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SELF-ADAPTIVE INERTIAL SUBGRADIENT-LIKE EXTRAGRADIENT METHOD FOR BILEVEL SPLIT PSEUDOMONOTONE VARIATIONAL INEQUALITY WITH CFPP CONSTRAINT IN BANACH SPACES

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Abstract. This paper introduces and analyzes a self-adaptive inertial subgradient-like extragradient method designed to solve the bilevel split pseudomonotone variational inequality problem within the context of a common fixed-point problem, constrained by finite Bregman relatively nonexpansive mappings in p-uniformly convex and uniformly smooth Banach spaces. The method incorporates a strongly monotone mapping for the upper-level problem and a pseudomonotone operator for the lower-level. We establish the strong convergence of the proposed method under mild conditions on the algorithm parameters without requiring prior knowledge of the operator norm or the coefficient of the underlying operator. Finally, we present numerical experiments to demonstrate the practicality and applicability of the proposed method. Our findings extend and improve existing results in the literature.

Keywords and Phrases: Subgradient-like extragradient method, finite Bregman relatively nonexpansive mappings, bilevel split pseudomonotone variational inequality problem, common fixed point, Bregman projection.
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1. Introduction

Consider a nonempty, closed, convex subsets C and Q of real Hilbert spaces H_1 and H_2, respectively. Here, H_i (for $i = 1, 2$) is equipped with the inner product $\langle \cdot, \cdot \rangle$ and the induced norm $\| \cdot \|$. Let $\mathcal{T} : H_1 \to H_2$ be a nonzero linear bounded operator, and let $A, F : H_1 \to H_1$ and $B : H_2 \to H_2$ be nonlinear mappings. We denote $\mathrm{VI}(C, A)$ and $\mathrm{VI}(Q, B)$ as the solution sets for the following variational inequality problems (VIPs), respectively:

(i) The VIP seeks $x^* \in C$ such that $\langle Ax^*, x - x^* \rangle \geq 0$ for all $x \in C$.
(ii) The VIP aims to find $y^* \in Q$ such that $\langle By^*, y - y^* \rangle \geq 0$ for all $y \in Q$.

To the best of our knowledge, Korpelevich's extragradient approach [20], developed in 1976, remains one of the most extensively utilized techniques for solving the VIP. In other words, given any starting point $x_0 \in C$, the sequence $\{x_m\}$ is constructed using this method.

$$\begin{cases} u_m = P_C(x_m - \epsilon A x_m), \\ x_{m+1} = P_C(x_m - \epsilon A u_m) \ \forall \, m \geq 0, \end{cases}$$

where $\epsilon \in (0, \frac{1}{L})$ and L is the Lipschitz constant of A. If $\mathrm{VI}(C, A) \neq \emptyset$, it is well known that the sequence $\{x_m\}$ converges weakly to an element of $\mathrm{VI}(C, A)$. The literature on the problem (VIP) is extensive, and Korpelevich's extragradient approach has garnered widespread attention from numerous scholars. This method has undergone various improvements, as evidenced by the works in [6-12, 15, 17-19, 21, 25, 27, 29-31, 33-37].

In contrast, our focus shifts to the *Bilevel Split Variational Inequality Problem* (BSVIP) [2] formulated as follows: Seek $z^* \in \Omega$ such that $\langle Fz^*, z - z^* \rangle \geq 0 \ \forall z \in \Omega$, where $\Omega := \{z \in \mathrm{VI}(C, A) : \mathcal{T}z \in \mathrm{VI}(Q, B)\}$ denotes the solution set of the *Split Variational Inequality Problem* (SVIP), introduced by Censor et al. [13]. An iterative method is proposed in [13] for approximating a solution to the SVIP. For any given $x_1 \in H_1$, the sequence $\{x_n\}$ is formulated as

$$x_{n+1} = P_C(I - \lambda A)(x_n + \gamma \mathcal{T}^*(P_Q(I - \lambda B) - I)\mathcal{T}x_n) \ \forall n \geq 1,$$

where both A and B are inverse-strongly monotone, and $\mathcal{T}$ is a nonzero bounded linear operator. As illustrated in [13], the sequence $\{x_n\}$ converges weakly to a solution of the (SVIP). Importantly, the problem (VIP) can be restated as a Fixed-Point Problem (FPP):

$$Sy = P_Q(y - \mu By), \ \mu > 0,$$

where $\mathrm{VI}(Q, B) = \mathrm{Fix}(S)$, and $\mathrm{Fix}(S)$ designates the fixed-point set of S. Consequently, the (BSVIP) can be reformulated as follows: consider $A : H_1 \to H_1$ as an L-Lipschitzian quasimonotonicity mapping, $F : H_1 \to H_1$ as a κ-Lipschitzian and η-strongly monotone mapping, $\mathcal{T} : H_1 \to H_2$ as a nonzero linear bounded operator, and $S : H_2 \to H_2$ as a τ-demimetric mapping with $\tau \in (-\infty, 1)$. The problem is to find $z^* \in \Omega$ such that $\langle Fz^*, z - z^* \rangle \geq 0 \ \forall z \in \Omega$, where $\Omega := \{z \in \mathrm{VI}(C, A) : \mathcal{T}z \in \mathrm{Fix}(S)\}$. This problem is identified as a *Bilevel Split Quasimonotone Variational Inequality Problem* (BSQVIP).

Recently, Abuchu et al. [1] introduced a modified relaxed inertial subgradient extragradient algorithm to address the *Bilevel Split Quasimonotone Variational Inequality Problem* (BSQVIP). In their work, they demonstrated that the sequence generated by this algorithm converges strongly to a unique solution of the BSQVIP [1]. Subsequently, Ceng et al. [5] developed a triple-adaptive subgradient extragradient method with extrapolation to tackle a *Bilevel Split Pseudomonotone Variational Inequality Problem* (BSPVIP). The problem (BSPVIP) involves the *Common Fixed-point Problem* (CFPP) constraint of finitely many nonexpansive mappings in real Hilbert spaces. Notably, the BSPVIP incorporates a fixed-point problem of demimetric mapping. Consider a nonzero bounded linear operator $\mathcal{T}: H_1 \to H_2$ along with its adjoint $\mathcal{T}^*$, and a τ-demimetric mapping $S : H_2 \to H_2$, where $\tau \in (-\infty, 1)$. Additionally, let $A : H_1 \to H_1$ be a pseudomonotone and L-Lipschitz continuous mapping. Assume a finite set of nonexpansive self-mappings $\{S_i\}_{i=1}^N$ on H_1, and define

$$\varOmega := \{z \in \mathrm{VI}(C, A) : \mathcal{T}z \in \mathrm{Fix}(S)\} \text{ and } \varXi := \bigcap_{i=1}^{N} \mathrm{Fix}(S_i) \cap \varOmega \neq \emptyset.$$

If necessary, denote $S_n := S_{n \bmod N}$ for $n = 1, 2, 3, \ldots$. Introduce a contraction mapping $f : H_1 \to H_1$ with a constant $\delta \in [0, 1)$, and a mapping $F : H_1 \to H_1$ that is both η-strongly monotone and κ-Lipschitzian, satisfying $\delta < \zeta := 1 - \sqrt{1 - \rho(2\eta - \rho\kappa^2)}$ for $\rho \in (0, 2\eta/\kappa^2)$. Assume sequences $\{\beta_n\}, \{\gamma_n\}, \{\varepsilon_n\} \subset (0, \infty)$ such that $\beta_n + \gamma_n < 1$, $\sum_{n=1}^{\infty} \beta_n = \infty$, $\lim_{n\to\infty} \beta_n = 0$, $\liminf_{n\to\infty} \gamma_n(1 - \gamma_n) > 0$, and $\varepsilon_n = o(\beta_n)$. The specification of the triple-adaptive subgradient extragradient method is as follows.

Algorithm 1.1 (Refer to [7, Algorithm 3.1]).
Initialization: Choose arbitrary values for $\lambda_1 > 0, \epsilon > 0, \sigma \geq 0, \mu \in (0, 1), \alpha \in [0, 1)$, and select x_0 and x_1 from H_1.

Iterative steps: Calculate x_{n+1} as follows:
Step 1. Given the iterates x_{n-1} and x_n ($n \geq 1$), determine α_n such that $0 \leq \alpha_n \leq \overline{\alpha}_n$, where

$$\overline{\alpha}_n = \begin{cases} \min\left\{\alpha, \frac{\varepsilon_n}{\|x_n - x_{n-1}\|}\right\} & \text{if } x_n \neq x_{n-1}, \\ \alpha & \text{otherwise.} \end{cases}$$

Step 2. Compute $w_n = S_n x_n + \alpha_n(S_n x_n - S_n x_{n-1})$ and $y_n = P_C(w_n - \lambda_n A w_n)$.

Step 3. Construct $C_n := \{y \in H_1 : \langle w_n - \lambda_n A w_n - y_n, y_n - y\rangle \geq 0\}$, and find $v_n = P_{C_n}(w_n - \lambda_n A y_n)$ and $z_n = v_n - \sigma_n \mathcal{T}^*(I - S)\mathcal{T} v_n$.

Step 4. Calculate $x_{n+1} = \beta_n f(x_n) + \gamma_n x_n + ((1 - \gamma_n)I - \beta_n \rho F)z_n$ and update

$$\lambda_{n+1} = \begin{cases} \min\{\mu \dfrac{\|w_n - y_n\|^2 + \|v_n - y_n\|^2}{2\langle Aw_n - Ay_n, v_n - y_n\rangle}, \lambda_n\} & \text{if } \langle Aw_n - Ay_n, v_n - y_n\rangle > 0, \\ \lambda_n & \text{otherwise.} \end{cases}$$

For any fixed $\epsilon > 0$, σ_n is selected as a bounded sequence that satisfies

$$0 < \epsilon \leq \sigma_n \leq \frac{(1-\tau)\|\mathcal{T}v_n - S\mathcal{T}v_n\|^2}{\|\mathcal{T}^*(\mathcal{T}v_n - S\mathcal{T}v_n)\|^2} \quad \text{if } \mathcal{T}v_n \neq S\mathcal{T}v_n,$$

otherwise set $\sigma_n = \sigma \geq 0$. Return to Step 1 after setting $n := n+1$.

In reference [5], it was established that the sequence $\{x_n\}$ strongly converges to a point $z \in \Xi$, which serves as the unique solution to the problem (VIP):

$$\langle (\rho F - f)z, y - z \rangle \geq 0 \ \forall y \in \Xi.$$

Moving to a different perspective, for p and q in the range $(1, \infty)$ with $\frac{1}{p} + \frac{1}{q} = 1$, let E be a Banach space characterized by p-uniform convexity and uniform smoothness. Consider C as a nonempty, closed, and convex subset of E. The dual space of E is denoted as E^*. The norm and duality pairing between elements in E and E^* are represented by $\|\cdot\|$ and $\langle \cdot, \cdot \rangle$, respectively. Consider the duality mappings of E and E^* denoted as J_E^p and $J_{E^*}^q$, respectively. Define the function $f_p(x) = \|x\|^p/p$ for all $x \in E$ and let D_{f_p} represent the Bregman distance with respect to f_p. Additionally, denote Π_C as the Bregman projection of E onto C with respect to f_p. Eskandani et al. [15] introduced a Mann-type subgradient-like extragradient method with a line search process designed to find a common solution to the problem (VIP) associated with a uniformly continuous pseudomonotonicity mapping $F : E \to E^*$ and the Fixed-Point Problem (FPP) of a Bregman relatively nonexpansive mapping $T : C \to C$. It is assumed that there exist sequences $\{\alpha_n\}$ and $\{\beta_n\}$ in the interval $(0,1)$ such that $\lim_{n\to\infty} \alpha_n = 0$, $\liminf_{n\to\infty} \beta_n(1-\beta_n) > 0$, and $\sum_{n=1}^{\infty} \alpha_n = \infty$.

Algorithm 1.2 (Refer to [15]).

Initialization: Given $\mu > 0$, $l \in (0,1)$, $\lambda \in \left(0, \frac{1}{\mu}\right)$, choose an arbitrary point $x_1 \in C$.

Iterative steps: For the current iterate x_n, calculate x_{n+1} as follows:

Step 1. Compute $y_n = \Pi_C(J_{E^*}^q(J_E^p x_n - \lambda F x_n))$ and $r_\lambda(x_n) := x_n - y_n$. If $r_\lambda(x_n) = 0$ and $Tx_n = x_n$, then stop; $x_n \in \Omega = \mathrm{Fix}(T) \cap \mathrm{VI}(C, F)$. Otherwise, go to the next step.

Step 2. Compute $t_n = x_n - \tau_n r_\lambda(x_n)$, where $\tau_n = l^{j_n}$ and j_n is the smallest nonnegative integer j satisfying $\langle Fx_n - F(x_n - l^j r_\lambda(x_n)), r_\lambda(x_n) \rangle \leq \frac{\mu}{2} D_{f_p}(x_n, y_n)$.

Step 3. Compute

$$v_n = J_{E^*}^q(\beta_n J_E^p x_n + (1-\beta_n) J_E^p(T\Pi_{C_n} x_n))$$

and

$$x_{n+1} = \Pi_C(J_{E^*}^q(\alpha_n J_E^p u + (1-\alpha_n) J_E^p v_n)),$$

where $C_n = \{x \in C : h_n(x) \leq 0\}$ and $h_n(x) = \langle Ft_n, x - x_n \rangle + \frac{\tau_n}{2\lambda} D_{f_p}(x_n, y_n)$.

In [15], the strong convergence of the sequence $\{x_n\}$ to $\hat{u} = \Pi_\Omega u$ is established, representing the unique solution to the problem (VIP): $\langle J_E^P(\hat{u}) - J_E^P(u), v - \hat{u} \rangle \geq 0$

for all $v \in \Omega$. In this study, let H and E denote a Hilbert space and a p-uniformly convex and uniformly smooth Banach space, respectively. Consider $\mathcal{T} : E \to H$ as a nonzero bounded linear operator. Drawing inspiration from prior research, we propose a self-adaptive inertial subgradient-like extragradient method to address the BSPVIP with a CFPP constraint involving a finite set of Bregman relatively nonexpansive self-mappings $\{S_i\}_{i=1}^N$ on C. The BSPVIP encompasses a lower-level VIP associated with a uniformly continuous pseudomonotonicity operator $F : E \to E^*$ and an additional FPP related to a demimetric mapping $S : H \to H$. Consider $\varXi = \bigcap_{i=1}^N \mathrm{Fix}(S_i) \cap \varOmega \neq \emptyset$, where $\varOmega = \{z \in \mathrm{VI}(C, F) : \mathcal{T}z \in \mathrm{Fix}(S)\}$. Subject to mild conditions, we establish the strong convergence of the proposed method to $\hat{u} = \Pi_{\varXi} u$, representing a unique solution to the upper-level problem (VIP): $\langle J_E^P(\hat{u}) - J_E^P(u), v - \hat{u} \rangle \geq 0$ for all $v \in \varXi$. Finally, we present an illustrative example to demonstrate the practical applicability of the proposed method.

The paper is organized as follows: Section 2 introduces essential concepts and basic tools for subsequent discussions. In Section 3, we delve into the convergence analysis of the proposed algorithm. Section 4 applies our key findings to address the BSPVIP with a CFPP constraint through an illustrative example and multiple numerical examples. Importantly, our results significantly enhance and extend the findings presented in [15], [5], [1]. Section 5 concludes the paper.

2. Preliminaries

Let E denote a real Banach space, with its dual denoted as E^*. Consider a sequence $\{x_n\}$ in E. We use the notation $x_n \rightharpoonup x$ (or $x_n \to x$) to signify weak (or strong) convergence of the sequence $\{x_n\}$ to x. Additionally, let $\omega_w(x_n)$ represent the weak limit point set of $\{x_n\}$, defined as

$$\omega_w(x_n) = \{x^\dagger \in E : x_{n_k} \rightharpoonup x^\dagger \text{ for certain } \{x_{n_k}\} \subset \{x_n\}\}.$$

Define $U = \{x \in E : \|x\| = 1\}$, and let $q \in (1, 2]$ and $p \in [2, \infty)$ satisfy $\frac{1}{p} + \frac{1}{q} = 1$. E is considered strictly convex if, for all x and y in U such that $x \neq y$, $\|x + y\|/2 < 1$. It is termed uniformly convex if, for every $\ell \in (0, 2]$, there exists $\bar{\delta} > 0$ such that for all x and y in U with $\|x - y\| \geq \ell$, $\|x + y\|/2 \leq 1 - \bar{\delta}$. It is evident that the uniform convexity of E implies reflexivity and strict convexity. The convexity modulus of E is a mapping $\delta : [0, 2] \to [0, 1]$, defined by

$$\delta(\ell) = \inf\{1 - \|x + y\|/2 : x, y \in U \text{ with } \|x - y\| \geq \ell\}.$$

The space E is labeled uniformly convex if $\delta(\ell) > 0$ for all $\ell \in (0, 2]$. Furthermore, E is termed p-uniformly convex if there exists $c > 0$ such that $\delta(\ell) \geq c\ell^p$ for all $\ell \in [0, 2]$.

