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A NEW GENERALIZATION OF ĆIRIĆ'S MULTI-VALUED OPERATORS

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Abstract. The aim of this paper is to introduce a new type of multi-valued operators and to present some basic problems of the fixed points and strict fixed points for them. Obtained results generalize, complement and extend classical results given by Ćirić (Mat. Vesnik 9 (24): 265-272, (1972)) or Nadler (Pacific J. Math. 30: 475-488 (1969)), as well as recent results given by Alecsa and Petruşel (Anal. Univ. Vest Timisoara, LVII (1): 23-42 (2019)).

Key Words and Phrases: Metric space, fixed point, Ćirić's multi-valued operator, data dependence, Ulam-Hyers stability, Ostrowski property, strict fixed point.
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1. Introduction

Let (X, d) be a metric space and $P(X)$ the family of all nonempty subsets of X. We denote by $P_{cl}(X)$ the family of nonempty, closed subsets of X, by $P_b(X)$ the family of nonempty, bounded subsets of X, and by $P_{cp}(X)$ the family of nonempty, compact subsets of X. Also, by $B(x_0, r) := \{x \in X : d(x_0, x) < r\}$, we denote the open ball with radius $r > 0$ and center $x_0 \in X$ and by $\bar{B}(x_0, r) := \{x \in X : d(x_0, x) \leq r\}$ we denote the closed ball centered in $x_0 \in X$ and with radius $r > 0$.

The following important functionals will be used throughout the paper:
- the gap functional $D : P(X) \times P(X) \to \mathbb{R}_+, D(A, B) := \inf_{a\in A, b\in B} \{d(a, b)\}$,
- the excess functional $\rho : P(X) \times P(X) \to \mathbb{R}_+, \rho(A, B) := \sup_{a\in A} \{D(a, B)\}$,
- the generalized Pompeiu-Haussdorf functional $H : P(X) \times P(X) \to \mathbb{R}_+ \cup \{+\infty\}$,

$$H(A, B) := \max \left\{ \sup_{a\in A} D(a, B), \sup_{b\in B} D(b, A) \right\}.$$

It is well known that $(P_{cl}(X), H)$ is a complete generalized metric space provided (X, d) is a complete metric space [14]. Furthermore, an element $x \in X$ is a fixed point (strict fixed point or an endpoint) for a multi-valued operator $T : X \to P(X)$ if and only if $x \in Tx(\{x\} = Tx)$. We denote by F_T the set of all fixed points of T and by $(SF)_T$ the set of all strict fixed points of T. A single-valued mapping $t : X \to X$ is called a selection of T if for each $x \in X$, we have $tx \in Tx$.

A sequence (x_n) satisfying the following conditions:
(i) $x_0 = x$,
(ii) $x_{n+1} \in Tx_n$ for each $n \in \mathbb{N} \cup \{0\}$,
(iii) $x_n \to x^* \in F_T$ as $n \to \infty$,
is called a sequence of successive approximations (for short, s.s.a.) of T starting from $x \in X$.
We denote by $V(Y;\epsilon) := \{x \in X : D(x,Y) < \epsilon\}$ the ϵ - neighborhood of the set $Y \in P(X)$.

In 1969, Nadler [14] proved a multi-valued extension of the Banach contraction principle.

Theorem 1.1 *Let (X,d) be a complete metric space and let T be a mapping from X into $P_b(X)$. Assume that there exists $\alpha \in [0,1)$ such that $H(Tx,Ty) \leq \alpha d(x,y)$ for all $x,y \in X$. Then, there exists $z \in X$ such that $z \in Tz$.*

Many fixed point theorems have been proved by various authors as generalizations of Nadler's theorem (see [4], [7], [13], [22]). One of the general fixed point theorems for a generalized multi-valued mappings belongs to Ćirić [5].

Theorem 1.2 *Let (X,d) be a complete metric space and let T be a mapping from X into $P_{cl}(X)$. Assume that there exists $\alpha \in [0,1)$ such that $H(Tx,Ty) \leq \alpha M(x,y)$ for all $x,y \in X$, where*

$$M(x,y) := \max\{d(x,y), D(x,Tx), D(y,Ty), [D(x,Ty) + D(y,Tx)]/2\},$$

Then, there exists $z \in X$ such that $z \in Tz$.

