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ON GRAPHICAL FUZZY METRIC SPACES AND RELATED FIXED POINT THEOREMS

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Abstract. The notion of triangular inequality plays an important role in determining the structure of distance spaces. In particular, the structure of fuzzy metric spaces depends on the triangular inequality and the concerned t-norm. In most of the fixed point theorems in fuzzy metric spaces both the triangular inequality and the concerned t-norm have a major impact on the proof of fixed point theorems. Inspired by the concept of graphical metric space, it was recently introduced in N. Saleem et al., *On Graphical Fuzzy Metric Spaces with Application to Fractional Differential Equations*, Fractal and Fract., **6:5** (2022), 238:1-12, the notion of graphical fuzzy metric space and proved some fixed point results. The triangular inequality in such spaces is replaced by a weaker one which is directly associated with the graphical structure affine with the space. In this paper some observations on the recent results of Saleem et al. are made and so the results are revisited. Some related topological properties with some new fixed point results in graphical fuzzy metric spaces are also proved. The results of this paper generalize and extend Banach contraction principle and some other known results in this new setting. Several examples are given which support the claims and illustrate the significance of the new concepts and results.
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1. Introduction

The deportment of systems with uncertain nature was considered by Zadeh [20] with the help of fuzzy sets. A fuzzy version of metric spaces was introduced by Kramosil and Michalek [10]. They considered the distance between two points as a fuzzy set. George and Veeramani [1] considered the fuzzy distance of two points as a measure of degree of nearness of points with respect to a parameter $t \in (0, \infty)$, and gave some modifications to the definition of fuzzy metric spaces due to Kramosil and Michalek. Existence of fixed point of mappings with contractive nature in fuzzy metric spaces was considered by Grabiec [2] which has become of interest for several authors and nowadays constitutes an intensive field of research (see, for instance, [4, 8, 11, 12, 14, 21] and references therein).

On the other hand, Shukla et al. [18] announced the notion of graphical metric space and generalized the concept of metric space for the sets possessing a graphical structure. In such spaces the usual triangular inequality satisfied by the metric is replaced by a weaker one satisfied by only those points which are situated on some path formed by graphical structure associated with the space. Inspired by Shukla et al. [18], recently, Saleem et al. [15] introduced a fuzzy version of graphical metric spaces. Then, they established some fixed point results and presented some applications.

The purpose of this paper is two fold: on the one hand, by giving some counter examples to [15] we show that one should take care of complex structure of graphical fuzzy metric spaces when considering the properties and fixed point results in graphical fuzzy metric spaces. On the other hand, we discuss some topological properties of graphical fuzzy metric spaces and identify two different kind of completeness in this context, which are usually known as G-completeness and M-completeness, respectively. Then, we prove some new fixed point results in G-complete graphical fuzzy metric spaces, which improve the fixed point results of Saleem et al. [15]. Moreover, we establish some new fixed point results in M-complete graphical fuzzy metric spaces. In addition we show that our results generalize the celebrated fixed point theorem given by Grabiec in [2] and a fixed point result recently established in [4].

The remaining of the paper is organized as follows. Section 2 is devoted to recall the main basics on fuzzy metric spaces and graphical metric spaces. Then, Section 3 is dedicated to show some observations on the concepts and results provided in [15]. Section 4 contains the main results of the paper. Finally, Section 5 exposes the conclusions of the paper.

2. Preliminaries

We recall some known definitions and the properties about the fuzzy metric spaces, graphs and graphical metric spaces. By $\mathbb{R}$, $\mathbb{Q}$ and $\mathbb{N}$ we shall denote the set of all real numbers, the set of all rational numbers and the set of all positive integer numbers, respectively.

Definition 2.1 (Schweizer and Sklar[16]). A binary operation $*\colon [0,1]\times[0,1]\to[0,1]$ is called a t-norm if the following conditions are satisfied:

(T1) $a*b=b*a$;
(T2) $a*b\leq c*d$ for $a\leq c, b\leq d$;
(T3) $(a*b)*c=a*(b*c)$;
(T4) $a*0=0, a*1=a$;

for all $a,b,c,d\in[0,1]$.

A t-norm $*$ is said to be positive if $a*b > 0$ whenever $a, b \in (0, 1]$. For $a_1, a_2, ..., a_n \in [0, 1]$ and $n \in \mathbb{N}$, the product $a_1 * a_2 * \cdots * a_n$ will be denoted by $\prod_{i=1}^{n} a_i$. For the details concerning t-norms the reader is referred to [6, 9].

Definition 2.2 (George and Veeramani [1])**.** A triple $(X, M, *)$ is called a fuzzy metric space if X is a nonempty set, $*$ is a continuous t-norm and $M : X \times X \times (0, \infty) \to [0, 1]$ is a fuzzy set satisfying following conditions:

(GV1) $M(x, y, t) > 0$;
(GV2) $M(x, y, t) = 1$ if and only if $x = y$;
(GV3) $M(x, y, t) = M(y, x, t)$;
(GV4) $M(x, z, t + s) \geq M(x, y, t) * M(y, z, s)$;
(GV5) $M(x, y, .) : (0, \infty) \to [0, 1]$ is a continuous mapping;

for all $x, y, z \in X$ and $s, t > 0$.

For various properties of a fuzzy metric space the reader is referred to [1].

For a nonempty set X, let $\Delta = \{(x, x) : x \in X\}$ (see, [7, 18]) and consider a directed graph H, such that $V(H) = X$, $E(H) \supseteq \Delta$ and H is without parallel edges. In this case the set X is said to be endowed with the graph $H = (V(H), E(H))$. By H^{-1}, we define the graph such that:

$$V(H^{-1}) = V(H),\ E(H^{-1}) = \{(x, y) \in X \times X : (y, x) \in E(H)\}.$$

The graph H induces an undirected graph H_I such that $V(H_I) = V(H)$ and $E(H_I) = E(H) \cup E(H^{-1})$. For two vertices u and v in H, a path from u to v of length $n \in \mathbb{N}$ in H is a sequence $\{x_i\}_{i=0}^{n}$ of $n + 1$ vertices such that $x_0 = u, x_n = v$ and $(x_{i-1}, x_i) \in E(H)$ for $i = 1, 2, \ldots, n$. A graph H is called connected if there is a path between any two vertices. H is weakly connected if the induced undirected graph H_I is connected. We define a relation P on X by:

$$P = \{(u, v) \in X \times X : \text{ there is a directed path from } u \text{ to } v \text{ in } H\}.$$

We write $(uPv)_{\mathrm{H}}$ if $(u, v) \in P$. We say that a vertex w falls on some directed path joining u and v and we write $w \in (uPv)_{\mathrm{H}}$, if w is contained in some directed path from u to v in H. For an $n \in \mathbb{N}$, we set

$$[u]_H^n = \{v \in X : \text{ there is a directed path from } u \text{ to } v \text{ of length } n\}.$$

If for a sequence $\{x_n\}$ in X we have $(x_n P x_{n+1})_{\mathrm{H}}$ for all $n \in \mathbb{N}$, then the sequence $\{x_n\}$ is said to be an H-termwise connected sequence.

All the graphs considered in this paper are directed and with nonempty sets of vertices and edges.

Definition 2.3 (Shukla et al. [18])**.** Let X be a nonempty set endowed with a graph H and $d_{\mathrm{H}} : X \times X \to \mathbb{R}$ be a function satisfying the following conditions:

(GM_1) $d_{\mathrm{H}}(x, y) \geq 0$ for all $x, y \in X$;
(GM_2) $d_{\mathrm{H}}(x, y) = 0$ if and only if $x = y$;
(GM_3) $d_{\mathrm{H}}(x, y) = d_{\mathrm{H}}(y, x)$ for all $x, y \in X$;

(GM_4) $(xPy)_{\mathrm{H}}, z \in (xPy)_{\mathrm{H}}$ implies $d_{\mathrm{H}}(x,y) \leq d_{\mathrm{H}}(x,z) + d_{\mathrm{H}}(z,y)$ for all $x,y,z \in X$.

Then, the mapping d_{H} is called a graphical metric on X, and the pair (X, d_{H}) is called a graphical metric space.

For examples and properties of graphical metric spaces, we refer to [18, 17].

In the next section we point out some inappropriate results and proofs of Saleem at al. [15].

3. Some comments on graphical fuzzy metric spaces and fixed point theorems

Saleem et al. [15] introduced the graphical fuzzy metric spaces as follows:

Definition 3.1. Let X be a nonempty set endowed with a graph H, $*$ be a continuous t-norm and $M_{\mathrm{H}}\colon X \times X \times (0,\infty) \to [0,1]$ be a fuzzy set. Then, the triple $(X, M_{\mathrm{H}}, *)$ is called a graphical fuzzy metric space and M_{H} is called graphical fuzzy metric on X if the following conditions are satisfied:

(GFM1) $M_{\mathrm{H}}(x,y,t) > 0$;
(GFM2) $M_{\mathrm{H}}(x,y,t) = 1$ if and only if $x = y$;
(GFM3) $M_{\mathrm{H}}(x,y,t) = M_{\mathrm{H}}(y,x,t)$;
(GFM4) $(xPy)_{\mathrm{H}}, z \in (xPy)_{\mathrm{H}}$ implies $M_{\mathrm{H}}(x,y,t+s) \geq M_{\mathrm{H}}(x,z,t) * M_{\mathrm{H}}(z,y,s)$;
(GFM5) $M_{\mathrm{H}}(x,y,\cdot) : (0,\infty) \to [0,1]$ is a continuous mapping;

for all $x,y,z \in X$ and $s,t > 0$.

Definition 3.2 (Saleem et al. [15])**.** Let $(X, M_{\mathrm{H}}, *)$ be a graphical fuzzy metric space. A sequence $\{x_n\}$ in X is called convergent and converges to $x \in X$ if for every given $r \in (0,1)$ there is $n_0 \in \mathbb{N}$ such that $M_{\mathrm{H}}(x_n, x, t) > 1 - r$ for all $n \geq n_0$. The sequence $\{x_n\}$ is called a Cauchy sequence if for every given $r \in (0,1)$ there is $n_0 \in \mathbb{N}$ such that $M_{\mathrm{H}}(x_n, x_m, t) > 1 - r$ for all $n, m > n_0$. The space $(X, M_{\mathrm{H}}, *)$ is called complete if every Cauchy sequence in X converges to some $x \in X$. Suppose H' is a graph such that $V(H') = X$, then $(X, M_{\mathrm{H}}, *)$ is called H'-complete if every H'-termwise connected Cauchy sequence in X converges to some $x \in X$.