The smoothness modulus, denoted by $\rho_E : [0, \infty) \to [0, \infty)$, is defined as

$$\rho_E(\ell) = \sup\{(\|x + \ell y\| + \|x - \ell y\|)/2 - 1 : x, y \in U\}.$$

The space E is considered uniformly smooth if and only if $\lim_{\ell \to 0} \rho_E(\ell)/\ell = 0$, and q-uniformly smooth if there exists $C_q > 0$ such that $\rho_E(\ell) \leq C_q \ell^q$ for all $\ell > 0$. The p-uniform convexity of E is stated equivalently as the q-uniform smoothness of its dual, E^*; for more details, please refer to [28].

Let $B(0,\ell) = \{x \in E : \|x\| \leq \ell\}$ for each $\ell > 0$. A function $f : E \to \mathbb{R}$ is termed uniformly convex on bounded sets (see [15]) if $\rho_\ell(t) > 0$ for all $\ell, t > 0$, where $\rho_\ell(t) : [0,\infty) \to [0,\infty]$ is defined by

$$\begin{aligned}\rho_\ell(t) \;=\; &\inf\{[\alpha f(x) + (1-\alpha)f(y) - f(\alpha x + (1-\alpha)y)]/\alpha(1-\alpha) : \\ &\alpha \in (0,1) \text{ and } x, y \in B(0,\ell) \text{ with } \|x - y\| = t\}.\end{aligned}$$

This function ρ_ℓ is referred to as the gauge of uniform convexity of f and is known to be a nondecreasing function.

Consider the function $f : E \to \mathbb{R}$ exhibiting convexity. If the limit

$$\lim_{\ell \to 0^+} \frac{f(v + \ell y) - f(v)}{\ell}$$

exists for each $y \in E$, then f is referred to as Gâteaux differentiable at v. In this context, the gradient of f at v is represented by the linear function $\nabla f(v)$, defined as

$$\langle \nabla f(v), y \rangle := \lim_{\ell \to 0^+} \frac{f(v + \ell y) - f(v)}{\ell}$$

for each $y \in E$. The function f is characterized as Gâteaux differentiable if and only if it is Gâteaux differentiable at each $v \in E$. When the limit $\lim_{\ell \to 0^+} \frac{f(v+\ell y) - f(v)}{\ell}$ is uniformly attained for any $y \in U$, it is asserted that f is Fréchet differentiable at v. Moreover, f is designated as uniformly Fréchet differentiable on a subset $K \subset E$ if $\lim_{\ell \to 0^+} \frac{f(v+\ell y) - f(v)}{\ell}$ is uniformly achieved for $(v, y) \in K \times U$. A Banach space E is classified as smooth if its norm is Gâteaux differentiable.

For p and q in the interval $(1,\infty)$ such that $\frac{1}{p} + \frac{1}{q} = 1$, the duality mapping $J_E^p : E \to E^*$ is defined as follows:

$$J_E^p(v) = \{\psi \in E^* : \langle \psi, v \rangle = \|v\|^p \text{ and } \|\psi\| = \|v\|^{p-1}\} \quad \forall v \in E.$$

It is clear that the smoothness of E is equivalent to $J_E^p : E \to E^*$ being a single-valued mapping. Similarly, the reflexivity of E corresponds to the surjectivity of J_E^p, while the strict convexity of E is linked to the injectivity of J_E^p. Consequently, if E is a Banach space that is smooth, strictly convex, and reflexive, then J_E^p forms a single-valued bijection. In this particular scenario, it also holds that $J_E^p = (J_{E^*}^q)^{-1}$, where $J_{E^*}^q$ represents the duality mapping of E^*. Additionally, it is evident that the uniform smoothness of E is equivalent to the uniform Fréchet differentiability of the function $f_p(v) = \|v\|^p/p$ on bounded sets, which, in turn, is synonymous with the single-valued and uniform continuity of J_E^p on bounded sets. Furthermore, the uniform convexity of E aligns with the uniform convexity of the function f_p (see [28]).

Consider the function $f : E \to \mathbb{R}$, which possesses both Gâteaux's differentiability and convexity. The Bregman distance with respect to f is expressed as

$$D_f(v, y) := f(v) - f(y) - \langle \nabla f(y), v - y \rangle \quad \forall v, y \in E.$$

The Bregman distance is notable for not conforming to conventional metric criteria. Although it is clear that $D_f(v, v) = 0$, the condition $D_f(v, y) = 0$ does not necessarily imply $v = y$. In general, D_f lacks symmetry and fails to satisfy the triangle inequality. However, it does adhere to the three-point identity:

$$D_f(v, y) + D_f(y, z) = D_f(v, z) - \langle \nabla f(y) - \nabla f(z), v - y \rangle.$$

For a more comprehensive understanding of Bregman functions and distances, we refer to [23].

It is crucial to emphasize that the duality mapping J_E^p on the smooth Banach space E functions as the Gâteaux derivative of f_p. Subsequently, the Bregman distance with respect to f_p is expressed as follows:

$$\begin{aligned} D_{f_p}(v, y) &= \|v\|^p/p - \|y\|^p/p - \langle J_E^p(y), v - y \rangle \\ &= \|v\|^p/p + \|y\|^p/q - \langle J_E^p(y), v \rangle \\ &= (\|y\|^p - \|v\|^p)/q - \langle J_E^p(y) - J_E^p(v), v \rangle. \end{aligned}$$

For $p \geq 2$, a significant relationship exists between the metric and Bregman distance in the smooth and p-uniformly convex Banach space E:

$$\tau \|v - y\|^p \leq D_{f_p}(v, y) \leq \langle J_E^p(v) - J_E^p(y), v - y \rangle, \tag{2.1}$$

where $\tau > 0$ is a fixed constant (refer to [26]). Using (2.1), it becomes evident that for any bounded sequence $\{v_n\} \subset E$, the convergence $v_n \to v$ is equivalent to $D_{f_p}(v, v_n) \to 0$ as $n \to \infty$.

Consider a nonempty, closed, convex subset C of a reflexive, smooth, and strictly convex Banach space E. Bregman projections are defined as the minimizers of Bregman distances. The Bregman projection of $v \in E$ onto C with respect to f_p is the unique element $\Pi_C v \in C$ such that $D_{f_p}(\Pi_C v, v) = \min_{y \in C} D_{f_p}(y, v)$. In Hilbert spaces, the Bregman projection with respect to f_2 reduces to the metric projection. Employing [4, Corollary 4.4] and [3, Theorem 2.1], in uniformly convex Banach spaces, Bregman projections can be characterized by the following inequality:

$$\langle J_E^p(v) - J_E^p(\Pi_C v), y - \Pi_C v \rangle \leq 0 \quad \forall y \in C. \tag{2.2}$$

Furthermore, this inequality corresponds to the descent property:

$$D_{f_p}(y, \Pi_C v) + D_{f_p}(\Pi_C v, v) \leq D_{f_p}(y, v) \quad \forall y \in C. \tag{2.3}$$

In the case where $p = 2$, the duality mapping J_E^p reduces to the normalized duality mapping, denoted by J. The function $\phi : E^2 \to \mathbb{R}$ is formulated as:

$$\phi(v, y) = \|v\|^2 - 2\langle Jy, v \rangle + \|y\|^2 \quad \forall v, y \in E,$$

and $\Pi_C(v) = \text{argmin}_{y \in C} \phi(y, v)$ for all $v \in E$.

According to [15], a function $V_{f_p} : E \times E^* \to [0, \infty)$ linked via f_p is formulated below

$$V_{f_p}(v, v^*) = \|v\|^p/p - \langle v^*, v\rangle + \|v^*\|^q/q \quad \forall (v, v^*) \in E \times E^*. \tag{2.4}$$

Hence, $V_{f_p}(v, v^*) = D_{f_p}(v, J^q_{E^*}(v^*))$ $\forall (v, v^*) \in E \times E^*$. Furthermore, using the subdifferential inequality, we get

$$V_{f_p}(v, v^*) + \langle y^*, J^q_{E^*}(v^*) - v\rangle \le V_{f_p}(v, v^* + y^*) \quad \forall v \in E,\ v^*, y^* \in E^*. \tag{2.5}$$

The second variable of V_{f_p} is also convex. So you have

$$D_{f_p}(z, J^q_{E^*}(\sum_{i=1}^n \ell_i J^p_E(v_i))) \le \sum_{i=1}^n \ell_i D_{f_p}(z, v_i) \tag{2.6}$$

for all $z \in E, \{v_i\}_{i=1}^n \subset E, \{\ell_i\}_{i=1}^n \subset [0, 1]$ with $\sum_{i=1}^n \ell_i = 1$.

Lemma 2.1 (See [3]). Consider a uniformly convex Banach space E, and let $\{v_n\}$ and $\{y_n\}$ be two sequences in E, where the first sequence is bounded. If $\lim_{n\to\infty} D_{f_p}(y_n, v_n) = 0$, then it follows that $\lim_{n\to\infty} \|y_n - v_n\| = 0$.

Given a mapping $T : C \to C$, we define $\mathrm{Fix}(T)$ as the fixed point set of T, represented by $\mathrm{Fix}(T) = \{v \in C : v = Tv\}$. A point $v \in C$ is considered an asymptotic fixed point of T if there exists a sequence $\{v_n\} \subset C$ such that $v_n \rightharpoonup v$ and $v_n - Tv_n \to 0$. The set of asymptotic fixed points of T is denoted by $\widehat{\mathrm{Fix}}(T)$. The concept of an asymptotic fixed point was introduced by Reich [24]. A mapping $T : C \to C$ is regarded as Bregman relatively nonexpansive with respect to f_p if $\mathrm{Fix}(T) = \widehat{\mathrm{Fix}}(T) \neq \emptyset$, and $D_{f_p}(u, Tv) \le D_{f_p}(u, v)$ for all $v \in C$ and $u \in \mathrm{Fix}(T)$.

A mapping $F : C \to E^*$ is called

(i) monotone on C if $\langle Fv - Fy, v - y\rangle \ge 0$ for all $v, y \in C$,
(ii) pseudomonotone if $\langle Fv, y - v\rangle \ge 0$ implies $\langle Fy, y - v\rangle \ge 0$ for all $v, y \in C$,
(iii) L-Lipschitz continuous or L-Lipschitzian if there exists $L > 0$ such that $\|Fv - Fy\| \le L\|v - y\|$ for all $v, y \in C$, and
(iv) weakly sequentially continuous if, for every sequence $\{v_n\} \subset C$, the weak convergence of $\{v_n\}$ to v implies the weak convergence of $\{Fv_n\}$ to Fv.

Lemma 2.2 (See [15]). Consider a constant $r > 0$. Let E be a Banach space, and let $f : E \to \mathbb{R}$ be a uniformly convex function on bounded subsets of E. For any $i, j \in \{1, 2, \ldots, n\}$, $\{v_k\}_{k=1}^n \subset B(0, r)$, and $\{\ell_k\}_{k=1}^n \subset (0, 1)$ with $\sum_{k=1}^n \ell_k = 1$, the inequality

$$f\left(\sum_{k=1}^n \ell_k v_k\right) \le \sum_{k=1}^n \ell_k f(v_k) - \ell_i \ell_j \rho_r(\|v_i - v_j\|),$$

holds, where ρ_r represents the gauge of uniform convexity of f.

Lemma 2.3 (See [18]). Let E_1 and E_2 be two Banach spaces. Suppose that the mapping $F : E_1 \to E_2$ is uniformly continuous on bounded subsets of E_1, and let M

be a bounded subset of E_1. The conclusion is that $F(M)$ is also bounded.

Lemma 2.4 (See [14]). Consider a nonempty closed convex subset C of a real Banach space E, and let $F : C \to E^*$ be pseudomonotone and continuous. Then, $x^\dagger \in C$ is a solution to the variational inequality problem (VIP) $\langle Fx^\dagger, x - x^\dagger \rangle \geq 0$ for all $x \in C$ if and only if $\langle Fx, x - x^\dagger \rangle \geq 0$ for all $x \in C$.

The following lemma has been previously established in $\mathbb{R}^n$ and is documented in [16]. It is evident that the proof of this lemma in Banach spaces closely parallels that in $\mathbb{R}^n$. Consequently, we present the lemma here while abstaining from providing the proof within the context of Banach spaces.

Lemma 2.5. Let C be a nonempty closed convex subset of a Banach space E. Consider a real-valued function h defined on E, and let $K := \{x \in C : h(x) \leq 0\}$. Assuming K is nonempty and h is Lipschitz continuous on C with a modulus of $\theta > 0$, then it holds that

$$\theta \operatorname{dist}(v, K) \geq \max\{h(v), 0\}$$

for all $v \in C$, where $\operatorname{dist}(v, K)$ denotes the distance from v to K.

Lemma 2.6 (See [32]). Let $\{a_n\}$ be a sequence in $[0, \infty)$ satisfying the recurrence relation $a_{n+1} \leq (1 - \beta_n)a_n + \beta_n \gamma_n$ for all $n \geq 1$, where $\{\beta_n\}$ and $\{\gamma_n\}$ are real sequences. Suppose the following conditions are met:

(i) $\{\beta_n\} \subset [0, 1]$ and $\sum_{n=1}^{\infty} \beta_n = \infty$, and
(ii) $\limsup_{n\to\infty} \gamma_n \leq 0$ or $\sum_{n=1}^{\infty} |\beta_n \gamma_n| < \infty$.

Then, $\lim_{n\to\infty} a_n = 0$.

Lemma 2.7 (See [22]). Consider a sequence of real numbers $\{\mathbf{\Phi}_n\}$ that does not decrease at infinity, meaning there exists a subsequence $\{\mathbf{\Phi}_{n_k}\} \subset \{\mathbf{\Phi}_n\}$ such that $\mathbf{\Phi}_{n_k} < \mathbf{\Phi}_{n_k+1}$ for all $k \geq 1$. Define the sequence of integers $\{\psi(n)\}_{n \geq n_0}$ as

$$\psi(n) = \max\{k \leq n : \mathbf{\Phi}_k < \mathbf{\Phi}_{k+1}\},$$

where $n_0 \geq 1$ is an integer satisfying $\{k \leq n_0 : \mathbf{\Phi}_k < \mathbf{\Phi}_{k+1}\} \neq \emptyset$. Then, the following assertions hold:

(i) $\psi(n_0) \leq \psi(n_0 + 1) \leq \cdots$ and $\psi(n) \to \infty$;
(ii) $\mathbf{\Phi}_{\psi(n)} \leq \mathbf{\Phi}_{\psi(n)+1}$ and $\mathbf{\Phi}_n \leq \mathbf{\Phi}_{\psi(n)+1}$ for all $n \geq n_0$.

3. Main results

In this section, let H be a real Hilbert space, and let the feasible set C be a nonempty closed convex subset of a real, p-uniformly convex, and uniformly smooth Banach space E. We are now poised to present and analyze our iterative method for solving the BSPVIP with the CFPP constraint of finite Bregman relatively nonexpansive self-mappings $\{S_i\}_{i=1}^N$ on C. We assume throughout that the following conditions are satisfied:

(C1) The mapping $S : H \to H$ is a τ-demimetric mapping with $\tau \in (-\infty, 1)$. Additionally, $I - S$ is demiclosed at zero, denoted as τ-demimetric, if there exists $\tau \in (-\infty, 1)$ such that

$$\langle (I-S)v, v-y \rangle \geq \tfrac{1-\tau}{2} \|(I-S)v\|^2$$

for all $v \in C$ and $y \in \mathrm{Fix}(S) \neq \emptyset$. Furthermore, $I - S$ is called demiclosed at zero if, for any sequence $\{v_n\} \subset H$ where $v_n \rightharpoonup v$ and $(I-S)v_n \to 0$, it implies $v \in \mathrm{Fix}(S)$.

