Fixed Point Theory, 25(2024), No. 2, 747-772
DOI: 10.24193/fpt-ro.2024.2.20
http://www.math.ubbcluj.ro/~nodeacj/sfptcj.html

TWO NOVEL ALGORITHMS FOR SOLVING VARIATIONAL INEQUALITY PROBLEMS GOVERNED BY FIXED POINT PROBLEMS AND THEIR APPLICATIONS

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Abstract. We study the problem of finding a common solution to the variational inequality problem with a pseudomonotone and Lipschitz continuous operator and the fixed point problem with a demicontractive mapping in real Hilbert spaces. Inspired by the inertial method and the subgradient extragradient method, two improved viscosity-type efficient iterative methods with a new adaptive non-monotonic step size criterion are proposed. We prove that the strong convergence theorems of these new methods hold under some standard and mild conditions. Numerical examples in finite- and infinite-dimensional spaces are provided to illustrate the effectiveness and potential applicability of the suggested iterative methods compared to some known ones.
Key Words and Phrases: Variational inequality problem, fixed point problem, subgradient extragradient method, inertial method, optimal control.
2020 Mathematics Subject Classification: 47H05, 47J20, 47J25, 47J30, 47H10, 65K15.

1. Introduction

In recent decades, bilevel optimization has received a lot of attention in mathematics and industry due to the proliferation of practical applications and the potential of algorithms in solving these problems. Bilevel optimization is a special type of optimization where one optimization problem contains the other optimization problem as a constraint. This means that the decision of the upper leader is influenced by the decision of the lower leader. Bilevel optimization problems are usually found in a number of real-world problems, which include problems in areas such as transportation, engineering, environmental ecology, and economics; see, e.g., [8, 9]. In this paper, we aim to propose efficient adaptive numerical algorithms to solve variational inequality problems and fixed point problems in real Hilbert spaces. The reason for studying such problems is that the ideas and techniques of variational inequalities and fixed point problems are being applied in various scientific fields and the theory can provide a straightforward and cohesive framework for the consideration of a variety of unconnected problems; see, for example, [1, 27, 32, 36, 37].

We begin by reviewing the two involved problems. Let $\mathcal{C}$ be a nonempty, closed, and convex subset of a real Hilbert space $\mathcal{H}$ with inner product $\langle \cdot , \cdot \rangle$ and induced norm $\| \cdot \|$, and $B : \mathcal{H} \to \mathcal{H}$ be a given operator. The classical variational inequality problem (shortly, VIP) is formulated as

$$\text{find } \ x \in \mathcal{C} \ \text{ such that } \ \langle Bx, y - x \rangle \geq 0, \quad \forall y \in \mathcal{C}. \tag{VIP}$$

For simplicity, its solution set is written as Ω. Next we analyze some existing classical and effective algorithms for solving (VIP). The most fundamental of them is the following extragradient method (shortly, EGM) proposed by Korpelevich [15] with the form:

$$\begin{cases} u_n = Proj_{\mathcal{C}} \left(x_n - \lambda B x_n \right), \\ x_{n+1} = Proj_{\mathcal{C}} \left(x_n - \lambda B u_n \right), \end{cases} \tag{1.1}$$

where the operator B is monotone and L-Lipschitz continuous, $Proj_{\mathcal{C}}$ is denoted by the metric projection from $\mathcal{H}$ onto $\mathcal{C}$, and $\lambda \in (0, 1/L)$. It is observed that EGM involves two projections on the set $\mathcal{C}$. This affects the execution efficiency of the method when the projection on the feasible set $\mathcal{C}$ is difficult to compute. Thus, one of the main improvements of EGM is to minimize the number of $Proj_{\mathcal{C}}$ computations in each iteration. Censor, Gibali and Reich [6] tried in this direction and proposed the subgradient extragradient method (shortly, SEGM), which modifies EGM by replacing the second projection with a projection on the half-space (it can be calculated by a closed-form formula, see [2, Example 29.20]). The algorithm is as follows:

$$\begin{cases} u_n = Proj_{\mathcal{C}} \left(x_n - \lambda B x_n \right), \\ T_n = \left\{ x \in \mathcal{H} \mid \langle x_n - \lambda B x_n - u_n, x - u_n \rangle \leq 0 \right\}, \\ x_{n+1} = Proj_{T_n} \left(x_n - \lambda B u_n \right), \end{cases} \tag{1.2}$$

where the operator B and step size λ are the same as in (1.1). It is worth noting that the sequences generated by (1.1) and (1.2) converge weakly to an element in Ω when the solution set is nonempty. In recent years, an increasing number of researchers have focused on developing efficient and implementable extragradient-type algorithms for solving (VIP), and continue to attract in-depth research; see, e.g., [10, 12, 22, 24, 33] and the extensive references therein.

On the other hand, the topical problem of finding fixed points in functional analysis is closely related to variational inequalities. The following is a description of the fixed point problem (shortly, FPP):

$$\text{find } \quad x \in \mathcal{C} \text{ such that } \quad Qx = x, \tag{FPP}$$

where $Q : \mathcal{C} \to \mathcal{C}$ is a nonlinear mapping. The solution set of (FPP) is denoted as Ψ. We want to find a common solution of (VIP) and (FPP) in this paper, i.e.,

$$\text{find } \quad q \ \text{ such that } \quad q \in \Psi \cap \Omega. \tag{VIP-FPP}$$

In the past few decades, many algorithms have been proposed to solve (VIP-FPP); see, e.g., [18, 7, 16, 25, 4, 30, 5, 35, 36]. Among them, Nadezhkina and Takahashi [18], and Censor, Gibali and Reich [7] proposed the following two algorithms inspired by EGM and SEGM, respectively,

$$\begin{cases} u_n = Proj_{\mathcal{C}}\left(x_n - \lambda B x_n\right), \\ x_{n+1} = \sigma_n x_n + \left(1 - \sigma_n\right) Q Proj_{\mathcal{C}}\left(x_n - \lambda B u_n\right), \end{cases} \tag{1.3}$$

and

$$\begin{cases} u_n = Proj_{\mathcal{C}}\left(x_n - \lambda B x_n\right), \\ T_n = \left\{x \in \mathcal{H} \mid \langle x_n - \lambda B x_n - u_n, x - u_n \rangle \le 0\right\}, \\ x_{n+1} = \sigma_n x_n + \left(1 - \sigma_n\right) Q Proj_{T_n}\left(x_n - \lambda B u_n\right), \end{cases} \tag{1.4}$$

where the operator B and step size λ are the same as in (1.1), and the mapping Q is nonexpansive. We note that the sequences generated by such iterative schemes as (1.3) and (1.4) both converge only weakly to the solution of (VIP-FPP). It is seen from some practical applications that algorithms with strong convergence usually obtain better results than algorithms with weak convergence in infinite-dimensional spaces. Therefore, algorithms that generate strongly convergent sequences need to be developed for solving the (VIP-FPP). Kraikaew and Saejung [16] proposed a strongly convergent iterative scheme using a combination of SEGM and the Halpern method, which is specified as follows:

$$\begin{cases} u_n = Proj_{\mathcal{C}}\left(x_n - \lambda B x_n\right), \\ T_n = \left\{x \in \mathcal{H} \mid \langle x_n - \lambda B x_n - u_n, x - u_n \rangle \le 0\right\}, \\ v_n = Proj_{T_n}\left(x_n - \lambda B u_n\right), \\ x_{n+1} = \eta_n x_n + \left(1 - \eta_n\right) Q\left[\sigma_n x_0 + \left(1 - \sigma_n\right) v_n\right], \end{cases}$$

where the operator B and step size λ are the same as in (1.1), the mapping $Q : \mathcal{H} \to \mathcal{H}$ is quasi-nonexpansive, $\{\sigma_n\} \subset (0,1)$, $\lim_{n\to\infty} \sigma_n = 0$, $\sum_{n=1}^{\infty} \sigma_n = +\infty$, and $\{\eta_n\} \subset [a,b] \subset (0,1)$. Subsequently, some scholars further investigated (VIP-FPP) with weaker constraints, assuming that the Lipschitz constant L of mapping B unknown, and Q is a demicontractive mapping. Recently, Thong and Hieu [31] introduced an iterative algorithm based on the viscosity-type extragradient method with a simpler step size update criterion and proved its strong convergence. In recent years, inertial extrapolation techniques received much interest and study by scholars who use inertial methods to accelerate the proposed iterative schemes and apply them to solve variational inequality problems, fixed point problems, and split feasibility problems; see, e.g., [11, 20, 23, 26, 34, 38]. Very recently, Tan, Zhou and Li [29] presented a new inertial subgradient extragradient algorithm to find a solution of (VIP-FPP). More precisely, their iterative scheme is described as follows:

$$\begin{cases} d_n = x_n + \gamma_n\left(x_n - x_{n-1}\right), \\ u_n = Proj_{\mathcal{C}}\left(d_n - \lambda_n B d_n\right), \\ T_n = \left\{x \in \mathcal{H} \mid \langle d_n - \lambda_n B d_n - u_n, x - u_n \rangle \le 0\right\}, \\ v_n = Proj_{T_n}\left(d_n - \lambda_n B u_n\right), \\ x_{n+1} = \sigma_n g(x_n) + \left(1 - \sigma_n\right)\left[(1 - \eta_n) v_n + \eta_n Q v_n\right], \end{cases}$$

where the updating formulas for inertial coefficient γ_n is

$$\gamma_n = \begin{cases} \min\left\{\dfrac{\epsilon_n}{\|x_n - x_{n-1}\|}, \gamma\right\}, & \text{if } x_n \neq x_{n-1}; \\ \gamma, & \text{otherwise,} \end{cases}$$

and for step size λ_n is

$$\lambda_{n+1} = \begin{cases} \min\left\{\dfrac{\delta\,\|d_n - u_n\|}{\|Bd_n - Bu_n\|}, \lambda_n\right\}, & \text{if } Bd_n - Bu_n \neq 0; \\ \lambda_n, & \text{otherwise,} \end{cases}$$

and the mapping B is monotone and Lipschitz continuous, the mapping Q is τ-demicontractive, the mapping g is contraction, $\gamma > 0$, $\lambda_1 > 0$, $\delta \in (0,1)$, $\sigma_n \subset (0,1)$, and $\eta_n \subset (a, 1-\tau) \subset (0,1)$. Their proposed iterative scheme guarantees strong convergence under some suitable conditions and obtains competitive convergence speed in some numerical experiments and applications.

