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AN INERTIAL METHOD FOR SOLVING SPLIT EQUALITY FIXED POINT PROBLEMS FOR NON-LIPSCHITZ PSEUDOCONTRACTIVE MAPPINGS

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Abstract. In this study, we introduce an inertial algorithm for solving the split equality fixed point problem in real Hilbert spaces under the assumption that the underlying mappings are uniformly continuous pseudocontractive self-mappings. We prove a strong convergence theorem under some mild conditions. We also provide numerical examples to demonstrate the applicability of the algorithm. The results in this paper extend and generalize many of the results in the literature as the problem considered is a more general split equality problem with an inertial approach.
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1. Introduction

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and induced norm $\| \cdot \|$ and let C be a nonempty, closed and convex subset of H. A mapping $T\colon C \to H$ is said to be

i. $\gamma-$ *strictly pseudocontractive* (Browder [4]) if there exists a positive real number γ such that

$$\langle Tx - Ty, x - y \rangle \leq \|x - y\|^2 - \gamma \|(x - y) - (Tx - Ty)\|^2, \text{ for all } x, y \in C;$$

ii. *pseudocontractive* if

$$\langle Tx - Ty, x - y \rangle \leq \|x - y\|^2, \text{ for all } x, y \in C; \tag{1.1}$$

iii. *Lipschitz continuous* if there exists a constant $L \geq 0$ such that

$$\|Tx - Ty\| \leq L\|x - y\|, \text{ for all } x, y \in C.$$

Remark 1.1. We observe from the above relations that every $\gamma-$strictly pseudocontractive mapping is pseudocontractive.

The class of pseudocontractive mappings has a close connection with the class of monotone mappings, where a mapping $A\colon D(T) \subset H \to H$ is said to be *monotone* if for all $x, y \in D(A)$, we have

$$\langle Ax - Ay, x - y\rangle \geq 0. \tag{1.2}$$

Remark 1.2. One can see from the relations (1.1) and (1.2) that the mapping T is pseudocontractive if and only if the mapping $A := I - T$ is monotone, where I is the identity mapping on H. Thus, it can be observed that the set of fixed points of T, denoted by $F(T)$, is equal to the set of zero points of A, $N(A)$.

The theory of fixed points has been serving as a very powerful and important tool in the study of nonlinear phenomena. Fixed point techniques have been applied in, for instance, biology, chemical reactions, chemistry, etc.

Many authors have proposed and studied different iterative algorithms involving fixed points of pseudocontractive mappings in Hilbert spaces (see, for instance, [23, 24, 25, 27] and the references therein).

In 2008, Zhou [27] introduced the following iterative algorithm and proved strong convergence of the method. Let C be a nonempty, closed and convex subset of a real Hilbert space H and let $T : C \to H$ be a $\kappa-$strictly pseudocontractive non-self mapping with $F(T) \neq \emptyset$. For $, x_1, u \in C$, let $\{x_n\}$ be the sequence generated by

$$\begin{cases} y_n = P_C\left[\alpha_n x_n + \beta_n T x_n\right], \\ x_{n+1} = \beta_n u + (1-\beta_n) T y_n, n \geq 1, \end{cases} \tag{1.3}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0, 1)$ satisfying some control conditions. He proved that the sequence $\{x_n\}$ generated by (1.3) converges strongly to a point $x^* \in F(T)$ with $x^* = P_{F(T)}u$, where $P_{F(T)}$ is the metric projection onto $F(T)$.

In 2013, Zegeye and Shahzad [24] introduced the following iterative algorithm: Let C be a nonempty, closed and convex subset of a real Hilbert space H and let T: $C \to C$ be a Lipschitz pseudocontractive mapping. Assume that $F(T) \neq \emptyset$. Let $\{x_n\}$ be the sequence generated from arbitrary $x_0, x \in C$ by

$$\begin{cases} u_n = (1 - c_n)x_n + c_n T x_n, \\ x_{n+1} = \alpha_n x + (1 - \alpha_n)(\theta_n x_n + \gamma_n T u_n),\ n \geq 0, \end{cases} \tag{1.4}$$

where $\{c_n\}$, $\{\theta_n\}$, $\{\gamma_n\} \subset (a, b) \subset (0, 1)$ and $\{\alpha_n\} \subset (0, c) \subset (0, 1)$ are control sequences satisfying some appropriate conditions. Then they proved that the sequence generated by (1.4) converges strongly to some $x^* \in F(T)$.

The other problem related to fixed point problems is the Split Fixed Point Problem (SFPP), which was introduced by Censor and Segal [5]. It is defined as finding a point

$$x^* \in F(T) \ \text{ such that } \ Ax^* \in F(S), \tag{1.5}$$

where H_1 and H_2 are real Hilbert spaces, $T\colon H_1 \to H_1$ and $S\colon H_2 \to H_2$ are nonlinear mappings, $A\colon H_1 \to H_2$ is a bounded linear mapping.

Several authors have proposed different iterative algorithms for approximating solutions of SFPP (see, for instance, [8, 15, 16]) involving different types of mappings.

Another problem which is related to the fixed point problems is the *Split Equality Fixed Point Problem* (SEFPP), which was introduced by Moudafi and Al-Shemas [17]. The SEFPP is defined as finding a point

$$(x^*, y^*) \in F(T_1) \times F(T_2) \text{ such that } B_1 x^* = B_2 y^*, \tag{1.6}$$

where H_1 and H_2 are real Hilbert spaces, T_1: $H_1 \to H_1$ and T_2: $H_2 \to H_2$ are mappings, B_1: $H_1 \to H_3$ and B_2: $H_2 \to H_3$ are bounded linear mappings with adjoints B_1^* and B_2^*, respectively, where H_3 is another real Hilbert space.

The SEFPP is a more general problem which contains several other problems. In fact, if, in (1.6), $B_2 = I$, then the SEFPP reduces to SFPP (1.5). If we take $S_1 = I_1 - T_1$ and $S_2 = I_2 - T_2$ in (1.6), then the SEFPP reduces to the Split Equality Null Point Problem (SENPP).

Several authors have proposed and studied different iterative algorithms for solving SEFPP (1.6) (see, for instance, [6, 7]).

In 2011, Moudafi and Al- Shemas [17] proposed the following algorithm which approximates a solution of SEFPP (1.6): Let H_1, H_2 and H_3 be real Hilbert spaces and let T: $H_1 \to H_1$ and S: $H_2 \to H_2$ be firmly quasi-nonexpansive mappings. Let $\{(x_n, y_n)\}$ be the sequence obtained by the following iteration:

$$\begin{cases} x_{n+1} = T(x_n - \beta_n A^*(Ax_n - By_n)), \\ y_{n+1} = S(y_n + \beta_n B^*(Ax_n - By_n)), \end{cases} \tag{1.7}$$

where $\{\beta_n\}$ is a real sequence satisfying some conditions and A: $H_1 \to H_3$ and B: $H_2 \to H_3$ are bounded linear mappings. Then they proved that the sequence $\{(x_n, y_n)\}$ converges weakly to a solution of the SEFPP (1.6).

In 2014, Ma et al. [12] proposed a strongly convergent iterative algorithm which approximates a solution of SEFPP (1.6) involving $\kappa-$strictly pseudocontractive mappings. We now raise the following important question.

Question 1.1. *Can we find a method for approximating solutions of split equality fixed point problems which involve uniformly continuous pseudocontractive self mappings in real Hilbert spaces?*

Motivated and inspired by the aforementioned results, it is our purpose in this paper to introduce and study a strongly convergent inertial algorithm for approximating solutions of the split equality fixed point problems that involve uniformly continuous pseudocontractive self-mappings in real Hilbert spaces.

2. Preliminaries

This section is devoted to present some basic definitions and important results that will be used in the sequel.

Consider the bi-function ϕ: $H \times H \to \mathbb{R}$, introduced by Alber [2] and defined as

$$\phi(y, x) = \|y\|^2 - 2\langle y, x\rangle + \|x\|^2, \text{ for all } x, y \in H. \tag{2.1}$$

The function ϕ in (2.1) is called the Lyapunov function and it satisfies the following identities:

$$\phi(x,y)+\phi(y,w)-\phi(x,w)=2\langle y-w,y-x\rangle, \text{ for all } x,y,w\in H; \quad (2.2)$$

$$\phi(y,z)+\phi(x,w)-\phi(y,w)-\phi(x,z)=2\langle w-z,y-x\rangle, \text{ for all } x,y,w,z\in H. \quad (2.3)$$

Moreover, the Lyapunov function has the following important property (see, [11]):

$$\phi(x,x^*)+2\langle y^*,x^*-x\rangle\leq\phi(x,x^*+y^*), \text{ for all } x,x^*,y^*\in H. \quad (2.4)$$

Let C be a nonempty, closed and convex subset of H. The metric projection of the point $x\in H$ onto C is the unique point, P_Cx, of C which satisfies:

$$\phi(P_Cx,x)=inf\left\{\phi(y,x):y\in C\right\}.$$

Moreover, this projection has the following two properties:

$$z=P_Cx \; if \; and \; only \; if \; \langle x-z,y-z\rangle\leq 0, \text{ for all } y\in C, \; and \quad (2.5)$$

$$\phi(y,P_Cx)+\phi(P_Cx,x)\leq\phi(y,x), \text{ for all } x\in H, y\in C. \quad (2.6)$$