Following the approach given in [15], Alecsa and Petruşel [1] gave a fully comprehensive study on Ćirić type multi-valued operators, i.e. operators which satisfy the inequality from Theorem 1.2. They have studied qualitative properties, namely data dependence, well-posedness, Ulam-Hyers stability, Ostrowski property. In [3], Boriceanu studied the existence and uniqueness of the fixed point and data dependence for multi-valued operators in the context of b-metric spaces. Also, Ćirić type multi-valued operators have been studied in [16] - [20].

The following lemma from [2] will be necessary in the future results.

Lemma 1.3 *Let $(a_n)_{n\in\mathbb{N}}, (b_n)_{n\in\mathbb{N}}$ two sequences of nonnegative numbers and $0 \leq k < 1$ such that*

$$a_{n+1} \leq k a_n + b_n$$

for all $n \geq 1$. If $\lim_{n\to\infty} b_n = 0$, then $\lim_{n\to\infty} a_n = 0$.

Now, we recall the basic concepts for the qualitative properties of the fixed point inclusion and of the fixed point iteration. The first two definitions are related to the concept of well-posedness of the fixed point problem, see [11] and [19].

Definition 1.4 Let (X,d) be a metric space and $T : Y \to P_{cl}(X)$ be a multi-valued operator. Then, the fixed point problem is well-posed for T with respect to the gap functional D if and only if
(i) $F_T = \{x^*\}$,
(ii) if $(x_n) \subset X$ has the property that $D(x_n, Tx_n) \to 0$, then $x_n \to x^*$.

Definition 1.5 Let (X,d) be a metric space and $T : Y \to P_{cl}(X)$ be a multi-valued operator. Then, the fixed point problem is well-posed for T with respect to the Pompeiu-Haussdorf functional if and only if
(i) $(SF)_T = \{x^*\}$,
(ii) if $(x_n) \subset X$ has the property that $H(x_n, Tx_n) \to 0$, then $x_n \to x^*$.

Another important concept related to the fixed point problem is Ostrowski property or limit shadowing, see [10], [11].

Definition 1.6 Let (X,d) be a metric space and $T : Y \to P(X)$ be a multi-valued operator. Then, the fixed point problem has the Ostrowski property if and only if
(i) $F_T = \{x^*\}$,
(ii) if $(y_n) \subset X$ has the property that $D(y_{n+1}, Ty_n) \to 0$, then $y_n \to x^*$.

The next two definitions are related to the concept of generalized Ulam-Hyers stability, see [15].

Definition 1.7 Let (X,d) be a metric space and $T : X \to P(X)$ be a multi-valued operator. The fixed point inclusion $x \in Tx$ is called generalized Ulam-Hyers stable if and only if there exists an increasing, continuous in 0 function $\psi : \mathbb{R}_+ \to \mathbb{R}_+, \psi(0) = 0$, such that for every $\epsilon > 0$ and for each $y^* \in X$ with $D(y^*, Ty^*) \leq \epsilon$, there exists a solution x^* of the fixed point inclusion such that $d(x^*, y^*) \leq \psi(\epsilon)$.

Definition 1.8 Let (X,d) be a metric space and $T : X \to P(X)$ be a multi-valued operator. The strict fixed point inclusion $\{x\} = Tx$ is called generalized Ulam-Hyers stable if and only if there exists an increasing, continuous in 0 function $\psi : \mathbb{R}_+ \to \mathbb{R}_+, \psi(0) = 0$, such that for every $\epsilon > 0$ and for each $y^* \in X$ with $H(y^*, Ty^*) \leq \epsilon$, there exists a solution x^* of the strict fixed point inclusion such that $d(x^*, y^*) \leq \psi(\epsilon)$.

Finally, following [8], [9], [17] and [18], we recall the last important concepts.

