***Fixed Point Theory***, 26(2025), No. 1, 3-14
DOI: 10.24193/fpt-ro.2025.1.01
http://www.math.ubbcluj.ro/~nodeacj/sfptcj.html

# ON COINCIDENCE POINTS OF MAPS IN PROBABILISTIC METRIC SPACES

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**Abstract.** We define the notion of a covering (single-valued and multivalued) map of probabilistic metric spaces and prove analogs of the results of papers [2], [3] on coincidence points of Lipschitz and covering maps. As application, we consider the solvability of a system of equations and the existence of a solution for a feedback control system in a probabilistic normed space.

**Key Words and Phrases**: Probabilistic metric space, Menger space, covering map, Lipschitz map, multivalued map, coincidence point, fixed point.

**2020 Mathematics Subject Classification**: 54H25, 47H09, 46S50.

## 1. Introduction

The notion of a probabilistic metric space was introduced by K. Menger [9] and its main idea consists in the substitution of the distance between two points with the distribution function assigning to each nonnegative number the probability that the distance between these points does not exceed mentioned number. Since that time, this notion was systematically studied in many works and found significant applications in various branches of mathematics, in the theory of information and physics (see monographs [5], [6], [12] and the bibliography therein). In particular, let us mention the interesting notion of a probabilistic normed space suggested by A.N. Sherstnev [15].

In paper [14] the notion of a contractive map in a probabilistic metric space was defined and the extension of the Banach fixed point principle was proved. Later on, a significant number of fixed point theorems in probabilistic metric spaces for single-valued and multivalued maps of contractive type was investigated (see, e.g., [1], [4] - [7], [10], [11], [12] and the bibliography therein).

In the present paper we define the notion of a covering (single-valued and multivalued) map of probabilistic metric spaces and prove analogs of the results of papers [2], [3] on coincidence points of Lipschitz and covering maps. As application, we consider the solvability of a system of equations and the existence of a solution for a feedback control system in a probabilistic normed space.

## 2. Preliminaries

In this section we will collect necessary definitions and results from the theory of probabilistic metric spaces (details can be found, for example, in [5], [6], [12], [13]).

A function $F\colon \mathbb{R} \to [0,1]$ is called the distribution function if it is nondecreasing, left continuous and $\inf F = 0$, $\sup F = 1$. The set of all distribution functions will be denoted by $\mathcal{D}$. Choose $\mathcal{D}_+ = \{F\colon F \in \mathcal{D}, F(0) = 0\}$ and denote by $H_0 \in \mathcal{D}_+\colon H_0(t) = 1$, if $t > 0$.

**Definition 1.** *A map $T\colon I \times I \to I$, where $I = [0,1]$ is called $t$-norm if it satisfies the following conditions:*

- $(T1)$ $T(a,1) = a$;
- $(T2)$ $T(a,b) = T(b,a)$;
- $(T3)$ $T(a,b) \le T(c,d)$, *if* $a \le c$ *and* $b \le d$;
- $(T4)$ $T(T(a,b),c) = T(a,T(b,c))$.

As an example, let us take the $t$-norm

$$T_M(a,b) = \min\{a,b\}. \tag{2.1}$$

This $t$-norm will be used in the sequel.

**Definition 2.** *A probabilistic metric space of Menger type (or simply Menger space) is a triple $(X,\mathcal{F},T)$, where $X$ is a nonempty set; $\mathcal{F}\colon X \times X \to \mathcal{D}_+$ is a map (we will denote $\mathcal{F}(x,y) = F_{x,y}$); $T$ is a $t$-norm and the following conditions are satisfied:*

- $(F1)$ $F_{x,y} = H_0$ *is equivalent to* $x = y$;
- $(F2)$ $F_{x,y} = F_{y,x}, \quad \forall x,y \in X$;
- $(F3)$ $F_{x,y}(t_1+t_2) \ge T(F_{x,z}(t_1), F_{z,y}(t_2)), \quad \forall x,y,z \in X,\ t_1,t_2 \ge 0$.

An important example of a Menger space is the probabilistic normed space which means a triple $(E,\mathfrak{F},T)$, where $E$ is a real linear space; $\mathfrak{F}\colon E \to \mathcal{D}_+$ is a map ($\mathfrak{F}(x)$ is denoted by $F_x$); $T$ is a $t$-norm and the following conditions are satisfied:

- $(N1)$ $F_x = H_0$ is equivalent to $x = 0$;
- $(N2)$ $F_{\lambda x}(t) = F_x(\frac{t}{|\lambda|}), \quad \forall t > 0,\ \lambda \ne 0,\ x \in E$;
- $(N3)$ $F_{x+y}(t_1+t_2) \ge T(F_x(t_1), F_y(t_2)), \quad \forall x,y \in E,\ t_1,t_2 > 0$.