(C2) For $i = 1, \ldots, N$, $S_i : C \to C$ is a uniformly continuous and Bregman relatively nonexpansive mapping. The sequence $\{S_n\}_{n=1}^{\infty}$ is defined as $S_n := S_{n \bmod N}$ for an integer $n \geq 1$, where the mod function takes values in the set $\{1, 2, \ldots, N\}$. Specifically, if $n = jN + m$ for some integers $j \geq 0$ and $0 \leq m < N$, then $S_n = S_N$ if $m = 0$ and $S_n = S_m$ if $0 < m < N$.

(C3) $\mathcal{T}\mathrm{VI}(C, G) \subset \mathrm{Fix}(S)$, where $G = \mathcal{T}^*(I-S)\mathcal{T} : E \to E^*$ is pseudomonotone and uniformly continuous on C. This is such that $\|Gv\| \leq \liminf_{n\to\infty} \|Gv_n\|$ for each $\{v_n\} \subset C$ with $v_n \rightharpoonup v$.

(C4) The mapping $F : E \to E^*$ is pseudomonotone and uniformly continuous on C. Specifically, $\|Fv\| \leq \liminf_{n\to\infty} \|Fv_n\|$ for each $\{v_n\} \subset C$ with $v_n \rightharpoonup v$.

(C5) The intersection $\varXi = \bigcap_{i=1}^{N} \mathrm{Fix}(S_i) \cap \varOmega \neq \emptyset$ with

$$\varOmega = \{v \in \mathrm{VI}(C, F) : \mathcal{T}v \in \mathrm{Fix}(S)\}.$$

Algorithm 3.1.

Initialization: Given arbitrarily chosen x_1 and x_0 from the set C, consider $\epsilon > 0$, $\mu > 0$, $\lambda \in (0, \frac{1}{\mu})$, and $l \in (0,1)$. Choose sequences $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, and $\{\ell_n\}$, where α_n, β_n, and γ_n are within the interval $(0,1)$, and ℓ_n is within the interval $(0,\infty)$. Ensure that $\lim_{n\to\infty} \ell_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$, $\lim_{n\to\infty} \alpha_n = 0$, $\liminf_{n\to\infty} \beta_n(1-\beta_n) > 0$, and $\liminf_{n\to\infty} \gamma_n(1-\gamma_n) > 0$. Additionally, for given iterates x_n and x_{n-1} where $n \geq 1$, select ϵ_n such that $0 \leq \epsilon_n \leq \tilde{\epsilon}_n$, where $\sup_{n\geq 1} \frac{\epsilon_n}{\alpha_n} < \infty$, and

$$\tilde{\epsilon}_n = \begin{cases} \min\left\{\epsilon, \frac{\ell_n}{\|J_E^p x_n - J_E^p(2x_n - x_{n-1})\|}\right\} & \text{if } x_n \neq x_{n-1}, \\ \epsilon & \text{otherwise.} \end{cases}$$

Iterative steps: Calculate x_{n+1} in the following way:

Step 1. Put $u_n = J_{E^*}^q((1-\epsilon_n)J_E^p x_n + \epsilon_n J_E^p(2x_n - x_{n-1}))$, and calculate $g_n = J_{E^*}^q(\gamma_n J_E^p x_n + (1-\gamma_n)J_E^p u_n)$, $y_n = \Pi_C(J_{E^*}^q(J_E^p g_n - \lambda G g_n))$, $r_\lambda(g_n) := g_n - y_n$ and $s_n = g_n - \tau_n r_\lambda(g_n)$, here, $\tau_n := l^{i_n}$, where i_n represents the smallest nonnegative integer i such that

$$\langle G g_n - G(g_n - l^i r_\lambda(g_n)), g_n - y_n \rangle \leq \frac{\mu}{2} D_{f_p}(g_n, y_n). \tag{3.1}$$

Step 2. Calculate $w_n = \Pi_{K_n}(g_n)$, with $K_n := \{x \in C : h_n(x) \le 0\}$ and

$$h_n(x) = \langle Gs_n, x - g_n\rangle + \frac{\tau_n}{2\lambda} D_{f_p}(g_n, y_n). \tag{3.2}$$

Step 3. Calculate $\bar{y}_n = \Pi_C(J^q_{E^*}(J^p_E w_n - \lambda F w_n))$, $R_\lambda(w_n) := w_n - \bar{y}_n$ and $t_n = w_n - \bar{\tau}_n R_\lambda(w_n)$, where $\bar{\tau}_n := l^{j_n}$ and j_n is the smallest nonnegative integer j satisfying

$$\langle Fw_n - F(w_n - l^j R_\lambda(w_n)), w_n - \bar{y}_n\rangle \le \frac{\mu}{2} D_{f_p}(w_n, \bar{y}_n). \tag{3.3}$$

Step 4. Put $z_n = \Pi_{C_n}(w_n)$, and compute $v_n = J^q_{E^*}(\beta_n J^p_E w_n + (1-\beta_n) J^p_E(S_n z_n))$ and $x_{n+1} = \Pi_C(J^q_{E^*}(\alpha_n J^p_E u + (1-\alpha_n) J^p_E v_n)$, where $C_n := \{x \in C : \hbar_n(x) \le 0\}$ and

$$\hbar_n(x) = \langle Ft_n, x - w_n\rangle + \frac{\bar{\tau}_n}{2\lambda} D_{f_p}(w_n, \bar{y}_n). \tag{3.4}$$

Proceed to Step 1 after setting $n := n + 1$.

The lemmas presented below are crucial in deriving our main results in the subsequent discussion.

Lemma 3.1. Consider the sequence $\{x_n\}$ generated by Algorithm 3.1. The following inequalities hold: $\langle Gg_n, r_\lambda(g_n)\rangle \ge \frac{1}{\lambda} D_{f_p}(g_n, y_n)$ and $\langle Fw_n, R_\lambda(w_n)\rangle \ge \frac{1}{\lambda} D_{f_p}(w_n, \bar{y}_n)$.

Proof. Given the similarity of the last two inequalities, it is sufficient to demonstrate the validity of the latter. By utilizing the definition of $\bar{y}_n$ and the properties of Π_C, we can express it as follows:

$$\langle J^p_E w_n - \lambda F w_n - J^p_E \bar{y}_n, y - \bar{y}_n\rangle \le 0 \quad \forall y \in C.$$

Substituting $y = w_n$ into the above inequality and leveraging the properties from (2.1), we obtain:

$$D_{f_p}(w_n, \bar{y}_n) \le \langle J^p_E w_n - J^p_E \bar{y}_n, w_n - \bar{y}_n\rangle \le \lambda \langle Fw_n, w_n - \bar{y}_n\rangle.$$

Thus, the desired result is achieved.

Lemma 3.2. The Armijo-type search rules (3.1) and (3.3), along with the sequence $\{x_n\}$ generated in Algorithm 3.1, are well-defined.

Proof. Since rules (3.1) and (3.3) are analogous, it suffices to establish the validity of the latter rule (3.3). Given $l \in (0,1)$ and the uniform continuity of F on C, we have $\lim_{j\to\infty}\langle Fw_n - F(w_n - l^j R_\lambda(w_n)), R_\lambda(w_n)\rangle = 0$. If $R_\lambda(w_n) = 0$, then $j_n = 0$. In the case where $R_\lambda(w_n) \ne 0$, there exists an integer $j_n \ge 0$ satisfying (3.3).

For every $n \ge 1$, it is evident that C_n is both closed and convex. We claim that $\varXi \subset C_n$. Let $z \in \varXi = \bigcap_{i=1}^N \mathrm{Fix}(S_i) \cap \varOmega$, with $\varOmega = \{z \in \mathrm{VI}(C, F) : \mathcal{T} z \in \mathrm{Fix}(S)\}$. By utilizing Lemma 2.4, we obtain $\langle Ft_n, t_n - z\rangle \ge 0$, and hence

$$\begin{aligned}\hbar_n(z) &= \langle Ft_n, z - w_n\rangle + \tfrac{\bar{\tau}_n}{2\lambda} D_{f_p}(w_n, \bar{y}_n) \\ &= -\langle Ft_n, w_n - t_n\rangle - \langle Ft_n, t_n - z\rangle + \tfrac{\bar{\tau}_n}{2\lambda} D_{f_p}(w_n, \bar{y}_n) \\ &\le -\bar{\tau}_n \langle Ft_n, R_\lambda(w_n)\rangle + \tfrac{\bar{\tau}_n}{2\lambda} D_{f_p}(w_n, \bar{y}_n).\end{aligned} \tag{3.5}$$

Using (3.3), we have

$$\langle Fw_n - Ft_n, R_\lambda(w_n)\rangle \leq \frac{\mu}{2} D_{f_p}(w_n, \bar{y}_n).$$

This, along with Lemma 3.1, leads to

$$\begin{aligned}\langle Ft_n, R_\lambda(w_n)\rangle \quad &\geq \langle Fw_n, R_\lambda(w_n)\rangle - \tfrac{\mu}{2} D_{f_p}(w_n, \bar{y}_n) \\ &\geq (\tfrac{1}{\lambda} - \tfrac{\mu}{2}) D_{f_p}(w_n, \bar{y}_n).\end{aligned}$$

Combining this with (3.5) yields

$$\hbar_n(z) \leq -\frac{\bar{\tau}_n}{2}\left(\frac{1}{\lambda} - \mu\right) D_{f_p}(w_n, \bar{y}_n) \leq 0.$$

Consequently, $\Xi \subset C_n$. Hence, the sequence $\{x_n\}$ is well-defined.

Lemma 3.3. Consider the sequences $\{y_n\}$ and $\{\bar{y}_n\}$ generated by Algorithm 3.1. If $\lim_{n\to\infty} \|g_n - y_n\| = 0$ and $\lim_{n\to\infty} \|w_n - \bar{y}_n\| = 0$, then $\omega_w(g_n) \subset \mathrm{VI}(C, G)$ and $\omega_w(w_n) \subset \mathrm{VI}(C, F)$.

Proof. As the last two relations are analogous, it suffices to demonstrate the validity of the latter relation. Suppose $z \in \omega_w(w_n)$. Then, there exists a subsequence $\{w_{n_k}\} \subset \{w_n\}$ such that $w_{n_k} \rightharpoonup z$ and $\lim_{n\to\infty} \|w_{n_k} - \bar{y}_{n_k}\| = 0$. Consequently, $\bar{y}_{n_k} \rightharpoonup z$. Since C is convex and closed, and $\{\bar{y}_n\} \subset C$ and $\bar{y}_{n_k} \rightharpoonup z$, it implies that $z \in C$.

Now, we consider two cases. If $Fz = 0$, then $z \in \mathrm{VI}(C, F)$ because $\langle Fz, y - z\rangle \geq 0$ for all $y \in C$. If $Fz \neq 0$, utilizing the assumption on F instead of the weakly sequential continuity of F, we obtain $0 < \|Fz\| \leq \liminf_{k\to\infty} \|Fw_{n_k}\|$. Thus, we can assume that $\|Fw_{n_k}\| \neq 0$ for all $k \geq 1$. Using (2.2), we have

$$\langle J_E^p w_{n_k} - \lambda F w_{n_k} - J_E^p \bar{y}_{n_k}, x - \bar{y}_{n_k}\rangle \leq 0 \quad \forall x \in C,$$

and consequently,

$$\tfrac{1}{\lambda}\langle J_E^p w_{n_k} - J_E^p \bar{y}_{n_k}, x - \bar{y}_{n_k}\rangle + \langle Fw_{n_k}, \bar{y}_{n_k} - w_{n_k}\rangle \leq \langle Fw_{n_k}, x - w_{n_k}\rangle \quad \forall x \in C. \tag{3.6}$$

Given the uniform continuity of F, it is established that $\{Fw_{n_k}\}$ is bounded (as indicated by Lemma 2.3). Notably, the boundedness of $\{\bar{y}_{n_k}\}$ is also evident. Leveraging the uniform continuity of J_E^p on bounded subsets of E, we infer from (3.6) the following expression:

$$\liminf_{k\to\infty} \langle Fw_{n_k}, x - w_{n_k}\rangle \geq 0 \quad \forall x \in C. \tag{3.7}$$

To demonstrate that $z \in \mathrm{VI}(C, F)$, we now choose a sequence $\{\varsigma_k\} \subset (0, 1)$ such that $\varsigma_k \downarrow 0$ as $k \to \infty$. For each $k \geq 1$, let m_k denote the smallest positive integer satisfying the condition:

$$\langle Fw_{n_j}, y - w_{n_j}\rangle + \varsigma_k \geq 0 \quad \forall j \geq m_k. \tag{3.8}$$

Because $\{\varsigma_k\}$ is decreasing, it is easily known that $\{m_k\}$ is increasing. For simplicity, we indicate $\{Fw_{n_{m_k}}\}$ by $\{Fw_{m_k}\}$. Note that $Fw_{m_k} \neq 0$ $\forall k \geq 1$ (due to $\{Fw_{m_k}\} \subset$

$\{Fw_{n_k}\}$). Then one sets $\xi_{m_k} = \frac{Fw_{m_k}}{\|Fw_{m_k}\|^{\frac{q}{q-1}}}$, one gets $\langle Fw_{m_k}, J^q_{E^*}\xi_{m_k}\rangle = 1\ \forall k \ge 1$. In fact, it is evident that

$$\begin{aligned}\langle Fw_{m_k}, J^q_{E^*}\xi_{m_k}\rangle &= \langle Fw_{m_k}, (\frac{1}{\|Fw_{m_k}\|^{\frac{q}{q-1}}})^{q-1} J^q_{E^*} Fw_{m_k}\rangle \\ &= (\frac{1}{\|Fw_{m_k}\|^{\frac{q}{q-1}}})^{q-1}\|Fw_{m_k}\|^q = 1\ \forall k \ge 1.\end{aligned}$$

So, using (3.8) one has

$$\langle Fw_{m_k}, y + \varsigma_k J^q_{E^*}\xi_{m_k} - w_{m_k}\rangle \ge 0\ \forall k \ge 1.$$

Again, from the pseudomonotonicity of F one has

$$\langle F(y + \varsigma_k J^q_{E^*}\xi_{m_k}), y + \varsigma_k J^q_{E^*}\xi_{m_k} - w_{m_k}\rangle \ge 0 \quad \forall y \in C. \tag{3.9}$$

We assert that $\lim_{k\to\infty} \varsigma_k J^q_{E^*}\xi_{m_k} = 0$. Indeed, since $\{w_{m_k}\} \subset \{w_{n_k}\}$ and $\varsigma_k \downarrow 0$ as $k \to \infty$, we observe that

$$0 \le \limsup_{k\to\infty} \|\varsigma_k J^q_{E^*}\xi_{m_k}\| = \limsup_{k\to\infty} \frac{\varsigma_k}{\|Fw_{m_k}\|} \le \frac{\limsup_{k\to\infty}\varsigma_k}{\liminf_{k\to\infty}\|Fw_{m_k}\|} = 0.$$

Consequently, we obtain $\varsigma_k J^q_{E^*}\xi_{m_k} \to 0$ as $k \to \infty$. Subsequently, by taking the limit as $k \to \infty$ in (3.9) and utilizing condition (C3), we deduce $\langle Fy, y - z\rangle \ge 0$ for all $y \in C$. With the assistance of Lemma 2.4, we conclude that $z \in \mathrm{VI}(C, F)$.

Lemma 3.4. Consider the sequences $\{y_n\}$ and $\{\bar{y}_n\}$ generated by Algorithm 3.1. Then, then following statements hold true:

(i) If $\lim_{n\to\infty} \tau_n D_{f_p}(g_n, y_n) = 0$, then $\lim_{n\to\infty} D_{f_p}(g_n, y_n) = 0$;
(ii) If $\lim_{n\to\infty} \bar{\tau}_n D_{f_p}(w_n, \bar{y}_n) = 0$, then $\lim_{n\to\infty} D_{f_p}(w_n, \bar{y}_n) = 0$.

Proof. As assertions (i) and (ii) are analogous, it is sufficient to establish the validity of assertion (ii). To demonstrate assertion (ii), we consider two cases. In the scenario where $\liminf_{n\to\infty} \bar{\tau}_n > 0$, we assume the existence of a positive constant $\bar{\tau} > 0$ such that $\bar{\tau}_n \ge \bar{\tau} > 0$ for all $n \ge 1$. This assumption yields

$$D_{f_p}(w_n, \bar{y}_n) = \frac{1}{\bar{\tau}_n}\bar{\tau}_n D_{f_p}(w_n, \bar{y}_n) \le \frac{1}{\bar{\tau}} \cdot \bar{\tau}_n D_{f_p}(w_n, \bar{y}_n). \tag{3.10}$$

Combined with $\lim_{n\to\infty} \bar{\tau}_n D_{f_p}(w_n, \bar{y}_n) = 0$, this implies $\lim_{n\to\infty} D_{f_p}(w_n, \bar{y}_n) = 0$.