Inspired by the above-mentioned works, we propose two improved viscosity-type inertial subgradient extragradient algorithms, which are optimized at the projection stage and step size update. Under some mild conditions, strong convergence results are obtained for our algorithms when approximating the common solution of the (VIP) with a pseudomonotone and Lipschitz continuous operator and the (FPP) with a demicontractive mapping in a real Hilbert space. In addition, the step size used in this paper differs from the general adaptive step size by adding a nonnegative real sequence, which can improve the convergence efficiency of the algorithm without the prior knowledge of the Lipschitz constant L of the pseudomonotone operator. Finally, some numerical experiments and applications are provided to demonstrate that our algorithms are more competitive than other comparative ones in [31, 29].

The summary of this article is as follows. In Sect. 2, we introduce some basic definitions and lemmas for further use. In Sect. 3, we prove the convergence of the proposed algorithms. Numerical examples comparing the performance of our proposed methods with several related algorithms are presented in Sect. 4. Finally, we conclude this paper in Sect. 5.

2. Preliminaries

Let $\mathcal{C}$ be a nonempty, closed, and convex subset of a real Hilbert space $\mathcal{H}$. The weak convergence and strong convergence of $\{x_n\}$ to x are denoted as $x_n \rightharpoonup x$ and $x_n \to x$, respectively.

Definition 2.1. ([2, p. 535]) Recall that $Proj_{\mathcal{C}} : \mathcal{H} \to \mathcal{C}$ is called the *metric projection* from $\mathcal{H}$ onto $\mathcal{C}$ if for every point $x \in \mathcal{H}$, there exists a unique nearest point $Proj_{\mathcal{C}}(x) \in \mathcal{C}$ such that

$$\|x - Proj_{\mathcal{C}}(x)\| \leq \|x - y\|, \quad \forall y \in \mathcal{C}.$$

It is known that $Proj_{\mathcal{C}}$ is nonexpansive and satisfies the following characteristics:

$$\langle x - Proj_{\mathcal{C}}(x), y - Proj_{\mathcal{C}}(x)\rangle \leq 0, \ \ \forall x \in \mathcal{H}, y \in \mathcal{C}, \tag{2.1}$$

and

$$\|Proj_C(x) - y\|^2 \leq \|x - y\|^2 - \|x - Proj_C(x)\|^2, \ \forall x \in \mathcal{H}, y \in \mathcal{C}. \tag{2.2}$$

For each $x, y \in \mathcal{H}$ and $\sigma \in (0, 1)$, we have

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y\rangle \tag{2.3}$$

and

$$\|\sigma x + (1 - \sigma)y\|^2 = \sigma\|x\|^2 + (1 - \sigma)\|y\|^2 - \sigma(1 - \sigma)\|x - y\|^2. \tag{2.4}$$

Definition 2.2. ([2, Definition 4.26]) Let $Q : \mathcal{H} \to \mathcal{H}$ be a nonlinear operator and its set of fixed points be nonempty (i.e., $\Psi \neq \emptyset$). The mapping $I - Q$ is said to be demiclosed at zero if for any $\{x_n\} \in \mathcal{H}$, the following holds:

$$x_n \rightharpoonup x \text{ and } (I - Q)x_n \to 0 \Rightarrow x \in \Psi.$$

Definition 2.3. ([2, Definition 4.1]) Let $Q : \mathcal{H} \to \mathcal{H}$ be a mapping and its set of fixed points be nonempty (i.e., $\Psi \neq \emptyset$). Recall that Q is said to be:

(1) *L-Lipschitz continuous* with $L > 0$ if $\|Qx - Qy\| \leq L\|x - y\|$ for all $x, y \in \mathcal{H}$;
(2) *monotone* if $\langle Qx - Qy, x - y\rangle \geq 0$ for all $x, y \in \mathcal{H}$;
(3) *pseudomonotone* if $\langle Qx, y - x\rangle \geq 0 \Longrightarrow \langle Qy, y - x\rangle \geq 0$ for all $x, y \in \mathcal{H}$;
(4) *nonexpansive* if $\|Qx - Qy\| \leq \|x - y\|$ for all $x, y \in \mathcal{H}$;
(5) *quasi-nonexpansive* if $\|Qx - z\| \leq \|x - z\|$ for all $z \in \Psi$ and $x \in \mathcal{H}$;
(6) *τ-demicontractive* with $0 \leq \tau < 1$ if

$$\|Qx - z\|^2 \leq \|x - z\|^2 + \tau\|(I - Q)x\|^2, \quad \forall z \in \Psi, x \in \mathcal{H},$$

or equivalently

$$\langle Qx - x, x - z\rangle \leq \frac{\tau - 1}{2}\|x - Qx\|^2, \quad \forall z \in \Psi, x \in \mathcal{H}. \tag{2.5}$$

The following lemma is provided for proving the convergence of our algorithms.

Lemma 2.1. ([21, Lemma 2.6]) *Let $\{p_n\}$ be a positive sequence, $\{q_n\}$ be a sequence of real numbers, and $\{\sigma_n\}$ be a sequence such that $\sigma_n \in (0, 1)$ and $\sum_{n=1}^{\infty} \sigma_n = \infty$. Suppose that*

$$p_{n+1} \leq \sigma_n q_n + (1 - \sigma_n) p_n, \quad \forall n \geq 1.$$

If $\limsup_{k\to\infty} q_{n_k} \leq 0$ for every subsequence $\{p_{n_k}\}$ of $\{p_n\}$ satisfying

$$\liminf_{k\to\infty} (p_{n_k+1} - p_{n_k}) \geq 0,$$

then $\lim_{n\to\infty} p_n = 0$.

3. Main results

In this section, we introduce two new inertial subgradient extragradient algorithms for solving variational inequality problems and fixed point problems and analyze their convergence. The proposed algorithms can work without the prior knowledge of the Lipschitz constant of the mapping. Now, we present a modified viscosity-type inertial subgradient extragradient algorithm for solving (VIP-FPP). The first method is stated in Algorithm 3.1 below.

The following are the conditional assumptions satisfied by our proposed algorithms.

Algorithm 3.1

Initialization: Take $\gamma > 0$, $\lambda_1 > 0$, $\varphi \in (0, 2/(1+\delta))$ and $\delta \in (0,1)$. Select $\{\epsilon_n\}$, $\{\sigma_n\}$, $\{\eta_n\}$ and $\{\xi_n\}$ to satisfy Condition (C5). Let $x_0, x_1 \in \mathcal{H}$.

Iterative Steps: Given the iterates x_{n-1} and $x_n\,(n \geq 1)$. Calculate x_{n+1} as follows:

Step 1. Compute $d_n = x_n + \gamma_n (x_n - x_{n-1})$, where

$$\gamma_n = \begin{cases} \min\left\{\dfrac{\epsilon_n}{\|x_n - x_{n-1}\|}, \gamma\right\}, & \text{if } x_n \neq x_{n-1}; \\ \gamma, & \text{otherwise.} \end{cases} \tag{3.1}$$

Step 2. Compute $u_n = Proj_{\mathcal{C}}(d_n - \lambda_n B d_n)$.

Step 3. Compute $v_n = Proj_{T_n}(d_n - \varphi\lambda_n B u_n)$, where the half-space T_n is defined as

$$T_n = \{x \in \mathcal{H} \mid \langle d_n - \lambda_n B d_n - u_n, x - u_n\rangle \leq 0\}.$$

Step 4. Compute $x_{n+1} = \sigma_n g(d_n) + (1-\sigma_n)[(1-\eta_n)v_n + \eta_n Q v_n]$, and update

$$\lambda_{n+1} = \begin{cases} \min\left\{\delta\dfrac{\|d_n - u_n\|^2 + \|v_n - u_n\|^2}{2\langle Bd_n - Bu_n, v_n - u_n\rangle}, \lambda_n + \xi_n\right\}, & \text{if } \langle Bd_n - Bu_n, v_n - u_n\rangle > 0; \\ \lambda_n + \xi_n, & \text{otherwise.} \end{cases} \tag{3.2}$$

Set $n \leftarrow n+1$ and go to **Step 1**.

(C1) The solution set of (VIP-FPP) is nonempty, i.e., $\Psi \cap \Omega \neq \emptyset$.

(C2) The mapping $B : \mathcal{H} \to \mathcal{H}$ is pseudomonotone, L-Lipschitz continuous on $\mathcal{H}$, and sequentially weakly continuous on $\mathcal{C}$.

(C3) The mapping $Q : \mathcal{H} \to \mathcal{H}$ is τ-demicontractive such that $(I - Q)$ is demiclosed at zero.

(C4) The mapping $g : \mathcal{H} \to \mathcal{H}$ is ρ-contraction with $\rho \in (0,1)$.

(C5) Let $\{\epsilon_n\}$ be a positive sequence such that $\lim_{n\to\infty} \frac{\epsilon_n}{\sigma_n} = 0$, where $\{\sigma_n\} \subset (0,1)$ satisfies $\lim_{n\to\infty}\sigma_n = 0$ and $\sum_{n=1}^{\infty}\sigma_n = \infty$. Let $\{\eta_n\}$ be a real sequence such that $\eta_n \in (0,1)$ and $\{\eta_n\} \subset (a, 1-\tau)$ for some $a > 0$. Choose a nonnegative real sequence $\{\xi_n\}$ satisfying $\sum_{n=1}^{\infty}\xi_n < +\infty$.

The following lemmas are quite helpful to analyze the convergence of our algorithms.

Lemma 3.1. *Suppose that Condition (C2) holds. Then the sequence* $\{\lambda_n\}$ *generated by* (3.2) *is well defined and* $\lim_{n\to\infty}\lambda_n = \lambda$ *and* $\lambda \in \left[\min\{\delta/L, \lambda_1\}, \lambda_1 + \sum_{n=1}^{\infty}\xi_n\right]$.

Proof. The proof is very similar to Lemma 3.1 in [17]. So we omit the details. □

Lemma 3.2. *Assume that Condition (C2) holds. Let* $\{v_n\}$ *be a sequence created by Algorithm 3.1. Then, for all* $p \in \Omega$,

$$\|v_n - p\|^2 \leq \|d_n - p\|^2 - \varphi^*\left(\|d_n - u_n\|^2 + \|v_n - u_n\|^2\right),$$

where $\varphi^* = 2 - \varphi - \frac{\varphi\delta\lambda_n}{\lambda_{n+1}}$ *if* $\varphi \in [1, 2/(1+\delta))$ *and* $\varphi^* = \varphi - \frac{\varphi\delta\lambda_n}{\lambda_{n+1}}$ *if* $\varphi \in (0,1)$.