**Proposition 1.** *Let $(X,\mathcal{F},T)$ be a Menger space such that*

$$\sup_{0<t<1} T(t,t) = 1. \tag{2.2}$$

*Then $(X, \mathcal{F}, T)$ is a Hausdorff topological space whose topology $\mathcal{T}$ is generated by a family of $(\varepsilon, \lambda)$-neighborhoods $\{U_x(\varepsilon, \lambda)\colon x \in X, \varepsilon > 0, \lambda > 0\}$, where*

$$U_x(\varepsilon, \lambda) = \{y \in X\colon F_{x,y}(\varepsilon) > 1 - \lambda\}.$$

It is worth noting that, in general, a Menger space is not metrizable.

Let $(X, \mathcal{F}, T)$ be a Menger space for which condition (2.2) holds true. We will use the following notions.

**Definition 3.** (*i*) *A sequence $\{x_n\} \subset X$ is called convergent to $x \in X$ ($x_n \underset{\mathcal{T}}{\longrightarrow} x$), if for each $\varepsilon > 0$, $\lambda > 0$ there exists an integer $N = N(\varepsilon, \lambda)$ such that $F_{x,x_n}(\varepsilon) > 1 - \lambda$, $\forall n \geq N$;*

(*ii*) *A sequence $\{x_n\} \subset X$ is called fundamental if for every $\varepsilon > 0$, $\lambda > 0$ there exists an integer $N = N(\varepsilon, \lambda)$ such that $F_{x_n,x_m}(\varepsilon) > 1 - \lambda$, $\forall n, m \geq N$;*

(*iii*) *A Menger space $(X, \mathcal{F}, T)$ is called complete if every its fundamental sequence converges to a certain point $x \in X$.*

Let $(X, \mathcal{F}, T_M)$ is a Menger space, where the $t$-norm $T_M$ is defined by (2.1). For a given $x, y \in X$ and $c \in (0, 1)$, let

$$d_c(x, y) = \sup\{t \in \mathbb{R}\colon F_{x,y}(t) < c\}. \tag{2.3}$$

**Proposition 2.** *For each $c \in (0, 1)$ the function $d_c\colon X \times X \to \mathbb{R}$ is a pseudometric on $X$ (i.e., it obeys the symmetry condition and the triangle inequality but it can take zero value while $x \neq y$). For each $x, y \in X$, $x \neq y$ there exists $c \in (0, 1)$ such that $d_c(x, y) \neq 0$.*

The family of pseudometrics $\{d_c\}_{c \in (0,1)}$ generates the topology $\mathcal{T}$ in the following sense.

**Proposition 3.** *A sequence $\{x_n\} \subset X$ converges to $x \in X$ or is fundamental if and only if $d_c(x_n, x) \to 0$ or, respectively, $d_c(x_n, x_m) \to 0$ for each $c \in (0, 1)$.*

Notice that from this statement it does not follow that each of the pseudometric spaces $(X, d_c)$ is complete whenever the space $(X, \mathcal{F}, T_M)$ is complete.

## 3. Theorem on the coincidence for maps of Menger spaces

**Definition 4.** *Let $(X, \mathcal{F}^X, T_M)$, $(Y, \mathcal{F}^Y, T_M)$ be Menger spaces. For $\beta > 0$, a map $\varphi\colon X \to Y$ is called $\beta$-Lipschitz if for all $x, x' \in X$ and $t \geq 0$ we have*

$$F^Y_{\varphi(x),\varphi(x')}(t) \geq F^X_{x,x'}\left(\frac{t}{\beta}\right). \tag{3.1}$$

It is easy to see that if $d_c^X$ and $d_c^Y$ for $c \in (0, 1)$ are pseudometrics generated in the spaces $X$ and $Y$ by formula (2.3), then from (3.1) it follows that

$$d_c^Y(\varphi(x), \varphi(x')) \leq \beta d_c^X(x, x') \tag{3.2}$$

for all $x, x' \in X$ and $c \in (0, 1)$. Indeed, we have:

$$d_c^Y(\varphi(x), \varphi(x')) \overset{(2.3)}{=} \sup\{t\colon F^Y_{\varphi(x),\varphi(x')}(t) < c\} \overset{(3.1)}{\leq} \sup\{t\colon F^X_{x,x'}\left(\frac{t}{\beta}\right) < c\}$$

$$= \beta \sup\{\tau \colon F^X_{x,x'}(\tau) < c\} \overset{(2.3)}{=} \beta d^X_c(x,x'),$$

where $\tau = \frac{t}{\beta}$.

Notice that from here, by Proposition 3 it follows that a $\beta$-Lipschitz map is continuous.

Now, let us introduce the following notion.

