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MODIFIED SPLITTING ALGORITHMS FOR APPROXIMATING SOLUTIONS OF SPLIT VARIATIONAL INCLUSIONS IN HILBERT SPACES

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Abstract. The purpose of this paper is to explore the split variational inclusion problem in Hilbert spaces. A splitting algorithm is constructed for solving the split variational inclusion with the help of self-adaptive techniques. Convergence analysis of the proposed algorithm is provided under additional conditions.
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1. Introduction

Let H_1 be a real Hilbert space. Let $\Phi : H_1 \rightrightarrows 2^{H_1}$ be a set-valued maximal monotone operator and $\psi : H_1 \to H_1$ be a single-valued monotone operator. Recall that the variational inclusion problem is to seek $z \in H_1$ such that

$$0 \in (\Phi + \psi)(z). \tag{1.1}$$

Variational inclusion problem (1.1) can be used to settle numerous problems, for example, well-known minimization problem. If fact, suppose that $f, g : H_1 \to \mathbf{R} \cup \{+\infty\}$ are two proper, convex and lower semicontinuous functions. Assume that f is subdifferentiable and g is differentiable. Set $\partial f = \Phi$ and $\nabla g = \psi$. Then solving the following minimization problem

$$\min_{z \in H_1} (f(z) + g(z)). \tag{1.2}$$

can be converted into solving the variational inclusion problem (1.1).

There are many ways to solve variational inclusion (1.1) in the literature, see [2, 5, 6, 9, 10, 11, 14, 15]. An essential path for solving (1.1) is the well-known splitting method ([16]) defined by: for a fixed point $x_0 \in H_1$,

$$x_{k+1} = (I + \varsigma_k \Phi)^{-1}(I - \varsigma_k \psi)(x_k),\ k \geq 0. \tag{1.3}$$

The sequence $\{x_k\}$ converges to some point in the solution set $(\Phi + \psi)^{-1}(0)$ of the variational inclusion (1.1) provided ψ is (inverse) strongly monotone. To get rid of this strong monotonicity restriction, Cholamjiak, Hieu and Cho [3] suggested a Tseng-type splitting algorithm for solving (1.1) in which the operator ψ is monotone.

Let H_2 be any other real Hilbert space. Let $\varphi : H_2 \rightrightarrows 2^{H_2}$ be a maximal monotone operator and $\phi : H_2 \to H_2$ be a monotone operator. Let $\Psi \colon H_1 \to H_2$ be a nonzero bounded linear operator and Ψ^* be the adjoint of Ψ. Consider the following split variational inclusion problem of seeking $u \in H_1$ such that

$$u \in (\Phi + \psi)^{-1}(0) \quad \text{and} \quad \Psi(u) \in (\phi + \varphi)^{-1}(0). \tag{1.4}$$

The solution set of (1.4) is denoted by $\Omega := \{u | u \in (\Phi+\psi)^{-1}(0), \Psi(u) \in (\phi+\varphi)^{-1}(0)\}$.

There are several interesting methods for solving (1.4) in the literature, see [4, 7, 8]. For solving (1.4), Moudafi [12] proposed the following iterative algorithm: $\forall x_0 \in H_1$,

$$x_{n+1} = (I + \gamma\Phi)^{-1}(I - \gamma\psi)[x_n - \gamma\Psi^*(I - B)\Psi(x_n)], n \geq 0, \tag{1.5}$$

where $B = (I + \gamma\varphi)^{-1}(I - \gamma\phi)$ and $\gamma \in (0, \frac{1}{\|\Psi\|^2})$.

Yao et al. [17] suggested the following inertial algorithm for solving (1.4): for two initial points $x_0, x_1 \in H_1$, the sequence $\{x_n\}$ is generated by

$$\begin{cases} y_n = x_n + \tau_n(x_n - x_{n-1}), \\ x_{n+1} = (I + \gamma\Phi)^{-1}(I - \gamma\psi)[y_n - \theta_n\Psi^*(I - B)\Psi(y_n)], n \geq 1, \end{cases} \tag{1.6}$$

where

$$\theta_n = \begin{cases} \frac{\varepsilon_n\|(I-B)\Psi(y_n)\|^2}{\|\Psi^*(I-B)\Psi(y_n)\|^2}, & \text{if } (I - B)\Psi(y_n) \neq 0, \\ \theta, & \text{if } (I - B)\Psi(y_n) = 0, \end{cases}$$

where $B = (I + \gamma\varphi)^{-1}(I - \gamma\phi)$ and $\varepsilon_n \in (0, 1)$.