Proof. From the definition of v_n and (2.2), we have

$$\begin{aligned}
&\|v_n - p\|^2 \\
&= \|Proj_{T_n}(d_n - \varphi\lambda_n Bu_n) - p\|^2 \\
&\leq \|d_n - \varphi\lambda_n Bu_n - p\|^2 - \|d_n - \varphi\lambda_n Bu_n - v_n\|^2 \\
&= \|d_n - p\|^2 + (\varphi\lambda_n)^2 \|Bu_n\|^2 - 2\langle d_n - p, \varphi\lambda_n Bu_n\rangle - \|d_n - v_n\|^2 \\
&\quad - (\varphi\lambda_n)^2 \|Bu_n\|^2 + 2\langle d_n - v_n, \varphi\lambda_n Bu_n\rangle \\
&= \|d_n - p\|^2 - \|d_n - v_n\|^2 - 2\langle \varphi\lambda_n Bu_n, v_n - p\rangle \\
&= \|d_n - p\|^2 - \|d_n - v_n\|^2 - 2\langle \varphi\lambda_n Bu_n, v_n - u_n\rangle - 2\langle \varphi\lambda_n Bu_n, u_n - p\rangle .
\end{aligned} \tag{3.3}$$

Since $p \in \Omega$ and $u_n \in \mathcal{C}$, we have $\langle Bp, u_n - p\rangle \geq 0$. By the pseudomonotonicity of mapping B, we obtain $\langle Bu_n, u_n - p\rangle \geq 0$. Thus, (3.3) reduces to

$$\|v_n - p\|^2 \leq \|d_n - p\|^2 - \|d_n - v_n\|^2 - 2\langle \varphi\lambda_n Bu_n, v_n - u_n\rangle . \tag{3.4}$$

Now we estimate $2\langle \varphi\lambda_n Bu_n, v_n - u_n\rangle$. Note that

$$-\|d_n - v_n\|^2 = -\|d_n - u_n\|^2 - \|u_n - v_n\|^2 + 2\langle d_n - u_n, v_n - u_n\rangle . \tag{3.5}$$

In addition, we see that

$$\begin{aligned}
&\langle d_n - u_n, v_n - u_n\rangle \\
&= \langle d_n - u_n - \lambda_n Bd_n + \lambda_n Bd_n - \lambda_n Bu_n + \lambda_n Bu_n, v_n - u_n\rangle \\
&= \langle d_n - \lambda_n Bd_n - u_n, v_n - u_n\rangle + \lambda_n \langle Bd_n - Bu_n, v_n - u_n\rangle \\
&\quad + \langle \lambda_n Bu_n, v_n - u_n\rangle .
\end{aligned} \tag{3.6}$$

By $v_n \in T_n$ and the definition of T_n, we have

$$\langle d_n - \lambda_n Bd_n - u_n, v_n - u_n\rangle \leq 0. \tag{3.7}$$

According to the definition of λ_{n+1}, it is easy to obtain

$$\langle Bd_n - Bu_n, v_n - u_n\rangle \leq \frac{\delta}{2\lambda_{n+1}} \|d_n - u_n\|^2 + \frac{\delta}{2\lambda_{n+1}} \|v_n - u_n\|^2 . \tag{3.8}$$

Substituting (3.6), (3.7), and (3.8) into (3.5), we obtain

$$-\|d_n - v_n\|^2 \leq -\left(1 - \frac{\delta\lambda_n}{\lambda_{n+1}}\right)\left(\|d_n - u_n\|^2 + \|v_n - u_n\|^2\right) + 2\langle \lambda_n Bu_n, v_n - u_n\rangle ,$$

which implies that

$$\begin{aligned}
-2\langle \varphi\lambda_n Bu_n, v_n - u_n\rangle \leq{}& -\varphi\left(1 - \frac{\delta\lambda_n}{\lambda_{n+1}}\right)\left(\|d_n - u_n\|^2 + \|v_n - u_n\|^2\right) \\
&+ \varphi \|d_n - v_n\|^2 .
\end{aligned} \tag{3.9}$$

Combining (3.4) and (3.9), we conclude that

$$\begin{aligned}
\|v_n - p\|^2 \leq{}& \|d_n - p\|^2 - \varphi\left(1 - \frac{\delta\lambda_n}{\lambda_{n+1}}\right)\left(\|d_n - u_n\|^2 + \|v_n - u_n\|^2\right) \\
&- (1 - \varphi)\|d_n - v_n\|^2 .
\end{aligned} \tag{3.10}$$

Note that

$$\|d_n - v_n\|^2 \le 2\left(\|d_n - u_n\|^2 + \|v_n - u_n\|^2\right),$$

which yields that

$$-(1-\varphi)\|d_n - v_n\|^2 \le -2(1-\varphi)\left(\|d_n - u_n\|^2 + \|v_n - u_n\|^2\right), \quad \forall \varphi \ge 1.$$

This together with (3.10) implies

$$\|v_n - p\|^2 \le \|d_n - p\|^2 - \left(2 - \varphi - \frac{\varphi\delta\lambda_n}{\lambda_{n+1}}\right)\left(\|d_n - u_n\|^2 + \|v_n - u_n\|^2\right), \quad \forall \varphi \ge 1.$$

In addition, if $\varphi \in (0,1)$, then we obtain

$$\|v_n - p\|^2 \le \|d_n - p\|^2 - \varphi\left(1 - \frac{\delta\lambda_n}{\lambda_{n+1}}\right)\left(\|d_n - u_n\|^2 + \|v_n - u_n\|^2\right), \quad \forall \varphi \in (0,1).$$

The proof of Lemma 3.2 is completed. □

Remark 3.1. From Lemma 3.1 and the assumptions of the parameters δ and φ (i.e., $\delta \in (0,1)$ and $\varphi \in (0, 2/(1+\delta))$), we obtain that $\varphi^* > 0$ for all $n \ge n_0$ in Lemma 3.2 always holds.

According to a simple modification of [28, Lemma 3.3], we can obtain the following Lemma 3.3.

Lemma 3.3. *Let $B : \mathcal{H} \to \mathcal{H}$ be a pseudomonotone and L-Lipschitz continuous mapping on $\mathcal{C}$. Let $T = Proj_{\mathcal{C}}(I - \lambda B)$, where $\lambda > 0$. If $\{x_n\}$ is a sequence in $\mathcal{H}$ satisfying $x_n \rightharpoonup z$ and $x_n - Tx_n \to 0$, then $z \in \Omega$, where Ω denotes the set of variational inequality solutions of the operator B.*

Theorem 3.1. *Assume that Conditions (C1)–(C5) hold. Then the sequence $\{x_n\}$ generated by Algorithm 3.1 converges strongly to $x^* \in \Psi \cap \Omega$, where $x^* = Proj_{\Psi\cap\Omega}(g(x^*))$.*

Proof. Note that both Ω and Ψ are closed convex subsets. Hence, the mapping $Proj_{\Psi\cap\Omega}(g) : \mathcal{H} \to \mathcal{H}$ is a contraction. From the Banach contraction principle, there exists a unique point $x^* \in \mathcal{H}$ such that $x^* = Proj_{\Psi\cap\Omega}(g(x^*))$. In particular, $x^* \in \Psi \cap \Omega$ and

$$\langle g(x^*) - x^*, z - x^* \rangle \le 0, \quad \forall z \in \Psi \cap \Omega.$$

The above inequality is obtained by (2.1).

We divide the proof into three steps. First, we show that the sequence $\{x_n\}$ is bounded. Let $t_n = (1-\eta_n)v_n + \eta_n Q v_n$. From (2.5), we have

$$\begin{aligned}
\|t_n - x^*\|^2 &= \|(1-\eta_n)v_n + \eta_n Q v_n - x^*\|^2 \\
&= \|v_n - x^*\|^2 + 2\eta_n\langle v_n - x^*, Qv_n - v_n\rangle + \eta_n^2\|Qv_n - v_n\|^2 \\
&\le \|v_n - x^*\|^2 + \eta_n(\tau - 1)\|Qv_n - v_n\|^2 + \eta_n^2\|Qv_n - v_n\|^2 \\
&= \|v_n - x^*\|^2 - \eta_n(1 - \tau - \eta_n)\|(I - Q)v_n\|^2.
\end{aligned}$$

In view of Lemma 3.2 and $\{\eta_n\} \subset (a, 1-\tau)$, we obtain

$$\begin{aligned} \|t_n - x^*\|^2 \le \|d_n - x^*\|^2 - \varphi^* \left(\|d_n - u_n\|^2 + \|v_n - u_n\|^2\right) \\ - \eta_n (1 - \tau - \eta_n) \|Qv_n - v_n\|^2 . \end{aligned} \tag{3.11}$$

According to Remark 3.1, we obtain that there exists $n_0 \in \mathbb{N}$ such that $\varphi^* > 0$ for all $n \ge n_0$. By using (3.11), one has

$$\|t_n - x^*\| \le \|d_n - x^*\|, \quad \forall n \ge n_0. \tag{3.12}$$

From the definition of d_n, we can write

$$\begin{aligned} \|d_n - x^*\| &= \|x_n + \gamma_n (x_n - x_{n-1}) - x^*\| \\ &\le \|x_n - x^*\| + \sigma_n \cdot \frac{\gamma_n}{\sigma_n} \|x_n - x_{n-1}\| . \end{aligned} \tag{3.13}$$

It follows from (3.1) and Condition (C5) that

$$\lim_{n\to\infty} \frac{\gamma_n}{\sigma_n} \|x_n - x_{n-1}\| \le \lim_{n\to\infty} \frac{\epsilon_n}{\sigma_n} = 0. \tag{3.14}$$

Thus, there exists a constant $M_1 > 0$ such that

$$\frac{\gamma_n}{\sigma_n} \|x_n - x_{n-1}\| \le M_1, \quad \forall n \ge 1. \tag{3.15}$$

Combining (3.12), (3.13), and (3.15), we find that

$$\|t_n - x^*\| \le \|d_n - x^*\| \le \|x_n - x^*\| + \sigma_n M_1, \quad \forall n \ge n_0. \tag{3.16}$$

Using (3.16), we obtain

$$\begin{aligned} \|x_{n+1} - x^*\| &= \|\sigma_n g(d_n) + (1 - \sigma_n) t_n - x^*\| \\ &\le \sigma_n \|g(d_n) - g(x^*)\| + \sigma_n \|g(x^*) - x^*\| + (1 - \sigma_n) \|t_n - x^*\| \\ &\le \sigma_n \rho \|d_n - x^*\| + \sigma_n \|g(x^*) - x^*\| + (1 - \sigma_n) \|d_n - x^*\| \\ &\le [1 - \sigma_n(1-\rho)] \|x_n - x^*\| + \sigma_n(1-\rho) \frac{\|g(x^*) - x^*\| + M_1}{1-\rho} \\ &\le \max\left\{ \|x_n - x^*\|, \frac{\|g(x^*) - x^*\| + M_1}{1-\rho} \right\} (\forall n \ge n_0) \\ &\le \cdots \le \max\left\{ \|x_{n_0} - x^*\|, \frac{\|g(x^*) - x^*\| + M_1}{1-\rho} \right\}, \end{aligned}$$

which implies that the sequence $\{x_n\}$ is bounded. So the sequences $\{d_n\}$, $\{g(d_n)\}$, $\{u_n\}$, and $\{v_n\}$ are also bounded.