Lemma 2.1. [21] *Let H be a real Hilbert space. Then*

$$\phi(y,x)\geq\frac{1}{2}\|x-y\|^2, \; for \; all \; x,y\in H. \quad (2.7)$$

Let $\{w_i\}\subseteq H$ and $\{\alpha_i\}\subseteq(0,1)$ be such that $\sum_{i=1}^N\alpha_i=1$. Then, we have by [19] that for all $x\in H$,

$$\phi\left(x,\sum_{i=1}^N\alpha_iw_i\right)\leq\sum_{i=1}^N\alpha_i\phi(x,w_i). \quad (2.8)$$

Lemma 2.2. [14] *If the sequence $\{\phi(x_n,x_0)\}$ is bounded for any $x_0\in H$, then $\{x_n\}$ is bounded.*

Lemma 2.3. [18] *Let the sequences $\{x_n\}$ and $\{y_n\}$ be bounded in H. Then, $\lim_{n\to\infty}\phi(x_n,y_n)=0$ if and only if $\lim_{n\to\infty}\|x_n-y_n\|=0$.*

Lemma 2.4. [22] *Let $\{b_n\}$ be a sequence of non-negative real numbers such that $b_{n+1}\leq(1-\alpha_n)b_n+\alpha_nd_n$, where $\{\alpha_n\}\subset(0,1)$ with $\sum_{n=1}^{\infty}\alpha_n=\infty$ and $\{d_n\}$ is a sequence of real numbers with $\limsup_{n\to\infty}d_n\leq 0$. Then $\lim_{n\to\infty}b_n=0$.*

Lemma 2.5. [13] *Let $\{c_n\}$ be a sequence of non-negative real numbers. If $\{c_{n_i}\}$ is a sub-sequence of $\{c_n\}$ such that $c_{n_i}<c_{n_i+1}$ for all $i\in\mathbb{N}$, then there exists a non-decreasing sequence $\{m_k\}$ of $\mathbb{N}$ such that $\lim_{k\to\infty}m_k=\infty$ and the following properties are satisfied by all (sufficiently large) number $k\in\mathbb{N}$:*

$$c_{m_k}\leq c_{m_k+1} \quad and \quad c_k\leq c_{m_k+1}.$$

In fact, m_k=max$\{n\leq k:c_n<c_{n+1}\}$.

Lemma 2.6. [9] *Let H be a real Hilbert space and let C be a nonempty closed convex subset of H. For all $u\in H$ and $\alpha\geq\beta>0$, the inequalities hold:*

$$\left\|\frac{u - P_C(u-\alpha Au)}{\alpha}\right\| \le \left\|\frac{u - P_C(u-\beta Au)}{\beta}\right\| \text{ and}$$
$$\|u - P_C(u-\beta Au)\| \le \|u - P_C(u-\alpha Au)\|.$$

Lemma 2.7. [3] *If H_1 and H_2 are real Hilbert spaces, then $H = H_1 \times H_2$ is also a real Hilbert space with inner product*

$\langle (x_1, y_1), (x_2, y_2)\rangle = \langle x_1, x_2\rangle + \langle y_1, y_2\rangle$, for all (x_1, y_1), $(x_2, y_2) \in H = H_1 \times H_2$, and $(x_n, y_n) \rightharpoonup (x, y)$ implies $x_n \rightharpoonup x$ and $y_n \rightharpoonup y$.

Lemma 2.8. [3] *Let $H = H_1 \times H_2$, where H_1 and H_2 are real Hilbert spaces, and let C be a nonempty, closed and convex subset of H. If $(u, v) \in H$ and $(u^*, v^*) = P_C(u, v)$, then $\langle (u, v) - (u^*, v^*), (x, y) - (u^*, v^*)\rangle \le 0$, for all $(x, y) \in C$.*

If H is a real Hilbert space, then we have the following relation:

$$\|x + y\|^2 \le \|x\|^2 + 2\langle x + y, y\rangle, \quad \text{for all } x, y \in H. \tag{2.9}$$

Let H be a real Hilbert space, C be a nonempty, closed and convex subset of H. A mapping $T\colon\ C \to C$ is said to satisfy the demiclosedness property if $(I - T)$ is demiclosed at 0, that is, if $\{x_n\}$ is any sequence in C such that $x_n \rightharpoonup p$ and $\|(I - T)x_n\| \to 0$, then $Tp = p$.

Lemma 2.9. [26] *Let H be a real Hilbert space, C be a nonempty, closed and convex subset of H and let $T : C \to C$ be a continuous pseudocontractive mapping. Then*

(1) *$F(T)$ is a closed and convex subset of C;*
(2) *$(I - T)$ is demiclosed at zero.*

3. Main result

In this section, we state our algorithm and discuss its convergence analysis. We shall assume the following conditions in the sequel.

Conditions

(C1) Let C and D be nonempty, closed and convex subsets of real Hilbert spaces H_1 and H_2, respectively;
(C2) Let $T_1\colon\ C \to C$ and $T_2\colon\ D \to D$ be uniformly continuous pseudocontractive mappings;
(C3) Let $B_1\colon\ H_1 \to H_3$ and $B_2\colon\ H_2 \to H_3$ be bounded linear mappings with adjoints B_1^* and B_2^*, respectively, where H_3 is a real Hilbert space;
(C4) Let $\Omega = \{(x^*, y^*) \in F(T_1) \times F(T_2) : B_1x^* = B_2y^*\} \neq \emptyset$;
(C5) Let $\{\alpha_n\} \subset (0, 1)$ be such that $\lim\limits_{n\to\infty} \alpha_n = 0$ and $\sum\limits_{n=1}^{\infty} \alpha_n = \infty$;
(C6) Let $\{\zeta_n\}$ be a sequence of positive numbers such that $\zeta_n \in \left(0, \frac{1}{4}\right)$ for all $n \ge 0$, and $\frac{\zeta_n}{\alpha_n} \to 0$ as $n \to \infty$.

We now state our proposed algorithm.

Algorithm 3.1

Initialization: Let $x_0, x_1 \in H_1$; $y_0, y_1 \in H_2$; $\mu, \sigma \in \left(0, \frac{1}{2}\right)$, $l, \gamma \in (0,1)$ and $\theta > 0$. For arbitrary points $u \in H_1$ and $v \in H_2$, calculate $\{x_n\}$ and $\{y_n\}$ as follows:

Step 1: Given $x_{n-1}, x_n \in H_1$ and $y_{n-1}, y_n \in H_2$, choose θ_n such that $0 \le \theta_n \le \sigma_n$, where

$$\sigma_n = \begin{cases} \min\left\{\theta, \dfrac{\zeta_n}{\|x_n - x_{n-1}\| + \|y_n - y_{n-1}\|}\right\}, & if \ \ \|x_n - x_{n-1}\| + \|y_n - y_{n-1}\| \neq 0, \\ \theta, & \text{otherwise.} \end{cases}$$

Step 2: Compute

$$\begin{aligned} a_n &= P_C\big(x_n + \theta_n(x_n - x_{n-1})\big) \\ b_n &= P_D\big(y_n + \theta_n(y_n - y_{n-1})\big). \end{aligned} \tag{3.1}$$

Step 3: Compute

$$\begin{aligned} c_n &= P_C\big(a_n - \gamma_n B_1^* \left(B_1 a_n - B_2 b_n\right)\big), \\ d_n &= P_D\big(b_n - \gamma_n B_2^* \left(B_2 b_n - B_1 a_n\right)\big), \end{aligned} \tag{3.2}$$

where $0 < \rho \le \gamma_n \le \rho_n$ with

$$\rho_n = \min\left\{\rho + 1, \frac{\|B_1 a_n - B_2 b_n\|^2}{2[\|B_1^*(B_1 a_n - B_2 b_n)\|^2 + \|B_2^*(B_2 b_n - B_1 a_n)\|^2]}\right\},$$

for $n \in \Upsilon = \{m \in \mathbb{N} : B_1 a_m - B_2 b_m \neq 0\}$, otherwise $\gamma_n = \rho$, for some $\rho > 0$.
Step 4: Compute

$$\begin{aligned} e_n &= c_n - \lambda_n(I_1 - T_1)c_n, \\ h_n &= d_n - \eta_n(I_2 - T_2)d_n, \end{aligned} \tag{3.3}$$

where $\lambda_n = \gamma l^{j_n}$ and j_n is the smallest non negative integer j satisfying the relation

$$\gamma l^j \|(I_1 - T_1)e_n - (I_1 - T_1)c_n\| \le \mu\|e_n - c_n\|, \tag{3.4}$$

and $\eta_n = \gamma l^{m_n}$, where m_n is the smallest non negative integer m satisfying the relation

$$\gamma l^m \|(I_2 - T_2)h_n - (I_2 - T_2)d_n\| \le \sigma\|h_n - d_n\|. \tag{3.5}$$

Step 5: Compute

$$\begin{aligned} p_n &= e_n - \lambda_n((I_1 - T_1)e_n - (I_1 - T_1)c_n), \\ q_n &= h_n - \eta_n((I_2 - T_2)h_n - (I_2 - T_2)d_n). \end{aligned} \tag{3.6}$$

Step 6: Compute

$$\begin{aligned} x_{n+1} &= (1 - \alpha_n)p_n + \alpha_n u, \\ y_{n+1} &= (1 - \alpha_n)q_n + \alpha_n v. \end{aligned} \tag{3.7}$$

Set $n = n + 1$ and go to **Step 1**.