In the case where $\liminf_{n\to\infty} \bar{\tau}_n = 0$, let us assume that $\limsup_{n\to\infty} D_{f_p}(w_n, \bar{y}_n) = a > 0$. Consequently, there exists a subsequence $\{n_k\} \subset \{n\}$ such that

$$\lim_{k\to\infty} \bar{\tau}_{n_k} = 0 \quad \text{and} \quad \lim_{k\to\infty} D_{f_p}(w_{n_k}, \bar{y}_{n_k}) = a > 0.$$

For each $k \ge 1$, we define $\overline{t_{n_k}} = \frac{1}{l}\bar{\tau}_{n_k}\bar{y}_{n_k} + (1 - \frac{1}{l}\bar{\tau}_{n_k})w_{n_k}$. Applying (2.1) and noting $\lim_{k\to\infty} \bar{\tau}_{n_k} D_{f_p}(w_{n_k}, \bar{y}_{n_k}) = 0$, we obtain $\lim_{k\to\infty} \bar{\tau}_{n_k}\|w_{n_k} - \bar{y}_{n_k}\|^p = 0$, and hence

$$\lim_{k\to\infty} \|\overline{t_{n_k}} - w_{n_k}\|^p = \lim_{k\to\infty} \frac{\bar{\tau}^{p-1}_{n_k}}{l^p} \cdot \bar{\tau}_{n_k}\|w_{n_k} - \bar{y}_{n_k}\|^p = 0. \tag{3.11}$$

As F is uniformly continuous on bounded subsets of C, we deduce:

$$\lim_{k\to\infty} \|Fw_{n_k} - F\overline{t_{n_k}}\| = 0. \tag{3.12}$$

Utilizing the step-size rule (3.3) and the definition of $\overline{t_{n_k}}$, it follows that:

$$\langle Fw_{n_k} - F\overline{t_{n_k}}, w_{n_k} - \bar{y}_{n_k}\rangle > \frac{\mu}{2} D_{f_p}(w_{n_k}, \bar{y}_{n_k}). \tag{3.13}$$

Now, from (3.12) we have $\lim_{k\to\infty} D_{f_p}(w_{n_k}, \bar{y}_{n_k}) = 0$. However, this leads to a contradiction. Therefore, it follows that $\lim_{n\to\infty} D_{f_p}(w_n, \bar{y}_n) = 0$.

Now, we are ready to establish the strong convergence theorem for Algorithm 3.1.

Theorem 3.1. Assuming conditions (C1)-(C5), the sequence $\{x_n\}$ generated in Algorithm 3.1 converges strongly to $\Pi_{\Xi} u$ if and only if $\sup_{n\geq 1} \|x_n\| < \infty$.

Proof. The necessity of Theorem 3.1 is self-evident. Therefore, our sole focus is on proving the sufficiency. Suppose $\sup_{n\geq 0} \|x_n\| < \infty$. In the ensuing discussion, we present our proof through four distinct claims.

Claim 1. We prove that

$$(1-\alpha_n)\gamma_n(1-\gamma_n)\rho_b^*\|J_E^p x_n - J_E^p u_n\| \leq D_{f_p}(\hat{u}, x_n) - D_{f_p}(\hat{u}, x_{n+1}) + \ell_n M + \alpha_n D_{f_p}(\hat{u}, u).$$

Indeed, put $\hat{u} = \Pi_{\Xi} u$. Since $w_n = \Pi_{K_n} g_n$, using (2.1) and (2.3) we have

$$\begin{aligned} D_{f_p}(\hat{u}, w_n) &\leq D_{f_p}(\hat{u}, g_n) - D_{f_p}(w_n, g_n) \\ &= D_{f_p}(\hat{u}, g_n) - D_{f_p}(\Pi_{K_n} g_n, g_n) \\ &\leq D_{f_p}(\hat{u}, g_n) - \tau\|\Pi_{K_n} g_n - g_n\|^p \\ &\leq D_{f_p}(\hat{u}, g_n) - \tau\|P_{K_n} g_n - g_n\|^p \\ &= D_{f_p}(\hat{u}, g_n) - \tau[\mathrm{dist}(K_n, g_n)]^p. \end{aligned}$$

Since $z_n = \Pi_{C_n} w_n$, from (2.1) and (2.3) we get

$$\begin{aligned} D_{f_p}(\hat{u}, z_n) &\leq D_{f_p}(\hat{u}, w_n) - D_{f_p}(z_n, w_n) \\ &= D_{f_p}(\hat{u}, w_n) - D_{f_p}(\Pi_{C_n} w_n, w_n) \\ &\leq D_{f_p}(\hat{u}, w_n) - \tau\|\Pi_{C_n} w_n - w_n\|^p \\ &\leq D_{f_p}(\hat{u}, w_n) - \tau\|P_{C_n} w_n - w_n\|^p \\ &= D_{f_p}(\hat{u}, w_n) - \tau[\mathrm{dist}(C_n, w_n)]^p. \end{aligned}$$

Combining the last two inequalities, we get

$$\begin{aligned} D_{f_p}(\hat{u}, z_n) &\leq D_{f_p}(\hat{u}, w_n) - D_{f_p}(z_n, w_n) \\ &\leq D_{f_p}(\hat{u}, g_n) - D_{f_p}(w_n, g_n) - D_{f_p}(z_n, w_n) \\ &\leq D_{f_p}(\hat{u}, g_n) - \tau[\mathrm{dist}(K_n, g_n)]^p - \tau[\mathrm{dist}(C_n, w_n)]^p. \end{aligned} \tag{3.14}$$

Using (2.1), (2.6), the definition of ϵ_n and the three point identity of D_{f_p} we obtain that

$$\epsilon_n \|J_E^p x_n - J_E^p(2x_n - x_{n-1})\| \le \ell_n,$$

and hence

$$\begin{aligned} D_{f_p}(\hat{u}, u_n) \quad &= D_{f_p}(\hat{u}, J_{E^*}^q((1-\epsilon_n)J_E^p x_n + \epsilon_n J_E^p(2x_n - x_{n-1}))) \\ &\le (1-\epsilon_n)D_{f_p}(\hat{u}, x_n) + \epsilon_n D_{f_p}(\hat{u}, 2x_n - x_{n-1}) \\ &= D_{f_p}(\hat{u}, x_n) + \epsilon_n[D_{f_p}(\hat{u}, 2x_n - x_{n-1}) - D_{f_p}(\hat{u}, x_n)] \\ &= D_{f_p}(\hat{u}, x_n) + \epsilon_n[D_{f_p}(x_n, 2x_n - x_{n-1}) \\ &\quad + \langle J_E^p x_n - J_E^p(2x_n - x_{n-1}), \hat{u} - x_n\rangle] \\ &\le D_{f_p}(\hat{u}, x_n) + \epsilon_n[\langle J_E^p x_n - J_E^p(2x_n - x_{n-1}), x_{n-1} - x_n\rangle \\ &\quad + \langle J_E^p x_n - J_E^p(2x_n - x_{n-1}), \hat{u} - x_n\rangle] \\ &= D_{f_p}(\hat{u}, x_n) + \epsilon_n\langle J_E^p x_n - J_E^p(2x_n - x_{n-1}), \hat{u} + x_{n-1} - 2x_n\rangle \\ &\le D_{f_p}(\hat{u}, x_n) + \epsilon_n\|J_E^p x_n - J_E^p(2x_n - x_{n-1})\|\|\hat{u} + x_{n-1} - 2x_n\| \\ &\le D_{f_p}(\hat{u}, x_n) + \ell_n M, \end{aligned} \tag{3.15}$$

where $\sup_{n\ge 1}\|\hat{u} + x_{n-1} - 2x_n\| \le M$ for any $M > 0$. By Lemma 2.2 and due to the definition of g_n, we deduce that

$$\begin{aligned} &D_{f_p}(\hat{u}, g_n) = D_{f_p}(\hat{u}, J_{E^*}^q(\gamma_n J_E^p x_n + (1-\gamma_n)J_E^p u_n)) \\ &\le \gamma_n D_{f_p}(\hat{u}, x_n) + (1-\gamma_n)D_{f_p}(\hat{u}, u_n) - \gamma_n(1-\gamma_n)\rho_b^*\|J_E^p x_n - J_E^p u_n\| \\ &\le \gamma_n D_{f_p}(\hat{u}, x_n) + (1-\gamma_n)[D_{f_p}(\hat{u}, x_n) + \ell_n M] - \gamma_n(1-\gamma_n)\rho_b^*\|J_E^p x_n - J_E^p u_n\| \\ &\le D_{f_p}(\hat{u}, x_n) + \ell_n M - \gamma_n(1-\gamma_n)\rho_b^*\|J_E^p x_n - J_E^p u_n\|. \end{aligned} \tag{3.16}$$

Using (2.3), (2.6) and (3.16), we have

$$\begin{aligned} &D_{f_p}(\hat{u}, x_{n+1}) \le D_{f_p}(\hat{u}, J_{E^*}^q(\alpha_n J_E^p u + (1-\alpha_n)J_E^p v_n)) \\ &\le \alpha_n D_{f_p}(\hat{u}, u) + (1-\alpha_n)D_{f_p}(\hat{u}, v_n) \\ &\le \alpha_n D_{f_p}(\hat{u}, u) + (1-\alpha_n)[\beta_n D_{f_p}(\hat{u}, w_n) + (1-\beta_n)D_{f_p}(\hat{u}, S_n z_n)] \\ &\le \alpha_n D_{f_p}(\hat{u}, u) + (1-\alpha_n)[\beta_n D_{f_p}(\hat{u}, w_n) + (1-\beta_n)D_{f_p}(\hat{u}, z_n)] \\ &\le \alpha_n D_{f_p}(\hat{u}, u) + (1-\alpha_n)[\beta_n D_{f_p}(\hat{u}, w_n) + (1-\beta_n)D_{f_p}(\hat{u}, w_n)] \\ &= \alpha_n D_{f_p}(\hat{u}, u) + (1-\alpha_n)D_{f_p}(\hat{u}, w_n) \\ &\le (1-\alpha_n)D_{f_p}(\hat{u}, g_n) + \alpha_n D_{f_p}(\hat{u}, u) \\ &\le (1-\alpha_n)[D_{f_p}(\hat{u}, x_n) + \ell_n M - \gamma_n(1-\gamma_n)\rho_b^*\|J_E^p x_n - J_E^p u_n\|] + \alpha_n D_{f_p}(\hat{u}, u) \\ &\le D_{f_p}(\hat{u}, x_n) + \ell_n M - (1-\alpha_n)\gamma_n(1-\gamma_n)\rho_b^*\|J_E^p x_n - J_E^p u_n\| + \alpha_n D_{f_p}(\hat{u}, u). \end{aligned} \tag{3.17}$$

This promptly establishes the intended assertion. Additionally, it is evident that the sequences $\{g_n\}$, $\{u_n\}$, $\{w_n\}$, $\{y_n\}$, $\{\bar{y}_n\}$, $\{s_n\}$, $\{t_n\}$, $\{v_n\}$, and $\{S_n z_n\}$ are all bounded.

Claim 2. We prove that

$$D_{f_p}(w_n, g_n) + (1-\beta_n)D_{f_p}(z_n, w_n) \le D_{f_p}(\hat{u}, g_n) - D_{f_p}(\hat{u}, x_{n+1}) + \alpha_n\langle J_E^p u - J_E^p \hat{u}, \xi_n - \hat{u}\rangle.$$

Define $b = \sup_{n\geq 1}\{\|w_n\|^{p-1}, \|S_n z_n\|^{p-1}\}$. Employing Lemma 2.2, we obtain

$$\begin{aligned}
&D_{f_p}(\hat{u}, v_n) = V_{f_p}(\hat{u}, \beta_n J_E^p w_n + (1-\beta_n) J_E^p S_n z_n) \\
&\leq \tfrac{1}{p}\|\hat{u}\|^p - \beta_n\langle J_E^p w_n, \hat{u}\rangle - (1-\beta_n)\langle J_E^p S_n z_n, \hat{u}\rangle + \tfrac{\beta_n}{q}\|J_E^p w_n\|^q \\
&\ +\tfrac{(1-\beta_n)}{q}\|J_E^p S_n z_n\|^q - \beta_n(1-\beta_n)\rho_b^*\|J_E^p w_n - J_E^p S_n z_n\| \\
&= \tfrac{1}{p}\|\hat{u}\|^p - \beta_n\langle J_E^p w_n, \hat{u}\rangle - (1-\beta_n)\langle J_E^p S_n z_n, \hat{u}\rangle + \tfrac{\beta_n}{q}\|w_n\|^p \\
&\ +\tfrac{(1-\beta_n)}{q}\|S_n z_n\|^p - \beta_n(1-\beta_n)\rho_b^*\|J_E^p w_n - J_E^p S_n z_n\| \\
&= \beta_n D_{f_p}(\hat{u}, w_n) + (1-\beta_n) D_{f_p}(\hat{u}, S_n z_n) - \beta_n(1-\beta_n)\rho_b^*\|J_E^p w_n - J_E^p S_n z_n\| \\
&\leq \beta_n D_{f_p}(\hat{u}, w_n) + (1-\beta_n) D_{f_p}(\hat{u}, z_n) - \beta_n(1-\beta_n)\rho_b^*\|J_E^p w_n - J_E^p S_n z_n\| \\
&\leq D_{f_p}(\hat{u}, w_n) - \beta_n(1-\beta_n)\rho_b^*\|J_E^p w_n - J_E^p S_n z_n\|.
\end{aligned} \tag{3.18}$$

Set $\xi_n = J_{E^*}^q(\alpha_n J_E^p u + (1-\alpha_n) J_E^p v_n)$. Using (2.5), we have

$$\begin{aligned}
&D_{f_p}(\hat{u}, x_{n+1}) \leq D_{f_p}(\hat{u}, J_{E^*}^q(\alpha_n J_E^p u + (1-\alpha_n) J_E^p v_n)) \\
&= V_{f_p}(\hat{u}, \alpha_n J_E^p u + (1-\alpha_n) J_E^p v_n) \\
&\leq V_{f_p}(\hat{u}, \alpha_n J_E^p u + (1-\alpha_n) J_E^p v_n - \alpha_n(J_E^p u - J_E^p \hat{u})) \\
&\ +\alpha_n\langle J_E^p u - J_E^p \hat{u}, \xi_n - \hat{u}\rangle \\
&\leq \alpha_n D_{f_p}(\hat{u}, \hat{u}) + (1-\alpha_n) D_{f_p}(\hat{u}, v_n) + \alpha_n\langle J_E^p u - J_E^p \hat{u}, \xi_n - \hat{u}\rangle \\
&= (1-\alpha_n) D_{f_p}(\hat{u}, v_n) + \alpha_n\langle J_E^p u - J_E^p \hat{u}, \xi_n - \hat{u}\rangle \\
&\leq (1-\alpha_n)[D_{f_p}(\hat{u}, w_n) - \beta_n(1-\beta_n)\rho_b^*\|J_E^p w_n - J_E^p S_n z_n\|] \\
&\ +\alpha_n\langle J_E^p u - J_E^p \hat{u}, \xi_n - \hat{u}\rangle \\
&= (1-\alpha_n) D_{f_p}(\hat{u}, w_n) - (1-\alpha_n)\beta_n(1-\beta_n)\rho_b^*\|J_E^p w_n - J_E^p S_n z_n\| \\
&\ +\alpha_n\langle J_E^p u - J_E^p \hat{u}, \xi_n - \hat{u}\rangle \\
&\leq (1-\alpha_n) D_{f_p}(\hat{u}, w_n) + \alpha_n\langle J_E^p u - J_E^p \hat{u}, \xi_n - \hat{u}\rangle.
\end{aligned} \tag{3.19}$$