Next, it follows from (3.16) that

$$\begin{aligned} \|d_n - x^*\|^2 &\le (\|x_n - x^*\| + \sigma_n M_1)^2 \\ &= \|x_n - x^*\|^2 + \sigma_n \left(2M_1 \|x_n - x^*\| + \sigma_n M_1^2\right) \\ &\le \|x_n - x^*\|^2 + \sigma_n M_2, \end{aligned} \tag{3.17}$$

where $M_2 := \sup_{n\in\mathbb{N}} \left(2M_1 \|x_n - x^*\| + \sigma_n M_1^2\right)$. Using (2.4), (3.11), and (3.17), we have

$$\begin{aligned}\|x_{n+1} - x^*\|^2 &= \|\sigma_n (g(d_n) - x^*) + (1-\sigma_n)(t_n - x^*)\|^2 \\ &\le \sigma_n \|g(d_n) - x^*\|^2 + (1-\sigma_n)\|t_n - x^*\|^2 \\ &\le \sigma_n \|g(d_n) - x^*\|^2 + \|x_n - x^*\|^2 + \sigma_n M_2 \\ &\quad - (1-\sigma_n)\varphi^* \left(\|d_n - u_n\|^2 + \|v_n - u_n\|^2\right) \\ &\quad - (1-\sigma_n)\eta_n (1-\tau-\eta_n)\|Qv_n - v_n\|^2 .\end{aligned}$$

Thus, we obtain

$$\begin{aligned}&(1-\sigma_n)\left[\varphi^* \left(\|d_n - u_n\|^2 + \|v_n - u_n\|^2\right) + \eta_n (1-\tau-\eta_n)\|Qv_n - v_n\|^2\right] \\ &\le \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 + \sigma_n \|g(d_n) - x^*\|^2 + \sigma_n M_2.\end{aligned} \tag{3.18}$$

Then, by the definition of d_n, one obtains

$$\begin{aligned}\|d_n - x^*\|^2 &= \|x_n + \gamma_n (x_n - x_{n-1}) - x^*\|^2 \\ &= \|x_n - x^*\|^2 + 2\gamma_n \langle x_n - x^*, x_n - x_{n-1}\rangle + \gamma_n^2 \|x_n - x_{n-1}\|^2 \\ &\le \|x_n - x^*\|^2 + \gamma_n \|x_n - x_{n-1}\| (2\|x_n - x^*\| + \gamma \|x_n - x_{n-1}\|) \\ &\le \|x_n - x^*\|^2 + 3M\gamma_n \|x_n - x_{n-1}\| ,\end{aligned} \tag{3.19}$$

where $M := \sup_{n\in\mathbb{N}} \{\|x_n - x^*\|, \gamma\|x_n - x_{n-1}\|\} > 0$. Combining (2.3), (2.4), (3.16), and (3.19), we have

$$\begin{aligned}&\|x_{n+1} - x^*\|^2 \\ &= \|\sigma_n g(d_n) + (1-\sigma_n) t_n - x^*\|^2 \\ &= \|\sigma_n (g(d_n) - g(x^*)) + (1-\sigma_n)(t_n - x^*) + \sigma_n (g(x^*) - x^*)\|^2 \\ &\le \|\sigma_n (g(d_n) - g(x^*)) + (1-\sigma_n)(t_n - x^*)\|^2 + 2\sigma_n \langle g(x^*) - x^*, x_{n+1} - x^*\rangle \\ &\le \sigma_n \|g(d_n) - g(x^*)\|^2 + (1-\sigma_n)\|t_n - x^*\|^2 + 2\sigma_n \langle g(x^*) - x^*, x_{n+1} - x^*\rangle \\ &\le (1-(1-\rho)\sigma_n)\|d_n - x^*\| + 2\sigma_n \langle g(x^*) - x^*, x_{n+1} - x^*\rangle \\ &\le (1-(1-\rho)\sigma_n)\|x_n - x^*\| + 2\sigma_n \langle g(x^*) - x^*, x_{n+1} - x^*\rangle + 3M\gamma_n \|x_n - x_{n-1}\| \\ &= (1-(1-\rho)\sigma_n)\|x_n - x^*\|^2 + (1-\rho)\sigma_n \Big[\frac{2}{1-\rho}\langle g(x^*) - x^*, x_{n+1} - x^*\rangle \\ &\quad + \frac{3M\gamma_n}{(1-\rho)\sigma_n}\|x_n - x_{n-1}\|\Big], \quad \forall n \ge n_0.\end{aligned} \tag{3.20}$$

Finally, we prove that the sequence $\{\|x_n - x^*\|\}$ converges to zero. Indeed, from Lemma 2.1, (3.14), and (3.20), it suffices to show that

$$\limsup_{k\to\infty} \langle g(x^*) - x^*, x_{n_k+1} - x^*\rangle \le 0$$

for every subsequence $\{\|x_{n_k} - x^*\|^2\}$ of $\{\|x_n - x^*\|^2\}$ satisfying

$$\liminf_{k\to\infty} \left(\|x_{n_k+1} - x^*\|^2 - \|x_{n_k} - x^*\|^2\right) \geq 0. \tag{3.21}$$

It follows from (3.18), (3.21), and Condition (C5) that

$$\begin{aligned}
&\limsup_{k\to\infty} \Big\{ (1-\sigma_{n_k})\,\varphi^*\left(\|d_{n_k} - u_{n_k}\|^2 + \|v_{n_k} - u_{n_k}\|^2\right) \\
&\qquad\qquad + (1-\sigma_{n_k})\,\eta_{n_k}\,(1-\tau-\eta_{n_k})\,\|Qv_{n_k} - v_{n_k}\|^2 \Big\} \\
&\leq \limsup_{k\to\infty} \left[\|x_{n_k} - x^*\|^2 - \|x_{n_k+1} - x^*\|^2 + \sigma_{n_k}\,\|g\,(x_{n_k}) - x^*\|^2 + \sigma_{n_k} M_2\right] \\
&= -\liminf_{k\to\infty} \left[\|x_{n_k+1} - x^*\|^2 - \|x_{n_k} - x^*\|^2\right] \leq 0,
\end{aligned}$$

which implies that

$$\lim_{k\to\infty} \|d_{n_k} - u_{n_k}\| = 0, \; \lim_{k\to\infty} \|v_{n_k} - Qv_{n_k}\| = 0, \text{ and } \lim_{k\to\infty} \|v_{n_k} - u_{n_k}\| = 0. \tag{3.22}$$

Therefore, we obtain $\lim_{k\to\infty} \|v_{n_k} - d_{n_k}\| = 0$. According to the definition of d_n, one has

$$\|x_{n_k} - d_{n_k}\| = \gamma_{n_k}\,\|x_{n_k} - x_{n_k-1}\| = \sigma_{n_k} \cdot \frac{\gamma_{n_k}}{\sigma_{n_k}}\,\|x_{n_k} - x_{n_k-1}\| \to 0 \text{ as } k \to \infty. \tag{3.23}$$

This together with $\lim_{k\to\infty} \|v_{n_k} - d_{n_k}\| = 0$ yields that

$$\lim_{k\to\infty} \|v_{n_k} - x_{n_k}\| = 0. \tag{3.24}$$

From $t_{n_k} = (1-\eta_{n_k})\,v_{n_k} + \eta_{n_k} Qv_{n_k}$, one sees that

$$\|t_{n_k} - v_{n_k}\| = \eta_{n_k}\,\|Qv_{n_k} - v_{n_k}\| \leq (1-\tau)\,\|Qv_{n_k} - v_{n_k}\|\,.$$

In view of (3.22), we obtain

$$\lim_{k\to\infty} \|t_{n_k} - v_{n_k}\| = 0. \tag{3.25}$$

From (3.24) and (3.25), we deduce that

$$\begin{aligned}
&\|x_{n_k+1} - x_{n_k}\| \\
&= \|\sigma_{n_k} g\,(d_{n_k}) + (1-\sigma_{n_k})\,t_{n_k} - x_{n_k}\| \\
&\leq \sigma_{n_k}\,\|g\,(d_{n_k}) - x_{n_k}\| + (1-\sigma_{n_k})\,\|t_{n_k} - x_{n_k}\| \\
&\leq \sigma_{n_k}\,\|g\,(d_{n_k}) - x_{n_k}\| + \|t_{n_k} - v_{n_k}\| + \|v_{n_k} - x_{n_k}\| \to 0 \text{ as } k \to \infty.
\end{aligned} \tag{3.26}$$

Since the sequence $\{x_{n_k}\}$ is bounded, one infers that there exists a subsequence $\{x_{n_{k_j}}\}$ of $\{x_{n_k}\}$ such that $x_{n_{k_j}} \rightharpoonup z$. Moreover,

$$\begin{aligned}
\limsup_{k\to\infty} \langle g(x^*) - x^*, x_{n_k} - x^*\rangle &= \lim_{j\to\infty} \langle g(x^*) - x^*, x_{n_{k_j}} - x^*\rangle \\
&= \langle g(x^*) - x^*, z - x^*\rangle.
\end{aligned} \tag{3.27}$$

From (3.23), one obtains $d_{n_k} \rightharpoonup z$. Combining $\lim_{k\to\infty} \|d_{n_k} - u_{n_k}\| = 0$, $\lim_{n\to\infty} \lambda_n = \lambda$, and Lemma 3.3, one concludes that $z \in \Omega$. It follows from (3.24) that $v_{n_k} \rightharpoonup z$, which together with $\lim_{k\to\infty} \|v_{n_k} - Qv_{n_k}\| = 0$ and the demiclosedness of $(I-Q)$, we obtain that $z \in \Psi$. Thus, $z \in \Psi \cap \Omega$. Combining (2.1), (3.27), the definition of x^*, and $z \in \Psi \cap \Omega$, we obtain

$$\limsup_{k\to\infty} \langle g(x^*) - x^*, x_{n_k} - x^* \rangle = \langle g(x^*) - x^*, z - x^* \rangle \leq 0, \tag{3.28}$$

which together with (3.26) yields that

$$\begin{aligned} &\limsup_{k\to\infty} \langle g(x^*) - x^*, x_{n_k+1} - x^* \rangle \\ &\leq \limsup_{k\to\infty} \langle g(x^*) - x^*, x_{n_k+1} - x_{n_k} \rangle + \limsup_{k\to\infty} \langle g(x^*) - x^*, x_{n_k} - x^* \rangle \\ &= \langle g(x^*) - x^*, z - x^* \rangle \leq 0. \end{aligned} \tag{3.29}$$

Combining (3.14), (3.20), and (3.29), in the light of Lemma 2.1, we observe that $x_n \to x^*$ as $n \to \infty$. This completes the proof of Theorem 3.1. □

Next, we propose another modified version of the subgradient extragradient algorithm, where the main difference lies in the computation of the sequences $\{u_n\}$ and $\{v_n\}$. More precisely, the details of our second method are described in Algorithm 3.2 below.