Further, Abuchu et al. [1] introduced a modified relaxed inertial Mann-type iterative algorithm for solving (1.4): for two initial points $x_0, x_1 \in H_1$, choose τ_n such that $\tau_n \in (0, \bar{\tau_n})$ where $\bar{\tau_n} = \min\{\tau, \frac{\nu_n}{\|x_n - x_{n-1}\|}\}$ if $x_n \neq x_{n-1}$, otherwise, $\bar{\tau_n} = \tau$ and compute the next step $\{x_{n+1}\}$ by the following way

$$\begin{cases} z_n = x_n + \tau_n(x_n - x_{n-1}), \\ y_n = (1 - \alpha_n)z_n, \\ u_n = y_n - \tau_n(\Psi^*(I - B)\Psi(y_n)), \\ x_{n+1} = \rho_n u_n + (1 - \rho_n)(I + \gamma\Phi)^{-1}(I - \gamma\psi)u_n, n \geq 1. \end{cases} \tag{1.7}$$

where $B = (I + \gamma\varphi)^{-1}(I - \gamma\phi)$.

However, we observe that the operators ψ and ϕ in (1.5)-(1.7) are all inverse strongly monotone. It is an interesting work to relax this restrictive condition imposed on the operators ψ and ϕ. Our main purpose in this paper is to construct a new iterative algorithm in which the operators ψ and ϕ are plain monotone (not necessarily inverse strongly monotone) for solving (1.4). Our algorithm is based on splitting method and self-adaptive technique. Under some mild assumptions, we show that the constructed algorithm converges weakly to some solution of the split variational inclusion problem (1.4).

2. Preliminaries

In this section, we include several notations and some useful conclusions. Throughout in this section, assume that H is a real Hilbert space. In H, the following equality is well known: for all $z, z^\dagger \in H$ and $\forall \tau \in \mathbf{R}$,

$$\|\tau z + (1-\tau) z^\dagger\|^2 = \tau \|z\|^2 + (1-\tau)\|z^\dagger\|^2 - \tau(1-\tau)\|z - z^\dagger\|^2. \tag{2.1}$$

Definition 2.1. Let $T\colon H \to H$ be an operator. Recall that T is said to be

(i) L-Lipschitz if there is a positive constant L such that

$$\|T(x) - T(x^\dagger)\| \le L\|x - x^\dagger\|, \ \forall x, x^\dagger \in H.$$

(ii) strongly monotone if

$$\langle T(x) - T(x^\dagger), x - x^\dagger \rangle \ge \varepsilon \|x - x^\dagger\|^2, \ \forall x, x^\dagger \in H,$$

where $\varepsilon > 0$ is a constant.

(iii) inverse strongly monotone if

$$\langle T(x) - T(x^\dagger), x - x^\dagger \rangle \ge \varepsilon \|T(x) - T(x^\dagger)\|^2, \ \forall x, x^\dagger \in H,$$

where $\varepsilon > 0$ is a constant.

(iv) monotone if

$$\langle T(x) - T(x^\dagger), x - x^\dagger \rangle \ge 0, \ \forall x, x^\dagger \in H.$$

Definition 2.2. Let $S\colon H \rightrightarrows 2^H$ be a multi-valued operator. S is said to be monotone if and only if

$$\langle x - \hat{x}, p - \hat{p} \rangle \ge 0, \ \forall x, \hat{x} \in H,$$

where $p \in S(x)$ and $\hat{p} \in S(\hat{x})$.

A multi-valued monotone operator is said to be maximal monotone if and only if its graph is not strictly contained in the graph of any other monotone operator.

Let $S\colon H \rightrightarrows 2^H$ be a maximal monotone operator and $\gamma > 0$ be a constant. Define an operator $\mathrm{Res}_\gamma^S : H \to H$ by

$$\mathrm{Res}_\gamma^S(x) := (I + \gamma S)^{-1}(x), \ \forall x \in H,$$

which is called the resolvent of S. It is well-known tat the resolvent Res_γ^S is a single-valued firmly-nonexpansive and $S^{-1}(0) = \{x \in H : \mathrm{Res}_\gamma^S(x) = x\}$.

Lemma 2.1. ([13]) *Let Ω be a nonempty closed convex subset of a real Hilbert space H. Let $\{s_k\} \subset H$ be a sequence. If the following assumptions are satisfied*

(i) $\forall s^* \in \Omega$, $\lim_{k\to\infty} \|s_k - s^*\|$ exists;

(ii) $\omega_w(s_k) \subset \Omega$, where

$$\omega_w(s_k) := \{s \in H : \textit{there is a subsequence } \{s_{k_i}\} \textit{ of } \{s_k\} \textit{ such that } s_{k_i} \rightharpoonup s \textit{ as } i \to +\infty\}.$$

Then $s_k \rightharpoonup \hat{s} \in \Omega$ as $k \to +\infty$.