Definition 1.9 Let $X \neq \emptyset$ and $T : X \to P(X)$ be a multi-valued operator. Then, T has the approximate endpoint property if

$$\inf_{x \in X} \sup_{y \in Tx} d(x,y) = 0.$$

Definition 1.10 Let $X \neq \emptyset$ and $T : X \to P(X)$ be a multi-valued operator. We say that $t : X \to X$ is a Caristi selection of T if there exists a function $\varphi : X \to \mathbb{R}_+$ such that

$$d(x, tx) \leq \varphi(x) - \varphi(tx),$$

for each $x \in X$, where $tx \in Tx$ for each $x \in X$.

The aim of this paper is to introduce a new class of multi-valued operators which includes the Ćirić type multi-valued generalized contractions, and to study the metrical and topological properties for the fixed point problems. Our results generalize, complement and extend many classical results and also recent results, and open a new direction in this field of research.

2. Main results

Definition 2.1 Let (X, d) be a metric space and $T : X \to P(X)$ be a multi-valued operator. We say that T is a multi-valued $\mathcal{P}$- operator if there exists $\alpha \in (0, 1)$ such that

$$H(Tx, Ty) \leq \alpha P(x, y)$$

for each $x, y \in X$, where

$$P(x, y) := \max \{E_1(x, y), E_2(x, y), E_3(x, y), E_4(x, y)\},$$

$$E_1(x, y) := d(x, y) + D(x, Tx) - D(y, Ty),$$

$$E_2(x, y) := d(x, y) + D(y, Ty) - D(x, Tx),$$

$$E_3(x, y) := D(x, Tx) + D(y, Ty) - d(x, y)$$

and

$$E_4(x, y) := (1/2)[D(x, Ty) + D(y, Tx) + |D(x, Tx) - D(y, Ty)|].$$

Remark 2.2 Since

$$\max\left\{a, b, c, \frac{d+e}{2}\right\} \leq \max\left\{a + b - c, a + c - b, b + c - a, \frac{d + e + |b - c|}{2}\right\}$$

for every $a, b, c, d, e \in \mathbb{R}$, it is obvious that every multivalued α - Ćirić type operator is a multi-valued $\alpha - \mathcal{P}$ - operator.

In the following example, we will see that there exist multi-valued $\mathcal{P}$ - operators which are not multi-valued Ćirić type operators.

Example 2.3 Let $X = \{0, 1\}$ and $d : X \times X \to \mathbb{R}$, $d(x, y) = |x - y|$. Let $T : X \to P(X)$, $T0 = \{1\}$, $T1 = \{0, 1\}$. Since $D(0, T0) = 1$, $D(0, T1) = D(1, T0) = D(1, T1) = 0$, we have $H(T0, T1) = M(0, 1) = 1$ and $P(0, 1) = 2$. It is obvious that T is not a multivalued Ćirić type operator, but T is a $(1/2) - \mathcal{P}$ - operator.

Now we can prove the main result of this paper.

Theorem 2.4 *Let (X, d) be a complete metric space and $T : X \to P_{cl}(X)$ be a multi-valued $\alpha - \mathcal{P}$ - operator. Then, the following conclusions hold:*
(a) there exists $x^* \in F_T$;
(b) for each $x \in X$, there exists a s.s.a. (x_n) for T, starting from x, convergent to a fixed point of T;
(c) if (x_n) is a s.s.a. for T, starting from x_0, convergent to $x^* \in F_T$, then

$$d(x_0, x^*) \leq \frac{1+\alpha}{1-\alpha} d(x_0, x_1);$$

(d) if (x_n) is a s.s.a. for T, starting from x_0, convergent to $x^* \in F_T$, then for every $n \geq 1$

$$d(x_n, x^*) \leq \frac{\beta^n}{1-\beta} d(x_0, x_1),$$

where $\beta = 2\alpha/(1+\alpha)$*;*
(e) F_T *is closed in* (X, d)*;*
(f) if $G : X \to P_{cl}(X)$ *is a* $\beta - \mathcal{P}$ *- operator and* $H(Tx, Gx) \leq \eta$ *for all* $x \in X$*, then*

$$H(F_T, F_G) \leq \eta \max\left\{\frac{1+\alpha}{\alpha(1-\alpha)}, \frac{1+\beta}{\beta(1-\beta)}\right\};$$