Algorithm 3.2

Initialization: Take $\gamma > 0$, $\lambda_1 > 0$, $\varphi \in (1/(2-\delta), 1/\delta)$, $\delta \in (0,1)$. Select $\{\epsilon_n\}$, $\{\sigma_n\}$, $\{\eta_n\}$ and $\{\xi_n\}$ to satisfy Condition (C5). Let $x_0, x_1 \in \mathcal{H}$.
Iterative Steps: Given the iterates x_{n-1} and x_n $(n \geq 1)$, calculate x_{n+1} as follows:
Step 1. Compute $d_n = x_n + \gamma_n (x_n - x_{n-1})$, where γ_n is defined in (3.1).
Step 2. Compute $u_n = Proj_{\mathcal{C}} (d_n - \varphi\lambda_n B d_n)$.
Step 3. Compute $v_n = Proj_{T_n} (d_n - \lambda_n B u_n)$, where the half-space T_n is defined as

$$T_n = \{x \in \mathcal{H} \mid \langle d_n - \varphi\lambda_n B d_n - u_n, x - u_n \rangle \leq 0\}.$$

Step 4. Compute $x_{n+1} = \sigma_n g(d_n) + (1-\sigma_n)[(1-\eta_n) v_n + \eta_n Q v_n]$, and update λ_{n+1} by (3.2).
Set $n \leftarrow n+1$ and go to **Step 1**.

The following lemma plays a crucial role in the convergence analysis of Algorithm 3.2.

Lemma 3.4. *Assume that Condition (C2) holds. Let $\{v_n\}$ be a sequence generated by Algorithm 3.2. Then, for all $p \in \Omega$,*

$$\|v_n - p\|^2 \leq \|d_n - p\|^2 - \varphi^\dagger \left(\|d_n - u_n\|^2 + \|v_n - u_n\|^2 \right),$$

where $\varphi^\dagger = 2 - \frac{1}{\varphi} - \frac{\delta\lambda_n}{\lambda_{n+1}}$ if $\varphi \in (1/(2-\delta), 1]$ and $\varphi^\dagger = \frac{1}{\varphi} - \frac{\delta\lambda_n}{\lambda_{n+1}}$ if $\varphi \in (1, 1/\delta)$.

Proof. From (3.3) and (3.4), we obtain

$$\|v_n - p\|^2 \leq \|d_n - p\|^2 - \|d_n - v_n\|^2 - 2\langle \lambda_n B u_n, v_n - u_n \rangle. \tag{3.30}$$

Now we estimate $2\langle \lambda_n B u_n, v_n - u_n \rangle$. Note that

$$-\|d_n - v_n\|^2 = -\|d_n - u_n\|^2 - \|u_n - v_n\|^2 + 2\langle d_n - u_n, v_n - u_n \rangle. \tag{3.31}$$

One can show that

$$\begin{aligned}&\langle d_n - u_n, v_n - u_n\rangle \\ &= \langle d_n - u_n - \varphi\lambda_n B d_n + \varphi\lambda_n B d_n - \varphi\lambda_n B u_n + \varphi\lambda_n B u_n, v_n - u_n\rangle \\ &= \langle d_n - \varphi\lambda_n B d_n - u_n, v_n - u_n\rangle + \varphi\lambda_n \langle B d_n - B u_n, v_n - u_n\rangle \\ &\quad + \langle \varphi\lambda_n B u_n, v_n - u_n\rangle .\end{aligned} \tag{3.32}$$

Since $v_n \in T_n$, one has

$$\langle d_n - \varphi\lambda_n B d_n - u_n, v_n - u_n\rangle \leq 0. \tag{3.33}$$

Substituting (3.8), (3.32), and (3.33) into (3.31), we have

$$-\|d_n - v_n\|^2 \leq -\left(1 - \frac{\varphi\delta\lambda_n}{\lambda_{n+1}}\right)\left(\|d_n - u_n\|^2 + \|v_n - u_n\|^2\right) + 2\varphi\langle \lambda_n B u_n, v_n - u_n\rangle ,$$

which implies that

$$\begin{aligned}-2\langle \lambda_n B u_n, v_n - u_n\rangle \leq &-\left(\frac{1}{\varphi} - \frac{\delta\lambda_n}{\lambda_{n+1}}\right)\left(\|d_n - u_n\|^2 + \|v_n - u_n\|^2\right) \\ &+ \frac{1}{\varphi}\|d_n - v_n\|^2 .\end{aligned} \tag{3.34}$$

Combining (3.30) and (3.34), we conclude that

$$\begin{aligned}\|v_n - p\|^2 \leq \|d_n - p\|^2 &- \left(\frac{1}{\varphi} - \frac{\delta\lambda_n}{\lambda_{n+1}}\right)\left(\|d_n - u_n\|^2 + \|v_n - u_n\|^2\right) \\ &- \left(1 - \frac{1}{\varphi}\right)\|d_n - v_n\|^2 .\end{aligned} \tag{3.35}$$

Note that

$$\|d_n - v_n\|^2 \leq 2\left(\|d_n - u_n\|^2 + \|v_n - u_n\|^2\right),$$

which yields that

$$-\left(1 - \frac{1}{\varphi}\right)\|d_n - v_n\|^2 \leq -2\left(1 - \frac{1}{\varphi}\right)\left(\|d_n - u_n\|^2 + \|v_n - u_n\|^2\right), \quad \forall \varphi \in (0, 1].$$

This together with (3.35) obtains

$$\|v_n - p\|^2 \leq \|d_n - p\|^2 - \left(2 - \frac{1}{\varphi} - \frac{\delta\lambda_n}{\lambda_{n+1}}\right)\left(\|d_n - u_n\|^2 + \|v_n - u_n\|^2\right), \quad \forall \varphi \in (0, 1].$$

In addition, if $\varphi > 1$, then we have

$$\|v_n - p\|^2 \leq \|d_n - p\|^2 - \left(\frac{1}{\varphi} - \frac{\delta\lambda_n}{\lambda_{n+1}}\right)\left(\|d_n - u_n\|^2 + \|v_n - u_n\|^2\right), \quad \forall \varphi > 1.$$

The proof is completed. □

Remark 3.2. From Lemma 3.1 and the assumptions of the parameters δ and φ (i.e., $\delta \in (0, 1)$ and $\varphi \in (1/(2-\delta), 1/\delta)$), we obtain that $\varphi^{\dagger} > 0$ for all $n \geq n_1$ in Lemma 3.4 always holds.

Theorem 3.2. *Assume that Conditions (C1)–(C5) hold. Then the sequence* $\{x_n\}$ *generated by Algorithm 3.2 converges strongly to* $x^* \in \Psi \cap \Omega$, *where* $x^* = Proj_{\Psi\cap\Omega}(g(x^*))$.

Proof. We can use Lemma 3.4 to replace the necessary conclusions about Lemma 3.2 in the proof of Theorem 3.1. We omit the details of the proof to avoid repetition. □

In particular, considering that Q in the proposed Algorithms 3.1 and 3.2 of this paper is an identity operator, i.e., $Q = I$, we can obtain two new modified inertial subgradient extragradient algorithms to solve the (VIP). More precisely, we have the following corollary.

Corollary 3.1. *Assume that Conditions (C2), (C4), and (C5) hold, and the solution set* Ω *of the variational inequality problem* (VIP) *is nonempty. Let* $x_0, x_1 \in \mathcal{H}$ *and the sequence* $\{x_n\}$ *be generated by*

$$\begin{cases} d_n = x_n + \gamma_n (x_n - x_{n-1}), \\ u_n = Proj_{\mathcal{C}} (d_n - \lambda_n B d_n), \\ v_n = Proj_{T_n} (d_n - \varphi\lambda_n B u_n), \\ T_n = \{x \in \mathcal{H} \mid \langle d_n - \lambda_n B d_n - u_n, x - u_n\rangle \leq 0\}, \\ x_{n+1} = \sigma_n g(d_n) + (1 - \sigma_n) v_n, \end{cases} \tag{3.36}$$

or

$$\begin{cases} d_n = x_n + \gamma_n (x_n - x_{n-1}), \\ u_n = Proj_{\mathcal{C}} (d_n - \varphi\lambda_n B d_n), \\ v_n = Proj_{T_n} (d_n - \lambda_n B u_n), \\ T_n = \{x \in \mathcal{H} \mid \langle d_n - \varphi\lambda_n B d_n - u_n, x - u_n\rangle \leq 0\}, \\ x_{n+1} = \sigma_n g(d_n) + (1 - \sigma_n) v_n, \end{cases} \tag{3.37}$$

where γ_n *and* λ_n *are defined in* (3.1) *and* (3.2), *respectively. Then the iterative sequence* $\{x_n\}$ *generated by Algorithm* (3.36) *(or Algorithm* (3.37)*) converges strongly to* $x^* \in \Omega$, *where* $x^* = Proj_{\Omega}(g(x^*))$.

Remark 3.3. We have the following comments for the proposed algorithms.

(1) The suggested methods 3.1 and 3.2 are equivalent when $\varphi = 1$.
(2) The sequences generated by the two algorithms proposed in this paper in the infinite-dimensional Hilbert space converge strongly to the solution of variational inequalities and fixed points. In contrast, the results in the literature [18, 30, 36] can only obtain weak convergence.
(3) It is known that computing the projection on a general nonempty closed convex set is not an easy task. Our two algorithms require calculating the projection on the feasible set only once in each iteration, which improves the methods proposed in the literature [18, 4, 30, 36] that need to compute the projection on the feasible set at least twice in each iteration.
(4) To improve the convergence speed of the algorithm, our two iterative schemes use different step sizes in each iteration and employ a non-monotonic step size criterion. These changes allow them to improve the algorithms in the

literature [18, 7, 25, 16, 4, 30, 5, 36, 31, 29] that use Armijo-type step sizes (or fixed step sizes, or non-increasing step sizes) and the algorithms in the literature [31, 29] that employ the same step size in each iteration. On the other hand, our algorithms embed inertial terms which improve the convergence speed of the non-inertial algorithms. For more details on these findings, see the numerical experiments in Section 4.