Definition 3.3 (Saleem et al. [15])**.** Let $(X, M_{\mathrm{H}}, *)$ be a graphical fuzzy metric space, $T\colon X \to X$ a mapping and H' be a subgraph of H such that $E(H') \supseteq \Delta$. Then, T is called an (H, H')-fuzzy graphical contraction if the following conditions hold:

(FGC1) $(x,y) \in E(H')$ implies $(Tx, Ty) \in E(H')$, i.e., T is edge-preserving in H';
(FGC2) there exists $0 < k < 1$ such that $M_{\mathrm{H}}(Tx, Ty, kt) \geq M_{\mathrm{H}}(x,y,t)$ for all $x, y \in X$ with $(x,y) \in E(H')$.

A sequence $\{x_n\}$ in X with initial value $x_0 \in X$ is said to be a T-Picard sequence if $x_n = Tx_{n-1}$ for all $n \in \mathbb{N}$.

Apart from ordinary metric, a graphical fuzzy metric has involvement of the graphical structure and t-norms, therefore the structure of fuzzy metric spaces has a combinatorial nature. In particular, when proving fixed point theorems in such spaces, one has to follow a sophisticated way, otherwise one can make consequential misinterpretations. By giving some counterexamples to Saleem et al. [15], we next verify this fact.

Saleem et al. [15] stated the following theorem:

Theorem 3.1 (Theorem 1 of Saleem et al. [15])**.** *Let $(X, M_H, *)$ be an H'-complete graphical fuzzy metric space (where H' is a subgraph of H such that $E(H') \supseteq \Delta$) and $T: X \to X$ be an (H, H')-fuzzy graphical contraction. Suppose that the following conditions hold:*

- *(i) there exists $x_0 \in X$ such that $Tx_0 \in [x_0]^q_{H'}$ for some $q \in \mathbb{N}$;*
- *(ii) if an H'-termwise connected T-Picard sequence $\{x_n\}$ converges in X, then there exist a limit $z \in X$ of $\{x_n\}$ and $n_0 \in \mathbb{N}$ such that $(x_n, z) \in E(H')$ or $(z, x_n) \in E(H')$ for all $n > n_0$.*

Then there exists $x^\star \in X$ such that the T-Picard sequence $\{x_n\}$, with initial value $x_0 \in X$, is H'-termwise connected and converges to both $x^\star$ and $Tx^\star$.

Next example shows that the necessary conditions for the convergence of sequence $\{x_n\}$ in the above theorem are not adequate, and so, the above theorem is not correct.

Example 3.1. Let $X = \{0, 1\}$ and H, H' be graphs defined by $V(H) = V(H') = X, E(H) = E(H') = X \times X$. Consider the Lukasievicz t-norm $*_L$, i.e., $a *_L b = \max\{a + b - 1, 0\}$, and the graphical fuzzy set $M_{\mathrm{H}}: X \times X \times (0, \infty) \to [0, 1]$ given by $M_{\mathrm{H}}(1, 0, t) = M_{\mathrm{H}}(0, 1, t) = \dfrac{2}{3}$, $M_{\mathrm{H}}(0, 0, t) = M_{\mathrm{H}}(1, 1, t) = 1$, for all $t > 0$. Then, $(X, M_{\mathrm{H}}, *_L)$ is an H'-complete graphical fuzzy metric space. Define a mapping $T: X \to X$ by $T0 = 1, T1 = 0$. Then it is easy to see that the mapping T is an (H, H')-fuzzy graphical contraction with arbitrary $k \in (0, 1)$, and all other conditions of Theorem 1 of [15] are satisfied. But note that no T-Picard sequence in X converges.

Remark 3.1. The above example suggests that the conditions used to ensure the convergence of underlying Picard sequence in Theorem 1 of [15] are not adequate. Note that in the above example the condition "for all $x, y \in X$ we have $\lim_{t\to\infty} M_{\mathrm{H}}(x, y, t) = 1$" does not hold. In the next section, we will show that if this condition is included in the list of assumptions of Theorem 1 of [15], it assures the above mentioned convergence. Apart from this, in the same theorem, authors of [15] made a mistake when proving the Picard sequence to be a Cauchy sequence (see Remark 4.5 below). Later, we will give a corrected version of the statement and the proof of this theorem (see Theorem 4.3 below).

Definition 3.4 (Saleem et al. [15])**.** Let $(X, M_{\mathrm{H}}, *)$ be a graphical fuzzy metric space, H' be a subgraph of graph H and $T: X \to X$ be a mapping. Then, the five-tuple $(X, M_{\mathrm{H}}, *, H', T)$ is said to have the property (S) if:

$$\text{whenever an } H'\text{-termwise connected } T\text{-Picard sequence } \{x_n\} \text{ has two limits } x^\star \text{ and } y^\star, \text{ where } x^\star \in X, y^\star \in T(X), \text{ then } x^\star = y^\star. \tag{S}$$

By $\mathrm{Fix}(T)$, we denote the set of all fixed points of a mapping T. Also we denote $X_T = \{x \in X : (x, Tx) \in E(H')\}$. Next, Saleem et al. [15] established the following theorem:

Theorem 3.2 (Theorem 3 of Saleem et al. [15])**.** *Let $(X, M_H, *)$ be an H'-complete graphical metric space and $T\colon X \to X$ be an (H, H')-fuzzy graphical contraction. Suppose that all the conditions of Theorem 3.1 are satisfied and the five tuple $(X, M_H, *, H', T)$ satisfies the property (S), then T has a fixed point. In addition, if X_T is weakly connected (as a subgraph of H'), then the fixed point of T is unique.*

Again, we give another example which shows that the condition "if X_T is weakly connected (as a subgraph of H')" is not sufficient to prove the uniqueness of fixed point in Theorem 3.2, and so the above theorem is not correct.

Example 3.2. Let $X = [-1, 0) \cup (0, 1] \cup \{a^-, a^+\}$ where a^- and a^+ are two distinct points such that $([-1, 0) \cup (0, 1]) \cap \{a^-, a^+\} = \phi$. Let $H = H'$ be the graph such that $V(H) = V(H') = X$ and

$$\begin{aligned} E(H) &= E(H') = \Delta \cup \{(x, y) \in X \times X \colon 0 < y < x \leq 1\} \\ &\cup\{(x, y) \in X \times X \colon -1 \leq x < y < 0\} \\ &\cup\{(x, -x) \in X \times X \colon 0 < x \leq 1\} \\ &\cup\{(a^-, x) \in X \times X \colon -1 \leq x < 0\} \\ &\cup\{(x, a^+) \in X \times X \colon 0 < x \leq 1\}. \end{aligned}$$

Let $d_{\mathrm{H}}\colon X \times X \to [0, \infty)$ be the function given by:

$$d_{\mathrm{H}}(x, y) = \begin{cases} 0, & \text{if } \ x = y; \\ 1, & \text{if } \ x, y \in [-1, 0) \cup (0, 1] \\ & \text{are such that } y < 0 < x < -y \text{ or } x < 0 < y < -x; \\ |x - y|, & \text{if } \ x, y \in [-1, 0) \cup (0, 1] \text{ and } x \neq y \text{ in other case;} \\ y, & \text{if } \ x = a^+ \text{ and } y \in (0, 1]; \\ x, & \text{if } \ y = a^+ \text{ and } x \in (0, 1]; \\ 1 + |y|, & \text{if } \ x = a^+ \text{ and } y \in [-1, 0); \\ 1 + |x|, & \text{if } \ y = a^+ \text{ and } x \in [-1, 0); \\ 1 + y, & \text{if } \ x = a^- \text{ and } y \in (0, 1]; \\ 1 + x, & \text{if } \ y = a^- \text{ and } x \in (0, 1]; \\ |y|, & \text{if } \ x = a^- \text{ and } y \in [-1, 0); \\ |x|, & \text{if } \ y = a^- \text{ and } x \in [-1, 0); \\ 2 & \text{if } \ \{x, y\} = \{a^-, a^+\}. \end{cases}$$

Then (X, d_{H}) is a graphical metric space (see [13]). Let $(X, M_{\mathrm{H}}, \wedge)$ be the standard graphical fuzzy metric space induced by d_{H} (see, Proposition 4.1 below). Let $T\colon X \to X$ be given by

$$Tx = \begin{cases} \dfrac{x}{2}, & \text{if } \ x \in [-1, 0) \cup (0, 1]; \\ x & \text{if } \ x \in \{a^-, a^+\}. \end{cases}$$

Now, it is easy to see that T is an (H, H')-fuzzy graphical contraction with $k = \dfrac{1}{2}$. Also, one can easily verify that a sequence in X is Cauchy (respectively converges to some $x \in X$) in $(X, M_{\mathrm{H}}, \wedge)$ if and only if it is Cauchy (respectively converges to same $x \in X$) in (X, d_{H}). Keeping this in mind, the calculations of [13] shows that all the conditions of Theorem 3.2 are satisfied, but T has two fixed points, namely, a^- and a^+.

Remark 3.2. The incorrectness of Theorem 3.2 is because of a wrong interpretation of weak connectedness of graphs. Precisely, in the proof of the said theorem authors assumed that $M_{\mathrm{H}}(a^-, a^+, t) = M_{\mathrm{H}}(T^n a^-, T^n a^+, kt) \geq \prod_{i=0}^{l-1} M_{\mathrm{H}}(T^n z_i, T^n z_{i+1}, kt)$, because there is an undirected path $\{z_i\}_{i=0}^{l-1}$ from a^- to a^+. However, this path is not directed, and so, one cannot use the property (GFM$_4$). Later we will show that (see Theorem 4.5 below) if the weak connectedness of X_T is replaced with connectedness of X_T, then this flaw can be removed.

Remark 3.3. In Lemma 2 of Saleem at al. [15] authors claim that the topology induced by a graphical fuzzy metric space is T_1 but not T_2, in general. Although, this conclusion is true, the proof is not (compare the proof as given in Saleem at al. [15] and the proof of Theorem 4.2 below). Also, in [15] there is no example given which verifies the fact that the said topology is not T_2 in general, hence proof is incomplete as well. In Theorem 4.2 and Example 4.3, we correct and complete the proof in a more appropriate way.