On the other hand, we have

$$\begin{aligned}
D_{f_p}(\hat{u}, v_n) \ &\leq \beta_n D_{f_p}(\hat{u}, w_n) + (1-\beta_n) D_{f_p}(\hat{u}, z_n) \\
&\leq \beta_n D_{f_p}(\hat{u}, w_n) + (1-\beta_n)[D_{f_p}(\hat{u}, w_n) - D_{f_p}(z_n, w_n)] \\
&= D_{f_p}(\hat{u}, w_n) - (1-\beta_n) D_{f_p}(z_n, w_n).
\end{aligned}$$

By substituting the aforementioned inequality into (3.19), we obtain

$$\begin{aligned}
&D_{f_p}(\hat{u}, x_{n+1}) \leq (1-\alpha_n) D_{f_p}(\hat{u}, v_n) + \alpha_n\langle J_E^p u - J_E^p \hat{u}, \xi_n - \hat{u}\rangle \\
&\leq D_{f_p}(\hat{u}, w_n) - (1-\beta_n) D_{f_p}(z_n, w_n) + \alpha_n\langle J_E^p u - J_E^p \hat{u}, \xi_n - \hat{u}\rangle \\
&\leq D_{f_p}(\hat{u}, g_n) - D_{f_p}(w_n, g_n) - (1-\beta_n) D_{f_p}(z_n, w_n) + \alpha_n\langle J_E^p u - J_E^p \hat{u}, \xi_n - \hat{u}\rangle.
\end{aligned}$$

This immediately arrives at

$$D_{f_p}(w_n, g_n) + (1-\beta_n) D_{f_p}(z_n, w_n) \leq D_{f_p}(\hat{u}, g_n) - D_{f_p}(\hat{u}, x_{n+1}) + \alpha_n\langle J_E^p u - J_E^p \hat{u}, \xi_n - \hat{u}\rangle. \tag{3.20}$$

Claim 3. Next, we prove that

$$\begin{aligned}
&(1-\alpha_n)(1-\beta_n)\{\tau[\tfrac{\tau_n}{2\lambda L} D_{f_p}(g_n, y_n)]^p + \tau[\tfrac{\bar{\tau}_n}{2\lambda \bar{L}} D_{f_p}(w_n, \bar{y}_n)]^p\} \\
&\leq \alpha_n D_{f_p}(\hat{u}, u) + D_{f_p}(\hat{u}, g_n) - D_{f_p}(\hat{u}, x_{n+1}).
\end{aligned}$$

Certainly, given that the sequence $\{Gs_n\}$ is bounded, there exists a positive constant L such that $\|Gs_n\| \leq L$. This ensures that for any x and y belonging to K_n,

$$|h_n(x) - h_n(y)| = |\langle Gs_n, x - y\rangle| \leq \|Gs_n\|\|x - y\| \leq L\|x - y\|,$$

which implies that $h_n(x)$ is L-Lipschitz continuous on K_n. According to Lemma 2.5, we obtain

$$\operatorname{dist}(K_n, g_n) \geq \frac{1}{L} h_n(g_n) = \frac{\tau_n}{2\lambda L} D_{f_p}(g_n, y_n). \tag{3.21}$$

Similarly, since $\{Ft_n\}$ is bounded, there exists another positive constant $\bar{L}$ such that $\|Ft_n\| \leq \bar{L}$. This ensures that for any x and y in C_n,

$$|\hbar_n(x) - \hbar_n(y)| = |\langle Ft_n, x - y\rangle| \leq \|Ft_n\|\|x - y\| \leq \bar{L}\|x - y\|,$$

demonstrating that $\hbar_n(x)$ is $\bar{L}$-Lipschitz continuous on C_n. By applying Lemma 2.5, we get

$$\operatorname{dist}(C_n, w_n) \geq \frac{1}{\bar{L}} \hbar_n(w_n) = \frac{\bar{\tau}_n}{2\lambda \bar{L}} D_{f_p}(w_n, \bar{y}_n). \tag{3.22}$$

Combining (3.14), (3.21) and (3.22), we get

$$\begin{aligned} D_{f_p}(\hat{u}, z_n) &\leq D_{f_p}(\hat{u}, g_n) - \tau[\operatorname{dist}(K_n, g_n)]^p - \tau[\operatorname{dist}(C_n, w_n)]^p \\ &\leq D_{f_p}(\hat{u}, g_n) - \tau[\tfrac{\tau_n}{2\lambda L} D_{f_p}(g_n, y_n)]^p - \tau[\tfrac{\bar{\tau}_n}{2\lambda \bar{L}} D_{f_p}(w_n, \bar{y}_n)]^p. \end{aligned} \tag{3.23}$$

Note that

$$x_{n+1} = \Pi_C(J^q_{E^*}(\alpha_n J^p_E u + (1-\alpha_n) J^p_E v_n)) \text{ and } v_n = J^q_{E^*}(\beta_n J^p_E w_n + (1-\beta_n) J^p_E(S_n z_n)).$$

Consequently, it concludes from (3.23) that

$$\begin{aligned} & D_{f_p}(\hat{u}, x_{n+1}) \leq D_{f_p}(\hat{u}, J^q_{E^*}(\alpha_n J^p_E u + (1-\alpha_n) J^p_E v_n)) \\ \leq\; & \alpha_n D_{f_p}(\hat{u}, u) + (1-\alpha_n) D_{f_p}(\hat{u}, v_n) \\ \leq\; & \alpha_n D_{f_p}(\hat{u}, u) + (1-\alpha_n)[\beta_n D_{f_p}(\hat{u}, w_n) + (1-\beta_n) D_{f_p}(\hat{u}, z_n)] \\ =\; & \alpha_n D_{f_p}(\hat{u}, u) + (1-\alpha_n)\beta_n D_{f_p}(\hat{u}, w_n) + (1-\alpha_n)(1-\beta_n) D_{f_p}(\hat{u}, z_n) \\ \leq\; & \alpha_n D_{f_p}(\hat{u}, u) + (1-\alpha_n)\beta_n D_{f_p}(\hat{u}, g_n) \\ & + (1-\alpha_n)(1-\beta_n)\{D_{f_p}(\hat{u}, g_n) - \tau[\tfrac{\tau_n}{2\lambda L} D_{f_p}(g_n, y_n)]^p - \tau[\tfrac{\bar{\tau}_n}{2\lambda \bar{L}} D_{f_p}(w_n, \bar{y}_n)]^p\} \\ \leq\; & \alpha_n D_{f_p}(\hat{u}, u) + D_{f_p}(\hat{u}, g_n) \\ & -(1-\alpha_n)(1-\beta_n)\{\tau[\tfrac{\tau_n}{2\lambda L} D_{f_p}(g_n, y_n)]^p + \tau[\tfrac{\bar{\tau}_n}{2\lambda \bar{L}} D_{f_p}(w_n, \bar{y}_n)]^p\}. \end{aligned} \tag{3.24}$$

Claim 4. We establish the convergence $x_n \to \hat{u}$ as $n \to \infty$. Given the reflexivity of the space E and the boundedness of the sequence $\{x_n\}$, it follows that $\omega_w(x_n) \neq \emptyset$. Let $z^\dagger \in \omega_w(x_n)$. Consequently, there exists a subsequence $\{x_{n_k}\} \subset \{x_n\}$ such that $x_{n_k} \rightharpoonup z^\dagger$. For each $n \geq 1$, let $\mathbf{\Phi}_n = D_{f_p}(\hat{u}, x_n)$.

In what follows, we establish the convergence of $\{\mathbf{\Phi}_n\}$ to zero in two distinct cases.

***Case* 1.** Assume there exists an integer $n_0 \geq 1$ such that $\{\mathbf{\Phi}_n\}_{n=n_0}^\infty$ is nonincreasing. In this scenario, $\lim_{n\to\infty} \mathbf{\Phi}_n = d < +\infty$, and $\lim_{n\to\infty}(\mathbf{\Phi}_n - \mathbf{\Phi}_{n+1}) = 0$. Utilizing equations (3.16) and (3.20), we obtain

$$\begin{aligned} & D_{f_p}(w_n, g_n) + (1-\beta_n) D_{f_p}(z_n, w_n) \\ & \leq D_{f_p}(\hat{u}, g_n) - D_{f_p}(\hat{u}, x_{n+1}) + \alpha_n\langle J^p_E u - J^p_E \hat{u}, \xi_n - \hat{u}\rangle \\ & \leq D_{f_p}(\hat{u}, x_n) + \ell_n M - D_{f_p}(\hat{u}, x_{n+1}) + \alpha_n\langle J^p_E u - J^p_E \hat{u}, \xi_n - \hat{u}\rangle \\ & = \mathbf{\Phi}_n - \mathbf{\Phi}_{n+1} + \ell_n M + \alpha_n\langle J^p_E u - J^p_E \hat{u}, \xi_n - \hat{u}\rangle. \end{aligned}$$

Given that

$$\lim_{n\to\infty} \ell_n = 0,\ \lim_{n\to\infty} \alpha_n = 0,\ \liminf_{n\to\infty} \beta_n(1-\beta_n) > 0,\ \lim_{n\to\infty} (\mathbf{\Phi}_n - \mathbf{\Phi}_{n+1}) = 0,$$

and the sequence $\{\xi_n\}$ is bounded, we deduce

$$\lim_{n\to\infty} D_{f_p}(w_n, g_n) = 0 \quad \text{and} \quad \lim_{n\to\infty} D_{f_p}(z_n, w_n) = 0.$$

So it follows from (2.1) that

$$\lim_{n\to\infty} \|w_n - g_n\| = 0 \quad \text{and} \quad \lim_{n\to\infty} \|z_n - w_n\| = 0. \tag{3.25}$$

Furthermore, from (3.19) we have

$$\begin{aligned}
&(1-\alpha_n)\beta_n(1-\beta_n)\rho_b^*\|J_E^p w_n - J_E^p S_n z_n\| \\
&\le (1-\alpha_n)D_{f_p}(\hat{u}, w_n) - D_{f_p}(\hat{u}, x_{n+1}) + \alpha_n\langle J_E^p u - J_E^p \hat{u}, \xi_n - \hat{u}\rangle \\
&\le D_{f_p}(\hat{u}, g_n) - D_{f_p}(\hat{u}, x_{n+1}) + \alpha_n\langle J_E^p u - J_E^p \hat{u}, \xi_n - \hat{u}\rangle \\
&\le D_{f_p}(\hat{u}, x_n) - D_{f_p}(\hat{u}, x_{n+1}) + \ell_n M + \alpha_n\langle J_E^p u - J_E^p \hat{u}, \xi_n - \hat{u}\rangle \\
&= \mathbf{\Phi}_n - \mathbf{\Phi}_{n+1} + \ell_n M + \alpha_n\langle J_E^p u - J_E^p \hat{u}, \xi_n - \hat{u}\rangle.
\end{aligned}$$

By similar arguments, we deduce that $\lim_{n\to\infty} \|J_E^p w_n - J_E^p S_n z_n\| = 0$, which hence leads to $\lim_{n\to\infty} \|J_E^p v_n - J_E^p w_n\| = 0$ (due to $v_n = J_{E^*}^q(\beta_n J_E^p w_n + (1-\beta_n)J_E^p S_n z_n)$). By leveraging the uniform continuity of $J_{E^*}^q$ on bounded subsets of E^*, we obtain

$$\lim_{n\to\infty} \|w_n - S_n z_n\| = \lim_{n\to\infty} \|v_n - w_n\| = 0. \tag{3.26}$$

This together with (3.25) implies that

$$\lim_{n\to\infty} \|z_n - S_n z_n\| = \lim_{n\to\infty} \|v_n - g_n\| = 0. \tag{3.27}$$

Note that

$u_n = J_{E^*}^q((1-\epsilon_n)J_E^p x_n + \epsilon_n J_E^p(2x_n - x_{n-1}))$ and $g_n = J_{E^*}^q(\gamma_n J_E^p x_n + (1-\gamma_n)J_E^p u_n)$.

Therefore, we deduce from the fact that $\lim_{n\to\infty} \ell_n = 0$ and the definition of ϵ_n that

$$\|J_E^p u_n - J_E^p x_n\| = \epsilon_n\|J_E^p(2x_n - x_{n-1}) - J_E^p x_n\| \le \ell_n \to 0 \quad (n\to\infty),$$

and hence

$$\|J_E^p g_n - J_E^p x_n\| = (1-\gamma_n)\|J_E^p u_n - J_E^p x_n\| \le \|J_E^p u_n - J_E^p x_n\| \to 0 \quad (n\to\infty).$$

Exploiting the uniform continuity of $J_{E^*}^q$ on bounded subsets of E^*, we attain

$$\lim_{n\to\infty} \|u_n - x_n\| = \lim_{n\to\infty} \|g_n - x_n\| = 0. \tag{3.28}$$

So, based on (3.25), (3.27), and (3.28) it concludes that

$$\|v_n - x_n\| \le \|v_n - g_n\| + \|g_n - x_n\| \to 0 \quad (n\to\infty), \tag{3.29}$$

and

$$\|z_n - x_n\| \le \|z_n - w_n\| + \|w_n - g_n\| + \|g_n - x_n\| \to 0 \quad (n\to\infty). \tag{3.30}$$

Since $\xi_n = J_{E^*}^q(\alpha_n J_E^p u + (1-\alpha_n)J_E^p v_n)$, it is clear from (3.29) that

$$\lim_{n\to\infty} \|\xi_n - x_n\| = 0. \tag{3.31}$$

Furthermore, considering (2.3), (3.16), and (3.24), we have

$$\begin{aligned}
&D_{f_p}(\hat{u}, x_{n+1}) \le D_{f_p}(\hat{u}, \xi_n) - D_{f_p}(x_{n+1}, \xi_n)\\
&= D_{f_p}(\hat{u}, J_{E^*}^q(\alpha_n J_E^p u + (1-\alpha_n) J_E^p v_n) - D_{f_p}(x_{n+1}, \xi_n)\\
&\le \alpha_n D_{f_p}(\hat{u}, u) + D_{f_p}(\hat{u}, g_n) - D_{f_p}(x_{n+1}, \xi_n)\\
&\le \alpha_n D_{f_p}(\hat{u}, u) + D_{f_p}(\hat{u}, x_n) + \ell_n M - D_{f_p}(x_{n+1}, \xi_n),
\end{aligned}$$

which immediately arrives at

$$\begin{aligned}
D_{f_p}(x_{n+1}, \xi_n) \quad &\le \alpha_n D_{f_p}(\hat{u}, u) + \ell_n M + D_{f_p}(\hat{u}, x_n) - D_{f_p}(\hat{u}, x_{n+1})\\
&= \alpha_n D_{f_p}(\hat{u}, u) + \ell_n M + \mathbf{\Phi}_n - \mathbf{\Phi}_{n+1}.
\end{aligned}$$

Thus, we get $\lim_{n\to\infty} D_{f_p}(x_{n+1}, \xi_n) = 0$, and hence $\lim_{n\to\infty} \|x_{n+1} - \xi_n\| = 0$. Note that $\|x_{n+1} - x_n\| \le \|x_{n+1} - \xi_n\| + \|\xi_n - x_n\|$ and $\|z_{n+1} - z_n\| \le \|z_{n+1} - x_{n+1}\| + \|x_{n+1} - x_n\| + \|x_n - z_n\|$. So it follows from (3.30), (3.31) and $x_{n+1} - \xi_n \to 0$ that

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = \lim_{n\to\infty} \|z_{n+1} - z_n\| = 0. \tag{3.32}$$

Subsequently, we demonstrate that $z^\dagger \in \mathrm{VI}(C, G) \cap \mathrm{VI}(C, F)$. Employing equations (3.16) and (3.24), we obtain

$$\begin{aligned}
&(1-\alpha_n)(1-\beta_n)\{\tau[\tfrac{\tau_n}{2\lambda L} D_{f_p}(g_n, y_n)]^p + \tau[\tfrac{\bar{\tau}_n}{2\lambda \bar{L}} D_{f_p}(w_n, \bar{y}_n)]^p\}\\
&\le \alpha_n D_{f_p}(\hat{u}, u) + D_{f_p}(\hat{u}, g_n) - D_{f_p}(\hat{u}, x_{n+1})\\
&\le \alpha_n D_{f_p}(\hat{u}, u) + D_{f_p}(\hat{u}, x_n) + \ell_n M - D_{f_p}(\hat{u}, x_{n+1})\\
&= \alpha_n D_{f_p}(\hat{u}, u) + \ell_n M + \mathbf{\Phi}_n - \mathbf{\Phi}_{n+1}.
\end{aligned}$$