(g) if $T_n : X \to P_{cl}(X)$ *is a sequence of* $\alpha - \mathcal{P}$ *- operators with* $\lim_{n\to\infty} H(T_n x, Tx) = 0$*, uniformly with respect to* $x \in X$*, then* $\lim_{n\to\infty} H(F_{T_n}, F_T) = 0$*;*
(h) if there exists $x_0 \in X$ *and* $r > 0$ *such that* $D(x_0, Tx_0) < \frac{1-\alpha}{1+\alpha} r$*, then there exists* $x^* \in F_T \cap B(x_0, r)$*;*
(i) if there exists $x_0 \in X$ *and* $r > 0$ *such that* $\rho(x_0, Tx_0) < \frac{1-\alpha}{1+\alpha} r$*, then* $T : \bar{B}(x_0, r) \to P(\bar{B}(x_0, \frac{1+\alpha+2\alpha^2}{1-\alpha^2} r))$ *and there exists* $x^* \in F_T \cap B(x_0, r)$*;*
(j) if X *is a Banach space,* U *an open subset of* X *and* $T : U \to P_{cl}(X)$ *is a multi-valued* $\mathcal{P}$*-operator, then the associated multivalued operator* $G : U \to P(X)$*,* $G(x) = x - Tx$ *is open;*
(k) there exists a Caristi selection of T*;*
(l) if, additionally, $T : X \to P_{cp}(X)$*, then the fixed point inclusion* $x \in Tx$ *is generalized Ulam-Hyers stable;*
(m) the multi-valued operator T *has the approximate fixed point property;*
(n) if the multi-valued operator T *is lower semicontinuous, then it has the approximate endpoint property if and only if it has a unique strict fixed point;*
(o) if $\alpha < 1/3$*, then* F_T *is compact;*
(p) if $T : X \to P_{b,cl}(X)$*, then for each* $p > 0$*, one has*

$$H(F_p^*, F_T) \leq \frac{1+\alpha}{1-\alpha} p,$$

where $F_p^* := \{x \in X : D(x, Tx) < p\}$*.*

Proof. (a), (b), (c) and (d) Let $x_0 \in X$ and $x_1 \in Tx_0$ such that $d(x_0, x_1) < \frac{1}{\alpha} D(x_0, Tx_0)$. By hypothesis, we have $H(Tx_0, Tx_1) \leq \alpha P(x_0, x_1)$. Suppose that $D(x_0, Tx_0) < D(x_1, Tx_1)$. Then, we have:

$$\begin{aligned} E_1(x_0, x_1) &= d(x_0, x_1) + D(x_0, Tx_0) - D(x_1, Tx_1) \\ &< d(x_0, x_1) + D(x_1, Tx_1) - D(x_0, Tx_0) \\ &= E_2(x_0, x_1), \end{aligned}$$

$$\begin{aligned} E_4(x_0, x_1) &= [D(x_0, Tx_1) + D(x_1, Tx_0) + |D(x_0, Tx_0) - D(x_1, Tx_1)|]/2 \\ &= [D(x_0, Tx_1) + D(x_1, Tx_1) - D(x_0, Tx_0)]/2 \\ &\leq [d(x_0, x_1) + D(x_1, Tx_1) + D(x_1, Tx_1) - D(x_0, Tx_0)]/2 \\ &= D(x_1, Tx_1) + [d(x_0, x_1) - D(x_0, Tx_0)]/2 \\ &\leq D(x_1, Tx_1) + d(x_0, x_1) - D(x_0, Tx_0) \\ &= E_2(x_0, x_1). \end{aligned}$$

Since $D(x_0, Tx_0) \le d(x_0, x_1)$, it follows that

$$\begin{aligned} E_2(x_0, x_1) &= d(x_0, x_1) + D(x_1, Tx_1) - D(x_0, Tx_0) \\ &\ge D(x_0, Tx_0) + D(x_1, Tx_1) - d(x_0, x_1) \\ &= E_3(x_0, x_1). \end{aligned}$$