In the next section we state some examples and properties of graphical fuzzy metric spaces. Some definitions of Saleem at al. [15] are revisited and the statements and the proofs of the results said above are corrected and modified, as well as we establish some new properties, and new fixed point results in graphical fuzzy metric spaces are presented.

4. Some properties of graphical fuzzy metric spaces and fixed point theorems

We first give some new examples for illustration and some properties of graphical fuzzy metric spaces which will be useful in the sequel.

It is obvious that every fuzzy metric space $(X, M, *)$ is a graphical fuzzy metric space with a graph H such that $V(H) = X$ and $E(H) = X \times X$. We give some examples of graphical fuzzy metric spaces which are not a fuzzy metric space. From now on, we will denote by $\wedge$ the minimum t-norm (i.e., $a \wedge b = \min\{a, b\}$).

Example 4.1. Let $X = \left\{x_n \colon x_n = \dfrac{1}{n}, n \in \mathbb{N}\right\}$, H be a graph defined by $V(H) = X$ and

$$E(H) = \Delta \cup \{(x_{n+1}, x_n) \colon n \in \mathbb{N}\}.$$

Define a fuzzy set $M_{\mathrm{H}} : X \times X \times (0, \infty) \to [0, 1]$ by

$$M_{\mathrm{H}}(x, y, t) = \begin{cases} 1, & \text{if} \quad x = y; \\ xy, & \text{otherwise}; \end{cases}$$

for all $t > 0$. Then $(X, M_{\mathrm{H}}, \wedge)$ is a graphical fuzzy metric space. Note that $(X, M_{\mathrm{H}}, \wedge)$ is not a fuzzy metric space, e.g., for any $x, y, z \in X$ with $x < y < z$ the inequality $M_{\mathrm{H}}(x, y, t + s) \geq M_{\mathrm{H}}(x, z, t) \wedge M_{\mathrm{H}}(z, y, s)$ does not hold.

Example 4.2. Let $X = [0, \infty)$ and H be a graph defined by $V(H) = X$ and

$$E(H) = \Delta \cup \{(x, y) \in X \times X : x \leq y\}.$$

For any fixed number $\alpha > 1$, define a fuzzy set $M_{\mathrm{H}} : X \times X \times (0, \infty) \to [0, 1]$ by

$$M_{\mathrm{H}}(x, y, t) = \begin{cases} 1, & \text{if} \quad x = y; \\ \frac{1}{\alpha^{x+y}}, & \text{if} \quad x \neq y; \end{cases}$$

for all $t > 0$. Then $(X, M_{\mathrm{H}}, \wedge)$ is a graphical fuzzy metric space. Note that $(X, M_{\mathrm{H}}, \wedge)$ is not a fuzzy metric space, e.g., for any $x, y, z \in X$ with $z < y < x$ the inequality $M_{\mathrm{H}}(x, y, t + s) \geq M_{\mathrm{H}}(x, z, t) \wedge M_{\mathrm{H}}(z, y, s)$ does not hold.

With the help of the following propositions one can construct several more examples of graphical fuzzy metric spaces.

Proposition 4.1. Let (X, d_{H}) be a graphical metric space. Suppose, k, m are positive real numbers and $n \in \mathbb{N}$, and define the fuzzy set $M_{\mathrm{H}} : X \times X \times (0, \infty) \to [0, 1]$ by

$$M_{\mathrm{H}}(x, y, t) = \frac{kt^n}{kt^n + m d_{\mathrm{H}}(x, y)} \text{ for all } x, y \in X, \ t > 0.$$

Then $(X, M_{\mathrm{H}}, \wedge)$ is a graphical fuzzy metric space. If $k = m = n = 1$, then this graphical fuzzy metric is called graphical fuzzy metric induced by the graphical metric d_{H} or the standard graphical fuzzy metric space induced by d_{H}.

Proof. From the definition, properties (GFM1), (GFM2), (GFM3) and (GFM5) are obvious. For property (GFM4), let $t, s > 0$, and $x, y, z \in X$ such that $z \in (xPy)_H$. Then, we have $d_{\mathrm{H}}(x, y) \leq d_{\mathrm{H}}(x, z) + d_{\mathrm{H}}(z, y)$. If $\min\{M(x, z, t), M(z, y, s)\} = M(x, z, t)$, then we must have $t^n d_{\mathrm{H}}(z, y) \leq s^n d_{\mathrm{H}}(x, z)$, and so, $t^n d_{\mathrm{H}}(x, y) \leq t^n d_{\mathrm{H}}(x, z) + s^n d_{\mathrm{H}}(x, z) \leq (t + s)^n d_{\mathrm{H}}(x, z)$. This shows that $M_{\mathrm{H}}(x, y, t + s) \geq M_{\mathrm{H}}(x, z, t) \wedge M_{\mathrm{H}}(z, y, s)$ and (GFM4) is also satisfied. Similar is true if $\min\{M(x, z, t), M(z, y, s)\} = M(z, y, s)$. Hence $(X, M_{\mathrm{H}}, \wedge)$ is a graphical fuzzy metric space. □

The same arguments to those used in the proof of previous proposition can be used to prove the next one.

Proposition 4.2. Let (X, d_{H}) be a graphical metric space and $n \in \mathbb{N}$. Define the fuzzy set $M_{\mathrm{H}} : X \times X \times (0, \infty) \to [0, 1]$ by

$$M_{\mathrm{H}}(x, y, t) = \left[\exp\left\{\frac{d_{\mathrm{H}}(x, y)}{t^n}\right\}\right]^{-1} \text{ for all } x, y \in X, \ t > 0.$$

Then $(X, M_{\mathrm{H}}, \wedge)$ is a graphical fuzzy metric space.

As the graphs concerned with the graphical fuzzy metric spaces are directed (in general), it follows that the direction of paths in graphical structure is important and it is reasonable to define the open balls and related topologies in a compatible way of directed graphs. Hence, following the ideas used in [17] for classical graphical metrics, we define the left and right open balls (L-open and R-open balls respectively) and their corresponding topologies as follows:

Definition 4.1. Let $(X, M_{\rm H}, *)$ be a graphical fuzzy metric space. The R-open ball (right open ball) $B^R_{\rm H}(x, r, t)$ with centre $x \in X$, radius $r \in (0, 1)$ and parameter $t > 0$ is defined by

$$B^{\rm R}_{\rm H}(x, r, t) = \{y \in X\colon (xPy)_H \text{ and } M_{\rm H}(x, y, t) > 1 - r\}$$

Similarly, the L-open ball (left open ball) is defined by

$$B^{\rm L}_{\rm H}(x, r, t) = \{y \in X\colon (yPx)_H \text{ and } M_{\rm H}(x, y, t) > 1 - r\}$$

Observe that the balls $B^{\rm R}_{\rm H}(x, r, t)$ and $B^{\rm L}_{\rm H}(x, r, t)$ are nonempty.

Now, we define the following collections of subsets of a graphical fuzzy metric space $(X, M_{\rm H}, *)$:

$$\tau^{\rm R}_{\rm H} = \{U \subset X\colon \text{ for all } x \in U \text{ there exist } r \in (0, 1), t > 0 \text{ such that } B^{\rm R}_{\rm H}(x, r, t) \subset U\}$$

and

$$\tau^{\rm L}_{\rm H} = \{U \subset X\colon \text{ for all } x \in U \text{ there exist } r \in (0, 1), t > 0 \text{ such that } B^{\rm L}_{\rm H}(x, r, t) \subset U\}.$$

It is not hard to check that both $\tau^{\rm R}_{\rm H}$ and $\tau^{\rm L}_{\rm H}$ define topologies on X, which will be called the R-topology and L-topology, respectively, induced by the graphical fuzzy metric $M_{\rm H}$ and the members of $\tau^{\rm R}_{\rm H}$, $\tau^{\rm L}_{\rm H}$ will be called R-open sets and L-open sets, respectively.

The proof of the following theorem is analogues to the proof of Lemma 1 of [15].

Theorem 4.1. *Every R-open ball (respectively L-open ball) is an R-open set (respectively L-open set).*

Next we show that both R-topology and L-topology are T_1 but not T_2, in general.

Theorem 4.2. *Every R-topology (L-topology) induced by a graphical fuzzy metric is* T_1.

Proof. Let $(X, M_{\rm H}, *)$ be a graphical fuzzy metric space. For every $x \in X$, we shall show that the singleton set $\{x\}$ is a closed subset of X, i.e., the set $X \setminus \{x\}$ is an open subset of X. Suppose, $y \in X \setminus \{x\}$, then $y \neq x$, i.e., $0 < M_{\rm H}(x, y, t) < 1$ for all $t > 0$. For a fixed $t_0 > 0$, let $0 < r = 1 - M_{\rm H}(x, y, t_0)$. If $x \in B^{\rm R}_{\rm H}(y, r, t_0)$, then $(yPx)_H$ and $M_{\rm H}(y, x, t_0) > 1 - [1 - M_{\rm H}(x, y, t_0)]$ which is a contradiction. Hence $x \notin B^{\rm R}_{\rm H}(y, r, t_0)$ and therefore $B^{\rm R}_{\rm H}(y, r, t_0) \subset X \setminus \{x\}$. □

The following example shows that, in general, R-topology is not T_2, i.e., Hausdorff.

Example 4.3. Let $X = [0, 1]$ and H be the graph defined by $V(H) = X$ and

$$E(H) = \Delta \cup \{(x, y) \in X \times X : x, y \in (0, 1]; y \leq x\}.$$

Let $(X, M_{\rm H}, \wedge)$ be the standard graphical fuzzy metric space induced by $d_{\rm H}$, where $d_{\rm H}$ is defined as

$$d_{\rm H}(x, y) = \begin{cases} 0, & \text{if } \; x = y; \\ xy, & \text{if } \; x, y \in (0, 1] \text{ and } x \neq y; \\ x + y, & \text{otherwise.} \end{cases}$$

Then for any $0 < a < 1$ and for every $r_1 > 0, r_2 > 0$ and fixed $t > 0$, we have $B^{\rm R}_{\rm H}(a, r_1, t) \cap B^{\rm R}_{\rm H}(1, r_2, t) \neq \emptyset$.