Lemma 3.1. *Assume that Conditions* $(C1)-(C6)$ *hold. Then the Armijo line-search rules* (3.4) *and* (3.5) *are well defined.*

Proof. The line search rule (3.4) can be rewritten as

$$\|(I_1 - T_1)(c_n - \gamma l^j (I_1 - T_1)c_n) - (I_1 - T_1)c_n\| \leq \mu \|(I_1 - T_1)c_n\|. \tag{3.8}$$

Thus, it is sufficient to show that (3.8) is well defined. We now consider two cases on c_n. If c_n is a fixed point of T_1, then obviously $j = 0$ satisfies the relation (3.8). Assume on the contrary that c_n is not a fixed point of T_1. Then the right hand side of (3.8) is always positive. On the other hand, we have from the continuity of T_1 and the fact $l \in (0,1)$ that $\lim_{j\to\infty} \|(I_1 - T_1)\left(c_n - \gamma l^j (I_1 - T_1)\, c_n\right) - (I_1 - T_1)c_n\| = 0$.

Therefore, there exists a non negative integer j which satisfies the inequality (3.8) and hence (3.4) is well defined. Similarly, there exists a non negative integer m which satisfies the relation (3.5) and hence the proof is complete. □

Theorem 3.1. *Assume that conditions $(C1) - (C6)$ hold. Then the sequences $\{x_n\}$ and $\{y_n\}$ generated by Algorithm 3.1 are bounded.*

Proof. Let $(\hat{x}, \hat{y}) \in \Omega$. From (3.7) and (2.8), we obtain

$$\begin{aligned} \phi(\hat{x}, x_{n+1}) &= \phi(\hat{x}, (1-\alpha_n)p_n + \alpha_n u) \\ &\leq (1-\alpha_n)\phi(\hat{x}, p_n) + \alpha_n \phi(\hat{x}, u). \end{aligned} \tag{3.9}$$

From (3.6) and (2.1), we get

$$\begin{aligned} \phi\left(\hat{x}, p_n\right) &= \phi\left(\hat{x}, e_n - \lambda_n((I_1 - T_1)e_n - (I_1 - T_1)c_n)\right) \\ &= \|\hat{x}\|^2 + \|p_n\|^2 - 2\langle e_n, x\rangle + 2\lambda_n \langle (I_1 - T_1)e_n - (I_1 - T_1)c_n, \hat{x}\rangle \\ &= \|\hat{x}\|^2 + \|e_n\|^2 - \|e_n\|^2 - 2\langle e_n, \hat{x}\rangle + \|p_n\|^2 \\ &\quad + 2\lambda_n \langle (I_1 - T_1)e_n - (I_1 - T_1)c_n, \hat{x}\rangle \\ &= \phi(\hat{x}, e_n) - \|e_n\|^2 + \|p_n\|^2 + 2\lambda_n \langle (I_1 - T_1)e_n - (I_1 - T_1)c_n, \hat{x}\rangle \\ &= \phi(\hat{x}, e_n) - \|e_n\|^2 - \|p_n\|^2 + 2\|p_n\|^2 - 2\langle e_n, p_n\rangle + 2\langle e_n, p_n\rangle \\ &\quad + 2\lambda_n \langle (I_1 - T_1)e_n - (I_1 - T_1)c_n, \hat{x}\rangle \\ &= \phi(\hat{x}, e_n) - \phi(p_n, e_n) + 2\|p_n\|^2 - 2\langle e_n, p_n\rangle \\ &\quad + 2\lambda_n \langle (I_1 - T_1)e_n - (I_1 - T_1)c_n, \hat{x}\rangle \\ &= \phi(\hat{x}, e_n) - \phi(p_n, e_n) + 2\langle p_n, p_n\rangle - 2\langle e_n, p_n\rangle \\ &\quad + 2\lambda_n \langle (I_1 - T_1)e_n - (I_1 - T_1)c_n, \hat{x}\rangle \\ &= \phi(\hat{x}, e_n) - \phi(p_n, e_n) - 2\langle p_n, \lambda_n \left[(I_1 - T_1)e_n - (I_1 - T_1)c_n\right]\rangle \\ &\quad + 2\lambda_n \langle (I_1 - T_1)e_n - (I_1 - T_1)c_n, \hat{x}\rangle, \end{aligned}$$

that is,

$$\phi\left(\hat{x}, p_n\right) = \phi(\hat{x}, e_n) - \phi(p_n, e_n) + 2\lambda_n \langle (I_1 - T_1)e_n - (I_1 - T_1)c_n, \hat{x} - p_n\rangle. \tag{3.10}$$

By (2.3), we have

$$\phi(\hat{x}, e_n) - \phi(p_n, e_n) = \phi(\hat{x}, c_n) - \phi(p_n, c_n) + 2\langle e_n - c_n, p_n - \hat{x}\rangle. \tag{3.11}$$

Substituting (3.11) into (3.10), we obtain

$$\begin{aligned}\phi\left(\hat{x},p_n\right)&=\phi(\hat{x},c_n)-\phi(p_n,c_n)+2\langle e_n-c_n,p_n-\hat{x}\rangle\\&\quad+2\lambda_n\langle(I_1-T_1)e_n-(I_1-T_1)c_n,\hat{x}-p_n\rangle.\end{aligned}\tag{3.12}$$

By (2.2), we have

$$\phi(p_n,c_n)=\phi(p_n,e_n)+\phi(e_n,c_n)-2\langle e_n-c_n,e_n-p_n\rangle.\tag{3.13}$$

Substituting (3.13) into (3.12), we obtain

$$\begin{aligned}\phi\left(\hat{x},p_n\right)&=\phi(\hat{x},c_n)-\phi(p_n,e_n)-\phi(e_n,c_n)+2\langle e_n-c_n,e_n-p_n\rangle\\&\quad+2\langle e_n-c_n,p_n-\hat{x}\rangle+2\lambda_n\langle(I_1-T_1)e_n-(I_1-T_1)c_n,\hat{x}-p_n\rangle.\end{aligned}\tag{3.14}$$

From (3.2), (2.6) and (2.4), we have

$$\begin{aligned}\phi\left(\hat{x},c_n\right)&=\phi\left(\hat{x},P_C\left(a_n-\gamma_nB_1^*(B_1a_n-B_2b_n)\right)\right)\\&\leq\phi\left(\hat{x},a_n\right)-2\langle\gamma_nB_1^*(B_1a_n-B_2b_n),s_n-\hat{x}\rangle,\end{aligned}\tag{3.15}$$

where $s_n=a_n-\gamma_nB_1^*(B_1a_n-B_2b_n)$. Combining (3.14) and (3.15), we obtain

$$\begin{aligned}\phi\left(\hat{x},p_n\right)&\leq\phi\left(\hat{x},a_n\right)-\phi(p_n,e_n)-\phi(e_n,c_n)+2\langle e_n-c_n,e_n-p_n\rangle\\&\quad+2\langle e_n-c_n,p_n-\hat{x}\rangle+2\lambda_n\langle(I_1-T_1)e_n-(I_1-T_1)c_n,\hat{x}-p_n\rangle\\&\quad-2\langle\gamma_nB_1^*(B_1a_n-B_2b_n),s_n-\hat{x}\rangle.\end{aligned}\tag{3.16}$$

From (3.1) and (2.1), we obtain

$$\begin{aligned}\phi\left(\hat{x},a_n\right)&=\phi\left(\hat{x},P_C\left[x_n+\theta_n(x_n-x_{n-1})\right]\right)\\&\leq\|\hat{x}\|^2-2\langle\hat{x},x_n\rangle-2\langle\hat{x},\theta_n(x_n-x_{n-1})\rangle+\|a_n\|^2\\&=\|\hat{x}\|^2-2\langle\hat{x},x_n\rangle+\|x_n\|^2-\|x_n\|^2-2\langle\hat{x},\theta_n(x_n-x_{n-1})\rangle+\|a_n\|^2\\&=\phi(\hat{x},x_n)-\|x_n\|^2-2\langle\hat{x},\theta_n(x_n-x_{n-1})\rangle+\|a_n\|^2\\&=\phi(\hat{x},x_n)-\|x_n\|^2-2\langle\hat{x},\theta_n(x_n-x_{n-1})\rangle\\&\quad-\|a_n\|^2+2\|a_n\|^2-2\langle x_n,a_n\rangle+2\langle x_n,a_n\rangle\\&=\phi(\hat{x},x_n)-\phi(a_n,x_n)-2\langle\hat{x},\theta_n(x_n-x_{n-1})\rangle+2\|a_n\|^2-2\langle x_n,a_n\rangle\\&=\phi(\hat{x},x_n)-\phi(a_n,x_n)-2\langle\hat{x},\theta_n(x_n-x_{n-1})\rangle-2\langle x_n-a_n,a_n\rangle\\&=\phi(\hat{x},x_n)-\phi(a_n,x_n)-2\langle\hat{x},\theta_n(x_n-x_{n-1})\rangle+2\langle\theta_n(x_n-x_{n-1}),a_n\rangle\\&=\phi(\hat{x},x_n)-\phi(a_n,x_n)-2\langle\theta_n(x_n-x_{n-1}),\hat{x}-a_n\rangle.\end{aligned}\tag{3.17}$$