3. Main results

Let H_1 and H_2 be two real Hilbert spaces. Let $\Phi : H_1 \rightrightarrows 2^{H_1}$ and $\varphi : H_2 \rightrightarrows 2^{H_2}$ be two maximal monotone operators. Let $\psi : H_1 \to H_1$ be an α_1-Lipschitz monotone operator and $\phi : H_2 \to H_2$ be an α_2-Lipschitz monotone operator. Let $\Psi\colon H_1 \to H_2$ be a nonzero bounded linear operator with its adjoint Ψ^*. In what follows, suppose that $\Omega := \{x|x \in (\Phi+\psi)^{-1}(0), \Psi(x) \in (\phi+\varphi)^{-1}(0)\} \neq \emptyset$.

Let $\{\eta_k\}$ and $\{\theta_k\}$ be two real number sequences satisfying $\eta_k \in [\underline{\eta}, \overline{\eta}] \subset (0,1]$ and $\theta_k \in [\underline{\theta}, \overline{\theta}] \subset (0,1]$ for all $k \geq 0$. Let $\varepsilon_1, \varepsilon_2, \tau_1, \tau_2, \rho_1, \rho_2$ be six constants in $(0,1)$ and σ be a constant in $(0, 1/\|\Psi\|^2)$.

Next, we present an iterative algorithm for finding a point in Ω.

Algorithm 3.1. Let $x_0 \in H_1$ be an initial point. Let β_0 and ζ_0 be two positive constants. Set $k = 0$.

Step 1. For given x_k, compute

$$z_k = \mathrm{Res}^{\Phi}_{\tau_1\beta_k}(x_k - \tau_1\beta_k\psi(x_k)), \tag{3.1}$$

where $\beta_k = \max\{1, \varepsilon_1, \varepsilon_1^2, \cdots\}$ satisfies

$$\tau_1\beta_k\|\psi(x_k) - \psi(z_k)\| \leq \rho_1\|x_k - z_k\|, \tag{3.2}$$

and compute

$$u_k = (1-\eta_k)x_k + \eta_k(z_k - \tau_1\beta_k(\psi(z_k) - \psi(x_k))). \tag{3.3}$$

Step 2. Compute

$$w_k = \mathrm{Res}^{\varphi}_{\tau_2\zeta_k}(\Psi(u_k) - \tau_2\zeta_k\phi(\Psi(u_k))), \tag{3.4}$$

where $\zeta_k = \max\{1, \varepsilon_2, \varepsilon_2^2, \cdots\}$ satisfies

$$\tau_2\zeta_k\|\phi(\Psi(u_k)) - \phi(w_k)\| \leq \rho_2\|\Psi(u_k) - w_k\|, \tag{3.5}$$

and compute

$$y_k = (1-\theta_k)\Psi(u_k) + \theta_k(w_k - \tau_2\zeta_k(\phi(w_k) - \phi(\Psi(u_k)))). \tag{3.6}$$

Step 3. Compute

$$x_{k+1} = u_k + \sigma\Psi^*(y_k - \Psi(u_k)). \tag{3.7}$$

Set $k := k+1$ and return to Step 1.

Remark 3.1. If $z^k = x^k$ for some k, then $x^k \in (\Phi+\psi)^{-1}(0)$. In this case, we can choose $\beta_k = 1$. If $z^k \neq x^k$, then there exists some $\beta_k \in \{1, \varepsilon_1, \varepsilon_1^2, \cdots\}$ satisfying $\beta_k \leq \frac{\rho_1}{\tau_1\alpha_1}$ because $\|\psi(x_k) - \psi(z_k)\| \leq \alpha_1\|x_k - z_k\|$.

Now, we state that there must has some $\beta_k = \varepsilon_1^n$ such that (3.2) holds. If not so, we have $\rho_1\|x_k - z_k\| < \tau_1\beta_k\|\psi(x_k) - \psi(z_k)\| < \tau_1\beta_k\alpha_1\|x_k - z_k\|$ for all $k \in \mathcal{N}$ which yields that $\beta_k > \frac{\rho_1}{\tau_1\alpha_1}(\forall k \in \mathcal{N})$. It results in a contradiction.

Similarly, we can prove that there exists $\zeta_k = \max\{1, \varepsilon_2, \varepsilon_2^2, \cdots\}$ such that (3.5) holds.

Next, we demonstrate the convergence of Algorithm 3.1.