One can prove by similar arguments that L-topology induced by the graphical fuzzy metric is T_1 but not T_2, i.e., not Hausdorff.

Remark 4.1. If we consider the standard graphical fuzzy metric space $(X, M_{\mathrm{H}}, *)$ induced by a given graphical metric space (X, d_{H}) then it is easy to see that the L-topology induced by d_{H} is same as the L-topology induced by M_{H}. Similarly the R-topology induced by d_{H} is same as the R-topology induced by M_{H} (for properties of L and R-topologies induced by a graphical metric d_{H}, we refer to [17]). By keeping this fact in mind, one can see easily that an L-open ball and an R-open ball with same center and radius may not be same (for reference, see Example 3.1 of [17]).

On account of Theorem 4.2 one can prove the next result which characterizes convergent sequences in R-topology (L-topology) induced by a graphical fuzzy metric.

Proposition 4.3. Let $(X, M_{\mathrm{H}}, *)$ be a graphical fuzzy metric space. A sequence $\{x_n\}$ in X is R-convergent (L-convergent) to $x \in X$, i.e., converges to $x \in X$ with respect to τ_H^R (τ_H^L), if and only if for each $r \in (0, 1)$ and $t > 0$ we can find $n_0 \in \mathbb{N}$ such that $x_n \in B_{\mathrm{H}}^{\mathrm{R}}(x, r, t)$ for all $n \geq n_0$.

By the last result and the definition of $B_{\mathrm{H}}^{\mathrm{R}}(x, r, t)$, we conclude that a sequence $\{x_n\}$ in a graphical fuzzy metric space $(X, M_H, *)$ is R-convergent to $x \in X$ if for each $r \in (0, 1)$ and $t > 0$ we can find $n_0 \in \mathbb{N}$ such that $(xPx_n)_H$ and $M_H(x, x_n, t) > 1 - r$, for all $n \geq n_0$ (i.e., $\lim_{n\to\infty} M_{\mathrm{H}}(x, x_n, t) = 1$, for all $t > 0$). This observation motivates the introduction of the different notions of convergence in graphical fuzzy metric spaces, which will be useful in developing fixed point results in such spaces.

First, we introduce the notion of "M_{H}-convergence" in context of a graphical fuzzy metric, which constitutes a slight modification of the concept of convergence introduced by Saleem et al. [15].

Definition 4.2. Let $(X, M_{\mathrm{H}}, *)$ be a graphical fuzzy metric space. A sequence $\{x_n\}$ in X will be called M_{H}-convergent to $x \in X$ if for each $r \in (0, 1)$ and $t > 0$ we can find $n_0 \in \mathbb{N}$ such that $M_{\mathrm{H}}(x_n, x, t) > 1 - r$, for all $n \geq n_0$ (i.e., $\lim_{n\to\infty} M_{\mathrm{H}}(x_n, x, t) = 1$, for all $t > 0$).

One can observe the difference between the notion of M_{H}-convergence and the former one given by Saleem et al. in [15] for graphical fuzzy metric spaces. Indeed, in the context of fuzzy metric spaces, M_{H}-convergent sequences coincide with those are convergent with respect to the topology induced by a fuzzy metric (see [1]). Nevertheless, in such a context, convergent sequences in the sense of Saleem et al. coincide with strong convergent sequences introduced in [3]. Taking into account that, in the context of fuzzy metric spaces, strong convergence is a concept stronger than topological convergence, one can easily conclude that M_{H}-convergence is a concept weaker than the concept of convergence due to Saleem et al. in [15].

On the other hand, we know that a convergent sequence in fuzzy metric spaces has unique limit but this is not the case for graphical fuzzy metric spaces.

Example 4.4. Let $(X, M_{\mathrm{H}}, *)$ be the graphical fuzzy metric space as considered in Example 4.3. Then, the sequence $\{x_n\}$, where $x_n = \dfrac{1}{n}$ for all $n \in \mathbb{N}$ have infinitely many limits in X, i.e. M_{H}-converges to infinitely many points in X. Indeed, all the points of X are limits of this sequence.

We next show that the two topological convergence, i.e., convergence with respect to left and right topologies are not identical concepts, and these two concepts may not agree with the M_{H}-convergence. Indeed, it is not hard to prove that both L-convergent sequences and R-convergent sequences are M_{H}-convergent. However, the converse of such an affirmation is not true, in general, as shows the next example.

Example 4.5. Let $(X, M_{\mathrm{H}}, *)$ be the graphical fuzzy metric space same as we have considered in the Example 4.2. Consider the sequence $\{x_n\}$ in X, where $x_n = \dfrac{1}{n}$ for all $n \in \mathbb{N}$. Then, it is easy to see that for every given $r \in (0,1)$ and $t > 0$ we can find $n_0 \in \mathbb{N}$ such that $M_{\mathrm{H}}(x_n, 0, t) > 1 - r$ for all $n \geq n_0$, hence $\{x_n\}$ M_{H}-converges to 0. On the other hand, one can verify that $B^L_H(0, r, t) = \{0\}$ for all $r \in (0,1)$ and $t > 0$, since $(xP0)_H$ if and only if $x = 0$. Therefore, $\{x_n\}$ is not L-convergent to 0. Indeed, there is no L-limit of the sequence $\{x_n\}$ in X.

Remark 4.2. In the example considered above, we see that for all $r \in (0,1)$ and $t > 0$, $B^R_H(0, r, t) = \left[0, -\frac{\ln(1-r)}{\ln \alpha}\right)$, therefore, if we choose $n \geq n_0 \in \mathbb{N}$, where $n_0 > -\frac{\ln \alpha}{\ln(1-r)}$ then $x_n \in B^R_H(0, r, t)$ for all $n \geq n_0$, therefore $\{x_n\}$ is R-convergent to 0. This shows that L-limits and R-limits of a sequence may differ. Also in a similar way one can show that R-convergence implies the M_{H}-convergence, but not conversely.

We continue introducing a notion of convergence motivated by condition (ii) in Theorem 3.1.

Definition 4.3. Let $(X, M_{\mathrm{H}}, *)$ be a graphical fuzzy metric space and let H' be a subgraph of H such that $E(H') \supseteq \Delta$. A sequence $\{x_n\}$ in X will be called $E(H'_I)$-convergent to $x \in X$ if for each $r \in (0,1)$ and $t > 0$ we can find $n_0 \in \mathbb{N}$ such that $(x, x_n) \in E(H'_I)$ and $M_{\mathrm{H}}(x_n, x, t) > 1 - r$, for all $n \geq n_0$.

Observe that from the preceding definition we conclude that there exists $n \in \mathbb{N}$ such that $(x, x_n) \in E(H'_I)$, for all $n \geq n_0$, and $\lim_{n\to\infty} M_{\mathrm{H}}(x_n, x, t) = 1$, for all $t > 0$. So, condition (ii) in Theorem 3.1 now can be rewritten as “if an H'-termwise connected T-Picard sequence $\{x_n\}$ converges to $x \in X$, then $E(H'_I)$-converges to x”.

Remark 4.3. Obviously, $E(H'_I)$-convergent sequences are M_{H}-convergent. Besides, following Example 4.5, $\{x_n\}$, where $x_n = \frac{1}{n}$ for all $n \in \mathbb{N}$, is an R-convergent sequence which, in addition, is $E(H'_I)$-convergent when consider $H' = H$. So, we have an instance of $E(H'_I)$-convergent sequence that is not L-convergent. Nevertheless, if we consider H' such that $V(H') = X$ and $E(H') = \Delta \cup \{(x, y) \in X \times X : x, y \in (0,1]; x \leq y \text{ and } x, y \in \mathbb{Q}\}$, then the sequence $\{y_n\}$, where $y_n = x_n$ whenever n is odd and $y_n = \pi \cdot x_n$ elsewhere, is not $E(H'_I)$-convergent. Indeed, $(0, y_n) \notin E(H'_I)$ for each even $n \in \mathbb{N}$. Besides, one can check that $\{y_n\}$ is an R-convergent sequence. Similar arguments can be taken to show that R-convergence (L-convergence) and $E(H'_I)$-convergence are independent notions, i.e. there exist R-convergent (L-convergent) sequences which are not $E(H'_I)$-convergent and, vice-versa.

The last notion of convergence introduced below is motivated by the property (S) given in Definition 3.4.

Definition 4.4. Let $(X, M_{\mathrm{H}}, *)$ be a graphical fuzzy metric space and let H' be a subgraph of H such that $E(H') \supseteq \Delta$. A sequence $\{x_n\}$ in X will be called $E(H'_S)$-convergent to $x \in X$ if for each $r \in (0,1)$ and $t > 0$ we can find $n_0 \in \mathbb{N}$ such that $(x, x_n) \in E(H') \cap E(H'^{-1})$ and $M_{\mathrm{H}}(x_n, x, t) > 1 - r$, for all $n \geq n_0$.

Following similar arguments to those used in Remark 4.3, it is not hard to check that $E(H'_S)$-convergent sequences are $E(H'_I)$-convergent, but the converse is not true, in general.

Now, we continue tackling the concept of Cauchyness in graphical fuzzy metric spaces. The concepts of Cauchy sequences and completeness in fuzzy metric spaces are defined in two different senses, namely, in Grabiec sense (see [2]) and in George and Veeramani sense (see [1]). In similar perspective, we define two types of Cauchy sequences and their respective completeness.

Definition 4.5. Let $(X, M_{\mathrm{H}}, *)$ be a graphical fuzzy metric space and $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ is called a G-Cauchy sequence if for all $t > 0$ and $p \in \mathbb{N}$ we have $\lim_{n\to\infty} M_{\mathrm{H}}(x_{n+p}, x_n, t) = 1$. Or equivalently, the sequence $\{x_n\}$ is G-Cauchy if for all $t > 0$, $p \in \mathbb{N}$ and $\varepsilon \in (0,1)$ there exists $n_0 \in \mathbb{N}$ such that $1 - M_{\mathrm{H}}(x_n, x_{n+p}, t) < \varepsilon$ for all $n \geq n_0$. A G-complete graphical fuzzy metric space is a graphical fuzzy metric space in which every G-Cauchy sequence is M_{H}-convergent.