Thus, it means that $\lim_{n\to\infty} \frac{\tau_n}{2\lambda L} D_{f_p}(g_n, y_n) = \lim_{n\to\infty} \frac{\bar{\tau}_n}{2\lambda \bar{L}} D_{f_p}(w_n, \bar{y}_n) = 0$, and hence

$$\lim_{n\to\infty} \tau_n D_{f_p}(g_n, y_n) = \lim_{n\to\infty} \bar{\tau}_n D_{f_p}(w_n, \bar{y}_n) = 0. \tag{3.33}$$

Applying Lemma 3.4, we deduce that

$$\lim_{n\to\infty} \|g_n - y_n\| = \lim_{n\to\infty} \|w_n - \bar{y}_n\| = 0. \tag{3.34}$$

From (3.25), (3.28) and $x_{n_k} \rightharpoonup z^\dagger$, we know that $z^\dagger \in \omega_w(g_n)$ and $z^\dagger \in \omega_w(w_n)$. Combining Lemma 3.3 and (3.34), we get that $z^\dagger \in \omega_w(g_n) \cap \omega_w(w_n) \subset \mathrm{VI}(C, G) \cap \mathrm{VI}(C, F)$. Also, by the definition of S_n we have that $S_n \in \{S_1, \dots, S_N\}$ $\forall n \ge 1$. So we get

$$\begin{aligned}
\|z_n - S_{n+i} z_n\| \quad &\le \|z_n - z_{n+i}\| + \|z_{n+i} - S_{n+i} z_{n+i}\| + \|S_{n+i} z_{n+i} - S_{n+i} z_n\|\\
&\le \|z_n - z_{n+i}\| + \|z_{n+i} - S_{n+i} z_{n+i}\| + \sum_{j=1}^{N} \|S_j z_{n+i} - S_j z_n\|.
\end{aligned}$$

By employing equations (3.27) and (3.32), along with the uniform continuity of each S_j on C, we deduce that $\lim_{n\to\infty} \|z_n - S_r z_n\| = 0$ holds for $r = 1, \dots, N$. Consequently, given $z_{n_k} \rightharpoonup z^\dagger$ (as per (3.30)), we establish that $z \in \widehat{\mathrm{Fix}}(S_r) = \mathrm{Fix}(S_r)$ for $r = 1, \dots, N$. Hence, $z \in \bigcap_{i=1}^N \mathrm{Fix}(S_i)$. Note that $\mathcal{T} z^\dagger \in \mathcal{T}\mathrm{VI}(C, G) \subset \mathrm{Fix}(S)$ (due to condition (C3)). As a result, $z^\dagger \in \Xi = \bigcap_{i=1}^N \mathrm{Fix}(S_i) \cap \Omega$ with $\Omega = \{z \in \mathrm{VI}(C, F) : \mathcal{T} z \in \mathrm{Fix}(S)\}$. This suggests that $\omega_w(x_n) \subset \Xi$. Finally, we establish

$\limsup_{n\to\infty}\langle J_E^p u - J_E^p \hat{u}, \xi_n - \hat{u}\rangle \leq 0$. By selecting a subsequence $\{x_{n_j}\}$ from $\{x_n\}$, we have

$$\limsup_{n\to\infty}\langle J_E^p u - J_E^p \hat{u}, x_n - \hat{u}\rangle = \lim_{j\to\infty}\langle J_E^p u - J_E^p \hat{u}, x_{n_j} - \hat{u}\rangle.$$

Given the reflexivity of E and the boundedness of $\{x_n\}$, we can assume, without loss of generality, that $x_{n_j} \rightharpoonup \tilde{z}$. Consequently, utilizing (2.2) and $\tilde{z} \in \Xi$, we obtain

$$\limsup_{n\to\infty}\langle J_E^p u - J_E^p \hat{u}, x_n - \hat{u}\rangle = \lim_{j\to\infty}\langle J_E^p u - J_E^p \hat{u}, x_{n_j} - \hat{u}\rangle = \langle J_E^p u - J_E^p \hat{u}, \tilde{z} - \hat{u}\rangle \leq 0. \tag{3.35}$$

This, combined with (3.31), ensures that

$$\limsup_{n\to\infty}\langle J_E^p u - J_E^p \hat{u}, \xi_n - \hat{u}\rangle \leq 0. \tag{3.36}$$

Using (3.15), (3.16) and (3.19), we get

$$\begin{aligned}
&D_{f_p}(\hat{u}, x_{n+1}) \leq (1-\alpha_n)D_{f_p}(\hat{u}, w_n) + \alpha_n\langle J_E^p u - J_E^p \hat{u}, \xi_n - \hat{u}\rangle \\
&\leq (1-\alpha_n)D_{f_p}(\hat{u}, g_n) + \alpha_n\langle J_E^p u - J_E^p \hat{u}, \xi_n - \hat{u}\rangle \\
&\leq (1-\alpha_n)[D_{f_p}(\hat{u}, x_n) + \epsilon_n\|J_E^p x_n - J_E^p(2x_n - x_{n-1})\|M] + \alpha_n\langle J_E^p u - J_E^p \hat{u}, \xi_n - \hat{u}\rangle \\
&= (1-\alpha_n)D_{f_p}(\hat{u}, x_n) + \alpha_n[\tfrac{\epsilon_n}{\alpha_n}\|J_E^p x_n - J_E^p(2x_n - x_{n-1})\|M + \langle J_E^p u - J_E^p \hat{u}, \xi_n - \hat{u}\rangle].
\end{aligned} \tag{3.37}$$

By leveraging the uniform continuity of J_E^p on bounded subsets of E, we deduce from (3.32) and the boundedness of $\{x_n\}$ that $\lim_{n\to\infty}\|J_E^p x_n - J_E^p(2x_n - x_{n-1})\| = 0$. Noticing $\sup_{n\geq 1}\frac{\epsilon_n}{\alpha_n} < \infty$ and $\limsup_{n\to\infty}\langle J_E^p u - J_E^p \hat{u}, \xi_n - \hat{u}\rangle \leq 0$, we infer that

$$\limsup_{n\to\infty}[\frac{\epsilon_n}{\alpha_n}\|J_E^p x_n - J_E^p(2x_n - x_{n-1})\|M + \langle J_E^p u - J_E^p \hat{u}, \xi_n - \hat{u}\rangle] \leq 0.$$

Given that $\{\alpha_n\} \subset (0,1)$ and $\sum_{n=1}^{\infty}\alpha_n = \infty$, applying Lemma 2.6 to (3.37) allows us to deduce that $\lim_{n\to\infty}D_{f_p}(\hat{u}, x_n) = 0$. Consequently, we establish $\lim_{n\to\infty}\|\hat{u} - x_n\| = 0$.

***Case* 2.** Suppose there exists a subsequence $\{\mathbf{\Phi}_{n_k}\} \subset \{\mathbf{\Phi}_n\}$ such that $\mathbf{\Phi}_{n_k} < \mathbf{\Phi}_{n_k+1}$ for all $k \in \mathcal{N}$, where $\mathcal{N}$ denotes the set of positive integers. Introduce the mapping $\psi : \mathcal{N} \to \mathcal{N}$ defined by

$$\psi(n) := \max\{k \leq n : \mathbf{\Phi}_k < \mathbf{\Phi}_{k+1}\}.$$

By employing Lemma 2.7, we establish

$$\mathbf{\Phi}_{\psi(n)} \leq \mathbf{\Phi}_{\psi(n)+1} \quad \text{and} \quad \mathbf{\Phi}_n \leq \mathbf{\Phi}_{\psi(n)+1}. \tag{3.38}$$

Proceeding from (3.16) and (3.20), we derive

$$\begin{aligned}
&D_{f_p}(w_{\psi(n)}, g_{\psi(n)}) + (1-\beta_{\psi(n)})D_{f_p}(z_{\psi(n)}, w_{\psi(n)}) \\
&\leq D_{f_p}(\hat{u}, g_{\psi(n)}) - D_{f_p}(\hat{u}, x_{\psi(n)+1}) + \alpha_{\psi(n)}\langle J_E^p u - J_E^p \hat{u}, \xi_{\psi(n)} - \hat{u}\rangle \\
&\leq \mathbf{\Phi}_{\psi(n)} - \mathbf{\Phi}_{\psi(n)+1} + \ell_{\psi(n)}M + \alpha_{\psi(n)}\langle J_E^p u - J_E^p \hat{u}, \xi_{\psi(n)} - \hat{u}\rangle.
\end{aligned}$$

Consequently, we conclude

$$\lim_{n\to\infty}\|w_{\psi(n)} - g_{\psi(n)}\| = 0 \quad \text{and} \quad \lim_{n\to\infty}\|z_{\psi(n)} - w_{\psi(n)}\| = 0. \tag{3.39}$$

Furthermore, from (3.19) we have

$$\begin{aligned}&(1-\alpha_{\psi(n)})\beta_{\psi(n)}(1-\beta_{\psi(n)})\rho_b^*\|J_E^p w_{\psi(n)} - J_E^p S_{\psi(n)} z_{\psi(n)}\| \\ &\le (1-\alpha_{\psi(n)})D_{f_p}(\hat{u}, w_{\psi(n)}) - D_{f_p}(\hat{u}, x_{\psi(n)+1}) + \alpha_{\psi(n)}\langle J_E^p u - J_E^p \hat{u}, \xi_{\psi(n)} - \hat{u}\rangle.\end{aligned}$$

Noting that $v_{\psi(n)} = J_{E^*}^q(\beta_{\psi(n)} J_E^p w_{\psi(n)} + (1-\beta_{\psi(n)}) J_E^p S_{\psi(n)} z_{\psi(n)})$ and employing arguments similar to those in Case 1, we deduce

$$\lim_{n\to\infty}\|w_{\psi(n)} - S_{\psi(n)} z_{\psi(n)}\| = \lim_{n\to\infty}\|v_{\psi(n)} - w_{\psi(n)}\| = 0.$$

This, combined with (3.39), implies that

$$\lim_{n\to\infty}\|z_{\psi(n)} - S_{\psi(n)} z_{\psi(n)}\| = \lim_{n\to\infty}\|v_{\psi(n)} - g_{\psi(n)}\| = 0. \tag{3.40}$$

Note that $u_{\psi(n)} = J_{E^*}^q((1-\epsilon_{\psi(n)})J_E^p x_{\psi(n)} + \epsilon_{\psi(n)} J_E^p(2x_{\psi(n)} - x_{\psi(n)-1}))$ and $g_{\psi(n)} = J_{E^*}^q(\gamma_{\psi(n)}\ J_E^p x_{\psi(n)} + (1-\gamma_{\psi(n)})J_E^p u_{\psi(n)})$. Thus, we infer from $\lim_{n\to\infty}\ell_n = 0$ and the definition of ϵ_n that

$$\|J_E^p u_{\psi(n)} - J_E^p x_{\psi(n)}\| = \epsilon_{\psi(n)}\|J_E^p(2x_{\psi(n)} - x_{\psi(n)-1}) - J_E^p x_{\psi(n)}\| \le \ell_{\psi(n)} \to 0 \quad (n\to\infty),$$

and hence

$$\|J_E^p g_{\psi(n)} - J_E^p x_{\psi(n)}\| = (1-\gamma_{\psi(n)})\|J_E^p u_{\psi(n)} - J_E^p x_{\psi(n)}\| \le \|J_E^p u_{\psi(n)} - J_E^p x_{\psi(n)}\| \to 0 \quad (n\to\infty).$$

Utilizing the uniform continuity of $J_{E^*}^q$ on bounded subsets of E^*, we conclude

$$\lim_{n\to\infty}\|u_{\psi(n)} - x_{\psi(n)}\| = \lim_{n\to\infty}\|g_{\psi(n)} - x_{\psi(n)}\| = 0. \tag{3.41}$$

So, from (3.39), (3.40), and (3.41), it follows that

$$\|v_{\psi(n)} - x_{\psi(n)}\| \le \|v_{\psi(n)} - g_{\psi(n)}\| + \|g_{\psi(n)} - x_{\psi(n)}\| \to 0 \quad (n\to\infty), \tag{3.42}$$

and

$$\|z_{\psi(n)} - x_{\psi(n)}\| \le \|z_{\psi(n)} - w_{\psi(n)}\| + \|w_{\psi(n)} - g_{\psi(n)}\| + \|g_{\psi(n)} - x_{\psi(n)}\| \to 0 \quad (n\to\infty). \tag{3.43}$$

Noticing $\xi_{\psi(n)} = J_{E^*}^q(\alpha_{\psi(n)} J_E^p u + (1-\alpha_{\psi(n)}) J_E^p v_{\psi(n)})$, from (3.42) we get

$$\lim_{n\to\infty}\|\xi_{\psi(n)} - x_{\psi(n)}\| = 0. \tag{3.44}$$

Applying analogous reasoning to that employed in Case 1, we arrive at the following conclusion:

$$\lim_{n\to\infty}\|x_{\psi(n)+1} - x_{\psi(n)}\| = \lim_{n\to\infty}\|g_{\psi(n)} - w_{\psi(n)}\| = \lim_{n\to\infty}\|w_{\psi(n)} - \bar{y}_{\psi(n)}\| = 0, \tag{3.45}$$

and

$$\limsup_{n\to\infty}\langle J_E^p u - J_E^p \hat{u}, \xi_{\psi(n)} - \hat{u}\rangle \le 0. \tag{3.46}$$

Using (3.37) we have

$$\begin{aligned}\mathbf{\Phi}_{\psi(n)+1} \quad &\le (1-\alpha_{\psi(n)})\mathbf{\Phi}_{\psi(n)} + \epsilon_{\psi(n)}\|J_E^p x_{\psi(n)} - J_E^p(2x_{\psi(n)} - x_{\psi(n)-1})\|M \\ &\quad + \alpha_{\psi(n)}\langle J_E^p u - J_E^p \hat{u}, \xi_{\psi(n)} - \hat{u}\rangle,\end{aligned} \tag{3.47}$$

which together with (3.38), leads to

$$\begin{aligned}&\alpha_{\psi(n)}\mathbf{\Phi}_{\psi(n)} \leq \mathbf{\Phi}_{\psi(n)} - \mathbf{\Phi}_{\psi(n)+1} + \epsilon_{\psi(n)}\|J_E^p x_{\psi(n)} - J_E^p(2x_{\psi(n)} - x_{\psi(n)-1})\|M \\ &+ \alpha_{\psi(n)}\langle J_E^p u - J_E^p \hat{u}, \xi_{\psi(n)} - \hat{u}\rangle \\ &\leq \alpha_{\psi(n)}[\frac{\epsilon_{\psi(n)}}{\alpha_{\psi(n)}}\|J_E^p x_{\psi(n)} - J_E^p(2x_{\psi(n)} - x_{\psi(n)-1})\|M + \langle J_E^p u - J_E^p \hat{u}, \xi_{\psi(n)} - \hat{u}\rangle].\end{aligned}$$

Therefore, by leveraging the uniform continuity of J_E^p on bounded subsets of E, we can infer from equations (3.45) and (3.46), combined with the condition $\sup_{n\geq 1}\frac{\epsilon_n}{\alpha_n} < \infty$, that

$$\lim_{n\to\infty} \mathbf{\Phi}_{\psi(n)} = 0. \tag{3.48}$$

From (3.47), (3.48) and the definition of ϵ_n, one has that

$$\begin{aligned}\mathbf{\Phi}_{\psi(n)+1} &\leq (1-\alpha_{\psi(n)})\mathbf{\Phi}_{\psi(n)} + \ell_{\psi(n)}M + \alpha_{\psi(n)}\langle J_E^p u - J_E^p \hat{u}, \xi_{\psi(n)} - \hat{u}\rangle \\ &\leq \mathbf{\Phi}_{\psi(n)} + \ell_{\psi(n)}M + \alpha_{\psi(n)}\|J_E^p u - J_E^p \hat{u}\|\|\xi_{\psi(n)} - \hat{u}\| \to 0 \quad (n\to\infty).\end{aligned} \tag{3.49}$$

From equation (3.38), it follows that $\lim_{n\to\infty} D_{f_p}(\hat{u}, x_n) = \lim_{n\to\infty}\mathbf{\Phi}_n = 0$. Consequently, $\lim_{n\to\infty}\|x_n - \hat{u}\| = 0$. This concludes the proof.