**Definition 5.** *Let $(X, \mathcal{F}^X, T_M)$, $(Y, \mathcal{F}^Y, T_M)$ be Menger spaces. For a given $\alpha > 0$, a map $\psi \colon X \to Y$ is called $\alpha$-covering if for every $x \in X$, $y \in Y$ there exists $x' \in X$, $y = \psi(x')$ such that*

$$F^X_{x,x'}(t) \geq F^Y_{\psi(x),y}(\alpha t), \quad \forall t \geq 0. \tag{3.3}$$

Notice that by definition, an $\alpha$-covering map is surjective. The geometrical sense of this definition is revealed by the following assertion,

**Lemma 1.** *If $\psi \colon X \to Y$ is an $\alpha$-covering map then for each $x \in X$, $y \in Y$ there exists $x' \in X$, $y = \psi(x')$ such that for every $c \in (0,1)$ we have*

$$\alpha d^X_c(x,x') \leq d^Y_c(\psi(x), y). \tag{3.4}$$

*Proof.* By using (3.3), we get

$$d^Y_c(\psi(x), y) \overset{(2.3)}{=} \sup\{t \colon F_{\psi(x),y}(t) < c\} \overset{(3.3)}{\geq} \sup\{t \colon F_{x,x'}\left(\frac{t}{\alpha}\right) < c\}$$

$$= \alpha \sup\{\tau \colon F_{x,x'}(\tau) < c\} \overset{(2.3)}{=} \alpha d^X_c(x,x'),$$

where $\tau = \frac{t}{\alpha}$. □

The following version of the theorem on a coincidence point ([2] of maps of metric spaces for the case of maps of probabilistic metric spaces holds true.

**Theorem 1.** *Let $(X, \mathcal{F}^X, T_M)$, $(Y, \mathcal{F}^Y, T_M)$ be Menger spaces. Let the space $X$ is complete, for given $\alpha > \beta$, a map $\psi \colon X \to Y$ is $\alpha$-covering and closed (i.e., it has the closed graph), a map $\varphi \colon X \to Y$ is $\beta$-Lipschitz. Then for each $x \in X$ there exists $\xi = \xi(x)$ such that*

$$\psi(\xi) = \varphi(\xi) \tag{3.5}$$

*and, moreover,*

$$d^X_c(x,\xi) \leq \frac{d^Y_c(\psi(x), \varphi(x))}{\alpha - \beta} \quad \text{for each } c \in (0,1), \tag{3.6}$$

*where $d^X_c$ and $d^Y_c$ are the pseudometrics in $X$ and $Y$ respectively.*

*Proof.* For a given $x \in X$, set $x_0 = x$. According to condition (3.4), we can find $x_1 \in X$ such that

$$\psi(x_1) = \varphi(x_0) \tag{3.7}$$

and

$$d^X_c(x_0, x_1) \overset{(3.4)}{\leq} \frac{1}{\alpha} d^Y_c(\psi(x_0), \psi(x_1)) \overset{(3.7)}{=} \frac{1}{\alpha} d^Y_c(\psi(x_0), \varphi(x_0)). \tag{3.8}$$

Further, by using conditions of $\alpha$-covering for $\psi$ and $\beta$-Lipschitzness for $\varphi$, we can find $x_2 \in X$ such that

$$\psi(x_2) = \varphi(x_1) \tag{3.9}$$

and

$$F^X_{x_1,x_2}(t) \overset{(3.4)}{\geq} F^Y_{\psi(x_1),\psi(x_2)}(\alpha t) \overset{(3.9)}{=} F^Y_{\varphi(x_0),\varphi(x_1)}(\alpha t) \overset{(3.1)}{\geq} F^X_{x_0,x_1}\Big(\frac{\alpha}{\beta}t\Big). \tag{3.10}$$

Applying the construction by induction, we find the sequence $\{x_n\} \subset X$ such that for $n \geq 1$ we have

$$\psi(x_n) = \varphi(x_{n-1}) \tag{3.11}$$

and

$$F^X_{x_{n-1},x_n}(t) \geq F^X_{x_0,x_1}\bigg(\Big(\frac{\alpha}{\beta}\Big)^{n-1} t\bigg), \quad \forall t \geq 0. \tag{3.12}$$

But then for every $c \in (0,1)$ and $i \geq 1$ we get

$$d^X_c(x_{i-1},x_i) \overset{(2.3)}{=} \sup\{t\colon F^X_{x_{i-1},x_i}(t) < c\} \overset{(3.12)}{\leq} \sup\{t\colon F^X_{x_0,x_1}\bigg(\Big(\frac{\alpha}{\beta}\Big)^{i-1} t\bigg) < c\}$$

$$= \Big(\frac{\beta}{\alpha}\Big)^{i-1} \sup\{\tau\colon F^X_{x_0,x_1}(\tau) < c\} = \Big(\frac{\beta}{\alpha}\Big)^{i-1} d^X_c(x_0,x_1),$$

where $\tau = \left(\frac{\alpha}{\beta}\right)^{i-1} t$.