Definition 4.6. Let $(X, M_{\mathrm{H}}, *)$ be a graphical fuzzy metric space and $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ is called an M-Cauchy sequence if for all $t > 0$ we have $\lim_{n,m\to\infty} M_{\mathrm{H}}(x_n, x_m, t) = 1$. Or equivalently, the sequence $\{x_n\}$ is M-Cauchy if for all $t > 0$ and $\varepsilon \in (0,1)$ there exists $n_0 \in \mathbb{N}$ such that $1 - M_{\mathrm{H}}(x_n, x_m, t) < \varepsilon$ for all $n, m > n_0$. An M-complete graphical fuzzy metric space is a graphical fuzzy metric space in which every M-Cauchy sequence is M_{H}-convergent.

Remark 4.4. Every M-Cauchy sequence is a G-Cauchy sequence in graphical fuzzy metric spaces, and every G-complete graphical fuzzy metric space is an M-complete graphical fuzzy metric space, but the converse is not true in general, as every fuzzy metric space is a graphical fuzzy metric space (see [1]).

To develop fixed point results in graphical fuzzy metric spaces, the next notions of completeness will be useful.

Definition 4.7. Let $(X, M_{\mathrm{H}}, *)$ be a graphical fuzzy metric space and let H' be a subgraph of H such that $E(H') \supseteq \Delta$. Then, the space $(X, M_{\mathrm{H}}, *)$ is called H'-G-complete, $E(H'_I)$-G-complete or $E(H'_S)$-G-complete (H'-M-complete, $E(H'_I)$-M-complete or $E(H'_S)$-M-complete) if every H'-termwise connected G-Cauchy sequence (M-Cauchy sequence) in X is M_{H}-convergent, $E(H'_I)$-convergent or $E(H'_S)$-convergent, respectively.

Example 4.6. Let $X = (0, \infty)$ and H, H' be two graphs defined by $V(H) = V(H') = X$ and

$$E(H') = E(H) = \Delta \cup \{(x, y) \in X \times X, x, y \in (0, 1]\colon x \leq y\}.$$

For any fixed number $\alpha > 1$, define a fuzzy set $M_{\mathrm{H}}\colon X \times X \times (0,\infty) \to [0,1]$ by

$$M_{\mathrm{H}}(x,y,t) = \begin{cases} 1, & \text{if } \quad x = y; \\ \frac{1}{\alpha^{x+y}}, & \text{if } \quad x \neq y; \end{cases}$$

for all $t > 0$. Then, $(X, M_{\mathrm{H}}, \wedge)$ is a graphical fuzzy metric space. Notice that if $\{x_n\}$ is a G-Cauchy (or M-Cauchy) sequence in X, then it is either eventually constant or $x_n \to 0$ (with respect to usual metric of $\mathbb{R}$) and if $\{x_n\}$ is convergent in X, then it is eventually constant. Hence, X is neither G-complete nor M-complete. On the other hand, an H'-termwise connected sequence in X is a nondecreasing sequence of positive numbers, therefore it is trivial to show that X is H'-G-complete, and so H'-M-complete.

We now prove some fixed point results in G-complete graphical fuzzy metric spaces.

Remark 4.5. It is clear that Cauchy sequences and completeness of graphical fuzzy metric spaces in the sense of Saleem et al. [15] are equivalent to M-Cauchy sequences and M-completeness respectively. In Theorem 1 of Saleem et al.[15] authors assumed that X is an M-complete graphical fuzzy metric space and in the lines of the proof of this theorem they claimed that the Picard sequence generated by (H,H')-fuzzy graphical contraction is M-Cauchy sequence. Therefore, must be convergent in X. While in their proof of the claim one can see that they actually proved the sequence is G-Cauchy (not M-Cauchy). Hence, the proof of Theorem 1 of [15] is not appropriate. In the next theorem we provide an appropriate proof of the same and justify Remark 3.1.

The following theorem is the fuzzy version of the main result of Shukla et al. [18] and a generalization of the fixed point theorem of Grabiec [2] in graphical fuzzy metric spaces.

Theorem 4.3. *Let $(X, M_H, *)$ be an H'-G-complete graphical fuzzy metric space and $T\colon X \to X$ be an (H,H')-fuzzy graphical contraction. Suppose that the following conditions hold:*

(i) there exists $x_0 \in X$ such that $Tx_0 \in [x_0]^q_{H'}$ for some $q \in \mathbb{N}$;
(ii) $\lim_{t\to\infty} M_H(x,y,t) = 1$ for all $x, y \in X$ such that $(x,y) \in E(H')$;
(iii) if an H'-termwise connected T-Picard sequence $\{x_n\}$ M_H-converges to some $x \in X$, then $E(H'_I)$-converges to x.

Then there exists $x^\star \in X$ such that the T-Picard sequence $\{x_n\}$, with initial value $x_0 \in X$, is H'-termwise connected and $E(H'_I)$-converges to both $x^\star$ and $Tx^\star$.

Proof. Let $x_0 \in X$ be the given point such that $Tx_0 \in [x_0]^q_{H'}$ and $q \in \mathbb{N}$. Suppose, $\{x_n\}$ is the T-Picard sequence with initial value x_0. Then, by definition, there must be a path $\{y_i\}_{i=0}^q$ such that $x_0 = y_0, Tx_0 = y_q$ and $(y_{i-1}, y_i) \in E(H')$ for $i = 1, 2, \ldots, q$. As, T is an (H,H')-fuzzy graphical contraction, by (FGC1) we have $(Ty_{i-1}, Ty_i) \in E(H')$ for $i = 1, 2, \ldots, q$. Therefore, $\{Ty_i\}_{i=0}^q$ is a path from $Ty_0 = Tx_0 = x_1$ to $Ty_q = T^2x_0 = x_2$ of length q, and so, $x_2 \in [x_1]^q_{H'}$. In this way, for each $n \in \mathbb{N}$, we construct a path $\{T^n y_i\}_{i=0}^q$ from $T^n y_0 = T^n x_0 = x_n$ to $T^n y_q = T^n T x_0 = x_{n+1}$ of length q, which shows that $x_{n+1} \in [x_n]^q_{H'}$ for all $n \in \mathbb{N}$. Thus $\{x_n\}$ is an H'-termwise connected sequence. Since, $(T^n y_{i-1}, T^n y_i) \in E(H')$ for $i = 1, 2, \ldots, q$ and $n \in \mathbb{N}$, by (FGC2) we have

$$M_{\mathrm{H}}(T^n y_{i-1}, T^n y_i, t) \geq M_{\mathrm{H}}\left(T^{n-1} y_{i-1}, T^{n-1} y_i, \frac{t}{k}\right) \geq \cdots \geq M_{\mathrm{H}}\left(y_{i-1}, y_i, \frac{t}{k^n}\right) \quad (1)$$

for all $t > 0$. Since, the sequence $\{x_n\}$ is an H'-termwise connected sequence, for any $n \in \mathbb{N}$ we obtain from (GFM4) and (1) that

$$\begin{aligned} M_{\mathrm{H}}(x_n, x_{n+1}, t) &= M_{\mathrm{H}}(T^n y_0, T^n y_q, t) \\ &\geq M_{\mathrm{H}}\left(T^n y_0, T^n y_1, \frac{t}{q}\right) * M_{\mathrm{H}}\left(T^n y_1, T^n y_2, \frac{t}{q}\right) \\ &* \cdots * M_{\mathrm{H}}\left(T^n y_{q-1}, T^n y_q, \frac{t}{q}\right) \\ &\geq M_{\mathrm{H}}\left(y_0, y_1, \frac{t}{qk^n}\right) * M_{\mathrm{H}}\left(y_1, y_2, \frac{t}{qk^n}\right) * \cdots * M_{\mathrm{H}}\left(y_{q-1}, y_q, \frac{t}{qk^n}\right) \\ &= \prod_{i=1}^{q} M_{\mathrm{H}}\left(y_{i-1}, y_i, \frac{t}{qk^n}\right). \end{aligned}$$

Again, since the sequence $\{x_n\}$ is an H'-termwise connected sequence then, for all $t > 0$ and $p \in \mathbb{N}$, we have

$$\begin{aligned} M_{\mathrm{H}}(x_n, x_{n+p}, t) &\geq M_{\mathrm{H}}\left(x_n, x_{n+1}, \frac{t}{p}\right) * M_{\mathrm{H}}\left(x_{n+1}, x_{n+2}, \frac{t}{p}\right) \\ &\quad * \cdots * M_{\mathrm{H}}\left(x_{n+p-1}, x_{n+p}, \frac{t}{p}\right) \\ &\geq \prod_{i=1}^{q} M_{\mathrm{H}}\left(y_{i-1}, y_i, \frac{t}{pqk^n}\right) * \prod_{i=1}^{q} M_{\mathrm{H}}\left(y_{i-1}, y_i, \frac{t}{pqk^{n+1}}\right) \\ &\quad * \cdots * \prod_{i=1}^{q} M_{\mathrm{H}}\left(y_{i-1}, y_i, \frac{t}{pqk^{n+p-1}}\right) \end{aligned}$$

As, $0 < k < 1$, letting $n \to \infty$ and using (ii), we obtain

$$\lim_{n\to\infty} M_{\mathrm{H}}(x_n, x_{n+p}, t) \geq 1 * 1 * 1 * \cdots * 1 = 1$$

Therefore, $\{x_n\}$ is an H'-termwise connected G-Cauchy sequence in X. Now by the H'-G-completeness of X, the sequence $\{x_n\}$ M_{H}-converges to some $x^\star \in X$ and, by the condition (iii), we have that $\{x_n\}$ $E(H'_I)$-converges to $x^\star$. Then, there exists $n_0 \in \mathbb{N}$ such that $(x^\star, x_n) \in E(H'_I)$ for all $n \geq n_0$ and

$$\lim_{n\to\infty} M_{\mathrm{H}}(x_n, x^\star, t) = 1 \text{ for all } t > 0.$$

Thus, by (FGC1) we obtain $(Tx^\star, x_{n+1}) = (Tx^\star, Tx_n) \in E(H'_I)$, for each $n \geq n_0$, and using (FGC2), we get

$$M_{\mathrm{H}}(x_{n+1}, Tx^\star, t) = M_{\mathrm{H}}(Tx_n, Tx^\star, t) \geq M_{\mathrm{H}}\left(x_n, x^\star, \frac{t}{k}\right), \text{ for all } n \geq n_0.$$

Since, $\lim_{n\to\infty} M_{\rm H}(x_n, x^\star, t) = 1$ for all $t > 0$, we get

$$\lim_{n\to\infty} M_{\rm H}(x_{n+1}, Tx^\star, t) = 1$$

Thus, the sequence $\{x_n\}$ $E(H'_I)$-converges to $Tx^\star \in X$ as well. Hence the sequence $\{x_n\}$ $E(H'_I)$-converges to both $Tx^\star$ and $x^\star$. □

An immediate corollary of the preceding theorem is provided below.