Remark 3.1. The proof of Theorem 3.1 reveals that substituting the assumption $\lim_{n\to\infty}\frac{\ell_n}{\alpha_n} = 0$ for $\lim_{n\to\infty}\ell_n = 0$ and $\sup_{n\geq 1}\frac{\epsilon_n}{\alpha_n} < \infty$ maintains the validity of Theorem 3.1. It is noteworthy that the BSPVIP under consideration encompasses the following VIPs:

(i) The upper-level VIP involves finding $\hat{u} \in \Xi$ such that $\langle \Gamma\hat{u}, v - \hat{u}\rangle \geq 0\ \forall v \in \Xi$. Here, $\Gamma x = J_E^p(x) - J_E^p(u)$ $(\forall x \in E)$ is strongly monotone (due to (2.1)) and uniformly continuous on bounded subsets of E.

(ii) The lower-level VIP consists of finding $z^\dagger \in C$ such that $\langle Fz^\dagger, y - z^\dagger\rangle \geq 0\ \forall y \in C$. Here, $F: E \to E^*$ is pseudomonotone and uniformly continuous on C. Setting $F = 0$ yields $\Xi = \bigcap_{i=1}^N \mathrm{Fix}(S_i) \cap \Omega = \{z \in \bigcap_{i=1}^N \mathrm{Fix}(S_i) : \mathcal{T}z \in \mathrm{Fix}(S)\}$, representing the solution set to the split fixed-point problem (SFPP) given by:

$$\text{Seek } z \in \bigcap_{i=1}^N \mathrm{Fix}(S_i) \text{ such that } \mathcal{T}z \in \mathrm{Fix}(S). \tag{3.50}$$

In this scenario, Algorithm 3.1 simplifies to the follwing iterative algorithm designed for solving SFPP (3.50).

Algorithm 3.2.

Initialization: By choosing arbitrary values for x_1 and x_0 from the set C. Specify positive values for ϵ, μ, λ within the range $(0, \frac{1}{\mu})$, and l within $(0,1)$. Subsequently, meticulously define sequences $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, and $\{\ell_n\}$ within the interval $(0,1)$. Ensure that the conditions $\lim_{n\to\infty}\ell_n = 0$, $\sum_{n=1}^\infty \alpha_n = \infty$, $\lim_{n\to\infty}\alpha_n = 0$, $\liminf_{n\to\infty}\beta_n(1-\beta_n) > 0$, and $\liminf_{n\to\infty}\gamma_n(1-\gamma_n) > 0$ are satisfied. Furthermore, given iterates x_n and x_{n-1} for $n \geq 1$, choose ϵ_n within the range $0 \leq \epsilon_n \leq \tilde{\epsilon}_n$, making certain that $\sup_{n\geq 1} \frac{\epsilon_n}{\alpha_n} < \infty$ and

$$\tilde{\epsilon}_n = \begin{cases} \min\left\{\epsilon, \frac{\ell_n}{\|J_E^p x_n - J_E^p(2x_n - x_{n-1})\|}\right\} & \text{if } x_n \neq x_{n-1}, \\ \epsilon & \text{otherwise.} \end{cases}$$

Iterative steps: Evaluate x_{n+1} as follows:

Step 1. Set $u_n = J_{E^*}^q((1-\epsilon_n)J_E^p x_n + \epsilon_n J_E^p(2x_n - x_{n-1}))$, and calculate $g_n = J_{E^*}^q(\gamma_n J_E^p x_n + (1-\gamma_n)J_E^p u_n)$, $y_n = \Pi_C(J_{E^*}^q(J_E^p g_n - \lambda G g_n))$, $r_\lambda(g_n) := g_n - y_n$ and $s_n = g_n - \tau_n r_\lambda(g_n)$, where $\tau_n := l^{i_n}$ and i_n is the smallest nonnegative integer i satisfying

$$\langle Gg_n - G(g_n - l^i r_\lambda(g_n)), g_n - y_n\rangle \leq \frac{\mu}{2} D_{f_p}(g_n, y_n).$$

Step 2. Calculate $w_n = \Pi_{K_n}(g_n)$, with $K_n := \{x \in C : h_n(x) \leq 0\}$ and

$$h_n(x) = \langle Gs_n, x - g_n\rangle + \frac{\tau_n}{2\lambda} D_{f_p}(g_n, y_n).$$

Step 3. Compute $v_n = J_{E^*}^q(\beta_n J_E^p w_n + (1-\beta_n)J_E^p(S_n w_n))$ and $x_{n+1} = \Pi_C(J_{E^*}^q(\alpha_n J_E^p u + (1-\alpha_n)J_E^p v_n))$.

Proceed to Step 1 after setting $n := n + 1$.

Drawing upon Theorem 3.1, we deduce the following strong convergence result.

Corollary 3.1. Assuming that conditions (C1)–(C3) are met, and

$$\varXi = \{z \in \bigcap_{i=1}^{N} \text{Fix}(S_i) : \mathcal{T}z \in \text{Fix}(S)\} \neq \emptyset,$$

the sequence $\{x_n\}$ generated by Algorithm 3.2 demonstrates strong convergence to $\Pi_{\varXi} u$ if and only if $\sup_{n\geq 1} \|x_n\| < \infty$.

4. Numerical experiments

In this section, we present a series of numerical experiments to demonstrate the effectiveness of the proposed methodologies. The primary goal of these experiments is to provide insights into the selection of optimal control settings and to conduct a thorough examination of control parameter configuration. Throughout this section, the error term is consistently symbolized as D_n, while essential parameters such as the total number of iterations and the required execution time are denoted as k and t, respectively. This section presents an illustrative example highlighting the proposed method's practical applicability.

Set $\mu = 1$ and $l = \lambda = \frac{1}{3}$. Initially, we consider a Lipschitz continuous and pseudomonotone mapping $F : E \to E^*$, a Bregman relatively nonexpansive mapping

$S_1 : C \to C$, a τ-demimetric mapping $S : H \to H$, and a nonzero bounded linear operator $\mathcal{T} : E \to H$ with $\Xi = \mathrm{Fix}(S_1) \cap \Omega \neq \emptyset$, where $\Omega = \{z \in \mathrm{VI}(C, F) : \mathcal{T}z \in \mathrm{Fix}(S)\}$.

Consider $C = [-2, 2]$ and $E = H = \mathbb{R}$ with the inner product $\langle a, b\rangle = ab$ and induced norm $\|\cdot\| = |\cdot|$. The initial point x_1 is randomly chosen from C. Define $F : H \to H$ and $S_1 : C \to C$ as $Fx := \frac{1}{1+|\sin x|} - \frac{1}{1+|x|}$ and $S_1 x := \sin x$ for all $x \in C$. We now proceed to demonstrate that F is Lipschitz continuous and pseudomonotone. Indeed, for all $x, y \in H$, we have

$$\begin{aligned}\|Fx - Fy\| &= \left|\frac{1}{1+\|\sin x\|} - \frac{1}{1+\|x\|} - \frac{1}{1+\|\sin y\|} + \frac{1}{1+\|y\|}\right| \\ &\leq \left|\frac{\|y\| - \|x\|}{(1+\|x\|)(1+\|y\|)}\right| + \left|\frac{\|\sin y\| - \|\sin x\|}{(1+\|\sin x\|)(1+\|\sin y\|)}\right| \\ &\leq \|x - y\| + \|\sin x - \sin y\| \leq 2\|x - y\|.\end{aligned}$$

This confirms the Lipschitz continuity of F. Furthermore, we illustrate the pseudomonotonicity of F. For any x and y in H, it is evident that

$$\begin{aligned}&\langle Fx, y - x\rangle = \left(\tfrac{1}{1+|\sin x|} - \tfrac{1}{1+|x|}\right)(y - x) \geq 0 \\ \implies\ &\langle Fy, y - x\rangle = \left(\tfrac{1}{1+|\sin y|} - \tfrac{1}{1+|y|}\right)(y - x) \geq 0.\end{aligned}$$

It is evident that $\mathrm{Fix}(S_1) = \{0\}$, and S_1 demonstrates Bregman relatively nonexpansiveness. Additionally, consider $Sx = \frac{1}{5}x + \frac{3}{5}\sin x$ for all $x \in H$. Let us assume $\mathcal{T}x = x$ for all $x \in H$. Consequently, $\mathcal{T}$ is a bounded linear operator on H. Remarkably, S emerges as a τ-demicontractive mapping with $\tau = \frac{1}{5}$, and $\mathrm{Fix}(S) = \{0\}$. In fact, S is τ-strictly pseudocontractive with $\tau = \frac{1}{5}$ because

$$\|Sx - Sy\|^2 = \left\|\tfrac{1}{5}(x - y) + \tfrac{3}{5}(\sin x - \sin y)\right\|^2 \leq \|x - y\|^2 + \tfrac{1}{5}\|(I - S)x - (I - S)y\|^2.$$

Therefore, $\Xi = \mathrm{Fix}(S_1) \cap \Omega = \{0\} \neq \emptyset$ with $\Omega = \{z \in \mathrm{VI}(C, F) : \mathcal{T}z \in \mathrm{Fix}(S)\}$. In this case, $G = \mathcal{T}^*(I - S)\mathcal{T} = I - S$ is strongly monotone and Lipschitz continuous. In fact, we have

$$\begin{aligned}\langle Gx - Gy, x - y\rangle &= \left\langle \tfrac{4}{5}x - \tfrac{3}{5}\sin x - (\tfrac{4}{5}y - \tfrac{3}{5}\sin y), x - y\right\rangle \\ &= \tfrac{4}{5}\|x - y\|^2 - \tfrac{3}{5}\langle \sin x - \sin y, x - y\rangle \geq \tfrac{1}{5}\|x - y\|^2.\end{aligned}$$

The Lipschitz continuity of $Gx = \frac{4}{5}x - \frac{3}{5}\sin x$ is evident. Consequently, $\mathrm{VI}(C, G) = \{0\}$. This ensures that $\mathcal{T}\mathrm{VI}(C, G) \subset \mathrm{Fix}(S_1)$. Consequently, conditions (C1)-(C5) are fulfilled.

Example 4.1. Consider the sequences $\ell_n = \frac{1}{2(n+1)^2}$, $\alpha_n = \frac{1}{2(n+1)}$, and $\beta_n = \gamma_n = \frac{n}{2(n+1)}$ for all $n \geq 1$. When provided with the iterates x_{n-1} and x_n $(n \geq 1)$, opt for ϵ_n such that $0 \leq \epsilon_n \leq \tilde{\epsilon}_n$, where

$$\tilde{\epsilon}_n = \begin{cases} \min\{\epsilon, \frac{\ell_n}{\|x_n - x_{n-1}\|}\} & \text{if } x_n \neq x_{n-1}, \\ \epsilon \quad \text{else.} \end{cases}$$

Algorithm 3.1 yields

$$\begin{cases} u_n = x_n + \epsilon_n(x_n - x_{n-1}), \\ g_n = \frac{n}{2(n+1)}x_n + \frac{n+2}{2(n+1)}u_n, \\ y_n = P_C(g_n - \frac{1}{3}(I-S)g_n), \\ s_n = (1-\tau_n)g_n + \tau_n y_n, \\ w_n = P_{K_n}g_n, \\ \bar{y}_n = P_C(w_n - \frac{1}{3}Fw_n), \\ t_n = (1-\bar{\tau}_n)w_n + \bar{\tau}_n\bar{y}_n, \\ v_n = \frac{n}{2(n+1)}w_n + \frac{n+2}{2(n+1)}S_1P_{C_n}w_n, \\ x_{n+1} = P_C(\frac{1}{2(n+1)}u + \frac{2n+1}{2(n+1)}v_n) \quad \text{for all } n \geq 1. \end{cases}$$

For each $n \geq 1$, the choice of sets K_n, C_n, and step sizes τ_n, $\bar{\tau}_n$ adheres to the specifications outlined in Algorithm 3.1. In accordance with Theorem 3.1, it can be concluded that the sequence $\{x_n\}$ converges to $0 \in \Xi = \mathrm{Fix}(S_1) \cap \Omega$.

Experiment 1. In the first experiment, we examine the computational effectiveness of Algorithm 3.1 using the problem described in Example 4.1 as a test case. Our aim is to carefully test the algorithm's performance across multiple threshold values, denoted as φ, while monitoring the behavior of the error term $\|x_{n+1} - x^*\|$. The main objective is to determine the number of iterations and execution time required to achieve convergence for various threshold values. To conduct the experiment, we chose different values for φ, specifically 10^{-4}, 10^{-3}, 10^{-2}, and 10^{-1}. The chosen stopping criterion is defined as $\|x_{n+1} - x^*\| \leq \varphi$. By varying the threshold, we aim to gain insight into the algorithm's accuracy and performance characteristics.

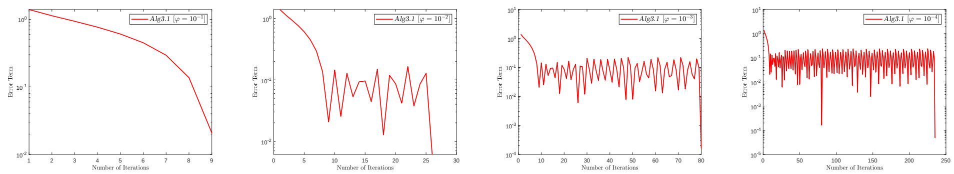
To understand the relationship between the selected threshold values, the number of iterations, and the accompanying execution times, we evaluate the data. This study clarifies Algorithm 3.1's stability and adaptability to a variety of convergence conditions in addition to contributing to understanding of the algorithm. To conduct this experiment, we will begin with the following parameter values:

$x_0 = x_1 = 2$, $\epsilon = \frac{1}{3}$, $\ell_n = \frac{1}{2(n+1)^2}$, $\gamma_n = \frac{n}{2(n+1)}$, $\beta_n = \frac{n}{2(n+1)}$, $\lambda = \frac{1}{3}$, $l = \frac{1}{3}$, $\mu = 1$, $u = 2$, and $\alpha_n = \frac{1}{2n+2}$.

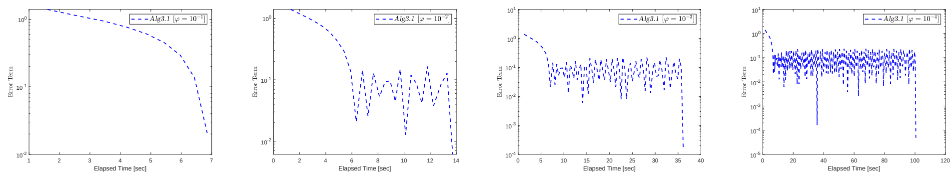
The numerical results regarding experiment are shown in Figure 1. It is important to note that when the value of φ, lowers, so does the number of iterations and the execution time in seconds in Example 4.1.

Experiment 2. The second experiment aims to evaluate the numerical efficiency of Algorithm 3.1 by selecting alternative initial values for x_0 and x_1. The chosen stopping condition is defined as $|x_{n+1} - x^*| \leq 10^{-2}$. Our primary objective is to precisely determine the number of iterations and associated execution time required for convergence. We are particularly interested in determining how the initial choice of starting points affects the algorithm's performance. To carry out this experiment, we will start the process with the following parameters:

$\epsilon = \frac{1}{3}$, $\ell_n = \frac{1}{2(n+1)^2}$, $\gamma_n = \frac{n}{2(n+1)}$, $\beta_n = \frac{n}{2(n+1)}$, $\lambda = \frac{1}{3}$, $l = \frac{1}{3}$, $\mu = 1$, $u = 1$, $\varphi = 10^{-2}$, and $\alpha_n = \frac{1}{2n+2}$.



(A) $\varphi = 10^{-1}$ and $k = 9$. (B) $\varphi = 10^{-2}$ and $k = 26$. (C) $\varphi = 10^{-3}$ and $k = 80$. (D) $\varphi = 10^{-4}$ and $k = 235$.



(E) $\varphi = 10^{-1}$ and $t = 6.8506539$. (F) $\varphi = 10^{-2}$ and $t = 13.721539$. (G) $\varphi = 10^{-3}$ and $t = 36.1309488$. (H) $\varphi = 10^{-4}$ and $t = 100.5667999$.

FIGURE 1. The numerical graph of Algorithm 3.1 enables us to analyze the impact of various values of φ on iteration count and execution time in seconds.

Figures 2 and 3 show a graph demonstrating the numerical results. It is worth noting that the computing performance in each situation is inextricably linked to the initial starting point selection. This emphasizes the importance of initial conditions for determining the algorithm's overall numerical performance.