Hence, we get

$$P(x_0, x_1) = E_2(x_0, x_1) = d(x_0, x_1) + D(x_1, Tx_1) - D(x_0, Tx_0),$$

so

$$D(x_1, Tx_1) \le H(Tx_0, Tx_1) \le \alpha[d(x_0, x_1) + D(x_1, Tx_1) - D(x_0, Tx_0)].$$

Then, we have

$$D(x_1, Tx_1) \le \frac{\alpha}{1-\alpha}[d(x_0, x_1) - D(x_0, Tx_0)],$$

by where

$$D(x_0, Tx_0) < \frac{\alpha}{1-\alpha}[d(x_0, x_1) - D(x_0, Tx_0)].$$

This yields to

$$D(x_0, Tx_0) < \alpha d(x_0, x_1) < D(x_0, Tx_0),$$

which is a contradiction. Therefore, we get $D(x_0, Tx_0) \ge D(x_1, Tx_1)$. In this case, it follows that:

$$\begin{aligned} E_1(x_0, x_1) &= d(x_0, x_1) + D(x_0, Tx_0) - D(x_1, Tx_1) \\ &\ge d(x_0, x_1) + D(x_1, Tx_1) - D(x_0, Tx_0) \\ &= E_2(x_0, x_1), \end{aligned}$$

$$\begin{aligned} E_4(x_0, x_1) &= [D(x_0, Tx_1) + D(x_1, Tx_0) + |D(x_0, Tx_0) - D(x_1, Tx_1)|]/2 \\ &= [D(x_0, Tx_1) + D(x_0, Tx_0) - D(x_1, Tx_1)]/2 \\ &\le [d(x_0, x_1) + D(x_1, Tx_1) + D(x_0, Tx_0) - D(x_1, Tx_1)]/2 \\ &= [d(x_0, x_1) + D(x_0, Tx_0)]/2 \\ &\le [d(x_0, x_1) + d(x_0, x_1)]/2 \\ &= d(x_0, x_1) \\ &\le d(x_0, x_1) + D(x_0, Tx_0) - D(x_1, Tx_1) \\ &= E_1(x_0, x_1). \end{aligned}$$

Since $E_2(x_0, x_1) \ge E_3(x_0, x_1)$, we obtain that

$$P(x_0, x_1) = E_1(x_0, x_1) = d(x_0, x_1) + D(x_0, Tx_0) - D(x_1, Tx_1).$$

Hence, we have

$$D(x_1, Tx_1) \le H(Tx_0, Tx_1) \le \alpha[d(x_0, x_1) + D(x_0, Tx_0) - D(x_1, Tx_1)].$$

It follows that

$$D(x_1, Tx_1) \le \frac{\alpha}{1+\alpha}[d(x_0, x_1) + D(x_0, Tx_0)],$$

by where

$$D(x_1, Tx_1) \leq \frac{2\alpha}{1+\alpha} d(x_0, x_1) = \beta d(x_0, x_1).$$

Furthermore, consider $q \in (1, \frac{1+\alpha}{2\alpha})$. Then, there exists $x_2 \in Tx_1$ such that $d(x_1, x_2) \leq q\beta d(x_0, x_1)$ and $d(x_1, x_2) \leq \frac{1}{\alpha} D(x_1, Tx_1)$. Let us denote by $\lambda := q\beta$. It is obvious that $\lambda \in (0, 1)$. Then, we have $d(x_1, x_2) \leq \lambda d(x_0, x_1)$. By induction, we can construct a sequence (x_n) such that

$$x_{n+1} \in Tx_n \quad \text{and} \quad d(x_n, x_{n+1}) \leq \lambda d(x_{n-1}, x_n)$$

for each $n \geq 0$. It follows that $d(x_n, x_{n+1}) \leq \lambda^n d(x_0, x_1)$, for each $n \geq 0$, so, by the triangle inequality, we get

$$d(x_n, x_{n+p}) \leq \lambda^n \frac{1-\lambda^p}{1-\lambda} d(x_0, x_1) \leq \frac{\lambda^n}{1-\lambda} d(x_0, x_1).$$