(5) Notice that the convergence condition of our algorithms requires that the variational inequality operator B is pseudomonotone rather than monotone. In other words, the two methods presented in this paper can find common solutions of pseudomonotone variational inequalities and fixed points involving a demicontractive mapping, which improves on the algorithms in the literature [18, 7, 16, 25, 4, 30, 36, 31, 29] where the variational inequality operator B is claimed to be monotone and the fixed point operator Q is required to be nonexpansive (or quasi-nonexpansive, or demicontractive).

Based on the above findings, the algorithms proposed in this paper are efficient and improve many known results in the field.

4. Numerical examples and applications

In this section, we provide some numerical examples to illustrate the numerical behavior of the proposed algorithms and also to compare them with some existing strongly convergent algorithms, which including the inertial-based viscosity-type subgradient extragradient method and the viscosity-type Tseng's extragradient method proposed by Tan, Zhou, and Li [29] (shortly, TZL Alg. 3.1 and TZL Alg. 3.2), and the general viscosity-type subgradient extragradient method and viscosity-type Tseng's extragradient method introduced by Thong and Hieu [31] (shortly, TH Alg. 3.1 and TH Alg. 3.2). In the next numerical experiments, we use "Time" to denote the running time of algorithms in seconds. All the programs are performed in MATLAB 2018a on a Intel(R) Core(TM) i5-8265U CPU @ 1.60GHz computer with RAM 8.00 GB.

4.1. Theoretical examples.

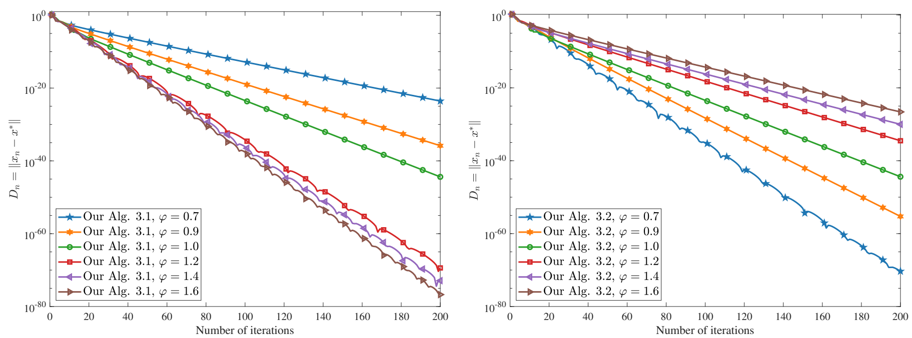
Example 4.1. Assume the nonlinear operator $B : \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$B(x, y) = (x + y + \sin x; -x + y + \sin y)$$

and the feasible set $\mathcal{C}$ is a box defined by $\mathcal{C} = [-1, 1] \times [-1, 1]$. It is easy to know that B is monotone and Lipschitz continuous with the constant $L = 3$. Let E be a 2×2 matrix, i.e.,

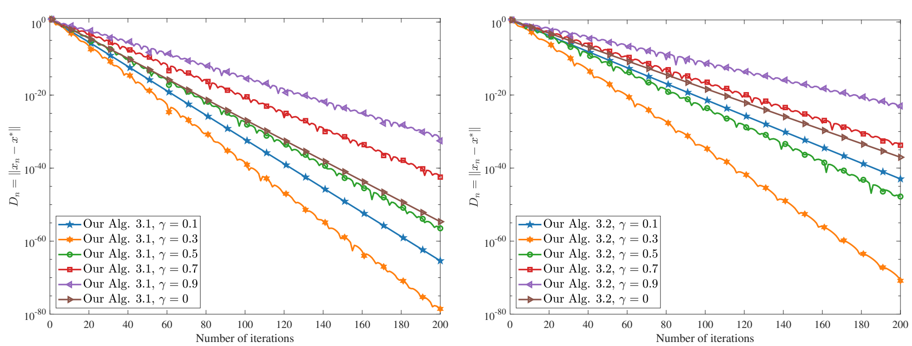
$$E = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}.$$

We consider the mapping $Q : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $Qz = \|E\|^{-1} Ez$, where $z = (x, y)^{\mathsf{T}}$. It is obvious to see that Q is 0-demicontractive and thus $\tau = 0$. The solution of the problem (VIP-FPP) is $x^* = (0, 0)^{\mathsf{T}}$. For all algorithms in the numerical experiment, we unify the parameters as $\sigma_n = 1/(n+1)$, $\eta_n = n/(2n+1)$, $\lambda_1 = 0.4$, $\delta = 0.2$, and $g(x) = 0.5x$. Take $\gamma = 0.3$, $\epsilon_n = 100/(n+1)^2$, $\xi_n = 1/(n+1)^5$ in our proposed algorithms and Tan et al. Alg. 3.1 and Tan et al. Alg. 3.2. In this experiment, the maximum number of iterations 200 as a common stopping criterion. We use $D_n = \|x_n - x^*\|$ to measure the n-th iteration error for all algorithms. First, we plot the convergence performance of the proposed algorithms with different parameters φ for initial values $x_0 = x_1 = 10\text{rand}(2,1)$ in Figure 1 to demonstrate the effect of the parameter φ on our algorithms. We then show the performance of our algorithms with different inertial parameters in Figure 2. According to Figure 1, we choose $\varphi = 1.6$ and $\varphi = 0.7$ for the proposed Algorithms 3.1 and 3.2, respectively. The numerical results of the suggested algorithms with some known schemes at different initial values $x_0 = x_1$ are shown in Table 1.



(a) Our Alg. 3.1 (b) Our Alg. 3.2

FIGURE 1. Computational performance of the proposed algorithms with different φ in Example 4.1



(a) Our Alg. 3.1 (b) Our Alg. 3.2

FIGURE 2. Computational performance of the proposed algorithms with different γ in Example 4.1

Table 1. Numerical results for all algorithms with different initial values in Example 4.1

Algorithms	$x_1 = 5\text{rand}(2,1)$		$x_1 = 10\text{rand}(2,1)$		$x_1 = 20\text{rand}(2,1)$		$x_1 = 50\text{rand}(2,1)$	
	D_n	Time	D_n	Time	D_n	Time	D_n	Time
Our Alg. 3.1	6.17E-72	0.0096	9.86E-81	0.0118	7.17E-75	0.0109	4.50E-79	0.0100
Our Alg. 3.2	1.11E-60	0.0108	3.62E-73	0.0137	2.98E-71	0.0100	1.21E-74	0.0130
TZL Alg. 3.1	6.66E-27	0.0095	4.40E-27	0.0146	2.45E-27	0.0103	1.80E-26	0.0113
TZL Alg. 3.2	3.97E-27	0.0076	3.36E-27	0.0079	4.19E-27	0.0113	1.84E-26	0.0099
TH Alg. 3.1	3.84E-17	0.0180	9.12E-18	0.0179	5.28E-17	0.0182	4.25E-17	0.0179
TH Alg. 3.2	3.69E-17	0.0068	1.18E-17	0.0069	5.30E-17	0.0074	4.76E-17	0.0069

Example 4.2. We consider our problem in the infinite-dimensional Hilbert space $\mathcal{H} = L^2([0,1])$ with inner product

$$\langle x, y\rangle = \int_0^1 x(t)y(t)\mathrm{d}t, \quad \forall x, y \in \mathcal{H}$$

and norm

$$\|x\| = \Big(\int_0^1 |x(t)|^2\mathrm{d}t\Big)^{1/2}, \quad \forall x \in \mathcal{H}.$$

Let the feasible set be the unit ball $\mathcal{C} = \{x \in \mathcal{H} : \|x\| \leq 1\}$. Define an operator $B : \mathcal{C} \to \mathcal{H}$ by

$$(Bx)(t) = \int_0^1 (x(t) - G(t,s)a(x(s)))\,\mathrm{d}s + h(t), \quad t \in [0,1],\, x \in \mathcal{C},$$

where

$$G(t,s) = \frac{2tse^{t+s}}{e\sqrt{\mathrm{e}^2-1}}, \quad a(x) = \cos x, \quad h(t) = \frac{2t\mathrm{e}^t}{e\sqrt{\mathrm{e}^2-1}}.$$

It is known that B is monotone and L-Lipschitz continuous with $L = 2$ (see [13, Example 2]). The mapping $Q : L^2([0,1]) \to L^2([0,1])$ is of the form

$$(Qx)(t) = \int_0^1 tx(s)\,\mathrm{d}s, \ \ t \in [0,1].$$

A simple calculation shows that Q is 0-demicontractive. The solution of the problem (VIP-FPP) is $x^*(t) = 0$. We use $D_n = \|x_n(t) - x^*(t)\|$ to measure the error of the n-th iteration step for all algorithms. The maximum number of iterations 50 is used as a common stopping criterion. The parameters of all algorithms remain the same as in Example 4.1. Table 2 shows the numerical behavior of D_n for the suggested algorithms with different parameters φ at the initial value $x_0 = x_1 = t^2$. Table 3 records the numerical results of each algorithm at four starting points.

Remark 4.1. From Examples 4.1 and 4.2, we have the following observations.

(1) It follows from Figure 1 and Table 2 that our Algorithms 3.1 and 3.2 have different performance with different parameters φ. Specifically, the proposed Algorithm 3.1 has better performance for $\varphi > 1$, while the suggested Algorithm 3.2 shows better behavior for $\varphi < 1$. Note that this is only a preliminary

TABLE 2. Computational behavior of D_n for the proposed algorithms with different φ at the initial values $x_0 = x_1 = t^2$

Parameter φ	$\varphi = 0.7$	$\varphi = 0.9$	$\varphi = 1.0$	$\varphi = 1.2$	$\varphi = 1.4$	$\varphi = 1.6$
Our Alg. 3.1	1.21E-18	6.27E-19	9.22E-20	3.47E-20	4.66E-21	1.46E-22
Our Alg. 3.2	3.41E-21	1.66E-19	9.22E-20	6.38E-19	1.10E-19	7.41E-19

TABLE 3. Numerical results of all algorithms at different initial values in Example 4.2

Algorithms	$x_0 = x_1 = 10e^t$		$x_0 = x_1 = 10\cos(t)$		$x_0 = x_1 = 10\log(t)$		$x_0 = x_1 = 10t^2$	
	D_n	Time	D_n	Time	D_n	Time	D_n	Time
Our Alg. 3.1	1.23E-20	30.4762	3.03E-21	30.1367	3.65E-21	29.6645	1.24E-21	39.3388
Our Alg. 3.2	8.11E-20	30.2945	3.72E-20	29.6247	5.00E-20	29.4172	1.10E-20	36.7991
TZL Alg. 3.1	1.75E-18	31.0261	2.75E-18	28.1855	2.25E-18	27.6887	2.39E-19	36.2258
TZL Alg. 3.2	8.27E-18	29.6310	2.82E-18	27.5188	3.04E-18	27.0733	2.19E-19	32.0952
TH Alg. 3.1	1.18E-09	28.3019	5.24E-10	27.7527	1.26E-10	27.5280	9.14E-10	24.9434
TH Alg. 3.2	1.90E-09	26.1001	1.36E-09	25.3795	2.30E-10	24.9872	1.05E-09	22.9963

conclusion from Examples 4.1 and 4.2. In order to obtain better results for the proposed algorithms, it is suggested that different parameters φ need to be tried in practical applications.