This estimate means that the sequence $\{x_n\}$ is fundamental with respect to each pseudometric $d^X_c$, $c \in (0,1)$, and hence, by Proposition 3 it is fundamental also in the space $(X,\mathcal{T})$. By virtue of the completeness of the space $(X,\mathcal{T})$ we have $x_n \underset{\mathcal{T}}{\to} \xi \in X$. Again applying Proposition 3, we have

$$d^X_c(x_n,\xi) \to 0 \quad \text{for each } c \in (0,1). \tag{3.13}$$

From the continuity of the map $\varphi$ we get

$$\varphi(x_n) \underset{\mathcal{T}}{\to} \varphi(\xi),$$

but then also

$$\psi(x_n) \underset{\mathcal{T}}{\to} \varphi(\xi).$$

The closedness of the map $\psi$ yields

$$\psi(\xi) = \varphi(\xi).$$

Applying the above estimate for $d^X_c(x_{i-1},x_i)$ and relation (3.8) we get for $k \geq 1$ for each $c \in (0,1)$ :

$$d^X_c(x,x_k) = d^X_c(x_0,x_k) < \frac{\alpha}{\alpha-\beta}\, d^X_c(x_0,x_1) \overset{(3.8)}{\leq} \frac{d^X_c(\psi(x),\varphi(x))}{\alpha-\beta}.$$

Passing to the limit as $k \to \infty$ and applying (3.13) we have the desired estimate

$$d^X_c(x,\xi) \leq \frac{d^Y_c(\psi(x),\varphi(x))}{\alpha-\beta}.$$

□

## 4. Theorem on the coincidence for multivalued maps of Menger spaces

Let $(X, \mathcal{F}^X, T_M)$, $(Y, \mathcal{F}^Y, T_M)$ be Menger spaces. By $C(Y)$ we denote the collection of all nonempty subsets of $Y$ closed in the topology $\mathcal{T}$.

**Definition 6.** *A multivalued map (multimap) $\Phi\colon X \to C(Y)$ is called $\beta$-Lipschitz, $\beta > 0$, if for each $x, x' \in X$ and every $y \in \Phi(x)$ there exists $y' \in \Phi(x')$ such that*

$$F^Y_{y,y'}(t) \geq F^X_{x,x'}\left(\frac{t}{\beta}\right), \quad t \geq 0. \tag{4.1}$$

In terms of pseudometrics $d^X_c$ and $d^Y_c$ in spaces $X$ and $Y$ the last condition can be written as

$$d^Y_c(y, y') \leq \beta d^X_c(x, x'), \quad \forall c \in (0, 1). \tag{4.2}$$

We will use the following property of $\beta$-Lipschitz maps.

**Lemma 2.** *If a multimap $\Phi\colon X \to C(Y)$ is $\beta$-Lipschitz, then it is closed, i.e., it has a closed graph.*

*Proof.* Consider sequences $\{x_n\} \subset X$, $\{y_n\} \subset Y$ such that $x_n \underset{\mathcal{T}}{\longrightarrow} x$, $y_n \in \Phi(x_n)$ and $y_n \underset{\mathcal{T}}{\longrightarrow} y$. Let us show that $y \in \Phi(x)$.

Supposing the contrary, by applying condition of $\beta$-Lipschitzness, select a sequence $\{y'_n\} \subset \Phi(x)$ such that

$$F^Y_{y_n,y'_n}(t) \geq F^X_{x_n,x}\left(\frac{t}{\beta}\right), \quad \forall t \geq 0,$$

from where, by Proposition 3 it follows that

$$d^Y_c(y_n, y'_n) \leq \beta d^X_c(x_n, x) \to 0, \quad \forall c \in (0, 1).$$

The condition $y_n \underset{\mathcal{T}}{\longrightarrow} y$ yields

$$d^Y_c(y_n, y) \to 0, \quad \forall c \in (0, 1),$$

implying

$$d^Y_c(y'_n, y) \to 0, \quad \forall c \in (0, 1),$$

and therefore $y'_n \underset{\mathcal{T}}{\longrightarrow} y$, in contradiction with the closedness of the set $\Phi(x)$. □

**Definition 7.** *A multimap $\Psi\colon X \to C(Y)$ is called $\alpha$-covering, $\alpha > 0$, if for each $x \in X$ and $y \in Y$ there exists $x' \in X$ such that $y \in \Psi(x')$ and, moreover, for each $z \in \Psi(x)$ we have*

$$F^X_{x,x'}(t) \geq F^Y_{z,y}(\alpha t), \quad t \geq 0. \tag{4.3}$$

Again we can express this condition in terms of pseudometrics $d^X_c$ and $d^Y_c$ :

$$\alpha d^X_c(x, x') \leq d^Y_c(z, y), \quad \forall c \in (0, 1). \tag{4.4}$$

Let us formulate now the following version of the theorem on a coincidence for multivalued maps proved in [3] for maps of metric spaces for the case of multimaps of probabilistic metric spaces.