Now, substitution of (3.17) into (3.16) gives

$$\begin{aligned}
\phi(\hat{x}, p_n) \leq\ & \phi(\hat{x}, x_n) - \phi(a_n, x_n) - \phi(p_n, e_n) - \phi(e_n, c_n) + 2\langle e_n - c_n, e_n - p_n\rangle \\
& + 2\langle e_n - c_n, p_n - \hat{x}\rangle + 2\lambda_n\langle (I_1 - T_1)e_n - (I_1 - T_1)c_n, \hat{x} - p_n\rangle \\
& - 2\langle \gamma_n B_1^*(B_1 a_n - B_2 b_n), s_n - \hat{x}\rangle - 2\langle \theta_n(x_n - x_{n-1}), \hat{x} - a_n\rangle \\
=\ & \phi(\hat{x}, x_n) - \phi(a_n, x_n) - \phi(p_n, e_n) - \phi(e_n, c_n) + 2\langle e_n - c_n, e_n - \hat{x}\rangle \\
& + 2\lambda_n\langle (I_1 - T_1)e_n - (I_1 - T_1)c_n, \hat{x} - e_n\rangle \\
& + 2\lambda_n\langle (I_1 - T_1)e_n - (I_1 - T_1)c_n, e_n - p_n\rangle \\
& - 2\langle \gamma_n B_1^*(B_1 a_n - B_2 b_n), s_n - \hat{x}\rangle - 2\langle \theta_n(x_n - x_{n-1}), \hat{x} - a_n\rangle \\
=\ & \phi(\hat{x}, x_n) - \phi(a_n, x_n) - \phi(p_n, e_n) - \phi(e_n, c_n) \\
& + 2\lambda_n\langle (I_1 - T_1)e_n - (I_1 - T_1)c_n - (e_n - c_n), \hat{x} - e_n\rangle \\
& + 2\lambda_n\langle (I_1 - T_1)e_n - (I_1 - T_1)c_n, e_n - p_n\rangle \\
& - 2\langle \gamma_n B_1^*(B_1 a_n - B_2 b_n), s_n - x\rangle - 2\langle \theta_n(x_n - x_{n-1}), \hat{x} - a_n\rangle.
\end{aligned} \tag{3.18}$$

But, $e_n = c_n - \lambda_n(I_1 - T_1)c_n$. Let $\tau_n = (I_1 - T_1)e_n$. Then we have

$$e_n + \lambda_n\tau_n = c_n + \lambda_n\left[(I_1 - T_1)e_n - (I_1 - T_1)c_n\right],$$

which implies that

$$\tau_n = \frac{1}{\lambda_n}\left[\lambda_n\left((I_1 - T_1)e_n - (I_1 - T_1)c_n\right) - (e_n - c_n)\right]. \tag{3.19}$$

Since $(I_1 - T_1)\hat{x} = 0$ and $I_1 - T_1$ is monotone, we have that

$$\langle \tau_n, e_n - \hat{x}\rangle \geq 0. \tag{3.20}$$

Thus, substituting (3.19) into (3.20), we obtain

$$\langle \lambda_n\left((I_1 - T_1)e_n - (I_1 - T_1)c_n\right) - (e_n - c_n), e_n - \hat{x}\rangle \geq 0. \tag{3.21}$$

From (3.18) and (3.21), we get

$$\begin{aligned}
\phi(\hat{x}, p_n) \leq\ & \phi(\hat{x}, x_n) - \phi(a_n, x_n) - \phi(p_n, e_n) - D(e_n, c_n) \\
& + 2\lambda_n\langle (I_1 - T_1)e_n - (I_1 - T_1)c_n, e_n - p_n\rangle \\
& - 2\langle \gamma_n B_1^*(B_1 a_n - B_2 b_n), s_n - \hat{x}\rangle - 2\langle \theta_n(x_n - x_{n-1}), \hat{x} - a_n\rangle.
\end{aligned} \tag{3.22}$$

Moreover, we obtain from the Cauchy Schwarz inequality and Lemma 2.7 that

$$\begin{aligned}
-\langle \theta_n(x_n - x_{n-1}), \hat{x} - a_n\rangle \leq\ & \theta_n\|x_n - x_{n-1}\|\,\|\hat{x} - a_n\| \\
\leq\ & \frac{\theta_n}{2}\|x_n - x_{n-1}\|\left[\|\hat{x} - a_n\|^2 + 1\right] \\
=\ & \frac{\theta_n}{2}\|x_n - x_{n-1}\|\left[\|\hat{x} - x_n + x_n - a_n\|^2 + 1\right] \\
\leq\ & \frac{\theta_n}{2}\|x_n - x_{n-1}\|\left[2\|\hat{x} - x_n\|^2 + 2\|x_n - a_n\|^2 + 1\right] \\
=\ & 2\theta_n\|x_n - x_{n-1}\|\phi(\hat{x}, x_n) + 2\theta_n\|x_n - x_{n-1}\|\phi(a_n, x_n) \\
& + \frac{\theta_n}{2}\|x_n - x_{n-1}\|.
\end{aligned} \tag{3.23}$$

From (3.22), (3.23), (3.19) and using the fact that $\theta_n\|x_n - x_{n-1}\| \leq \zeta_n$, we obtain

$$\begin{aligned}
\phi\left(\hat{x}, p_n\right) \leq\, & \phi(\hat{x}, x_n) - \phi(a_n, x_n) - \phi(p_n, e_n) - \phi(e_n, c_n) \\
& + 4\theta_n\|x_n - x_{n-1}\|\phi(\hat{x}, x_n) + 4\theta_n\|x_n - x_{n-1}\|\phi(a_n, x_n) \\
& + 2\lambda_n\langle (I-T_1)e_n - (I-T_1)c_n, e_n - p_n\rangle + \theta_n\|x_n - x_{n-1}\| \\
& - 2\langle \gamma_n B_1^*(B_1a_n - B_2b_n), s_n - \hat{x}\rangle \\
\leq\, & \phi(\hat{x}, x_n) - \phi(a_n, x_n) - \phi(p_n, e_n) - \phi(e_n, c_n) \\
& + 4\zeta_n\phi(\hat{x}, x_n) + 4\zeta_n\phi(a_n, x_n) + 2\mu\|e_n - c_n\|\,\|p_n - e_n\| + \zeta_n \\
& - 2\langle \gamma_n B_1^*(B_1a_n - B_2b_n), s_n - \hat{x}\rangle \\
\leq\, & \left(1 + 4\zeta_n\right)\phi(\hat{x}, x_n) - \left(1 - 4\zeta_n\right)\phi(a_n, x_n) - \left(1 - 2\mu\right)\phi(p_n, e_n) \\
& - \left(1 - 2\mu\right)\phi(e_n, c_n) + \zeta_n - 2\langle \gamma_n B_1^*(B_1a_n - B_2b_n), s_n - \hat{x}\rangle.
\end{aligned} \tag{3.24}$$

Since $\zeta_n \in \left(0, \dfrac{1}{4}\right)$ and $\mu \in \left(0, \dfrac{1}{2}\right)$, we obtain from (3.24) that

$$\phi\left(\hat{x}, p_n\right) \leq \left(1 + 4\zeta_n\right)\phi(\hat{x}, x_n) + \zeta_n - 2\langle \gamma_n B_1^*(B_1a_n - B_2b_n), s_n - \hat{x}\rangle. \tag{3.25}$$

Substituting (3.25) into (3.9), we obtain

$$\begin{aligned}
\phi(\hat{x}, x_{n+1}) \leq\, & \alpha_n\phi(\hat{x}, u) + \left(1 - \alpha_n\right)\left(1 + 4\zeta_n\right)\phi(\hat{x}, x_n) - \left(1 - \alpha_n\right)\left(1 - 4\zeta_n\right)\phi(a_n, x_n) \\
& - \left(1 - \alpha_n\right)\left(1 - 2\mu\right)\phi(p_n, e_n) - \left(1 - \alpha_n\right)\left(1 - 2\mu\right)\phi(e_n, c_n) \\
& + (1 - \alpha_n)\zeta_n - 2(1 - \alpha_n)\langle \gamma_n B_1^*(B_1a_n - B_2b_n), s_n - \hat{x}\rangle.
\end{aligned} \tag{3.26}$$

Similarly, we obtain

$$\begin{aligned}
\phi(\hat{y}, y_{n+1}) \leq\, & \alpha_n\phi(\hat{y}, v) + \left(1 - \alpha_n\right)\left(1 + 4\zeta_n\right)\phi(y, y_n) - \left(1 - \alpha_n\right)\left(1 - 4\zeta_n\right)\phi(b_n, y_n) \\
& - \left(1 - \alpha_n\right)\left(1 - 2\sigma\right)\phi(q_n, h_n) - \left(1 - \alpha_n\right)\left(1 - 2\sigma\right)\phi(h_n, d_n) \\
& + (1 - \alpha_n)\zeta_n - 2(1 - \alpha_n)\langle \gamma_n B_2^*(B_2b_n - B_1a_n), r_n - \hat{y}\rangle,
\end{aligned} \tag{3.27}$$

where $r_n = b_n - \gamma_n B_2^*(B_2b_n - B_1a_n)$.