Theorem 3.1. *The sequence $\{x_k\}$ generated by Algorithm 3.1 converges weakly to some point in Ω.*

Proof. Let $u^\dagger \in \Omega$. We have $u^\dagger \in (\Phi+\psi)^{-1}(0)$ and $\Psi(u^\dagger) \in (\varphi+\phi)^{-1}(0)$. Then,

$$\begin{aligned}\|z_k - u^\dagger + \tau_1\beta_k(\psi(x_k)-\psi(z_k))\|^2 = \|z_k - u^\dagger\|^2 + \tau_1^2\beta_k^2\|\psi(x_k)-\psi(z_k)\|^2 \\ + 2\tau_1\beta_k\langle \psi(x_k)-\psi(z_k), z_k - u^\dagger\rangle.\end{aligned} \tag{3.8}$$

Note that

$$\|z_k - u^\dagger\|^2 = \|x_k - u^\dagger\|^2 - \|z_k - x_k\|^2 + 2\langle z_k - x_k, z_k - u^\dagger\rangle.$$

This together with (3.8) implies that

$$\begin{aligned}\|z_k - u^\dagger + \tau_1\beta_k(\psi(x_k)-\psi(z_k))\|^2 &= \|x_k - u^\dagger\|^2 + 2\tau_1\beta_k\langle \psi(x_k)-\psi(z_k), z_k - u^\dagger\rangle \\ &\quad - \|z_k - x_k\|^2 + 2\langle z_k - x_k, z_k - u^\dagger\rangle \\ &\quad + \tau_1^2\beta_k^2\|\psi(x_k)-\psi(z_k)\|^2 \\ &= \|x_k - u^\dagger\|^2 + \tau_1^2\beta_k^2\|\psi(x_k)-\psi(z_k)\|^2 \\ &\quad + 2\langle z_k - x_k + \tau_1\beta_k(\psi(x_k)-\psi(z_k)), z_k - u^\dagger\rangle \\ &\quad - \|z_k - v_k\|^2.\end{aligned} \tag{3.9}$$

From (3.1), $z_k = (I+\tau_1\beta_k\Phi)^{-1}(x_k - \tau_1\beta_k\psi(x_k))$. Then

$$x_k - \tau_1\beta_k\psi(x_k) \in (I+\tau_1\beta_k\Phi)z_k. \tag{3.10}$$

This results in that

$$x_k - z_k - \tau_1\beta_k(\psi(x_k)-\psi(z_k)) \in \tau_1\beta_k(\Phi+\psi)z_k. \tag{3.11}$$

Owing to $0 \in \tau_1\beta_k(\Phi+\psi)u^\dagger$, by the monotonicity of $\tau_1\beta_k(\Phi+\psi)$ and (3.11) we obtain

$$\langle z_k - x_k + \tau_1\beta_k(\psi(x_k)-\psi(z_k)), z_k - u^\dagger\rangle \le 0. \tag{3.12}$$

By (3.2), $\tau_1^2\beta_k^2\|\psi(x_k)-\psi(z_k)\|^2 \le \rho_1^2\|x_k - z_k\|^2$. By virtue of (3.9) and (3.12), we receive

$$\|z_k - u^\dagger + \tau_1\beta_k(\psi(x_k)-\psi(z_k))\|^2 \le \|x_k - u^\dagger\|^2 - (1-\rho_1^2)\|z_k - x_k\|^2. \tag{3.13}$$

Applying equality (2.1) to (3.3) to deduce

$$\begin{aligned}\|u_k - u^\dagger\|^2 = (1-\eta_k)\|x_k - u^\dagger\|^2 + \eta_k\|z_k - u^\dagger + \tau_1\beta_k(\psi(x_k)-\psi(z_k))\|^2 \\ - (1-\eta_k)\eta_k\|z_k - x_k + \tau_1\beta_k(\psi(x_k)-\psi(z_k))\|^2.\end{aligned} \tag{3.14}$$

Thanks to (3.13) and (3.14), we have

$$\begin{aligned}\|u_k - u^\dagger\|^2 &\le \|x_k - u^\dagger\|^2 - (1-\eta_k)\eta_k\|z_k - x_k + \tau_1\beta_k(\psi(x_k)-\psi(z_k))\|^2 \\ &\quad - \eta_k(1-\rho_1^2)\|z_k - x_k\|^2 \\ &\le \|x_k - u^\dagger\|^2.\end{aligned} \tag{3.15}$$