(2) It can be seen in Figure 2 that our two inertial algorithms converge faster when choosing the appropriate inertial parameters γ than their corresponding algorithms without added inertial terms. However, since the sequences generated by our inertial algorithms do not have fejér monotonicity, this can result in inertial algorithms sometimes not converging faster than non-inertial algorithms as illustrated in Figure 2.

(3) Note that the proposed iterative schemes have a competitive advantage over the existing algorithms in [31, 29], especially in terms of accuracy (cf. Tables 1 and 3).

(4) In Example 4.1, it is obvious that our algorithms have a faster convergence speed and higher accuracy than the compared ones. However, for Example 4.2 in the infinite-dimensional Hilbert space, a slightly longer elapsed time is required to guarantee a higher accuracy, which can be explained by the fact that our algorithms require an updated inertial parameter and a better step size in each iteration compared to TH Alg. 3.1 and TH Alg. 3.2.

(5) In our experiments, it should be noted that we chose a maximum number of iterations of 200 and 50, respectively, which would require more iterations if higher accuracy requirements are to be met in practical applications.

4.2. **Applications to optimal control problems.** In this subsection, we use the proposed Algorithms (3.36) and (3.37) to solve the variational inequality problem (VIP) that appears in optimal control problems. The mathematical description of the optimal control problem can be seen in detail from the literature [32, 14], and we know that the optimal control problem can be transformed into a variational inequality problem. We first use the classical Euler discretization method to decompose the optimal control problem and then apply the proposed algorithms to solve the variational inequality problem corresponding to the discretized version of the problem (see [32] for more details).

Next, we illustrate the computational performance of the proposed Algorithms (3.36) and (3.37) with the schemes in the literature [31, 29] by solving two classical mathematical examples. We set $N = 100$, $\sigma_n = 10^{-4}/(n+1)$, $\lambda_1 = 0.4$, $\delta = 0.2$, and $g(x) = 0.1x$ for all algorithms. Choose $\gamma = 0.3$, $\epsilon_n = 100/(n+1)^2$, and $\xi_n = 1/(n+1)^2$ for the suggested Algorithms (3.36) and (3.37) and TZL Alg. 3.1 [29] and TZL Alg. 3.2 [29]. Take $\varphi = 1.6$ and $\varphi = 0.7$ for the suggested Algorithms (3.36) and (3.37), respectively. The initial controls $p_0(t) = p_1(t)$ are randomly generated in $[0, 1]$. The stopping criterion for all algorithms is $D_n = \|d_n - u_n\| \leq 10^{-4}$.

Example 4.3. (Control of a harmonic oscillator, see [19])

$$\begin{array}{ll} \text{minimize} & x_2(3\pi) \\ \text{subject to} & \dot{x}_1(t) = x_2(t), \\ & \dot{x}_2(t) = -x_1(t) + p(t), \quad \forall t \in [0, 3\pi], \\ & x(0) = 0, \\ & p(t) \in [-1, 1]. \end{array}$$

The exact optimal control of Example 4.3 is known:

$$p^*(t) = \begin{cases} 1, \text{ if } t \in [0, \pi/2) \cup (3\pi/2, 5\pi/2), \\ -1, \text{ if } t \in (\pi/2, 3\pi/2) \cup (5\pi/2, 3\pi]. \end{cases}$$

Consider Example 4.3 of the optimal control problem with a linear terminal function, our proposed Algorithm (3.36) and Algorithm (3.37) perform 34 and 52 iterations, respectively, to reach the stopping condition $D_n = \|d_n - u_n\| \leq 10^{-4}$, which take 0.0338 and 0.0321 seconds, respectively. The approximate optimal control and the corresponding trajectories of the proposed Algorithm (3.36) for Example 4.3 are shown in Figure 3.

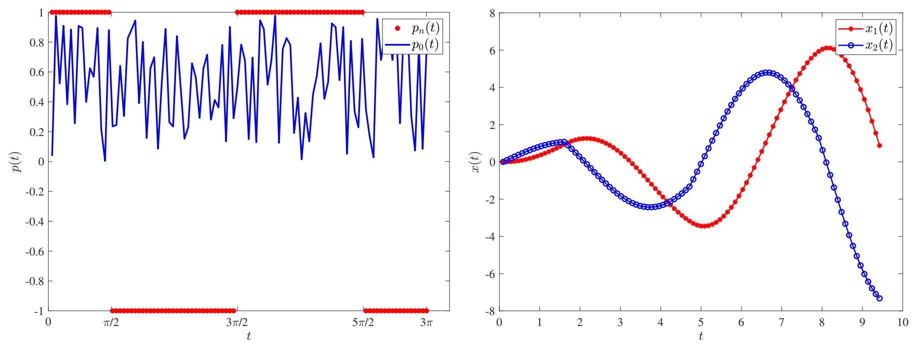
Example 4.4. (see [3])

$$\begin{array}{ll} \text{minimize} & -x_1(2) + (x_2(2))^2 \\ \text{subject to} & \dot{x}_1(t) = x_2(t), \\ & \dot{x}_2(t) = p(t), \quad \forall t \in [0, 2], \\ & x_1(0) = 0, \quad x_2(0) = 0, \\ & p(t) \in [-1, 1]. \end{array}$$

The exact optimal control of Example 4.4 is

$$p^*(t) = \begin{cases} 1, \text{ if } t \in [0, 1.2), \\ -1, \text{ if } t \in (1.2, 2]. \end{cases}$$

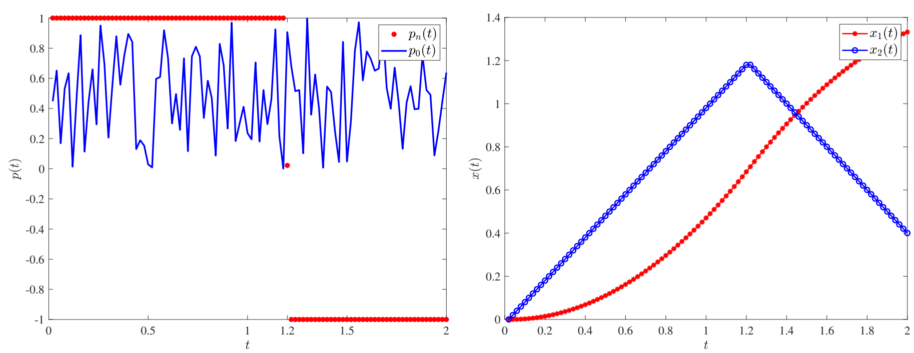
Considering Example 4.4 of the optimal control problem with a nonlinear terminal function, our proposed Algorithm (3.36) and Algorithm (3.37) perform 650 and 998 iterations, respectively, to reach the stopping condition $D_n = \|d_n - u_n\| \leq 10^{-4}$,



(a) Initial and optimal controls (b) Optimal trajectories

FIGURE 3. Numerical behavior of the proposed Algorithm (3.36) for Example 4.3

which take 0.2605 and 0.4008 seconds, respectively. The approximate optimal control and the corresponding trajectories of the suggested Algorithm (3.37) for Example 4.4 are shown in Figure 4. The numerical results of all algorithms for Example 4.3 and Example 4.4 are given in Table 4. Furthermore, we also test the numerical performance of the suggested Algorithms (3.36) and (3.37) with different parameters φ in Example 4.4, as shown in Figure 5 and Table 5.



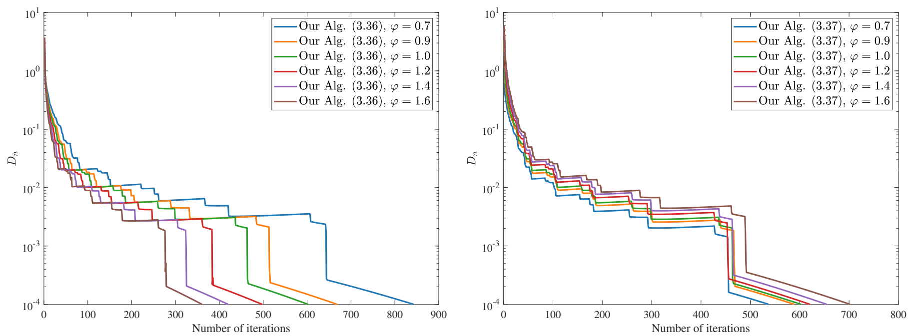
(a) Initial and optimal controls (b) Optimal trajectories

FIGURE 4. Numerical behavior of the proposed Algorithm (3.37) for Example 4.4

Remark 4.2. As shown in Examples 4.3 and 4.4, it can be observed that the algorithms proposed in this paper can be used to solve optimal control problems (cf. Figures 3 and 4). Moreover, our Algorithms (3.36) and (3.37) converge faster than the comparison methods in terms of number of iterations and execution CPU time (cf. Table 4). The results of Example 4.4 in Table 4 show that our algorithms with inertial terms are significantly faster than the algorithms without inertial terms introduced

Table 4. Numerical results of all algorithms for Examples 4.3 and 4.4

Algorithms	Example 4.3			Example 4.4		
	Iter.	D_n	Time	Iter.	D_n	Time
Our Alg. (3.36)	34	2.6230E-05	0.0338	650	9.8953E-05	0.2605
Our Alg. (3.37)	52	1.7206E-05	0.0321	998	9.9522E-05	0.4008
TZL Alg. 3.1	53	1.6883E-05	0.0289	1837	9.9787E-05	0.6634
TZL Alg. 3.2	53	1.6883E-05	0.0247	1854	9.9970E-05	0.6084
TH Alg. 3.1	73	1.2329E-05	0.0423	2239	9.9808E-05	0.8534
TH Alg. 3.2	73	1.2329E-05	0.0513	2248	9.9853E-05	0.7461



(a) Our Alg. (3.36) (b) Our Alg. (3.37)

Figure 5. Computational performance of the proposed algorithms with different φ in Example 4.4

Table 5. Numerical results of our Algorithms (3.36) and (3.37) with different φ in Example 4.4

Parameter φ	Our Alg. (3.36)			Our Alg. (3.37)		
	Iter.	D_n	Time	Iter.	D_n	Time
$\varphi = 0.7$	843	9.9488E-05	0.4261	536	9.9766E-05	0.1785
$\varphi = 0.9$	668	9.9751E-05	0.2805	588	9.9941E-05	0.1968
$\varphi = 1.0$	602	9.9524E-05	0.2177	602	9.9524E-05	0.2080
$\varphi = 1.2$	497	9.9802E-05	0.2006	620	9.9593E-05	0.2145
$\varphi = 1.4$	419	9.9984E-05	0.1615	654	9.9542E-05	0.2239
$\varphi = 1.6$	360	9.9582E-05	0.1269	701	9.9762E-05	0.2343

by Thong and Hieu [31] and the inertial-type algorithms with the same step size per iteration proposed by Tan et al. [29]. This indicates that the proposed methods are more efficient and outperform the implementation of the other algorithms in [31, 29].