Since $\dfrac{\zeta_n}{\alpha_n} \to 0$ as $n \to \infty$, for any $\varepsilon \in \left(0, \dfrac{1}{4}\right)$ there exits $n_0 \in \mathbb{N}$ such that $\zeta_n < \varepsilon\alpha_n$, for all $n \geq n_0$. This together with the properties of μ and σ implies that

$$\begin{aligned}
\phi(\hat{x}, x_{n+1}) \leq\, & \alpha_n\phi(\hat{x}, u) + \left(1 - \alpha_n\right)\left(1 + 4\varepsilon\alpha_n\right)\phi(\hat{x}, x_n) \\
& + \varepsilon\alpha_n - 2(1 - \alpha_n)\langle \gamma_n B_1^*(B_1a_n - B_2b_n), s_n - \hat{x}\rangle,
\end{aligned} \tag{3.28}$$

and

$$\begin{aligned}
\phi(\hat{y}, y_{n+1}) \leq\, & \alpha_n\phi(\hat{y}, v) + \left(1 - \alpha_n\right)\left(1 + 4\varepsilon\alpha_n\right)\phi(y, y_n) \\
& + \varepsilon\alpha_n - 2(1 - \alpha_n)\langle \gamma_n B_2^*(B_2b_n - B_1a_n), r_n - \hat{y}\rangle.
\end{aligned} \tag{3.29}$$

Denote $\Pi_n = \phi(\hat{x}, x_n) + \phi(\hat{y}, y_n)$ and $\Sigma = \phi(\hat{x}, u) + \phi(\hat{y}, v)$. Then the combination of (3.28) and (3.29) gives

$$\begin{aligned}
\Pi_{n+1} \leq\, & \alpha_n\Sigma + \left(1 - \alpha_n\right)\left(1 + 4\varepsilon\alpha_n\right)\Pi_n + 2\varepsilon\alpha_n \\
& - 2(1 - \alpha_n)\left[\gamma_n\langle B_1a_n - B_2b_n, B_1s_n - B_2r_n\rangle\right].
\end{aligned} \tag{3.30}$$

Furthermore, we obtain by the Cauchy Schwarz Inequality that

$$\begin{aligned}
-\langle B_1a_n - B_2b_n, B_1s_n - B_2r_n\rangle \\
&= -\langle B_1a_n - B_2b_n, B_1a_n - B_2b_n\rangle - \langle B_1a_n - B_2b_n, B_1s_n - B_1a_n\rangle \\
&\quad - \langle B_1a_n - B_2b_n, B_2b_n - B_2r_n\rangle \\
&= -\|B_1a_n - B_2b_n\|^2 - \langle B_1^*(B_1a_n - B_2b_n), s_n - a_n\rangle \\
&\quad - \langle B_2^*(B_1a_n - B_2b_n), b_n - r_n\rangle \\
&\leq -\|B_1a_n - B_2b_n\|^2 + \|s_n - a_n\| \, \|B_1^*(B_1a_n - B_2b_n)\| \\
&\quad + \|b_n - r_n\| \, \|B_2^*(B_1a_n - B_2b_n)\|. \qquad (3.31)
\end{aligned}$$

Moreover,

$$\|s_n - a_n\| = \|a_n - \gamma_n B_1^*(B_1a_n - B_2b_n) - a_n\| = \gamma_n\|B_1^*(B_1a_n - B_2b_n)\|. \qquad (3.32)$$

Similarly, we have

$$\|r_n - b_n\| = \gamma_n\|B_2^*(B_2b_n - B_1a_n)\|. \qquad (3.33)$$

Combining (3.31), (3.32) and (3.33), we obtain

$$\begin{aligned}
-2\gamma_n\langle B_1a_n - B_2b_n, B_1s_n - B_2r_n\rangle \\
&\leq -2\gamma_n\|B_1a_n - B_2b_n\|^2 + 2\gamma_n^2\|B_1^*(B_1a_n - B_2b_n)\|^2 \\
&\quad + 2\gamma_n^2\|B_2^*(B_2b_n - B_1a_n)\|^2 \\
&\leq -\rho\|B_1a_n - B_2b_n\|^2 - \gamma_n\|B_1a_n - B_2b_n\|^2 \\
&\quad + \gamma_n\left\{2\gamma_n\left[\|B_1^*(B_1a_n - B_2b_n)\|^2 + \|B_2^*(B_2b_n - B_1a_n)\|^2\right]\right\} \\
&\leq -\rho\|B_1a_n - B_2b_n\|^2. \qquad (3.34)
\end{aligned}$$

Substituting (3.34) into (3.30), we obtain

$$\begin{aligned}
\Pi_{n+1} &\leq \alpha_n\Sigma + (1-\alpha_n)(1+4\varepsilon\alpha_n)\Pi_n + 2\varepsilon\alpha_n - (1-\alpha_n)\rho\|B_1a_n - B_2b_n\|^2 \\
&\leq \alpha_n\Sigma + (1-\alpha_n)(1+4\varepsilon\alpha_n)\Pi_n + 2\varepsilon\alpha_n \\
&\leq [1-\alpha_n(1-4\varepsilon)]\Pi_n + \alpha_n[\Sigma + 2\varepsilon] \qquad (3.35) \\
&= [1-\alpha_n(1-4\varepsilon)]\Pi_n + \alpha_n(1-4\varepsilon)\left[\frac{\Sigma+2\varepsilon}{1-4\varepsilon}\right] \leq \max\left\{\Pi_n, \frac{\Sigma+2\varepsilon}{1-4\varepsilon}\right\},
\end{aligned}$$

which implies by the Principle of Mathematical Induction that

$$\Pi_n \leq \max\left\{\Pi_0, \frac{\Sigma+2\varepsilon}{1-4\varepsilon}\right\}.$$

Thus, we conclude that $\{\Pi_n\}$ is bounded and hence the sequences $\{\phi(\hat{x}, x_n)\}$ and $\{\phi(\hat{y}, y_n)\}$ are bounded, which implies together with Lemma 2.2 that $\{x_n\}$ and $\{y_n\}$ are bounded sequences. □

Theorem 3.2. *Assume that conditions $(C1)-(C6)$ hold. Then, the sequence $\{(x_n, y_n)\}$ generated by Algorithm 3.1 converges strongly to (x^*, y^*), where*

$$(x^*, y^*) = P_\Omega(u, v).$$

Proof. Let $(x^*, y^*) = P_\Omega(u, v)$. Then we have by Lemma 2.8 that

$$\langle (u,v) - (x^*, y^*),\ (s,t) - (x^*, y^*)\rangle \le 0,\ \text{for all } (s,t) \in \Omega. \tag{3.36}$$

From (3.7), (2.4) and (2.8) we get

$$\begin{aligned}
\phi(x^*, x_{n+1}) &= \phi(x^*, (1-\alpha_n)p_n + \alpha_n u) \\
&\le \phi(x^*, (1-\alpha_n)p_n + \alpha_n u - \alpha_n (u - x^*)) + 2\alpha_n \langle u - x^*, x_{n+1} - x^*\rangle \\
&= \phi(x^*, (1-\alpha_n)p_n + \alpha_n x^*) + 2\alpha_n \langle u - x^*, x_{n+1} - x^*\rangle \\
&\le \alpha_n \phi(x^*, x^*) + (1-\alpha_n)\phi(x^*, p_n) + 2\alpha_n \langle u - x^*, x_{n+1} - x^*\rangle \\
&= (1-\alpha_n)\phi(x^*, p_n) + 2\alpha_n \langle u - x^*, x_{n+1} - x^*\rangle.
\end{aligned} \tag{3.37}$$

Combining (3.25) and (3.37) with $x = x^*$, we obtain

$$\begin{aligned}
\phi(x^*, x_{n+1}) &\le (1-\alpha_n)\left[(1+4\zeta_n)\,\phi(x^*, x_n) + \zeta_n - 2\langle \gamma_n B_1^*(B_1 a_n - B_2 b_n), s_n - x^*\rangle\right] \\
&\quad + 2\alpha_n \langle u - x^*, x_{n+1} - x^*\rangle \\
&\le (1-\alpha_n)\phi(x^*, x_n) + 4\zeta_n \phi(x^*, x_n) + (1-\alpha_n)\zeta_n \\
&\quad - 2(1-\alpha_n)\langle \gamma_n B_1^*(B_1 a_n - B_2 b_n), s_n - x^*\rangle + 2\alpha_n \langle u - x^*, x_{n+1} - x^*\rangle.
\end{aligned} \tag{3.38}$$