Since

$$\begin{aligned}&\|w_k - \Psi(u^\dagger) + \tau_2\zeta_k(\phi(\Psi(u_k)) - \phi(w_k))\|^2 \\ &\quad = \|w_k - \Psi(u^\dagger)\|^2 + \tau_2^2\zeta_k^2\|\phi(\Psi(u_k)) - \phi(w_k)\|^2 \\ &\qquad + 2\tau_2\zeta_k\langle \phi(\Psi(u_k)) - \phi(w_k), w_k - \Psi(u^\dagger)\rangle.\end{aligned}$$

and

$$\|w_k - \Psi(u^\dagger)\|^2 = \|\Psi(u_k) - \Psi(u^\dagger)\|^2 + 2\langle w_k - \Psi(u_k), w_k - \Psi(u^\dagger)\rangle - \|w_k - \Psi(u_k)\|^2,$$

we attain

$$\begin{aligned}
&\|w_k - \Psi(u^\dagger) + \tau_2\zeta_k(\phi(\Psi(u_k)) - \phi(w_k))\|^2 \\
&\quad = \|\Psi(u_k) - \Psi(u^\dagger)\|^2 + 2\tau_2\zeta_k\langle \phi(\Psi(u_k)) - \phi(w_k), w_k - \Psi(u^\dagger)\rangle \\
&\qquad - \|w_k - \Psi(u_k)\|^2 + 2\langle w_k - \Psi(u_k), w_k - \Psi(u^\dagger)\rangle \\
&\qquad + \tau_2^2\zeta_k^2\|\phi(\Psi(u_k)) - \phi(w_k)\|^2 \\
&\quad = \|\Psi(u_k) - \Psi(u^\dagger)\|^2 + \tau_2^2\zeta_k^2\|\phi(\Psi(u_k)) - \phi(w_k)\|^2 \\
&\qquad + 2\langle w_k - \Psi(u_k) + \tau_2\zeta_k(\phi(\Psi(u_k)) - \phi(w_k)), w_k - \Psi(u^\dagger)\rangle \\
&\qquad - \|w_k - \Psi(u_k)\|^2.
\end{aligned} \tag{3.16}$$

Based on (3.4), we achieve

$$\Psi(u_k) - w_k - \tau_2\zeta_k(\phi(\Psi(u_k)) - \phi(w_k)) \in \tau_2\zeta_k(\varphi + \phi)w_k. \tag{3.17}$$

Using the monotonicity of $\tau_2\zeta_k(\varphi+\phi)$ and $0 \in \tau_2\zeta_k(\varphi+\phi)\Psi(u^\dagger)$, according to (3.17), we have

$$\langle w_k - \Psi(u_k) + \tau_2\zeta_k(\phi(\Psi(u_k)) - \phi(w_k)), w_k - \Psi(u^\dagger)\rangle \le 0. \tag{3.18}$$

Furthermore, by (3.5),

$$\tau_2^2\zeta_k^2\|\phi(\Psi(u_k)) - \phi(w_k)\|^2 \le \rho_2^2\|\Psi(u_k) - w_k\|^2.$$

So, from (3.16) and (3.18), we have

$$\begin{aligned}
\|w_k - \Psi(u^\dagger) + \tau_2\zeta_k(\phi(\Psi(u_k)) - \phi(w_k))\|^2 &\le \|\Psi(u_k) - \Psi(u^\dagger)\|^2 \\
&\quad - (1-\rho_2^2)\|w_k - \Psi(u_k)\|^2.
\end{aligned} \tag{3.19}$$

Utilizing (2.1) to (3.6) to get

$$\begin{aligned}
\|y_k - \Psi(u^\dagger)\|^2 = {}& (1-\theta_k)\|\Psi(u_k) - \Psi(u^\dagger)\|^2 + \theta_k\|w_k - \Psi(u^\dagger) \\
& + \tau_2\zeta_k(\phi(\Psi(u_k)) - \phi(w_k))\|^2 - (1-\theta_k)\theta_k\|w_k - \Psi(u_k) \\
& + \tau_2\zeta_k(\phi(\Psi(u_k)) - \phi(w_k))\|^2.
\end{aligned} \tag{3.20}$$

Substituting (3.19) into (3.20), we obtain

$$\begin{aligned}
\|y_k - \Psi(u^\dagger)\|^2 &\le \|\Psi(u_k) - \Psi(u^\dagger)\|^2 - \theta_k(1-\rho_2^2)\|w_k - \Psi(u_k)\|^2 \\
&\quad - (1-\theta_k)\theta_k\|w_k - \Psi(u_k) + \tau_2\zeta_k(\phi(\Psi(u_k)) - \phi(w_k))\|^2 \\
&\le \|\Psi(u_k) - \Psi(u^\dagger)\|^2.
\end{aligned} \tag{3.21}$$

Observe that

$$\begin{aligned}
\langle u_k - u^\dagger, \Psi^*(y_k - \Psi(u_k))\rangle &= \langle \Psi(u_k) - \Psi(u^\dagger), y_k - \Psi(u_k)\rangle \\
&= \frac{1}{2}[\|y_k - \Psi(u^\dagger)\|^2 - \|\Psi(u_k) - \Psi(u^\dagger)\|^2] \\
&\quad - \frac{1}{2}\|y_k - \Psi(u_k)\|^2,
\end{aligned}$$

which together with (3.21) yields that

$$\langle u_k - u^\dagger, \Psi^*(y_k - \Psi(u_k))\rangle \le -\frac{1}{2}\theta_k(1-\rho_2^2)\|w_k - \Psi(u_k)\|^2 - \frac{1}{2}\|y_k - \Psi(u_k)\|^2. \quad (3.22)$$

