frac{1}{4}\lambda\int_{\frac{1}{4}}^{\frac{3}{4}} H(1,\tau)h(\tau)d\tau\varphi(\theta_1 \cdot \frac{e}{16})\right)\int_{\frac{1}{4}}^{\frac{3}{4}} G(1,s)ds \\
&> \varphi^{-1}\left(\frac{1}{4}\underline{\lambda}\int_{\frac{1}{4}}^{\frac{3}{4}} H(1,\tau)h(\tau)d\tau\varphi(\theta_1 \cdot \frac{e}{16})\right)\int_{\frac{1}{4}}^{\frac{3}{4}} G(1,s)ds
\end{aligned}$$

$$= \varphi^{-1}\left(\gamma\left(\frac{16}{\min\{\theta_1,\theta_2\}\int_{\frac{1}{4}}^{\frac{3}{4}} G(1,s)ds}\right)\varphi(\theta_1\cdot\frac{e}{16})\right)\int_{\frac{1}{4}}^{\frac{3}{4}} G(1,s)ds$$

$$\geq \frac{16}{\min\{\theta_1,\theta_2\}\int_{\frac{1}{4}}^{\frac{3}{4}} G(1,s)ds}\cdot\theta_1\cdot\frac{e}{16}\cdot\int_{\frac{1}{4}}^{\frac{3}{4}} G(1,s)ds \geq e.$$

i.e.

$$\|T_\lambda(u)\| > \|u\|, \text{ for } u \in \partial K_e.$$

If follow from Lemma 2.1 that

$$i(T_\lambda, K_e, \overline{K_c}) = 0. \tag{3.36}$$

By the additivity of the fixed point index and (3.32)(3.36), we get

$$i(T_\lambda, K_d\backslash\overline{K_e}, \overline{K_c}) = 1. \tag{3.37}$$

Therefore, by (3.33)(3.34) and (3.37), we can derive that T_λ has at least three fixed points u_1, u_2, u_3 in $\overline{K_c}$ such that $e < \|u_1\| < d$, $a < \alpha(u_2)$, $d < \|u_3\|$ with $\alpha(u_3) < a$. That is to say, u_1, u_2, u_3 are three positive solutions of the boundary value problem (1.1) with $e < \|u_1\| < d$, $\min_{t\in[\frac{1}{4},\frac{3}{4}]} u_2(t) > a = k$, $\|u_3\| > d$ and $\min_{t\in[\frac{1}{4},\frac{3}{4}]} u_3(t) < a = k$ for any $\lambda \in (\underline{\lambda}, \overline{\lambda})$.

Example 3.7 Consider the following boundary value problem

$$\begin{cases} D_{0^+}^{\frac{3}{2}}(D_{0^+}^{\frac{5}{2}}u(t)) = \lambda t^{-1}f(u), \quad t\in(0,1), \\ u(0) = u'(0) = u'(1) = 0, \quad \varphi(D_{0^+}^{\frac{5}{2}}u(0)) = (\varphi(D_{0^+}^{\frac{5}{2}}u(1)))' = 0. \end{cases} \tag{3.38}$$

Here we take $\varphi(x) = x$, $x \in \mathbb{R}$, and

$$f(u) = \begin{cases} \frac{1}{2}u, \quad 0 \leq u < 1, \\ 333u^2 - \frac{1991}{6}u - \frac{2}{3}, \quad 1 \leq u < 4, \\ 1000u, \quad 4 \leq u < 64, \\ 64000 + \frac{1}{2}(u-64), \quad u \geq 64. \end{cases} \tag{3.39}$$

Using the similar arguments in Example 4.2 of [22] and Example 3.2 in this paper, we can choose $\psi(x) = \gamma(x) \equiv \varphi(x)$ and check that problem (3.38) satisfies conditions $(A)(H)$ (F_1) and

$$F_\infty = \limsup_{u\to+\infty} \frac{64000 + \frac{1}{2}(u-64)}{u} = \frac{1}{2} < 1.$$

From the representation of f, we can find a constant $d = 1$ satisfying

$$\max_{u\in[0,1]} f(u) = \max_{u\in[0,1]} \frac{1}{2}u = \frac{1}{2} < 1,$$

which means that condition (i) of Theorem 3.5 holds. Additionally, we can find two constants $k = 4$ and $\theta = 1000$ such that $f(u) = 1000u$ for all $4 \le u \le 64$ and

$$\frac{4\gamma\left(\frac{16}{\theta\int_{\frac{1}{4}}^{\frac{3}{4}}G(1,s)ds}\right)}{\int_{\frac{1}{4}}^{\frac{3}{4}}H(1,\tau)h(\tau)d\tau} < \psi\left(\frac{1}{\psi^{-1}(\int_0^1 H(\tau,\tau)h(\tau)d\tau)\int_0^1 G(1,s)ds}\right), \tag{3.40}$$

which implies that condition (ii) of Theorem 3.5 also holds. Substituting (3.13) (3.14) (3.15) (3.16) into (3.40), we can have

$$\frac{4\gamma\left(\frac{16}{\theta\int_{\frac{1}{4}}^{\frac{3}{4}}G(1,s)ds}\right)}{\int_{\frac{1}{4}}^{\frac{3}{4}}H(1,\tau)h(\tau)d\tau} \doteq 0.59,$$

$$\psi\left(\frac{1}{\psi^{-1}(\int_0^1 H(\tau,\tau)h(\tau)d\tau)\int_0^1 G(1,s)ds}\right) \doteq 1.41.$$

Therefore, by Theorem 3.5, we can conclude that problem (3.38) has at least one nonnegative solution u_1 and two positive solutions u_2, u_3 for $\lambda \in (0.59, 1.41)$. Particularly, we can also obtain $\|u_1\| < 1$, $\min_{t\in[\frac{1}{4},\frac{3}{4}]} u_2(t) > 4$, $\|u_3\| > 4$ and $\min_{t\in[\frac{1}{4},\frac{3}{4}]} u_3(t) < 4$.

However, if we take $f(u) = \frac{1}{2}\sqrt{u}$ for $0 \le u < 1$ in (3.39), we can also find constants $e = 9\times 10^{-6}$, $d = 1$, $k = 4$, $\theta_1 = 1500$ and $\theta_2 = 1000$ satisfying all conditions of Theorem 3.6 hold. Thus, by Theorem 3.6, we can conclude that problem (3.38) has at least three positive solutions u_1, u_2, u_3 for $\lambda \in (0.59, 1.41)$. It is worth to note that $9\times 10^{-6} < \|u_1\| < 1$, $\min_{t\in[\frac{1}{4},\frac{3}{4}]} u_2(t) > 4$, $\|u_3\| > 4$ and $\min_{t\in[\frac{1}{4},\frac{3}{4}]} u_3(t) < 4$.

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