**Theorem 2.** *Let $(X, \mathcal{F}^X, T_M)$, $(Y, \mathcal{F}^Y, T_M)$ be complete Menger spaces. Let for given $\alpha > \beta$ a multimap $\Psi \colon X \to C(Y)$ be $\alpha$-covering and closed, a multimap $\Phi \colon X \to C(Y)$ $\beta$-Lipschitz. Then for arbitrary $x \in X$ and $y \in \Phi(x)$ there exist such $\xi = \xi(x,y) \in X$, $\eta = \eta(x,y) \in Y$ that*

$$\eta \in \Phi(\xi) \bigcap \Psi(\xi) \tag{4.5}$$

*and, moreover for each $c \in (0,1)$ the following estimates hold true:*

$$d_c^X(x,\xi) \le \frac{D_c^Y(\Psi(x),\Phi(x))}{\alpha-\beta} \tag{4.6}$$

*and*

$$d_c^Y(y,\eta) \le \frac{\beta}{\alpha-\beta} D_c^Y(\Psi(x),\Phi(x)), \tag{4.7}$$

*where*

$$D_c^Y(A,B) := \sup\{d_c^Y(a,b) \colon a \in A, b \in B\}.$$

*Proof.* Denote $x_0 = x$ and $y_1 = y \in \Phi(x_0)$. From the condition of $\alpha$-covering of the multimap $\Psi$, it folllows that there exists $x_1 \in X$ such that

$$y_1 \in \Psi(x_1) \tag{4.8}$$

and

$$F^X_{x_0,x_1}(t) \ge F^Y_{z,y_1}(\alpha t), \quad \forall z \in \Psi(x_0) \quad \forall t \ge 0. \tag{4.9}$$

Further, using the condition of $\beta$-Lipschitzness of the multimap $\Phi$ we will find $y_2 \in \Phi(x_1)$ such that

$$F^Y_{y_1,y_2}(t) \ge F^X_{x_0,x_1}\Big(\frac{t}{\beta}\Big). \tag{4.10}$$

Again applying the $\alpha$-covering condition of $\Psi$, find $x_2 \in X$ such that $y_2 \in \Psi(x_2)$ and

$$F^X_{x_1,x_2}(t) \overset{(4.3)}{\ge} F^Y_{y_1,y_2}(\alpha t) \overset{(4.10)}{\ge} F^X_{x_0,x_1}\Big(\frac{\alpha}{\beta} t\Big).$$

Continuing this process, we will construct sequences $\{x_n\} \subset X$ and $\{y_n\} \subset Y$ for which we will have for $n \ge 1$ :

$$y_n \in \Phi(x_{n-1}) \bigcap \Psi(x_n), \tag{4.11}$$

$$F^X_{x_{n-1},x_n}(t) \ge F^X_{x_0,x_1}\Big(\Big(\frac{\alpha}{\beta}\Big)^{n-1} t\Big), \quad \forall t \ge 0, \tag{4.12}$$

$$F^Y_{y_{n-1},y_n}(t) \ge F^X_{x_0,x_1}\Big(\Big(\frac{\alpha}{\beta}\Big)^{n-2} \frac{1}{\beta} t\Big), \quad \forall t \ge 0. \tag{4.13}$$

Indeed, if relations (4.11) - (4.13) hold true till $n \ge 1$, then applying the $\beta$-Lipschitzness and the $\alpha$-covering conditions we will find at first $y_{n+1} \in \Phi(x_n)$ such that

$$F^Y_{y_n,y_{n+1}}(t) \overset{(4.1)}{\ge} F^X_{x_{n-1},x_n}\Big(\frac{t}{\beta}\Big) \overset{(4.12)}{\ge} F^X_{x_0,x_1}\Big(\Big(\frac{\alpha}{\beta}\Big)^{n-1} \frac{1}{\beta} t\Big), \quad t \ge 0,$$

and then $x_{n+1} \in X$ such that

$$y_{n+1} \in \Psi(x_{n+1})$$

and

$$F^X_{x_n,x_{n+1}}(t) \overset{(4.3)}{\geq} F^Y_{y_n,y_{n+1}}(\alpha t) \overset{(4.13)}{\geq} F^X_{x_0,x_1}\Big(\Big(\frac{\alpha}{\beta}\Big)^n t\Big).$$

Rewriting relations (4.12) - (4.13) in the form of estimates

$$d^X_c(x_{n-1},x_n) \leq \Big(\frac{\beta}{\alpha}\Big)^{n-1} d^X_c(x_0,x_1), \quad \forall c\in(0,1), \tag{4.14}$$

$$d^Y_c(y_{n-1},y_n) \leq \Big(\frac{\beta}{\alpha}\Big)^{n-2} \beta d^X_c(x_0,x_1), \quad \forall c\in(0,1), \tag{4.15}$$

from which, by Proposition 3 the fundamentality of sequences $\{x_n\}$ and $\{y_n\}$ follows and hence by virtue of the completeness of the spaces $X$ and $Y$, we get their convergence: $x_n \underset{\mathcal{T}}{\longrightarrow} \xi \in X$, $y_n \underset{\mathcal{T}}{\longrightarrow} \eta \in Y$. Since the multimaps $\Phi$ and $\Psi$ are closed, we can pass to the limit as $n\to\infty$ in (4.11) and to obtain desired relation (4.5).