Since $\dfrac{\zeta_n}{\alpha_n} \to 0$ as $n \to \infty$, for every $\varepsilon \in \left(0, \dfrac{1}{4}\right)$, there exists $n_0 \in \mathbb{N}$ such that $\zeta_n < \varepsilon\alpha_n$, $\forall n \ge n_0$. Thus, we obtain from (3.38) and the Cauchy Schwarz inequality that

$$\begin{aligned}
&\phi(x^*, x_{n+1}) \\
&\quad \le (1-\alpha_n)\phi(x^*, x_n) + 4\varepsilon\alpha_n \phi(x^*, x_n) \\
&\qquad - 2(1-\alpha_n)\langle \gamma_n B_1^*(B_1 a_n - B_2 b_n), s_n - x^*\rangle \\
&\qquad + 2\alpha_n \|u - x^*\|\, \|x_{n+1} - x_n\| + 2\alpha_n \langle u - x^*, x_n - x^*\rangle + \zeta_n \\
&\quad = \left[1 - \alpha_n (1 - 4\varepsilon)\right] \phi(x^*, x_n) - 2(1-\alpha_n)\langle \gamma_n B_1^*(B_1 a_n - B_2 b_n), s_n - x^*\rangle \\
&\qquad + 2\alpha_n \|u - x^*\|\, \|x_{n+1} - x_n\| + 2\alpha_n \langle u - x^*, x_n - x^*\rangle + \frac{\alpha_n \zeta_n}{\alpha_n} \\
&\quad = \left[1 - \alpha_n (1 - 4\varepsilon)\right] \phi(x^*, x_n) - 2(1-\alpha_n)\langle \gamma_n B_1^*(B_1 a_n - B_2 b_n), s_n - x^*\rangle \\
&\qquad + \alpha_n (1 - 4\varepsilon) \left[\frac{2\left[\|u - x^*\|\, \|x_{n+1} - x_n\| + \langle u - x^*, x_n - x^*\rangle\right]}{1 - 4\varepsilon} + \frac{\zeta_n}{(1-4\varepsilon)\alpha_n}\right].
\end{aligned} \tag{3.39}$$

Similarly, we have

$$\begin{aligned}
&D(y^*, y_{n+1}) \\
&\quad \le \left[1 - \alpha_n (1 - 4\varepsilon)\right] \phi(y^*, y_n) - 2(1-\alpha_n)\langle \gamma_n B_2^*(B_2 b_n - B_1 a_n), r_n - y^*\rangle \\
&\qquad + \alpha_n (1 - 4\varepsilon) \left[\frac{2\left[\|v - y^*\|\, \|y_{n+1} - y_n\| + \langle v - y^*, y_n - y^*\rangle\right]}{1 - 4\varepsilon} + \frac{\zeta_n}{(1-4\varepsilon)\alpha_n}\right].
\end{aligned} \tag{3.40}$$

Now, denote $\Pi_n^* = \phi(x^*, x_n) + \phi(y^*, y_n)$ and $\Sigma^* = \phi(x^*, u) + \phi(y^*, v)$. Combining (3.39) and (3.40) and using the relation (3.34), we obtain

$$\Pi_{n+1}^* \leq [1 - \alpha_n (1 - 4\varepsilon)] \Pi_n^* + \alpha_n (1 - 4\varepsilon) (\Delta_n + \Gamma_n), \tag{3.41}$$

where

$$\Delta_n = \frac{2 [\|u - x^*\| \, \|x_{n+1} - x_n\| + \langle u - x^*, x_n - x^* \rangle]}{1 - 4\varepsilon} + \frac{\zeta_n}{(1 - 4\varepsilon)\alpha_n}, \tag{3.42}$$

and

$$\Gamma_n = \frac{2 [\|v - y^*\| \, \|y_{n+1} - y_n\| + \langle v - y^*, y_n - y^* \rangle]}{1 - 4\varepsilon} + \frac{\zeta_n}{(1 - 4\varepsilon)\alpha_n}. \tag{3.43}$$

Combining (3.26) and (3.27) and using the relation (3.34) gives

$$\begin{aligned}
&(1 - \alpha_n) (1 - 4\zeta_n) \phi(a_n, x_n) + (1 - \alpha_n) (1 - 2\mu) \phi(p_n, e_n) \\
&+ (1 - \alpha_n) (1 - 2\mu) \phi(e_n, c_n) + (1 - \alpha_n) (1 - 4\zeta_n) \phi(b_n, y_n) \\
&+ (1 - \alpha_n) (1 - 2\sigma) \phi(q_n, h_n) + (1 - \alpha_n) (1 - 2\mu) \phi(h_n, d_n) \\
&(1 - \alpha_n)\rho\|B_1 a_n - B_2 b_n\|^2 \leq \Pi_n^* - \Pi_{n+1}^* + \alpha_n \left[\Sigma^* + (4\varepsilon - 1) \Pi_n^* + \frac{2\zeta_n}{\alpha_n}\right].
\end{aligned} \tag{3.44}$$

Now, we show that the sequence of real numbers $\{\Pi_n^*\}$ converges to zero by considering two cases:

Case I. If there exists a natural number n_0 such that $\Pi_{n+1}^* \leq \Pi_n^*$ for all $n \geq n_0$, then $\{\Pi_n^*\}$ converges. So, taking the limit of (3.44) as $n \to \infty$, we obtain

$$\lim_{n\to\infty} \|B_1 a_n - B_2 b_n\| = 0. \tag{3.45}$$

and

$$\begin{aligned}
&\lim_{n\to\infty} \phi(a_n, x_n) = \lim_{n\to\infty} \phi(p_n, e_n) = \lim_{n\to\infty} \phi(e_n, c_n) = 0, \\
&\lim_{n\to\infty} \phi(b_n, y_n) = \lim_{n\to\infty} \phi(q_n, h_n) = \lim_{n\to\infty} \phi(h_n, d_n) = 0,
\end{aligned} \tag{3.46}$$

which implies by Lemma 2.3 that

$$\begin{aligned}
&\lim_{n\to\infty} \|a_n - x_n\| = \lim_{n\to\infty} \|p_n - e_n\| = \lim_{n\to\infty} \|e_n - c_n\| = 0, \\
&\lim_{n\to\infty} \|b_n - y_n\| = \lim_{n\to\infty} \|q_n - h_n\| = \lim_{n\to\infty} \|h_n - d_n\| = 0.
\end{aligned} \tag{3.47}$$

From (3.2) and the property of metric projection, we have

$$\begin{aligned}
\|a_n - c_n\| &= \|a_n - P_C [a_n - \gamma_n B_1^* (B_1 a_n - B_2 b_n)] \| \\
&\leq \|a_n - [a_n - \gamma_n B_1^* (B_1 a_n - B_2 b_n)] \| \\
&\leq (\rho + 1)\|B_1^*\| \, \|B_1 a_n - B_2 b_n\| \to 0, \text{ as } n \to \infty.
\end{aligned} \tag{3.48}$$

From (3.7), we have

$$\lim_{n\to\infty} \|x_{n+1} - p_n\| = \lim_{n\to\infty} \|\alpha_n u + (1 - \alpha_n)p_n - p_n\| = \lim_{n\to\infty} \alpha_n\|u - p_n\| = 0. \tag{3.49}$$

From (3.47), (3.48) and (3.49), we obtain

$$\begin{aligned}
\|x_{n+1} - x_n\| &\leq \|x_{n+1} - p_n\| + \|p_n - e_n\| + \|e_n - c_n\| \\
&\quad + \|c_n - a_n\| + \|a_n - x_n\| \to 0, \quad \text{as } n \to \infty.
\end{aligned} \tag{3.50}$$

Similarly, we have

$$\lim_{n\to\infty}\|y_{n+1}-y_n\|=0. \tag{3.51}$$

Since the sequence $\{(x_n,y_n)\}$ is bounded in $H_1\times H_2$, there exists a sub-sequence $\{(x_{n_k},y_{n_k})\}$ of $\{(x_n,y_n)\}$ and an element $(\bar{x},\bar{y})$ of $H_1\times H_2$ such that $(x_{n_k},y_{n_k})\rightharpoonup(\bar{x},\bar{y})$ and

$$\begin{aligned}&\limsup_{n\to\infty}\langle(u,v)-(x^*,y^*),(x_n,y_n)-(x^*,y^*)\rangle\\&=\lim_{k\to\infty}\langle(u,v)-(x^*,y^*),(x_{n_k},y_{n_k})-(x^*,y^*)\rangle,\end{aligned} \tag{3.52}$$

Moreover, we have by Lemma 2.7 that $x_{n_k}\rightharpoonup\bar{x}$ and $y_{n_k}\rightharpoonup\bar{y}$. Now, we show that $(\bar{x},\bar{y})\in\Omega$. Put $s_n=c_n-\lambda_n l^{-1}(I_1-T_1)c_n$. From (3.47) and (3.48), we obtain $c_n\rightharpoonup\bar{x}$. By Lemma 2.6 and (3.47), we have

$$\|c_n-s_n\|\le\frac{1}{l}\|c_n-e_n\|\to0,\ \text{as } n\to\infty. \tag{3.53}$$

Therefore, $s_n\rightharpoonup\bar{x}$. Thus, we have that $\{s_n\}$ is bounded. Since I_1-T_1 is uniformly continuous, we have

$$\|(I_1-T_1)c_n-(I_1-T_1)s_n\|\to0,\ \text{as } n\to\infty. \tag{3.54}$$

By the Armijo line-search rule (3.4), we have

$$\lambda_n l^{-1}\|(I_1-T_1)(c_n-\lambda_n l^{-1}(I_1-T_1)c_n)-(I_1-T_1)c_n\|>\mu\|\lambda_n l^{-1}(I_1-T_1)c_n)\|,$$

which implies

$$\frac{1}{\mu}\|(I_1-T_1)(c_n-\lambda_n l^{-1}(I_1-T_1)c_n)-(I_1-T_1)c_n\|>\|(I_1-T_1)c_n)\|. \tag{3.55}$$