According to (3.7), (3.15) and (3.22), we receive

$$\begin{aligned}\|x_{k+1} - u^\dagger\|^2 &= \|u_k - u^\dagger + \sigma\Psi^*(y_k - \Psi(u_k))\|^2 \\ &= \|u_k - u^\dagger\|^2 + \|\sigma\Psi^*(y_k - \Psi(u_k))\|^2 \\ &\quad + 2\sigma\langle \Psi^*(y_k - \Psi(u_k)), u_k - u^\dagger\rangle \\ &\le \|u_k - u^\dagger\|^2 + \sigma^2\|\Psi\|^2\|y_k - \Psi(u_k)\|^2 - \sigma\|y_k - \Psi(u_k)\|^2 \\ &\quad - \sigma\theta_k(1-\rho_2^2)\|w_k - \Psi(u_k)\|^2 \\ &= \|u_k - u^\dagger\|^2 - \sigma(1-\sigma\|\Psi\|^2)\|y_k - \Psi(u_k)\|^2 \\ &\quad - \sigma\theta_k(1-\rho_2^2)\|w_k - \Psi(u_k)\|^2 \\ &\le \|x_k - u^\dagger\|^2 - \sigma(1-\sigma\|\Psi\|^2)\|y_k - \Psi(u_k)\|^2 \\ &\quad - \sigma\theta_k(1-\rho_2^2)\|w_k - \Psi(u_k)\|^2 \\ &\le \|x_k - u^\dagger\|^2,\end{aligned} \quad (3.23)$$

which implies that $\lim_{k\to+\infty}\|x_k - u^\dagger\|$ exists and

$$\lim_{k\to+\infty}\|u_k - u^\dagger\| = \lim_{k\to+\infty}\|x_k - u^\dagger\|. \quad (3.24)$$

It is easily seen that the sequences $\{x_k\}$, $\{y_k\}$, $\{z_k\}$, $\{u_k\}$ and $\{w_k\}$ are all bounded.

From (3.23), we have

$$\begin{aligned}\sigma(1-\sigma\|\Psi\|^2)\|y_k - \Psi(u_k)\|^2 + \sigma\theta_k(1-\rho_2^2)\|w_k - \Psi(u_k)\|^2 \\ \le \|x_k - u^\dagger\|^2 - \|x_{k+1} - u^\dagger\|^2 \to 0.\end{aligned}$$

It yields that

$$\lim_{k\to+\infty}\|y_k - \Psi(u_k)\| = 0, \quad (3.25)$$

and

$$\lim_{k\to+\infty}\|w_k - \Psi(u_k)\| = 0. \quad (3.26)$$

Thanks to (3.15), we obtain

$$\eta_k(1-\rho_1^2)\|z_k - x_k\|^2 \le \|x_k - u^\dagger\|^2 - \|u_k - u^\dagger\|^2 \to 0.$$

It results in that

$$\lim_{k\to+\infty}\|z_k - x_k\| = 0. \quad (3.27)$$

In addition, from (3.3), we have

$$\begin{aligned}\|u_k - x_k\| &= \|\eta_k[z_k - x_k - \tau_1\beta_k(\psi(z_k) - \psi(x_k))]\| \\ &\le \eta_k\|z_k - x_k\| + \eta_k\tau_1\beta_k\|\psi(z_k) - \psi(x_k))\|.\end{aligned}$$

Hence, we get from (3.27) and the Lipschitz continuity of ψ that

$$\lim_{k\to+\infty} \|u_k - x_k\| = 0. \tag{3.28}$$

In view of (3.7), we have

$$x_{k+1} - x_k = u_k - x_k + \sigma\Psi^*(y_k - \Psi(u_k)).$$

Hence, take into account of (3.25) and (3.28), we deduce

$$\lim_{k\to+\infty} \|x_{k+1} - x_k\| = 0. \tag{3.29}$$

Next, we show $\omega_w(x_k) \subset \Omega$. Choosing any $u^* \in \omega_w(x_k)$, there is a subsequence $\{x_{k_i}\}$ of $\{x_k\}$ such that $x_{k_i} \rightharpoonup u^*$ as $i \to \infty$. Let $(\hat{p}, \hat{q}) \in \text{Graph}(\Phi + \psi)$. Then, $\hat{q} - \psi(\hat{p}) \in \Phi(\hat{p})$. By (3.1),

$$z_{k_i} = \text{Res}^{\Phi}_{\tau_1\beta_{k_i}}(x_{k_i} - \tau_1\beta_{k_i}\psi(x_{k_i})) = (I + \tau_1\beta_{k_i}\Phi)^{-1}(x_{k_i} - \tau_1\beta_{k_i}\psi(x_{k_i})).$$