Writing now relation (4.9) in the form

$$d^X_c(x_0,x_1) \leq \frac{1}{\alpha} d^Y_c(z,y_1), \quad \forall c\in(0,1), z\in\Psi(x_0),$$

we get the estimate

$$d^X_c(x_0,x_1) \leq \frac{1}{\alpha} D^Y_c(\Psi(x_0),\Phi(x_0))), \quad \forall c\in(0,1). \tag{4.16}$$

Then estimates (4.14) and (4.15) imply for each $c\in(0,1)$ and $k\geq 1$ the following relations:

$$d^X_c(x,x_k) = d^X_c(x_0,x_k) < \frac{\alpha}{\alpha-\beta} d^X_c(x_0,x_1) \overset{(4.16)}{\leq} \frac{D^Y_c(\Psi(x),\Phi(x))}{\alpha-\beta}$$

and

$$d^Y_c(y,y_k) = d^Y_c(y_1,y_k) < \frac{\alpha\beta}{\alpha-\beta} d^X_c(x_0,x_1) \overset{(4.16)}{\leq} \frac{\beta}{\alpha-\beta} D^Y_c(\Psi(x),\Phi(x)).$$

Passing to the limit as $k\to\infty$ we get desired estimates (4.6) and (4.7). □

## 5. Examples

In this section we consider some applications to the solvability of systems of equations and feedback control systems in probabilistic normed spaces.

5.1. **System of equations in probabilistic normed spaces.** Let $(E_1,\mathfrak{F}^1,T_M)$, $(E_2,\mathfrak{F}^2,T_M)$ be complete probabilistic normed spaces. Then the product space $(E_1\times E_2,\mathfrak{F}^1\iota\mathfrak{F}^2,T_M)$, where

$$\Big(\mathfrak{F}^1\iota\mathfrak{F}^2\Big)(y_1,y_2)(t) = T_M\Big(\mathfrak{F}^1(y_1)(t),\mathfrak{F}^2(y_2)(t)\Big),\ y_1\in E_1, y_2\in E_2$$

is also a complete probabilistic normed space ([8]). We will denote

$$\Big(\mathfrak{F}^1\iota\mathfrak{F}^2\Big)(y_1,y_2) = F^\star_{y_1,y_2}.$$

Let $\gamma\colon E_1\to E_2$ be a $\beta_\gamma$-Lipschitz map ($\beta_\gamma\geq 0$) and $X\subset E_1\times E_2$ its graph. The structure of a Menger space with respect to $T_M$ can be naturally induced on $X$ if for $(y_1,y_2),(y_1',y_2')\in X$ we set $F^X_{(y_1,y_2),(y_1',y_2')} = F^\star_{(y_1-y_1'),(y_2-y_2')}$.

In accordance with (2.3) for each $c \in (0,1)$ and $(y_1, y_2), (y_1', y_2') \in X$ we define the pseudodistance between these points as

$$d_c^X((y_1,y_2),(y_1',y_2')) = \sup\{t \in \mathbb{R}\colon F^X_{(y_1,y_2),(y_1',y_2')}(t) < c\} = \\ = \sup\{t \in \mathbb{R}\colon F^\star_{(y_1-y_1'),(y_2-y_2')}(t) < c\}. \tag{5.1}$$

Then we get

$$d_c^X((y_1,y_2),(y_1',y_2')) = \sup\{t \in \mathbb{R}\colon T_M\left(\mathfrak{F}^1(y_1-y_1')(t), \mathfrak{F}^2(y_2-y_2')(t)\right) < c\} = \\ = \sup\{t \in \mathbb{R}\colon \min\{F^1_{y_1-y_1'}(t), F^2_{y_2-y_2'}(t)\} < c\} = \\ = \max\Big\{\sup\{t \in \mathbb{R}\colon F^1_{y_1-y_1'}(t) < c\}, \sup\{t \in \mathbb{R}\colon F^2_{y_2-y_2'}(t) < c\}\Big\} = \\ = \max\{d_c^{E_1}(y_1,y_1'), d_c^{E_2}(y_2,y_2')\}.$$

Consider the projection $\eta\colon X \to E_1$ defined as $\eta(y_1,y_2) = y_1$.