We conclude from (3.54) and (3.55) that $\lim_{n\to\infty}(I_1-T_1)c_n=0$ which implies from the demiclosedness of T_1 and the fact that $c_n\rightharpoonup\bar{x}$ that $(I_1-T_1)\bar{x}=0$ and hence $\bar{x}\in F(T_1)$. One can show in a similar fashion that $\bar{y}\in F(T_2)$. Next we show that $B_1\bar{x}=B_2\bar{y}$. Indeed, we have by (2.9) that

$$\begin{aligned}\|B_1\bar{x}-B_2\bar{y}\|^2&=\|B_1a_{n_k}-B_2b_{n_k}+B_1\bar{x}-B_1a_{n_k}+B_2b_{n_k}-B_2\bar{y}\|^2\\&\le\|B_1a_{n_k}-B_2b_{n_k}\|^2+2\langle B_1\bar{x}-B_2\bar{y},\ B_1\bar{x}-B_1a_{n_k}+B_2b_{n_k}-B_2\bar{y}\rangle,\end{aligned} \tag{3.56}$$

and from the fact that $B_1a_{n_k}\rightharpoonup B_1\bar{x}$ and $B_2b_{n_k}\rightharpoonup B_2\bar{y}$ as $k\to\infty$, we obtain that $B_1\bar{x}=B_2\bar{y}$ and so $(\bar{x},\bar{y})\in\Omega$. Consequently, we obtain using Lemma 2.8, (3.36) and (3.52) that

$$\begin{aligned}&\limsup_{n\to\infty}\langle(u,v)-(x^*,y^*),(x_n,y_n)-(x^*,y^*)\rangle\\&=\lim_{k\to\infty}\langle(u,v)-(x^*,y^*),(x_{n_k},y_{n_k})-(x^*,y^*)\rangle\\&=\langle(u,v)-(x^*,y^*),(\bar{x},\bar{y})-(x^*,y^*)\rangle\le0.\end{aligned} \tag{3.57}$$

Combining (3.42), (3.43) and (3.57), we obtain

$$\limsup_{n\to\infty}(\Delta_n+\Gamma_n)\le0. \tag{3.58}$$

From (3.41), (3.58) and Lemma 2.4, we conclude that $\lim_{n\to\infty} \Pi_n^* = 0$, which implies that

$$\lim_{n\to\infty} \phi(x^*, x_n) = \lim_{n\to\infty} \phi(y^*, y_n) = 0,$$

which in turn implies by Lemma 2.3 that

$$\lim_{n\to\infty} \|x_n - x^*\| = 0 \text{ and } \lim_{n\to\infty} \|y_n - y^*\| = 0.$$

Case II. Suppose now that there exists a sub-sequence $\{\Pi_{n_i}^*\}$ of $\{\Pi_n^*\}$ such that $\Pi_{n_i}^* < \Pi_{n_i+1}^*$, for all $i \geq 0$. Then by Lemma 2.5, there exists a non-decreasing sequence $\{m_k\}$ of positive integers such that $m_k \to \infty$ as $k \to \infty$ and

$$\Pi_{m_k}^* \leq \Pi_{m_k+1}^* \text{ and } \Pi_k^* \leq \Pi_{m_k+1}^*, \tag{3.59}$$

for all $k \in \mathbb{N}$. We have from (3.44) that

$$\begin{aligned}
&(1-\alpha_{m_k})(1-4\zeta_{m_k})\,\phi(a_{m_k}, x_{m_k}) + (1-\alpha_{m_k})(1-2\mu)\,\phi(p_{m_k}, e_{m_k}) \\
&+ (1-\alpha_{m_k})(1-2\mu)\,\phi(e_{m_k}, c_{m_k}) + (1-\alpha_{m_k})(1-4\zeta_{m_k})\,\phi(b_{m_k}, y_{m_k}) \\
&+ (1-\alpha_{m_k})(1-2\sigma)\,\phi(q_{m_k}, h_{m_k}) + (1-\alpha_{m_k})(1-2\mu)\,\phi(h_{m_k}, d_{m_k}) \\
&(1-\alpha_{m_k})\rho\|B_1 a_{m_k} - B_2 b_{m_k}\|^2 \leq \Pi_{m_k}^* - \Pi_{m_k+1}^* + \alpha_{m_k}\left[\Sigma^* + (4\varepsilon - 1)\,\Pi_{m_k}^* + \frac{2\zeta_{m_k}}{\alpha_{m_k}}\right].
\end{aligned} \tag{3.60}$$

Making use of (3.59) and the conditions on α_{m_k} and ζ_{m_k} and taking the limit as $k \to \infty$ of (3.60) we obtain that $\lim_{k\to\infty} \|B_1 a_{m_k} - B_2 b_{m_k}\| = 0$. Following similar steps as in Case I, we obtain that

$$\lim_{k\to\infty} \|x_{m_k+1} - x_{m_k}\| = \lim_{k\to\infty} \|y_{m_k+1} - y_{m_k}\| = 0 \text{ and } \limsup_{k\to\infty} (\Delta_{m_k} + \Gamma_{m_k}) \leq 0. \tag{3.61}$$

From (3.41) and (3.59), we have that

$$\begin{aligned}
\alpha_{m_k}(1-4\varepsilon)\,\Pi_{m_k}^* &\leq \Pi_{m_k}^* - \Pi_{m_k+1}^* + \alpha_{m_k}(1-4\varepsilon)(\Delta_{m_k} + \Gamma_{m_k}) \\
&\leq \alpha_{m_k}(1-4\varepsilon)(\Delta_{m_k} + \Gamma_{m_k}),
\end{aligned} \tag{3.62}$$

and thus we have $\Pi_{m_k}^* \leq \Delta_{m_k} + \Gamma_{m_k}$. Since $\limsup_{k\to\infty} (\Delta_{m_k} + \Gamma_{m_k}) \leq 0$, we have that $\lim_{k\to\infty} \Pi_{m_k}^* = 0$, which together with (3.41) implies that $\lim_{k\to\infty} \Pi_{m_k+1}^* = 0$. Since $\Pi_k^* \leq \Pi_{m_k+1}^*$, it follows that $\lim_{k\to\infty} \Pi_k^* = 0$. Thus, we have

$$\lim_{k\to\infty} \phi(x^*, x_k) = \lim_{k\to\infty} \phi(y^*, y_k) = 0,$$

which implies by Lemma 2.3 that $\lim_{k\to\infty} \|x_k - x^*\| = 0$ and $\lim_{k\to\infty} \|y_k - y^*\| = 0$. Thus, we have shown, in Cases I and II, that the sequence $\{(x_n, y_n)\}$ generated by Algorithm 3.1 converges strongly to $(x^*, y^=) = P_\Omega(u, v)$, and this completes the proof. □

Corollary 3.1. *Assume that conditions $(C1), (C3) - (C6)$ hold. If $T_1 : C \to C$ and $T_2 : D \to D$ are Lipschitz pseudocontractive mappings, then the sequence $\{(x_n, y_n)\}$ generated by Algorithm 3.1 converges strongly to (x^*, y^*), where $(x^*, y^*) = P_\Omega(u, v)$.*

4. Applications

In this section, we present some specific applications of the main result.

4.1. **Split fixed point problems.** If in Theorem 3.2, we consider $H_2 = H_3$ and $B_2 = I_{H_2}$, then the SEFPP reduces to *split fixed point problem* which is finding a point $(x^*, y^*) \in H_1 \times H_2$ such that $(x^*, y^*) \in F(T_1) \times F(T_2) : B_1x^* = y^*$. Denote $\Theta = \{(x^*, y^*) \in F(T_1) \times F(T_2) : B_1x^* = y^*\}$. Then we have the following corollaries.

Corollary 4.1. *Assume that conditions* $(C1)$, $(C2)$, $(C5)$ *and* $(C6)$, *with* $H_2 = H_3$ *and* $B_2 = I_{H_2}$ *hold. If* $\Theta^* \neq \emptyset$, *then the sequence* $\{(x_n, y_n)\}$ *generated by Algorithm 3.1 converges strongly to* (x^*, y^*), *where* $(x^*, y^*) = P_\Theta(u, v)$.

Corollary 4.2. *Assume that conditions* $(C1)$, $(C5)$ *and* $(C6)$, *with* $H_2 = H_3$ *and* $B_2 = I_{H_2}$ *hold. Let* $T_1 \colon C \to C$ *and* $T_2 \colon D \to D$ *be Lipschitz pseudocontractive mappings. If* $\Theta \neq \emptyset$, *then the sequence* $\{(x_n, y_n)\}$ *generated by Algorithm 3.1 converges strongly to* (x^*, y^*), *where* $(x^*, y^*) = P_\Theta(u, v)$.

4.2. **Split equality null point problem.** If in Theorem 3.2, we take $C = H_1$ and $D = H_2$, then the SEFPP reduces to *split equality null point problem* which is defined as finding a point $(x^*, y^*) \in H_1 \times H_2$ such that $(x^*, y^*) \in N(A_1) \times N(A_2) : B_1x^* = B_2y^*$, where $A_i = I_i - T_i$. Denote $\Phi = \{(x^*, y^*) \in N(A_1) \times N(A_2) : B_1x^* = B_2y^*\}$.