It follows that

$$\frac{x_{k_i} - z_{k_i}}{\tau_1\beta_{k_i}} - \psi(x_{k_i}) \in \Phi(z_{k_i}). \tag{3.30}$$

Combining (3.30) and the monotonicity of Φ, we acquire

$$\langle \hat{q} - \psi(\hat{p}) - (\frac{x_{k_i} - z_{k_i}}{\tau_1\beta_{k_i}} - \psi(x_{k_i})), \hat{p} - z_{k_i}\rangle \geq 0.$$

It follows that

$$\begin{aligned} \langle \hat{q}, \hat{p} - z_{k_i}\rangle &\geq \langle \psi(\hat{p}) - \psi(x_{k_i}) + \frac{x_{k_i} - z_{k_i}}{\tau_1\beta_{k_i}}, \hat{p} - z_{k_i}\rangle \\ &= \langle \psi(\hat{p}) - \psi(z_{k_i}), \hat{p} - z_{k_i}\rangle + \langle \psi(z_{k_i}) - \psi(x_{k_i}), \hat{p} - z_{k_i}\rangle \\ &\quad + \frac{1}{\tau_1\beta_{k_i}}\langle x_{k_i} - z_{k_i}, \hat{p} - z_{k_i}\rangle. \end{aligned} \tag{3.31}$$

Since ψ is monotone, $\langle \psi(\hat{p}) - \psi(z_{k_i}), \hat{p} - z_{k_i}\rangle \geq 0$. It follows from (3.31) that

$$\langle \hat{q}, \hat{p} - z_{k_i}\rangle \geq \langle \psi(z_{k_i}) - \psi(x_{k_i}), \hat{p} - z_{k_i}\rangle + \frac{1}{\tau_1\beta_{k_i}}\langle x_{k_i} - z_{k_i}, \hat{p} - z_{k_i}\rangle. \tag{3.32}$$

Note that $x_{k_i} \rightharpoonup u^*$ and $z_{k_i} \rightharpoonup u^*$. Combining (3.27) with (3.32), we obtain

$$\langle \hat{q}, \hat{p} - u^*\rangle \geq 0$$

for all $(\hat{p}, \hat{q}) \in \text{Graph}(\Phi + \psi)$. So, $u^* \in (\Phi + \psi)^{-1}(0)$.
Next, we prove $\Psi(u^*) \in (\varphi + \phi)^{-1}(0)$. Let $(p^\dagger, q^\dagger) \in \text{Graph}(\varphi + \phi)$.
Then, $q^\dagger - \phi(p^\dagger) \in \varphi(p^\dagger)$. By (3.4), we have

$$w_{k_i} = \text{Res}^{\varphi}_{\tau_2\zeta_{k_i}}(\Psi(u_{k_i}) - \tau_2\zeta_{k_i}\phi(\Psi(u_{k_i}))) = (I + \tau_2\zeta_{k_i}\varphi)^{-1}(\Psi(u_{k_i}) - \tau_2\zeta_{k_i}\phi(\Psi(u_{k_i}))).$$

It follows that

$$\frac{\Psi(u_{k_i}) - w_{k_i}}{\tau_2\zeta_{k_i}} - \phi(\Psi(u_{k_i})) \in \varphi(w_{k_i}). \tag{3.33}$$

With the help of the monotonicity of φ, from (3.33), we obtain

$$\langle q^\dagger - \phi(p^\dagger) - (\frac{\Psi(u_{k_i}) - w_{k_i}}{\tau_2\zeta_{k_i}} - \phi(\Psi(u_{k_i}))), p^\dagger - w_{k_i}\rangle \geq 0.$$

Then,

$$\begin{aligned}\langle q^\dagger, p^\dagger - w_{k_i}\rangle &\geq \langle \phi(p^\dagger) - \phi(\Psi(u_{k_i})) + \frac{\Psi(u_{k_i}) - w_{k_i}}{\tau_2\zeta_{k_i}}, p^\dagger - w_{k_i}\rangle \\ &= \langle \phi(p^\dagger) - \phi(w_{k_i}), p^\dagger - w_{k_i}\rangle + \langle \phi(w_{k_i}) - \phi(\Psi(u_{k_i})), p^\dagger - w_{k_i}\rangle \\ &\quad + \frac{1}{\tau_2\zeta_{k_i}}\langle \Psi(u_{k_i}) - w_{k_i}, p^\dagger - w_{k_i}\rangle.\end{aligned} \tag{3.34}$$