Corollary 4.1. *Let $(X, M_H, *)$ be a $E(H'_I)$-G-complete graphical fuzzy metric space and $T\colon X \to X$ be an (H, H')-fuzzy graphical contraction. Suppose that the following conditions hold:*

(i) there exists $x_0 \in X$ such that $Tx_0 \in [x_0]^q_{H'}$ for some $q \in \mathbb{N}$;
(ii) $\lim_{t\to\infty} M_H(x, y, t) = 1$ for all $x, y \in X$ such that $(x, y) \in E(H')$.

Then there exists $x^\star \in X$ such that the T-Picard sequence $\{x_n\}$, with initial value $x_0 \in X$, is H'-termwise connected and $E(H'_I)$-converges to both $x^\star$ and $Tx^\star$.

Observe that the above results ensure only the convergence of a T-Picard sequence, not the existence of fixed point of T, as illustrates the next example.

Example 4.7. Let $X = [0, \infty)$ and H be the graph defined by $V(H) = X$ and

$$E(H) = \Delta \cup \{(x, y) \in X \times X : x, y \in X \setminus \{0\}; y \leq x\}.$$

Consider the standard graphical fuzzy metric space $(X, M_{\rm H}, \wedge)$ induced by $d_{\rm H}$, where $d_{\rm H}$ is defined as:

$$d_{\rm H}(x, y) = \begin{cases} 0, & \text{if } \ x = y; \\ \min\{x, y\}, & \text{if } \ x, y \in X \setminus \{0\} \text{ and } x \neq y; \\ 1, & \text{otherwise.} \end{cases}$$

Then, $(X, M_{\rm H}, *)$ is an H'-G-complete graphical fuzzy metric space. Let the mapping $T\colon X \to X$ be defined by:

$$Tx = \begin{cases} \dfrac{x}{2}, & \text{if } \ x \neq 0; \\ 1, & \text{if } x = 0. \end{cases}$$

Suppose H' be a subgraph of graph H such that $V(H') = X, E(H') = E(H)$. Then, it is easy to see that T is an (H, H')-fuzzy graphical contraction with k such that $\frac{1}{2} \leq k < 1$. For each $x_0 \in X \setminus \{0\}$ we get $(x_0, Tx_0) \in E(H')$ such that $Tx_0 \in [x_0]^q_{H'}$, $q = 1$. It is easy to see that $\lim_{t\to\infty} M_{\rm H}(x, y, t) = 1$ for all $x, y \in X$. Also, any H'-termwise connected T-Picard sequence in X will be of the form $\{x_n\}$, $x_n = \dfrac{x_0}{2^n}$, $x_0 \in (0, \infty)$ and $M_{\rm H}$-converges to every $x^\star \in (0, \infty)$. Besides, fixed $x_0 \in (0, \infty)$, it is easy to verify that, for each $x^\star \in (0, \infty)$, we can find $n_0 \in \mathbb{N}$ such that $(x^\star, x_n) \in E(H') \subset E(H'_I)$. So, all the requirements of Theorem 4.3 are satisfied. Nevertheless, T has no fixed point.

To ensure the existence of fixed point, we involve the property (S) or the notion of $E(H'_S)$-convergence in the list of assumptions of Theorem 4.3.

Theorem 4.4. *Let $(X, M_H, *)$ be an H'-G-complete graphical fuzzy metric space and $T\colon X \to X$ be an (H, H')-fuzzy graphical contraction. Suppose that the following conditions hold:*

(i) there exists $x_0 \in X$ such that $Tx_0 \in [x_0]^q_{H'}$ for some $q \in \mathbb{N}$;
(ii) $\lim_{t\to\infty} M_H(x, y, t) = 1$ for all $x, y \in X$ such that $(x, y) \in E(H')$;
(iii) if an H'-termwise connected T-Picard sequence $\{x_n\}$ M_H-converges to some $x \in X$, then $E(H'_I)$-converges to x.

Then there exists $x^\star \in X$ such that the T-Picard sequence $\{x_n\}$, with initial value $x_0 \in X$, is H'-termwise connected and $E(H'_I)$-converges to both $x^\star$ and $Tx^\star$. In addition, if the five-tuple $(X, M_H, *, H', T)$ has the property (S), then T has a fixed point in X.

Proof. It follows from Theorem 4.3 that the T-Picard sequence $\{x_n\}$ with initial value x_0 $E(H'_I)$-converges to both $x^\star$ and $Tx^\star$. Since $x^\star \in X$ and $Tx^\star \in T(X)$, therefore by the property (S) we must have $Tx^\star = x^\star$. Thus, $x^\star$ is a fixed point of T. □

Next example vindicate that Theorem 4.4 provides a sufficient condition only for the existence of fixed point, but not for its uniqueness.

Example 4.8. Let $X = [0, \infty)$ and H a graph defined by $V(H) = X$ and

$$E(H) = \Delta \cup \{(x, y) \in X \times X, : y \le x\}.$$

Let H' be a subgraph of graph H such that $V(H') = X$

$$E(H') = \Delta \cup \{(x, y) \in [0, 1) \times [0, 1), : y \le x\}.$$

Then $(X, M_{\rm H}, \wedge)$ is a graphical fuzzy metric space, where $M_{\rm H}$ is the same as we have considered in Proposition 2 and $d_{\rm H}$ is defined as

$$d_{\rm H}(x, y) = \begin{cases} 0, & \text{if } \ x = y; \\ (x + y)^2, & \text{if } \ x \ne y. \end{cases}$$

Then, $(X, M_{\rm H}, *)$ is an H'-G-complete graphical fuzzy metric space. Define a mapping $T\colon X \to X$ by:

$$Tx = \begin{cases} \frac{x}{2}, & \text{if } x \in [0, 1); \\ x^2, & \text{if } x \in [1, \infty). \end{cases}$$

Then, it is easy to see that T is an (H, H')-fuzzy graphical contraction with k such that $\dfrac{1}{4} \le k < 1$. For each $x_0 \in [0, 1)$ we have $(x_0, Tx_0) \in E(H')$, so $Tx_0 \in [x_0]^q_{H'}$, $q = 1$. It is easy to see that $\lim_{t\to\infty} M_{\rm H}(x, y, t) = 1$ for all $x, y \in X$. Also, any H'-termwise connected T-Picard sequence in X is either a constant sequence (with each term equal to 1) or is in the form $\{x_n\}$, $x_n = \dfrac{x_0}{2^n}$, $x_0 \in [0, 1)$ and converges to 1 and 0 respectively, and $(1, 1), (x_n, 0) \in E(H')$ for all $n \in \mathbb{N}$. Note that, property (S) is also satisfied. Thus, all the conditions of Theorem 4.4 are satisfied and the mapping T has two fixed point in X with $\mathrm{Fix}(T) = \{0, 1\}$.

In the next two corollaries, we prove the existence of fixed point of an (H, H')-fuzzy graphical contraction using the notion of $E(H'_S)$-convergence.

Corollary 4.2. *Let $(X, M_H, *)$ be an H'-G-complete graphical fuzzy metric space and $T\colon X \to X$ be an (H, H')-fuzzy graphical contraction. Suppose that the following conditions hold:*

(i) there exists $x_0 \in X$ such that $Tx_0 \in [x_0]^q_{H'}$ for some $q \in \mathbb{N}$;
(ii) $\lim_{t\to\infty} M_H(x, y, t) = 1$ for all $x, y \in X$ such that $(x, y) \in E(H')$;
(iii') if an H'-termwise connected T-Picard sequence $\{x_n\}$ M_H-converges to some $x \in X$, then $E(H'_S)$-converges to x.

Then there exists $x^\star \in X$ such that the T-Picard sequence $\{x_n\}$, with initial value $x_0 \in X$, is H'-termwise connected and $E(H'_S)$-converges to both $x^\star$ and $Tx^\star$. In addition, T has a fixed point.

Proof. Obviously, conditions (i), (ii) and (iii) in Theorem 4.4 are satisfied. So, such a theorem ensures that there exists $x^\star \in X$ such that the T-Picard sequence $\{x_n\}$, with initial value $x_0 \in X$, is H'-termwise connected and $E(H'_I)$-converges to both $x^\star$ and $Tx^\star$. Taking into account that $E(H'_I)$-convergence implies M_{H}-convergence, by (iii') we conclude that $\{x_n\}$ $E(H'_S)$-converges to both $x^\star$ and $Tx^\star$. It remains to show that $x^\star = Tx^\star$. To this end, we claim that the five-tuple $(X, M_{\mathrm{H}}, *, H', T)$ has the property (S). Let $\{y_n\}$ be an H'-termewise connected T-Picard sequence M_{H}-converging to both $y^\star \in X$ and $z^\star \in T(X)$. By (iii') we have that $\{y_n\}$ $E(H'_S)$-converges to both $y^\star \in X$ and $z^\star \in T(X)$. So, there exists $n_0 \in \mathbb{N}$ such that $(x_n, y^\star), (x_n, z^\star) \in E(H') \cap E(H'^{-1})$ and $\lim_{n\to\infty} M_{\mathrm{H}}(x_n, y^\star, t) = 1 = \lim_{n\to\infty} M_{\mathrm{H}}(x_n, z^\star, t)$. Observe that, $x_n \in (y^\star P z^\star)_H$ for each $n \geq n_0$. Indeed, $(y^\star, x_n), (x_n, z^\star) \in E(H') \subset E(H)$ for each $n \geq n_0$. Therefore, fixed $t > 0$, we obtain by (GFM4) the following:

$$M_{\mathrm{H}}(y^\star, z^\star, t) \geq M_{\mathrm{H}}(y^\star, x_n, t/2) * M_{\mathrm{H}}(x_n, z^\star, t/2), \text{ for all } n \geq n_0.$$

Thus, taking limits in the above inequality we conclude $M_{\mathrm{H}}(y^\star, z^\star, t) = 1$ for all $t > 0$ and so, $y^\star = z^\star$. Thus, the five-tuple $(X, M_{\mathrm{H}}, *, H', T)$ has the property (S) and by Theorem 4.4 we conclude that T has a fixed point. □

Corollary 4.3. *Let $(X, M_H, *)$ be an $E(H'_S)$-G-complete graphical fuzzy metric space and $T\colon X \to X$ be an (H, H')-fuzzy graphical contraction. Suppose that the following conditions hold:*

(i) there exists $x_0 \in X$ such that $Tx_0 \in [x_0]^q_{H'}$ for some $q \in \mathbb{N}$;
(ii) $\lim_{t\to\infty} M_H(x, y, t) = 1$ for all $x, y \in X$ such that $(x, y) \in E(H')$.