Letting $n \to \infty$, we obtain that (x_n) is a Cauchy sequence, hence there exists $x^* \in X$ such that $x_n \to x^*$. Taking $p \to \infty$ in the above inequality, we get for each $n \geq 0$

$$d(x_n, x^*) \leq \frac{\lambda^n}{1-\lambda} d(x_0, x_1).$$

Making $q \searrow 1$, it follows the estimate

$$d(x_n, x^*) \leq \frac{\beta^n}{1-\beta} d(x_0, x_1).$$

For $n = 0$, we get

$$d(x_0, x^*) \leq \frac{1}{1-\beta} d(x_0, x_1) = \frac{1+\alpha}{1-\alpha} d(x_0, x_1).$$

Now, we prove that $x^* \in F_T$, i.e. $D(x^*, Tx^*) = 0$. Since $D(x_n, Tx_n) \leq d(x_n, x_{n+1})$ and $\lim_{n\to\infty} d(x_n, x^*) = \lim_{n\to\infty} d(x_n, x_{n+1}) = 0$, we have $\lim_{n\to\infty} D(x_n, Tx_n) = 0$ and $\lim_{n\to\infty} d(x^*, Tx_n) = 0$. Hence,

$$\lim_{n\to\infty} E_1(x_n, x^*) = -D(x^*, Tx^*),$$

$$\lim_{n\to\infty} E_2(x_n, x^*) = \lim_{n\to\infty} E_3(x_n, x^*) = \lim_{n\to\infty} E_4(x_n, x^*) = D(x^*, Tx^*).$$

Therefore, $\lim_{n\to\infty} P(x_n, x^*) = D(x^*, Tx^*)$. By hypothesis, we have for every $n \geq 0$ that:

$$D(x_{n+1}, Tx^*) \leq H(Tx_n, Tx^*) \leq \alpha P(x_n, x^*).$$

Taking $n \to \infty$, we obtain $D(x^*, Tx^*) \leq \alpha D(x^*, Tx^*)$, so $D(x^*, Tx^*) = 0$.
(e) Let $x_n \in F_T$ such that $x_n \to x^*$. We shall show that $x^* \in F_T$, i.e.
$x^* \in Tx^*$. Since $D(x_n, Tx_n) = 0, \lim_{n\to\infty} D(x_n, Tx^*) = D(x^*, Tx^*)$,
$\lim_{n\to\infty} D(x^*, Tx_n) = 0$, we have:

$$\lim_{n\to\infty} E_1(x_n, x^*) = -D(x^*, Tx^*),$$

$$\lim_{n\to\infty} E_2(x_n, x^*) = \lim_{n\to\infty} E_3(x_n, x^*) = \lim_{n\to\infty} E_4(x_n, x^*) = D(x^*, Tx^*).$$

Therefore, $\lim_{n\to\infty} P(x_n, x^*) = D(x^*, Tx^*)$. By hypothesis, we have for every $n \geq 0$ that:

$$D(x_n, Tx^*) \leq H(Tx_n, Tx^*) \leq \alpha P(x_n, x^*).$$

Taking $n \to \infty$, we obtain $D(x^*, Tx^*) \le \alpha D(x^*, Tx^*)$, so $D(x^*, Tx^*) = 0$. Since the operator T has closed values, it follows that $x^* \in Tx^*$, i.e. $x^* \in F_T$.

(f) By (a), (b), (c) and (d), we have that $d(x_0, x^*) \le \frac{1+\alpha}{1-\alpha} d(x_0, x_1)$, where $x_0 \in X$ is arbitrarly chosen, $x_1 \in Tx_0$ such that $d(x_0, x_1) < D(x_0, x_1)/\alpha$, and $x^* \in F_T$. Taking $x_0 = y^* \in F_G$, we obtain that $d(x^*, y^*) \le \frac{1+\alpha}{1-\alpha} d(y, y^*)$, where $y \in Ty^*$ such that $d(y^*, y) < D(y^*, Ty^*)/\alpha$. It follows that $d(x^*, y^*) \le \frac{1+\alpha}{\alpha(1-\alpha)} D(y^*, Ty^*)$. Since $D(y^*, Ty^*) \le H(Gy^*, Ty^*) \le \eta$, we get $d(x^*, y^*) \le \frac{1+\alpha}{\alpha(1-\alpha)}\eta$. Hence, $D(y^*, F_T) \le \frac{1+\alpha}{\alpha(1-\alpha)}\eta$ for each $y^* \in F_G$. Similarly, we have $D(x^*, F_G) \le \frac{1+\beta}{\beta(1-\beta)}\eta$ for each $x^* \in F_T$. Then, we obtain