Since $\langle \phi(p^\dagger) - \phi(w_{k_i}), p^\dagger - w_{k_i}\rangle \geq 0$, it follows from (3.34) that

$$\begin{aligned}\langle q^\dagger, p^\dagger - w_{k_i}\rangle &\geq \langle \phi(w_{k_i}) - \phi(\Psi(u_{k_i})), p^\dagger - w_{k_i}\rangle \\ &\quad + \frac{1}{\tau_2\zeta_{k_i}}\langle \Psi(u_{k_i}) - w_{k_i}, p^\dagger - w_{k_i}\rangle.\end{aligned} \tag{3.35}$$

Owing to $w_{k_i} \rightharpoonup \Psi(u^*)$ and $\|\Psi(u_{k_i}) - w_{k_i}\| \to 0$, from (3.35), we conclude that

$$\langle q^\dagger, p^\dagger - \Psi(u^*)\rangle \geq 0$$

for all $(p^\dagger, q^\dagger) \in \mathrm{Graph}(\phi+\varphi)$ which implies that $\Psi(u^*) \in (\phi+\varphi)^{-1}(0)$. Thus, $u^* \in \Omega$. So, $\omega_w(x_k) \subset \Omega$.

Note that (i) $\forall u^\dagger \in \Omega$, $\lim_{k\to\infty} \|x_k - u^\dagger\|$ exists and (ii) $\omega_w(x_k) \subset \Omega$. Applying Lemma 2.1, we can conclude that $\{x_k\}$ converges weakly to some point in Ω. This completes the proof.

Let H be a real Hilbert spaces. Let $\Phi : H \rightrightarrows 2^H$ and $\varphi : H \rightrightarrows 2^H$ be two maximal monotone operators. Let $\psi : H \to H$ be an α_1-Lipschitz monotone operator and $\phi : H \to H$ be an α_2-Lipschitz monotone operator. Suppose that $\Omega_1 := \{x | x \in (\Phi + \psi)^{-1}(0) \cap (\phi + \varphi)^{-1}(0)\} \neq \emptyset$. Let $\{\eta_k\}$ and $\{\theta_k\}$ be two real number sequences satisfying $\eta_k \in [\underline{\eta}, \overline{\eta}] \subset (0, 1]$ and $\theta_k \in [\underline{\theta}, \overline{\theta}] \subset (0, 1]$ for all $k \geq 0$. Let $\varepsilon_1, \varepsilon_2, \tau_1, \tau_2, \rho_1, \rho_2$ and σ be seven constants in $(0, 1)$.

Algorithm 3.2. Let $x_0 \in H$ be an initial point. Let β_0 and ζ_0 be two positive constants. Set $k = 0$.

Step 1. For given x_k, compute

$$z_k = \mathrm{Res}^{\Phi}_{\tau_1\beta_k}(x_k - \tau_1\beta_k\psi(x_k)),$$

where $\beta_k = \max\{1, \varepsilon_1, \varepsilon_1^2, \cdots\}$ satisfies

$$\tau_1\beta_k\|\psi(x_k) - \psi(z_k)\| \leq \rho_1\|x_k - z_k\|,$$

and compute

$$u_k = (1 - \eta_k)x_k + \eta_k(z_k - \tau_1\beta_k(\psi(z_k) - \psi(x_k))).$$

Step 2. Compute

$$w_k = \mathrm{Res}^{\varphi}_{\tau_2\zeta_k}(u_k - \tau_2\zeta_k\phi(u_k)),$$

where $\zeta_k = \max\{1, \varepsilon_2, \varepsilon_2^2, \cdots\}$ satisfies

$$\tau_2\zeta_k\|\phi(u_k) - \phi(w_k)\| \leq \rho_2\|u_k - w_k\|,$$

and compute

$$y_k = (1 - \theta_k)u_k + \theta_k(w_k - \tau_2\zeta_k(\phi(w_k) - \phi(u_k))).$$

Step 3. Compute

$$x_{k+1} = u_k + \sigma(y_k - u_k).$$

Set $k := k + 1$ and return to Step 1.

Corollary 3.1. *The sequence $\{x_k\}$ generated by Algorithm 3.2 converges weakly to some point in Ω_1.*

4. Conclusions

In this paper, we investigate the split variational inclusion problem (1.4) where the involved operators ψ and ϕ are all plain monotone. To solve this split monotone variational inclusion problem, we suggest an iterative algorithm by using the splitting method and self-adaptive rules. We show the proposed algorithm converges weakly to a solution of the split variational inclusion (1.4) provided the involved parameters fulfil some appropriate restrictions.

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