On the other hand, the information in Figure 5 and Table 5 again verifies that the parameter φ plays an important role in the algorithms proposed in this paper.

4.3. **Applications to signal processing problems.** In many real-world scenarios, signals may be distorted during acquisition, transmission, or storage due to various factors such as noise interference, measurement errors, limited sensor functionality, or data loss. Signal recovery is a fundamental problem in signal processing. It involves the task of reconstructing an original signal from its degraded or corrupted version, typically affected by noise, distortions, or missing data. Signal recovery has applications in various fields such as image processing, audio enhancement, communication systems, and medical imaging. It plays a vital role in improving the quality, accuracy, and reliability of signals in different fields, and ultimately in better analysis, interpretation, and decision-making based on the recovered signals.

Let us consider the signal processing problem, which can be represented by the following model:

$$\mathbf{y} = \mathbf{A}\mathbf{x} + \mathbf{e}. \tag{4.1}$$

In this model, $\mathbf{x}$ represents the original signal, which is a vector in $\mathbb{R}^n$ with k non-zero elements. The observed signal, denoted by $\mathbf{y}$, is a vector in $\mathbb{R}^m$ and contains noise. The bounded linear operator $\mathbf{A}$ is a matrix of size $m \times n$, and $\mathbf{e}$ represents the noisy observation, which is a vector in $\mathbb{R}^m$. This model captures the relationship between the original signal $\mathbf{x}$ and the observed signal $\mathbf{y}$ in the presence of noise. The linear operator $\mathbf{A}$ defines how the original signal is transformed or mapped to the observed signal. The term $\mathbf{e}$ represents the noise or disturbance present in the observation. By understanding and analyzing this signal processing model, we can develop algorithms to estimate or reconstruct the original signal $\mathbf{x}$ from the observed signal $\mathbf{y}$.

In signal processing, it is often desirable to recover signals that have a sparse representation, meaning that they possess a small number of significant non-zero coefficients. The LASSO (Least Absolute Shrinkage and Selection Operator) model addresses this problem by adding a penalty term to the ordinary least squares objective function. Mathematically, the LASSO model can be formulated as follows:

$$\hat{\mathbf{x}} = \underset{\mathbf{x}\in\mathbb{R}^n}{\arg\min} \left(\|\mathbf{y} - \mathbf{A}\mathbf{x}\|_2^2 + \lambda \|\mathbf{x}\|_1 \right), \tag{4.2}$$

where λ is a regularization parameter that controls the trade-off between data fidelity and sparsity. The first term in the objective function measures the discrepancy between the observed signal and the reconstructed signal, while the second term introduces a penalty on the ℓ_1 norm of the estimated signal, encouraging sparsity. To solve the unconstrained optimization problem (4.2), we can convert to solve the following constrained problem

$$\min_{\mathbf{x}\in\mathbb{R}^n} f(\mathbf{x}) := \frac{1}{2}\|\mathbf{A}\mathbf{x} - \mathbf{y}\|_2^2 \quad \text{subject to } \|\mathbf{x}\|_1 \le t, \quad t > 0. \tag{4.3}$$

Notice that the above problem (4.3) can be reduced to the following variational inequality problem

$$\text{find } \hat{\mathbf{x}} \in \mathcal{C} \text{ such that } \nabla f\left(\hat{\mathbf{x}}\right)^{\mathsf{T}} \left(\mathbf{x} - \hat{\mathbf{x}}\right) \ge 0, \quad \forall \hat{\mathbf{x}} \in \mathcal{C}.$$

where $\nabla f(\mathbf{x}) = \mathbf{A}^{\mathsf{T}}(\mathbf{A}\mathbf{x} - \mathbf{y})$ and $\mathcal{C} = \{\mathbf{x} \in \mathbb{R}^n \mid \|\mathbf{x}\|_1 \le t\}$.

Now we can use the algorithms proposed in this paper to solve the problem (4.1). We pick $\sigma_n = 1/(n+1)$, $\lambda_1 = 0.1$, $\delta = 0.2$, $g(x) = 0.5x$, $\gamma = 0.3$, and $\epsilon_n = 100/(n+1)^2$ in our Algorithms (3.36) and (3.37) and TZL Alg. 3.1 [29]. Choose $\varphi = 1.6$ and $\xi_n = 1/(n+1)^5$ in our Algorithm (3.36). Take $\varphi = 0.7$ and $\xi_n = 1/(n+1)^5$ in our Algorithm (3.37). In equation (4.3) we select $t = k$ for all algorithms.

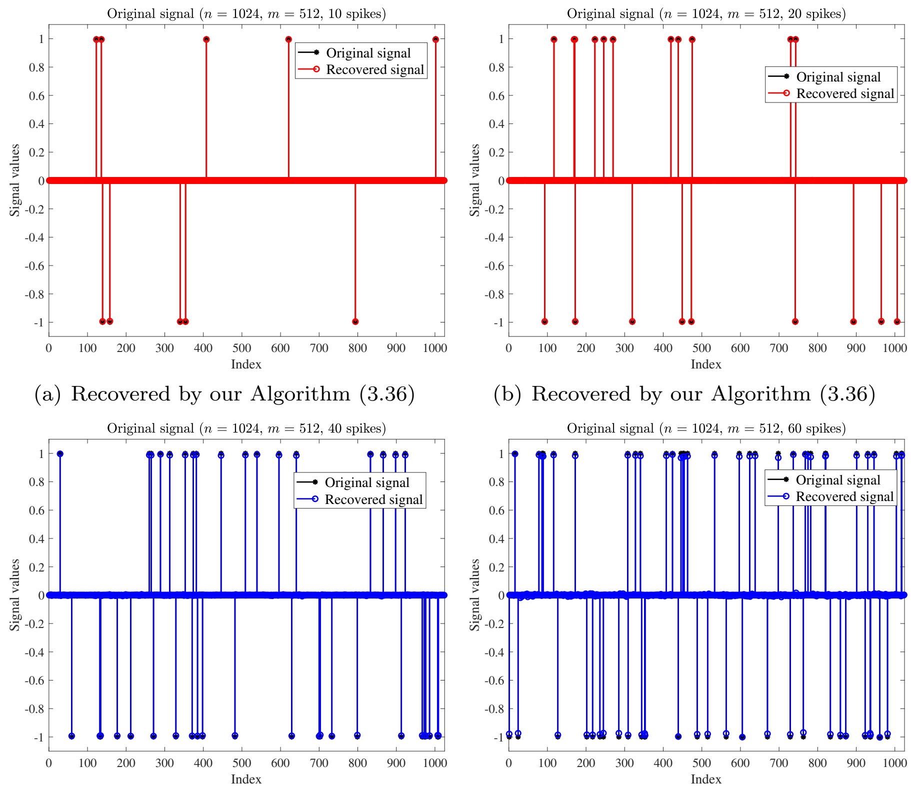
Example 4.5. The original signal $\mathbf{x} \in \mathbb{R}^n$ containing $k\,(k \ll n)$ non-zero values is randomly generated by ± 1 spikes. $\mathbf{A} : \mathbb{R}^{m\times n}$ is the matrix created by the standard normal distribution. $\mathbf{e}$ is the vector created by the function `0.001*randn(m,1)` in MATLAB. We employ the mean squared error (MSE) metric, defined as MSE $= \frac{1}{n}\|\hat{\mathbf{x}} - \mathbf{x}\|^2$, to assess the accuracy of signal recovery algorithms by comparing the recovered signal $\hat{\mathbf{x}}$ with the original signal $\mathbf{x}$. The recovery process for all algorithms begins with initial signals $\mathbf{x}_0 = \mathbf{x}_1 = \mathbf{0}$ and terminates when a maximum of 1000 iterations is reached. For our test, we set $n = 1024, m = 512$, and consider different levels of sparsity $k = \{10, 20, 40, 60\}$. Table 6 shows the computation time in seconds required for all algorithms to reach the stopping criterion at various sparsity levels, as well as the final iteration error. The recovered results obtained by our algorithms for different sparse signals are displayed in Figure 6. From Table 6 and Figure 6, it can be seen that the proposed algorithms can handle the signal recovery problem well at different sparsity and they perform better than the algorithm of Tan et al. [29].

Table 6. Numerical results for all algorithms at different sparsity k in Example 4.5 ($n = 1024, m = 512$)

Algorithms	$k = 10$		$k = 20$		$k = 40$		$k = 60$	
	Time	MSE	Time	MSE	Time	MSE	Time	MSE
Our Alg. (3.36)	1.3592	3.49E-07	1.3309	7.04E-07	1.4795	4.08E-06	1.4952	1.62E-05
Our Alg. (3.37)	1.5445	4.43E-07	1.3957	9.66E-07	1.3704	5.21E-06	1.3224	2.64E-05
TZL Alg. 3.1	1.2821	9.36E-03	1.3626	2.56E-02	1.1993	8.13E-02	1.2919	1.22E-01

5. Conclusions

The paper proposed two improved viscosity-type inertial subgradient extragradient algorithms for finding the common solutions of the pseudomonotone variational inequality problem and the fixed point problem with a demicontractive mapping in a real Hilbert space. We proved the strong convergence theorems for the sequences generated by the algorithms under suitable assumptions. In particular, the proposed algorithms have a new adaptive non-monotonic step size update criterion that can work without knowing the Lipschitz constant of the mapping. Finally, the computational efficiency and advantages of the suggested algorithms over previously known ones are illustrated with some numerical examples in finite- and infinite-dimensions and two applications in optimal control problems and signal processing problems.



(a) Recovered by our Algorithm (3.36) (b) Recovered by our Algorithm (3.36)

(c) Recovered by our Algorithm (3.37) (d) Recovered by our Algorithm (3.37)

FIGURE 6. Signals with different sparsity recovered by our algorithms

Acknowledgements. The authors are very grateful to the editor and the anonymous referees for their valuable comments that improved the initial manuscript. The corresponding author was supported by NNSF of China (No. 12371131), GuangDong Basic and Applied Basic Research Foundation (Nos. 2022A1515010317 and 2023A1515010614), STU Scientific Research Initiation Grant (No. NTF23004).

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Received: June 27, 2022; Accepted: August 8, 2023.