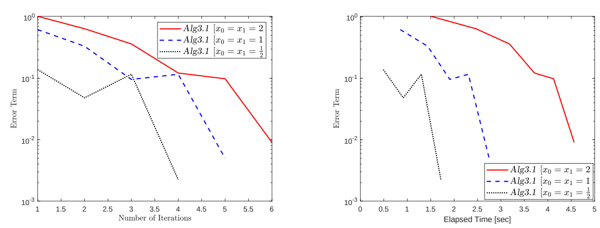


FIGURE 2. A numerical graph with iteration count and execution time of Algorithm 3.1 $[x_0 = x_1 = 2, k = 6, t = 4.5610639]$, $[x_0 = x_1 = 1, k = 5, t = 2.7475078]$, and $[x_0 = x_1 = \frac{1}{2}, k = 4, t = 1.7185153]$, respectively.

$\boxed{\textit{Experiment 3.}}$ The third experiment aims to assess the numerical efficiency of Algorithm 3.1 by varying the value of ϵ. The chosen termination criterion is defined as $|x_{n+1} - x^*| \leq 10^{-2}$. Our primary goal is to investigate how various parameter ϵ values affect the algorithm's efficiency. Figure 4 depicts a visual illustration of the numerical results. It is important to note that the computational performance in each scenario

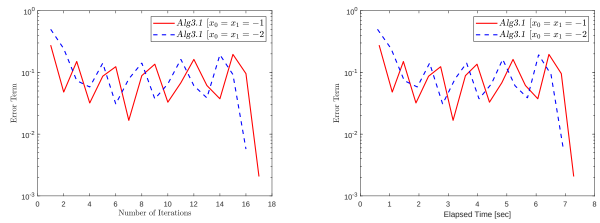
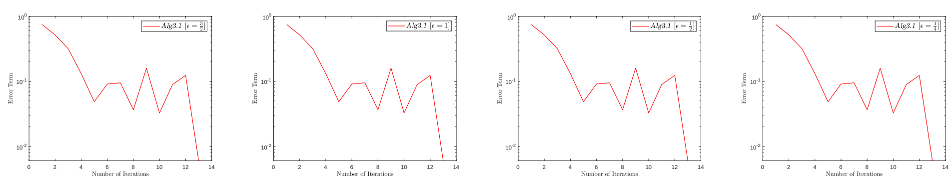


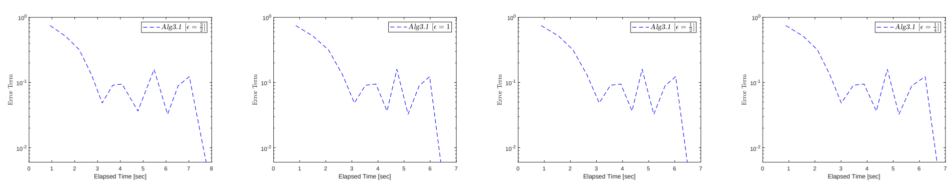
FIGURE 3. A numerical graph with iteration count and execution time of Algorithm 3.1 $[x_0 = x_1 = -1, k = 17, t = 7.2834759]$, and $[x_0 = x_1 = -2, k = 16, t = 6.9293856]$, respectively.

does not effected by the variation of parameter ϵ. To carry out this experiment, we start with the following parameters:

$x_0 = x_1 = 2$, $\ell_n = \frac{1}{2(n+1)^2}$, $\gamma_n = \frac{n}{2(n+1)}$, $\beta_n = \frac{n}{2(n+1)}$, $\lambda = \frac{1}{3}$, $l = \frac{1}{3}$, $\mu = 1$, $u = 2$, $\varphi = 10^{-2}$, and $\alpha_n = \frac{1}{2n+2}$.



(A) $\epsilon = \frac{3}{2}$ and $k = 13$. (B) $\epsilon = 1$ and $k = 13$. (C) $\epsilon = \frac{1}{2}$ and $k = 13$. (D) $\epsilon = \frac{1}{4}$ and $k = 13$.

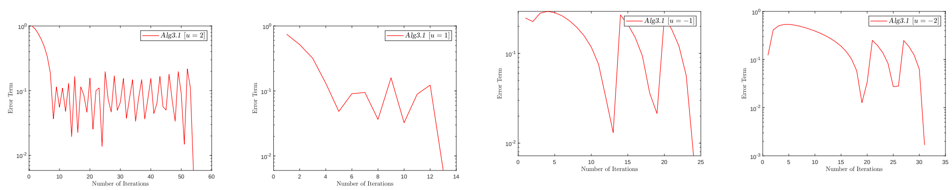


(E) $\epsilon = \frac{3}{2}$ and $t = 7.7553817$. (F) $\epsilon = 1$ and $t = 6.4066396$. (G) $\epsilon = \frac{1}{2}$ and $t = 6.4808668$. (H) $\epsilon = \frac{1}{4}$ and $t = 6.6772184$.

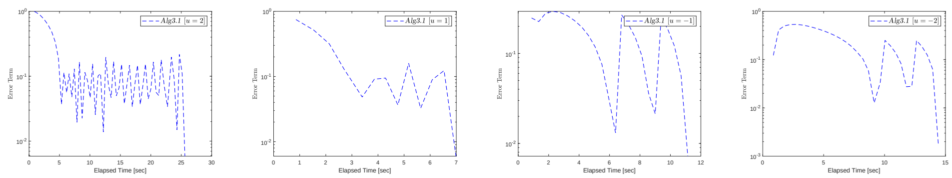
FIGURE 4. A numerical graph of Algorithm 3.1 allows us to examine how different values ϵ affect iteration count and execution time in seconds.

$\boxed{\textit{Experiment 4.}}$ The main objective of the this experiment is to examine the computational effectiveness of Algorithm 3.1 by varying the vector u and analyzing its impact on the overall performance of Algorithm 3.1. The termination criterion is defined as $|x_{n+1} - x^*| \leq 10^{-2}$. We are particularly interested in understanding how the vector u selection affects the performance of Algorithm 3.1. The numerical results are graphically shown in Figure 5. It is important to highlight that the computational performance in every case is strongly related to the vector u selection. To begin the numerical experiment, we set the following parameters:

$x_0 = x_1 = 1$, $\epsilon = \frac{1}{3}$, $\ell_n = \frac{1}{2(n+1)^2}$, $\gamma_n = \frac{n}{2(n+1)}$, $\beta_n = \frac{n}{2(n+1)}$, $\lambda = \frac{1}{3}$, $l = \frac{1}{3}$, $\mu = 1$, and $\alpha_n = \frac{1}{2n+2}$.



(A) $u = 2$ and $k = 54$. (B) $u = 1$ and $k = 13$. (C) $u = -1$ and $k = 24$. (D) $u = -2$ and $k = 31$.

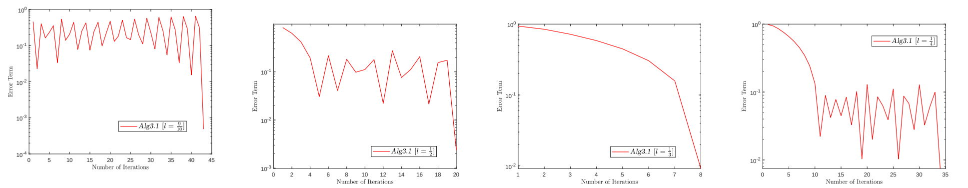


(E) $u = 2$ and $t = 25.5955706$. (F) $u = 1$ and $t = 6.9874447$. (G) $u = -1$ and $t = 11.1402337$. (H) $u = -2$ and $t = 14.4237914$.

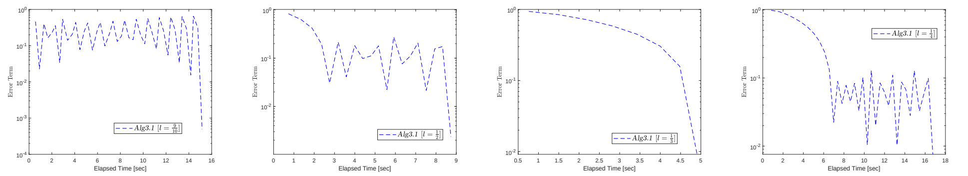
FIGURE 5. A numerical graph of Algorithm 3.1 allows us to examine how different values of vector u affect iteration count and execution time in seconds.

Experiment 5. In this experiment, we analyze the computational efficacy of Algorithm 3.1 through changing the parameter l and examining its impact on the overall performance of Algorithm 3.1. The termination criteria is specified as $|x_{n+1} - x^*| \leq 10^{-2}$. It is significant to note that the computing performance in every case is directly connected to the parameter l, and the numerical findings are graphically illustrated in Figure 6. To begin the numerical experiment, we set the following parameters: $x_0 = x_1 = 1$, $\epsilon = \frac{1}{3}$, $\ell_n = \frac{1}{2(n+1)^2}$, $\gamma_n = \frac{n}{2(n+1)}$, $\beta_n = \frac{n}{2(n+1)}$, $\lambda = \frac{1}{3}$, $\mu = 1$, $u = 2$, and $\alpha_n = \frac{1}{2n+2}$.

Experiment 6. In this experiment, our aim is to assess the computational efficiency of Algorithm 3.1 by varying the parameter λ and investigating its impact on the overall performance of Algorithm 3.1. The termination criterion is specified as $|x_{n+1} - x^*| \leq 10^{-2}$. It is crucial to highlight that the computing performance in each case is directly linked to the parameter λ, as illustrated in Figure 7. Notably, we observe that values such as $\lambda = \frac{1}{2}$ and $\lambda = \frac{1}{3}$ outperform other λ values. To initiate the numerical experiment, we have set the following parameters:



(A) $l = \frac{9}{10}$ and $k = 43$. (B) $l = \frac{1}{2}$ and $k = 20$. (C) $l = \frac{1}{3}$ and $k = 8$. (D) $l = \frac{1}{4}$ and $k = 34$.



(E) $l = \frac{9}{10}$ and $t = 15.1554477$. (F) $l = \frac{1}{2}$ and $t = 8.7339292$. (G) $l = \frac{1}{3}$ and $t = 4.9063413$. (H) $l = \frac{1}{4}$ and $t = 16.7598393$.

FIGURE 6. A numerical graph of Algorithm 3.1 allows us to examine how different values of parameter l affect iteration count and execution time in seconds.

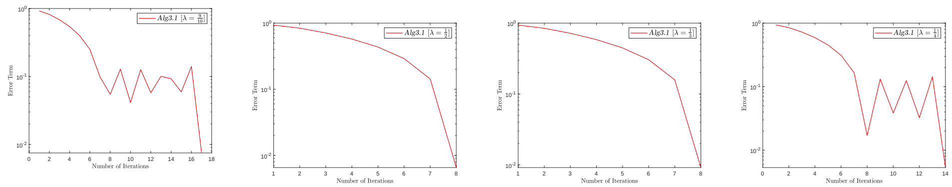
$x_0 = x_1 = 1$, $\epsilon = \frac{1}{3}$, $\ell_n = \frac{1}{2(n+1)^2}$, $\gamma_n = \frac{n}{2(n+1)}$, $\beta_n = \frac{n}{2(n+1)}$, $l = \frac{1}{3}$, $\mu = 1$, $u = 1$, and $\alpha_n = \frac{1}{2n+2}$.

$\boxed{\textit{Experiment 7.}}$ The objective of this experiment is to determine the computation effectiveness of Algorithm 3.1 by changing the parameter sequence α_n and examining the effect on the overall performance of the algorithm. It is critical to note that the computational performance in each scenario is tightly connected to the parameter sequence α_n. The numerical results are shown graphically in Figure 8. Notably, our analysis shows that sequences α_n with slow convergence to zero have a tendency to perform better in most situations. The termination criterion is defined as $|x_{n+1} - x^*| \leq 10^{-2}$. To initiate the numerical experiment, we have established the following parameters:

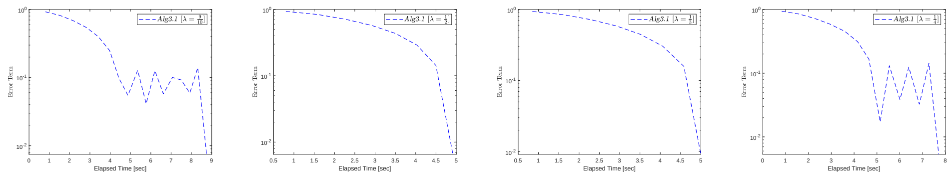
$x_0 = x_1 = 1$, $\epsilon = \frac{1}{3}$, $\ell_n = \frac{1}{2(n+1)^2}$, $\gamma_n = \frac{n}{2(n+1)}$, $\beta_n = \frac{n}{2(n+1)}$, $\lambda = \frac{1}{3}$, $l = \frac{1}{3}$, $\mu = 1$, and $u = 1$.

5. Conclusions

This article introduces and analyzes iterative algorithms designed to address the problem (BSPVIP), incorporating a problem (CFPP) constraint for finite Bregman relatively nonexpansive mappings in p-uniformly convex and uniformly smooth Banach spaces. By employing a self-adaptive inertial subgradient-like extragradient method, we develop an algorithm to approximate a common solution for both the BSPVIP and the CFPP of finite Bregman relatively nonexpansive mappings. The

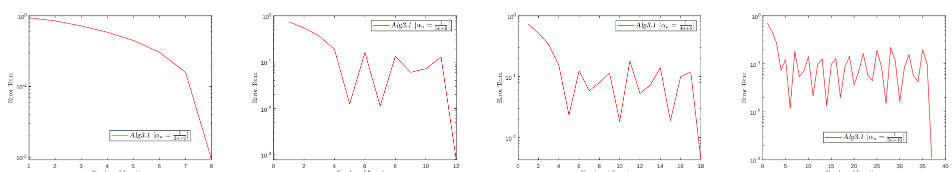


(A) $\lambda = \frac{9}{10}$ and $k = 17$. (B) $\lambda = \frac{1}{2}$ and $k = 8$. (C) $\lambda = \frac{1}{3}$ and $k = 8$. (D) $\lambda = \frac{1}{4}$ and $k = 14$.

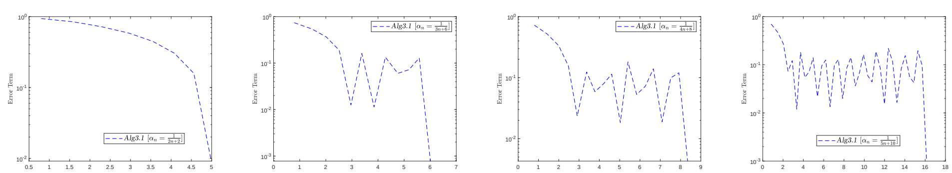


(E) $\lambda = \frac{9}{10}$ and $t = 8.7462596$. (F) $\lambda = \frac{1}{2}$ and $t = 4.9166412$. (G) $\lambda = \frac{1}{3}$ and $t = 4.9989945$. (H) $\lambda = \frac{1}{4}$ and $t = 7.7053504$.

FIGURE 7. A numerical graph of Algorithm 3.1 allows us to examine how different values λ affect iteration count and execution time in seconds.



(A) $\alpha_n = \frac{1}{2n+2}$ and $k = 8$. (B) $\alpha_n = \frac{1}{3n+6}$ and $k = 12$. (C) $\alpha_n = \frac{1}{4n+8}$ and $k = 18$. (D) $\alpha_n = \frac{1}{5n+10}$ and $k = 37$.



(E) $\alpha_n = \frac{1}{2n+2}$ and $t = 4.9895273$. (F) $\alpha_n = \frac{1}{3n+6}$ and $t = 6.0085349$. (G) $\alpha_n = \frac{1}{4n+8}$ and $t = 8.3464485$. (H) $\alpha_n = \frac{1}{5n+10}$ and $t = 16.1345931$.

FIGURE 8. A numerical graph of Algorithm 3.1 allows us to examine how different values α_n affect iteration count and execution time in seconds.

BSPVIP encompasses the upper-level VIP for a strongly monotone operator and the lower-level VIP for a pseudomonotone operator. Our focus lies in discussing the strong convergence of the proposed algorithm, utilizing standard conditions and innovative techniques. Furthermore, we establish the strong convergence outcome for the proposed method under mild conditions on the algorithm parameters, without prior knowledge of the operator norm or the coefficient of the underlying operator. Additionally, an illustrative example is provided to support the practicality and applicability of the proposed method. Finally, it is noteworthy that part of our future research aims to establish a strong convergence result for the modified version of our proposed method with Nesterov double inertial extrapolation steps (see [34]) and adaptive step sizes.

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