**Lemma 3.** *The map $\eta$ is $\alpha$-covering, where $\alpha > 0$ is such that $\alpha \cdot \max\{1, \beta_\gamma\} < 1$.*

*Proof.* Take any $x = (y_1, \gamma(y_1)) \in X$ and $y_1' \in E_1$. Then for $x' = (y_1', \gamma(y_1')) \in X$ and any $c \in (0,1)$ we have

$$\alpha d_c^X(x,x') = \alpha d_c^X((y_1,\gamma(y_1)),(y_1',\gamma(y_1'))) = \\ = \alpha \max\{d_c^{E_1}(y_1,y_1'), d_c^{E_2}(\gamma(y_1),\gamma(y_1'))\} \le \max\{\alpha d_c^{E_1}(y_1,y_1'), \alpha\beta_\gamma d_c^{E_1}(y_1,y_1')\} < \\ < d_c^{E_1}(y_1,y_1') = d_c^{E_1}(\eta(x),\eta(x')).$$

□

Now, let $\theta\colon (E_1 \times E_2, \mathfrak{F}^1\iota\mathfrak{F}^2, T_M) \to (E_1, \mathfrak{F}^1, T_M)$ be a $\beta_\theta$-Lipschitz map in the sense that for all $c \in (0,1)$ and $(y_1,y_2),(y_1',y_2') \in E_1 \times E_2$ we have

$$d_c^{E_1}(\theta(y_1,y_2),\theta(y_1',y_2')) \le \beta_\theta d_c^{E_1\times E_2}((y_1,y_2),(y_1',y_2')) = \\ = \beta_\theta \max\{d_c^{E_1}(y_1,y_1'), d_c^{E_2}(y_2,y_2')\},$$

where $d_c^{E_1\times E_2}$ is defined analogously to $d_c^X$ (see (5.1)).

Consider now the following system of equations:

$$y_1 = \theta(y_1,y_2), \tag{5.2}$$

$$y_2 = \gamma(y_1). \tag{5.3}$$

If we denote by $\widetilde{\theta}$ the restriction of $\theta$ to $X$ then it is clear that system (5.2) – (5.3) is equivalent to the equation

$$\eta(y_1,y_2) = \widetilde{\theta}(y_1,y_2). \tag{5.4}$$

Applying Lemma 3 and Theorem 1 we obtain the following result.

**Theorem 3.** *If $\beta_\theta \cdot \max\{1,\beta_\gamma\} < 1$, then system of equations (5.2) – (5.3) has a solution $y_1 \in E_1$, $y_2 \in E_2$. Moreover, for an arbitrary $x^\star = (y_1^\star, \gamma(y_1^\star)) \in X$ the pseudodistances from $x^\star$ to the set $\Delta \subset X$ of all solutions to (5.2) – (5.3) satisfy the following estimates:*

$$dist_c^X(x^\star,\Delta) := \inf\{d_c^X(x^\star,x) : x \in \Delta\} \le \frac{\mu d_c^{E_1}(y_1^\star, \theta(y_1^\star,\gamma(y_1^\star)))}{1-\beta_\theta\mu} \tag{5.5}$$

*for all $c \in (0,1)$, where $\mu = \max\{1,\beta_\gamma\}$.*

*Proof.* Taking $\alpha > 0$ such that $\beta_\theta < \alpha < \frac{1}{\mu}$, from Lemma 3 we get that the map $\eta$ is $\alpha$-covering. Then the existence of a solution to equation (5.4) follows from Theorem 1. Further, take $\varepsilon > 0$ so small that $\frac{1}{\mu} - \varepsilon > \beta_\theta$ and choose $\alpha$ so that $\frac{1}{\mu} - \varepsilon < \alpha < \frac{1}{\mu}$. Then $\alpha - \beta_\theta > \frac{1}{\mu} - \varepsilon - \beta_\theta$ and from estimate (3.6) we obtain the existence of a solution $x \in \Delta$ for which

$$d_c^X(x^\star, x) \leq \frac{d_c^{E_1}(y_1^\star, \theta(y_1^\star, \gamma(y_1^\star)))}{\frac{1}{\mu} - \varepsilon - \beta_\theta}$$

for each $c \in (0,1)$. Now we get estimate (5.5) taking $\varepsilon \to 0$. □

5.2. **Existence of a solution to a feedback control system in a probabilistic normed space.** In this section we will use the above result to show the existence of a solution to a class of control systems defined in probabilistic normed spaces.

Let $(E, \mathfrak{F}, T_M)$ be a complete probabilistic normed space and for $\delta > 0$ let $C([0,\delta], E)$ be a space of functions $x\colon [0,\delta] \to E$ which are continuous in the topology $\mathcal{T}$ on $(E, \mathfrak{F}, T_M)$. Define a map $\widetilde{\mathfrak{F}}\colon C([0,\delta], E) \to \mathcal{D}_+$ by

$$\widetilde{\mathfrak{F}}(x)(t) = \lim_{r \to t-0} \inf_{s \in [0,\delta]} F_{x(s)}(r)$$

and denote the distribution function $\widetilde{\mathfrak{F}}(x)$ by $\widetilde{F}_x$. Then $(C([0,\delta], E), \widetilde{\mathfrak{F}}, T_M)$ is also a complete probabilistic normed space (see [4]).