Corollary 4.3. *Let* H_1, H_2 *and* H_3 *be real Hilbert spaces and let* $A_1 : H_1 \to H_1$ *and* $A_2 : H_2 \to H_2$ *be uniformly continuous monotone mappings. Assume that the set* $\Phi \neq \emptyset$. *If the conditions* $(C3)$, $(C5)$ *and* $(C6)$ *hold, then the sequence* $\{(x_n, y_n)\}$ *generated by Algorithm 3.1, with* $T_i = I_i - A_i$, *converges strongly to an element* (x^*, y^*), *where* $(x^*, y^*) = P_\Phi(u, v)$.

5. Numerical examples

In this section, we provide examples of uniformly continuous pseudocontractive mappings which satisfy the conditions of Theorem 3.2. Besides, a numerical experiment is provided to exhibit the applicability of the method.

Example 5.1. Let $H_1 = [0, \infty) = C$ and $H_2 = H_3 = \mathbb{R} = D$. Let $T_1 \colon C \to C$ and $T_2 \colon D \to D$, be defined by $T_1(x) = x - \sqrt{x} + \frac{1}{2}$, and $T_2(y) = y - \sqrt[3]{y} - 2$. The mapping T_1 is uniformly continuous pseudocontractive on C which is not Lipschitz continuous. In fact, let $K > 0$ be given and choose $y = 0$ and $0 < x < \frac{1}{K^2}$ so that $K < \frac{1}{\sqrt{x}}$. Now,

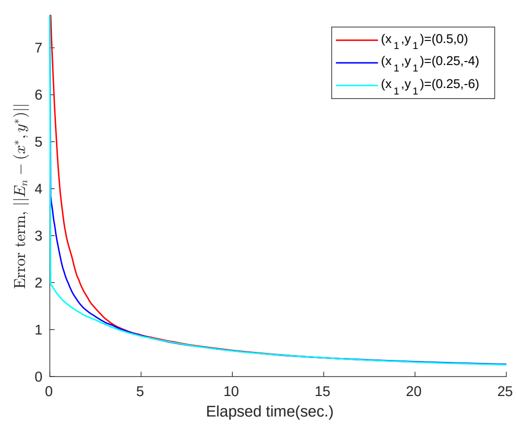
$$\frac{|T_1(x) - T_1(y)|}{|x - y|} = \frac{|x - \sqrt{x}|}{|x|} = \Big| \frac{1}{\sqrt{x}} - 1 \Big| > K - 1.$$

Since K is arbitrary, one concludes that T_1 is not Lipschitz continuous. Similarly, it can be shown that T_2 is uniformly continuous pseudocontractive on D which is not Lipschitz continuous. Clearly, $x^* = \frac{1}{4} \in F(T_1)$ and $y^* = -8 \in F(T_2)$. Now, define

B_1: $H_1 \to H_3$ and B_2: $H_2 \to H_3$ by $B_1(x) = 8x$ and $B_2(y) = -\frac{1}{4}y$. Then B_1 and B_2 are bounded linear mappings with adjoints $B_1^*(x) = 8x$ and $B_2^*(y) = -\frac{1}{4}y$. Moreover, we have that $B_1\left(\frac{1}{4}\right) = B_2(-8) = 2$. Thus, $(x^*,\ y^*) = \left(\frac{1}{4}, -8\right) \in \Omega$. For the control sequences $\zeta_n = \frac{1}{n^2+5}$, $\alpha_n = \frac{1}{n+100}$ and the parameters $\sigma = 0.4,\ \gamma = 0.5,\ l = 0.5,\ \mu = 0.4, \theta = 0.4$, the conditions $(C1)-(C6)$ are satisfied. A numerical experiment was carried out by taking the points $(u, v) = (-1, 0)$, $(x_0, x_0 = (0, 0)$ and $(x_1, y_1) = (0.5, 0)$ and the results are shown in Table 1 and Figure 1.

Table 1. Comparison of rates of convergence for the inertial and non inertial algorithms.

	Non inertial ($\theta_n = 0$)	Inertial ($\theta_n \neq 0$)
Error (E)	Time(sec.)	Time(sec.)
4.9	03568	0.3113
4.4	0.4279	0.3877
3.9	0.5432	0.4912
1.8	1.5955	1.4970
1.5	1.9915	1.9111
1.0	3.8809	3.6960
0.9	4.0038	3.8018



Parameters: $\gamma = l = 0.5, \quad \mu = \sigma = 0.4,\ \theta = 0.4.$

Figure 1. Convergence of the sequence $\{(x_n, y_n)\}$ for different initial points (x_1, y_1).

Remark 5.1. Table 1 shows the comparison of rates of convergence for the inertial and non inertial versions of the algorithm. The time taken for the inertial version to reach the error value $E = 1.0$ is 3.6960 seconds, whereas the time taken by the non inertial version to reach the same error value is about 3.8809 seconds. Thus, we conclude that the inertial version of the algorithm is faster than that of the non inertial version. Figure 1 shows the sequence $\{(x_n, y_n)\}$ generated by Algorithm 3.1 converges to the solution $(x^*, y^*) = \left(\frac{1}{4}, -8\right)$ for different initial points and the convergence gets relatively faster as the initial point gets closer to the solution of the problem.

Example 5.2. Let $H = H_1 = H_2 = H_3 = L_2[0, 1]$ with the inner product

$$\langle x(t), y(t)\rangle = \int_0^1 x(t)y(t)dt$$

and norm

$$\|x\|_{L_2} = \left(\int_0^1 |x(t)|^2 dt\right)^{\frac{1}{2}}.$$

Let $T_1\colon C \to C$ and $T_2\colon D \to D$ be the mappings defined by

$$(T_1x)(t) = x(t) - \max\{0, x(t)\}, \text{ and } (T_2y)(t) = \min\{0, y(t)\},$$

where

$$C = D = \{x \in H : \|x\|_{L_2} \leq 1\}.$$

It is proved in Tian and Xu [20] that the mapping $(Ax)(t) = \max\{0, x(t)\}$ a monotone mapping. Thus, we have by Remark 1.2 that the mapping

$$(T_1x)(t) = x(t) - (Ax)(t) = x(t) - \max\{0, x(t)\}$$

is a pseudocontractive mapping. It is also shown in [20] that T_1 is a uniformly continuous mapping. Similarly, one can show that T_2 is a uniformly continuous pseudocontractive mapping. Clearly,

$$x^*(t) = 0 \in F(T_1) \text{ and } y^*(t) = 0 \in F(T_2).$$

Let $B_1,\ B_2\colon H \to H$ be the bounded linear mappings defined by

$$(B_1x)(t) = 3x(t) \text{ and } (B_2y)(t) = 5y(t).$$

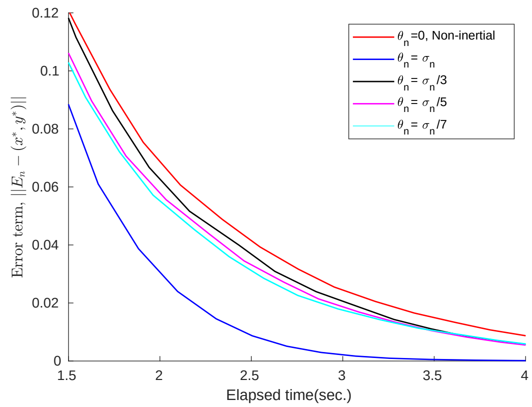
We note that $B_1(0) = B_2(0) = 0$. Therefore,

$$(x^*(t), y^*(t)) = (0, 0) \in \Omega.$$

If we take

$$\zeta_n = \frac{1}{n^2+5},\ \alpha_n = \frac{1}{n+100},\ \sigma = 0.4,\ \gamma = 0.5,\ l = 0.5,\ \mu = 0.4,\ \theta = 0.4$$

and the conditions $(C1) - (C6)$ are satisfied. Let $E_n = (x_n, y_n)$ be the sequence generated by Algorithm 3.1. For the points $(u, v) = (0, 0)$, $(x_0, y_0) = (0, 0)$ and $(x_1, y_1) = (0.5, 0)$ and for different choices of the inertial parameter θ_n, the figure below reveals that the error term sequence $\{E_n - (x^*, y^*)\}$ converges to zero.



Parameters: $\gamma = l = 0.5,\ \mu = \sigma = 0.4.$

FIGURE 2. Convergence of the sequence $\{(x_n, y_n)\}$ for different values of the inertial parameter θ_n.

Remark 5.2. From Figure 2 we observe that the method with nonzero inertial parameter ($\theta_n \neq 0$) has a faster rate of convergence than that of the non-inertial ($\theta_n = 0$) version.

6. Conclusions

In this paper, we introduced an inertial algorithm for solving the split equality fixed point problems in real Hilbert spaces. A strong convergence theorem is established under the assumption that the mappings under consideration are pseudocontractive self mappings which are uniformly continuous. A numerical example is also provided to demonstrate the applicability of the method.

The main result in this paper generalizes and extends many of the results in the literature in the sense that the problem considered is inertial and split equality, which is more general than all of the results discussed in the literature.

Specifically, our result is more general than the results obtained by Jung [10] and Zhou [12] because the mappings under consideration are relaxed from strictly pseudocontractive to uniformly continuous pseudocontractive. Moreover, our result generalizes the works of Akuchu [1] and Zegeye and Shahzad [24] in the sense that the problem is a more general problem and the class of Lipschitz continuous mappings is extended to the class of uniformly continuous mappings.

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