Then there exists $x^\star \in X$ such that the T-Picard sequence $\{x_n\}$, with initial value $x_0 \in X$, is H'-termwise connected and $E(H'_S)$-converges to both $x^\star$ and $Tx^\star$. In addition, T has a fixed point.

We now state a theorem for the uniqueness of fixed point of an (H, H')-fuzzy graphical contraction.

Theorem 4.5. *Let $(X, M_H, *)$ be an H'-G-complete graphical fuzzy metric space and $T\colon X \to X$ be an (H, H')-fuzzy graphical contraction. If all the conditions of Theorem 4.4 are satisfied and X_T is connected (as a subgraph of H'), then T has a unique fixed point.*

Proof. The existence of fixed point of T follows from Theorem 4.4. Suppose that X_T is connected (as a subgraph of H') and $x^\star, y^\star$ are two distinct fixed points of T. Since $E(H') \supseteq \Delta$, therefore $\mathrm{Fix}(T) \subseteq X_T$ and so $x^\star, y^\star \in X_T$. Again X_T is connected, then we have $(x^\star P y^\star)_{H'}$, that is, there exists a sequence $\{x_i\}_{i=0}^q, x_0 = x^\star, x_q = y^\star$ and $(x_{i-1}, x_i) \in E(H')$ for $i = 1, 2, \ldots, q$. Since, T is an (H, H')-graphical fuzzy contraction, by successive use of (FGC1) we have $(T^n x_{i-1}, T^n x_i) \in E(H')$ for $i = 1, 2, \ldots, q$ and for all $n \in \mathbb{N}$. Therefore, by (FGC2) we obtain

$$M_{\mathrm{H}}(T^n x_{i-1}, T^n x_i, t) \geq M_{\mathrm{H}}\left(T^{n-1} x_{i-1}, T^{n-1} x_i, \frac{t}{k}\right) \geq \cdots \geq M_{\mathrm{H}}\left(x_{i-1}, x_i, \frac{t}{k^n}\right) \quad (2)$$

for $i = 1, 2, \ldots, q$ and for all $n \in \mathbb{N}$. Therefore, by (GFM4) we obtain

$$M_{\mathrm{H}}(T^n x^\star, T^n y^\star, t) \geq \prod_{i=1}^{q} M_{\mathrm{H}}\left(T^n x_{i-1}, T^n x_i, \frac{t}{q}\right) \geq \prod_{i=1}^{q} M_{\mathrm{H}}\left(x_{i-1}, x_i, \frac{t}{qk^n}\right)$$

Since $x^\star, y^\star \in \mathrm{Fix}(T)$, we have $T^n x^\star = x^\star, T^n y^\star = y^\star$; therefore letting $n \to \infty$, it follows from the above inequality that $M_{\mathrm{H}}(x^\star, y^\star, t) = 1$ for all $t > 0$, that is, $x^\star = y^\star$. This contradiction proves the uniqueness of the fixed point of T. □

Below, we show that the celebrated fixed point theorem proved by Grabiec in [2] can be obtained as a corollary of the preceding result.

Corollary 4.4 (Grabiec [2]). *Let $(X, M, *)$ be a G-complete fuzzy metric space such that $\lim_{t\to\infty} M(x, y, t) = 1$ for all $x, y \in X$ and $T\colon X \to X$ be a mapping such that $M(Tx, Ty, kt) \geq M(x, y, t)$ for all $x, y \in X$, where $0 < k < 1$. Then T has a unique fixed point.*

Proof. Define the graphs H and H' such that $V(H) = V(H') = X$ and $E(H) = E(H') = X \times X$, then it is easy to see that all the conditions of Theorem 4.5 are satisfied, and so, the conclusion follows. □

The preceding fixed point results have been obtained in the framework of G-completeness with the aim of overcoming some flaws in the main results exposed in [15]. However, different authors have exposed some disadvantages on the notion of G-completeness (see [1, 5, 19]). So, the remaining of the paper is devoted to obtain a version of the celebrated Banach fixed point theorem in the context of graphical fuzzy metrics considering the notion of M-completeness. With this aim, we have focused in the contractive condition introduced in [4] which is defined by means of t-conorms. Below, we recall the notion of t-conorm.

Definition 4.8. A binary operation $\diamond$ on $[0, 1]$ is called a t-co*norm* if, for each $a, b, c \in [0, 1]$, the following four axioms are satisfied:

(S1) $a \diamond b = b \diamond a$;
(S2) $a \diamond (b \diamond c) = (a \diamond b) \diamond c$;
(S3) $a \diamond b \leq a \diamond c$ whenever $b \leq c$;
(S4) $a \diamond 0 = a$.

If in addition, the function $\diamond : [0,1]^2 \to [0,1]$ is continuous, we will say that $\diamond$ is a continuous t-conorm.

An essential class of continuous t-conorms in the following study is the so called Archimedeans, which is defined as follows.

Definition 4.9. A t-conorm $\diamond$ is said to be Archimedean if for each $a, b \in (0,1)$ there exists $n \in \mathbb{N}$ such that $a_\diamond^{(n)} > b$, where $a_\diamond^{(n)}$ denotes (throughout the rest of the paper) $\underbrace{a \diamond \cdots \diamond a}_{n \text{ times}}$.

Below, we recall some key properties of continuous Archimedean t-conorms. We refer the reader to [9] for a deeper treatment on t-conorms.

Proposition 4.4. Let $\diamond$ be an Archimedean t-conorm. Then, $\lim_{n\to\infty} a_\diamond^{(n)} = 1$ for each $a \in (0,1)$. Besides, $a \diamond a > a$ for each $a \in (0,1)$.

Observe that the preceding proposition implies $a \diamond b > a$ for each $a, b \in (0,1)$, whenever $\diamond$ is a continuous Archimedean t-conorm.

Now, we are able to define a new notion of contractivity in the context of graphical fuzzy metric spaces, based on the one introduced in [4].

Definition 4.10. Let $(X, M_{\mathrm{H}}, *)$ be a graphical fuzzy metric space, $\diamond$ a continuous t-conorm, $T\colon X \to X$ a mapping and H' be a subgraph of H such that $E(H') \supseteq \Delta$. Then, T is called a $\diamond$-(H, H')-fuzzy graphical contraction if the following conditions hold:

(FGC1) $(x, y) \in E(H')$ implies $(Tx, Ty) \in E(H')$ i.e., T is edge-preserving in H';

($\diamond$FGC2) there exists $0 < k < 1$ such that $M_{\mathrm{H}}(Tx, Ty, t) \geq k \diamond M_{\mathrm{H}}(x, y, t)$ for all $x, y \in X$ with $(xPy)_{H'}$.

Below, we establish a fixed point theorem for $\diamond$-(H, H')-fuzzy graphical contraction in H'-M-complete graphical fuzzy metric spaces.

Theorem 4.6. *Let $(X, M_H, *)$ be an H'-M-complete graphical fuzzy metric space, $\diamond$ be a continuous Archimedean t-conorm and $T\colon X \to X$ be a $\diamond$-(H, H')-fuzzy graphical contraction. Suppose that the following conditions hold:*

(i) there exists $x_0 \in X$ such that $Tx_0 \in [x_0]_{H'}^q$, for some $q \in \mathbb{N}$;

(ii) if an H'-termwise connected T-Picard sequence $\{x_n\}$ M_H-converges to some $x \in X$, then $E(H'_I)$-converges to x.

Then there exists $x^\star \in X$ such that the T-Picard sequence $\{x_n\}$, with initial value $x_0 \in X$, is H'-termwise connected and $E(H'_I)$-converges to both $x^\star$ and $Tx^\star$. In addition, if the five-tuple $(X, M_H, *, H', T)$ has the property (S), then T has a fixed point in X.

Proof. Let $x_0 \in X$ such that $Tx_0 \in [x_0]_{H'}^q$, with $q \in \mathbb{N}$ and consider $\{x_n\}$ the T-Picard sequence with initial value x_0. The same arguments to those used at the beginning of the proof of Theorem 4.3 ensure $\{x_n\}$ is an H'-termwise connected sequence. Besides, if $\{y_i\}_{i=0}^q$ is a path such that $x_0 = y_0, x_1 = Tx_0 = y_q$ and $(y_{i-1}, y_i) \in E(H')$, for

$i = 1, 2, \ldots, q$, such arguments lead to $(T^n y_{i-1}, T^n y_i) \in E(H')$ for $i = 1, 2, \ldots, q$ and $n \in \mathbb{N}$. Then, $(x_n P x_{n+1})_{H'}$ for each $n \in \mathbb{N}$. So, by ($\diamond$FGC2) we have, for all $n \in \mathbb{N}$ and $t > 0$, the following

$$\begin{aligned} M_{\mathrm{H}}(x_n, x_{n+1}, t) &= M_{\mathrm{H}}(T^n x_0, T^{n+1} x_0, t) \geq k \diamond M_{\mathrm{H}}\left(T^{n-1} x_0, T^n x_0, t\right) \\ &\geq k \diamond^{(2)} M_{\mathrm{H}}\left(T^{n-2} x_0, T^{n-1} x_0, t\right) \geq \cdots \geq k_\diamond^{(n)} \diamond M_{\mathrm{H}}\left(x_0, x_1, t\right) \geq k_\diamond^{(n)}. \end{aligned} \tag{3}$$

Then, taking into account that $\diamond$ is Archimedean we conclude that

$$\lim_{n\to\infty} M_{\mathrm{H}}(x_n, x_{n+1}, t) \geq \lim_{n\to\infty} k_\diamond^{(n)} = 1, \text{ for all } t > 0. \tag{4}$$

Hence, $\lim_{n\to\infty}\left(\lim_{t\to 0^+} M_{\mathrm{H}}(x_n, x_{n+1}, t)\right) = 1$, where $\lim_{t\to 0^+}$ denotes the right sided limit to 0.