Let $(E_1, \mathfrak{F}^1, T_M)$, $(E_2, \mathfrak{F}^2, T_M)$ be complete probabilistic normed spaces. For $a > 0$, let $f\colon [0,a] \times E_1 \times E_2 \to E_1$ and $g\colon [0,a] \times E_1 \to E_2$ be continuous maps. We will consider a feedback control system governed by the following relations:;

$$\dot{x}(t) = f(t, x(t), y(t)), \tag{5.6}$$

$$x(0) = x_0 \in E_1, \tag{5.7}$$

$$y(t) = g(t, x(t)). \tag{5.8}$$

By a solution of this system on some interval $[0, \delta_0]$, where $0 < \delta_0 \leq a$ we mean a pair $(x, y)$ with a trajectory function $x \in C([0,\delta_0], E_1)$ and a control function $y \in C([0,\delta_0], E_2)$ satisfying relations (5.6)–(5.8).

We will suppose that functions $f$ and $g$ satisfy Lipschitz conditions of the following form:

$(H_f)$ there exists $L > 0$ such that for all $t \in [0,a]$ and $x, x' \in E_1$, $y, y' \in E_2$ we have

$$F^1_{f(t,x,y)-f(t,x',y')}(\nu) \geq \min\left\{F^1_{x-x'}\left(\frac{\nu}{L}\right), F^2_{y-y'}\left(\frac{\nu}{L}\right)\right\},\ \nu \in \mathbb{R};$$

$(H_g)$ there exists $\beta_g > 0$ such that for all $t \in [0,a]$ and $x, x' \in E_1$ we have

$$F^2_{g(t,x)-g(t,x')}(\nu) \geq F^1_{x-x'}\left(\frac{\nu}{\beta_g}\right).$$

Introduce now the following operators. Let

$$\theta\colon \Big(C([0,\delta_0], E_1) \times C([0,\delta_0], E_2), \widetilde{\mathfrak{F}}^1 \iota \widetilde{\mathfrak{F}}^2, T_M\Big) \to \Big(C([0,\delta_0], E_1), \widetilde{\mathfrak{F}}^1, T_M\Big),$$

be defined as

$$\theta(x,y)(t) = x_0 + \int_0^t f(s, x(s), y(s))ds,\ t \in [0, \delta_0]$$

and $\gamma\colon \Big(C([0,\delta_0], E_1), \widetilde{\mathfrak{F}}^1, T_M\Big) \to \Big(C([0,\delta_0], E_2), \widetilde{\mathfrak{F}}^2, T_M\Big)$ be given as

$$\gamma(x)(t) = g(t, x(t)).$$

It is clear that the solvability of system (5.6)–(5.8) is equivalent to the existence of a solution to the system of equations

$$x = \theta(x,y), \tag{5.9}$$

$$y = \gamma(x). \tag{5.10}$$

Following the lines of [4] one can verify that the map $\theta$ is $\beta_\theta$-Lipschitz, where $\beta_\theta = L\delta_0$.

**Lemma 4.** *The map $\gamma$ is $\beta_\gamma$-Lipschitz with $\beta_\gamma = \beta_g$.*

*Proof.* Take $x(\cdot), x'(\cdot) \in \Big(C([0,\delta_0], E_1), \widetilde{\mathfrak{F}}^1, T_M\Big)$ and let $y \in \gamma(x),\ y' = \gamma(x')$. Then we have

$$\widetilde{F}^2_{y(\cdot)-y'(\cdot)}(t) = \lim_{r\to t-0} \inf_{s\in[0,\delta_0]} F^2_{g(s,x(s))-g(s,x'(s))}(r) \geq$$

$$\geq \lim_{r\to t-0} \inf_{s\in[0,\delta_0]} F^1_{x(s)-x'(s)}\Big(\frac{r}{\beta_g}\Big) = \widetilde{F}^1_{x(\cdot)-x'(\cdot)}\Big(\frac{t}{\beta_g}\Big).$$

□

Basing on Theorem 3 and Lipschitz properties of maps $\theta$ and $\gamma$ we can formulate now a sufficient condition for the existence of a solution to control system (5.6)–(5.8). It takes the following form.

**Theorem 4.** *Let conditions $(H_f)$, $(H_g)$ hold true and $L\delta_0 \cdot \max\{1, \beta_g\} < 1$. Then control system (5.6)–(5.8) has a solution on the interval $[0, \delta_0]$.*

**Acknowledgement.** The authors are grateful to the anonymous referee for his/her valuable remarks and suggestions. The research is supported by the Russian Science Foundation (the first author by Project No. 24-21-00012 and the second author by Project No. 22-71-10008).

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*Received: December 11, 2022; Accepted: October 3, 2023.*