We will see that $\{x_n\}$ is an M-Cauchy sequence by contradiction. So assume that $\{x_n\}$ is not an M-Cauchy sequence. Therefore, there exist $\varepsilon_0 \in (0, 1)$ and $t_0 > 0$ such that for all $n \in \mathbb{N}$ we can find $l > n$ satisfying $M_{\mathrm{H}}(x_n, x_l, t_0) \leq 1 - \varepsilon_0$. With this assumption, we construct a subsequence $\{x_{n_l}\}$ of $\{x_n\}$ as follows.

Let $n_1 = 1$. Now, for all $l \in \mathbb{N}$ we take n_{l+1} as the least integer greater than n_l satisfying $M_{\mathrm{H}}(x_{n_l}, x_{n_{l+1}}, t_0) \leq 1 - \varepsilon_0$, i.e. $M_{\mathrm{H}}(x_{n_l}, x_p, t_0) > 1 - \varepsilon_0$ for each $p \in \{n_l, \ldots, n_{l+1} - 1\}$. Note that our previous assumption allows to get such a construction. Then, for all $l \in \mathbb{N}$ and $0 < s < t_0$, we have by (GFM4)

$$1 - \varepsilon_0 \geq M_{\mathrm{H}}(x_{n_l}, x_{n_{l+1}}, t_0) \geq M_{\mathrm{H}}(x_{n_l}, x_{n_{l+1}-1}, t_0 - s) * M_{\mathrm{H}}(x_{n_{l+1}-1}, x_{n_{l+1}}, s).$$

So, by continuity of $*$ and (GFM5) we get, for all $l \in \mathbb{N}$

$$\begin{aligned} 1 - \varepsilon_0 \geq M_{\mathrm{H}}(x_{n_l}, x_{n_{l+1}}, t_0) \geq \\ \lim_{s\to 0^+}\left(M_{\mathrm{H}}(x_{n_l}, x_{n_{l+1}-1}, t_0 - s) * M_{\mathrm{H}}(x_{n_{l+1}-1}, x_{n_{l+1}}, s)\right) = \\ M_{\mathrm{H}}(x_{n_l}, x_{n_{l+1}-1}, t_0) * \lim_{s\to 0^+} M_{\mathrm{H}}(x_{n_{l+1}-1}, x_{n_{l+1}}, s) \geq \\ (1 - \varepsilon_0) * \lim_{s\to 0^+} M_{\mathrm{H}}(x_{n_{l+1}-1}, x_{n_{l+1}}, s). \end{aligned}$$

Taking limit as l tends to ∞ in the above inequality we conclude, again by continuity of $*$, that $\lim_{l\to\infty} M_{\mathrm{H}}(x_{n_l}, x_{n_{l+1}}, t_0) = 1 - \varepsilon_0$.

On the other hand, by (GFM4) and ($\diamond$FGC2) we have, for each $l \in \mathbb{N}$ and $0 < s < t_0$ we have

$$\begin{aligned} M_{\mathrm{H}}(x_{n_l}, x_{n_{l+1}}, t_0) \geq \\ M_{\mathrm{H}}(x_{n_l}, x_{n_l+1}, s/2) * M_{\mathrm{H}}(x_{n_l+1}, x_{n_{l+1}+1}, t_0 - s) * M_{\mathrm{H}}(x_{n_{l+1}+1}, x_{n_{l+1}}, s/2) \geq \\ M_{\mathrm{H}}(x_{n_l}, x_{n_l+1}, s/2) * \left(k \diamond M_{\mathrm{H}}(x_{n_l}, x_{n_{l+1}}, t_0 - s)\right) * M_{\mathrm{H}}(x_{n_{l+1}+1}, x_{n_{l+1}}, s/2). \end{aligned}$$

Again, by continuity of $*$ and axiom (GFM5) we conclude

$$\begin{aligned} M_{\mathrm{H}}(x_{n_l}, x_{n_{l+1}}, t_0) \geq \\ \left(\lim_{t\to 0^+} M_{\mathrm{H}}(x_{n_l}, x_{n_l+1}, t)\right) * \left(k \diamond M_{\mathrm{H}}(x_{n_l}, x_{n_{l+1}}, t_0)\right) * \left(\lim_{t\to 0^+} M_{\mathrm{H}}(x_{n_{l+1}+1}, x_{n_{l+1}}, t)\right). \end{aligned}$$

Now, taking limit as l tends to ∞ we obtain, by continuity of $*$ and $\diamond$, the next inequality

$$1-\varepsilon_0 = \lim_{l\to\infty} M_{\mathrm{H}}(x_{n_l}, x_{n_{l+1}}, t_0) \geq k \diamond \left(\lim_{l\to\infty} M_{\mathrm{H}}(x_{n_l}, x_{n_{l+1}}, t_0)\right) = k \diamond (1-\varepsilon_0).$$

On account of $\diamond$ is Archimedean we get $1-\varepsilon_0 \geq k\diamond(1-\varepsilon_0) > 1-\varepsilon_0$, a contradiction. Such a contradiction becomes from the assumption that $\{x_n\}$ is not M-Cauchy, then $\{x_n\}$ is an H'-termwise connected M-Cauchy sequence in X. Now by the H'-M-completeness of X, the sequence $\{x_n\}$ M_{H}-converges to some $x^\star \in X$ and, by the condition (ii), we have that $\{x_n\}$ $E(H'_I)$-converges to $x^\star$. Therefore, there exists $n_0 \in \mathbb{N}$ such that $(x^\star, x_n) \in E(H'_I)$ for all $n \geq n_0$ and

$$\lim_{n\to\infty} M_{\mathrm{H}}(x_n, x^\star, t) = 1 \text{ for all } t > 0.$$

Thus, by (FGC1) we obtain $(Tx^\star, x_{n+1}) = (Tx^\star, Tx_n) \in E(H'_I)$, for each $n \geq n_0$, and using ($\diamond$FGC2), we get

$$M_{\mathrm{H}}(x_{n+1}, Tx^\star, t) = M_{\mathrm{H}}(Tx_n, Tx^\star, t) \geq k \diamond M_{\mathrm{H}}\left(x_n, x^\star, t\right), \text{ for all } n \geq n_0.$$

Since, $\lim_{n\to\infty} M_{\mathrm{H}}(x_n, x^\star, t) = 1$ for all $t > 0$, we get $\lim_{n\to\infty} M_{\mathrm{H}}(x_{n+1}, Tx^\star, t) = 1$ for all $t > 0$.

Hence, the sequence $\{x_n\}$ $E(H'_I)$-converges to $Tx^\star \in X$ as well. Hence the sequence $\{x_n\}$ $E(H'_I)$-converges to both $Tx^\star$ and $x^\star$. In addition, if the five-tuple $(X, M_{\mathrm{H}}, *, H', T)$ has the property (S), it is clear that $x^\star$ is a fixed point of T in X. □

Following the same arguments used in the proof of Corollary 4.2 we obtain the next result.

Corollary 4.5. *Let $(X, M_H, *)$ be an H'-M-complete graphical fuzzy metric space, $\diamond$ be a continuous Archimedean t-conorm and $T\colon X \to X$ be a $\diamond$-(H, H')-fuzzy graphical contraction. Suppose that the following conditions hold:*

(i) there exists $x_0 \in X$ such that $Tx_0 \in [x_0]^q_{H'}$ for some $q \in \mathbb{N}$;

(ii) if an H'-termwise connected T-Picard sequence $\{x_n\}$ M_H-converges to some $x \in X$, then $E(H'_S)$-converges to x.

Then there exists $x^\star \in X$ such that the T-Picard sequence $\{x_n\}$, with initial value $x_0 \in X$, is H'-termwise connected and $E(H'_S)$-converges to both $x^\star$ and $Tx^\star$. In addition, T has a fixed point.

On account of Theorem 4.5, we are able to provide a theorem for the uniqueness of fixed point of a $\diamond$-(H, H')-fuzzy graphical contraction.

Theorem 4.7. *Let $(X, M_H, *)$ be an H'-M-complete graphical fuzzy metric space and $T\colon X \to X$ be a $\diamond$-(H, H')-fuzzy graphical contraction. If all the conditions of Theorem 4.6 are satisfied and X_T is connected (as a subgraph of H'), then T has a unique fixed point.*

Proof. The existence of fixed point of T follows from Theorem 4.6. Suppose that X_T is connected (as a subgraph of H') and $x^\star, y^\star$ are two fixed points of T. The same arguments to those used at the beginning of the proof of Theorem 4.5 allow to show that $(x^\star P y^\star)_{H'}$ and $(Tx^\star PTy^\star)_{H'}$. Then, by ($\diamond$FGC2) we obtain

$$M_{\rm H}(x^\star, y^\star, t) = M_{\rm H}\left(Tx^\star, Ty^\star, t\right) \geq k \diamond M_{\rm H}\left(x^\star, y^\star, t\right), \text{ for all } t > 0. \tag{5}$$

So, since $\diamond$ is Archimedean we deduce that $M(x^\star, y^\star, t) = 1$, for each $t > 0$, which concludes that $x^\star = y^\star$. □

Again, the same arguments exposed in the proof of Corollary 4.4 lead to obtain, as a corollary of Theorem 4.7 the main result provided in [4].

Theorem 4.8 (Gregori and Miñana [4])**.** *Let $(X, M, *)$ be an M-complete fuzzy metric space and let $T : X \to X$ be a fuzzy k-$\diamond$-contraction. If $\diamond$ is Archimedean, then T has a unique fixed point.*

5. Conclusions

Graph theory deals with the problems occur in computer science and many areas of applied mathematics, hence in a natural way the graphical structures have been incorporated with the metric and fuzzy metric spaces. Here, we have revisited the notion of graphical fuzzy metric spaces with some improvements and generalizations of related definitions, properties and fixed point results. We have also proved some new fixed point results in such spaces which generalize some celebrated results. Examples are included so that the claims are verified and the results illustrated.

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