$$H(F_T, F_G) \le \eta \max\left\{\frac{1+\alpha}{\alpha(1-\alpha)}, \frac{1+\beta}{\beta(1-\beta)}\right\}.$$

(g) Since $\lim_{n\to\infty} H(T_n x, Tx) = 0$, uniformly with respect to $x \in X$, then for $\epsilon > 0$, arbitrarly chosen, there exists $N(\epsilon) \in \mathbb{N}$ such that

$$\sup_{x \in X} H(T_n x, Tx) < \epsilon$$

for all $n \ge N(\epsilon)$. By (f) we get that $H(F_{T_n}, F_T) < \frac{1+\alpha}{\alpha(1-\alpha)}\epsilon$, hence $\lim_{n\to\infty} H(F_{T_n}, F_T) = 0$.

(h) Let $s \in (0, r)$ such that $\bar{B}(x_0, s) \subset B(x_0, r)$, where $D(x_0, Tx_0) < \frac{1-\alpha}{1+\alpha} s < \frac{1-\alpha}{1+\alpha} r$. Then, there exists $x_1 \in Tx_0$ such that $d(x_0, x_1) < \frac{1}{\alpha} D(x_0, Tx_0)$ and $d(x_0, x_1) < \frac{1-\alpha}{1+\alpha} s$. Hence, $d(x_0, x_1) < s$, so $x_1 \in \bar{B}(x_0, s)$. From the hypothesis, we have that $H(Tx_0, Tx_1) \le \alpha P(x_0, x_1)$. Like in the proof of (a) it follows that $D(x_1, Tx_1) \le D(x_0, Tx_0)$ and $P(x_0, x_1) = d(x_0, x_1) + D(x_0, Tx_0) - D(x_1, Tx_1)$. Thus,

$$D(x_1, Tx_1) \le H(Tx_0, Tx_1) \le \alpha[d(x_0, x_1) + D(x_0, Tx_0) - D(x_1, Tx_1)].$$

It follows that

$$D(x_1, Tx_1) \le \frac{\alpha}{1+\alpha}[d(x_0, x_1) + D(x_0, Tx_0)],$$

by where

$$D(x_1, Tx_1) \le \frac{2\alpha}{1+\alpha} d(x_0, x_1) < \frac{2\alpha}{1+\alpha}\frac{1-\alpha}{1+\alpha} s = \frac{2\alpha}{1+\alpha}\left(1 - \frac{2\alpha}{1+\alpha}\right) s.$$

By the triangle inequality, we obtain

$$\begin{aligned} d(x_0, x_2) \le d(x_0, x_1) + d(x_1, x_2) &< \frac{1-\alpha}{1+\alpha} s + \frac{2\alpha}{1+\alpha}\frac{1-\alpha}{1+\alpha} s \\ &= \left(1 - \frac{2\alpha}{1+\alpha}\right)\left(1 + \frac{2\alpha}{1+\alpha}\right) s = \left[1 - \left(\frac{2\alpha}{1+\alpha}\right)^2\right] s, \end{aligned}$$

by where $d(x_0, x_2) < s$, so $x_2 \in Tx_1 \cap \bar{B}(x_0, s)$. By induction, we can construct a sequence (x_n) such that for each $n \in \mathbb{N}, x_n \in Tx_{n-1} \cap \bar{B}(x_0, s)$,

$$d(x_0, x_n) \le \left(1 - \frac{2\alpha}{1+\alpha}\right)^n s$$

and

$$d(x_{n-1}, x_n) \leq \left(\frac{2\alpha}{1+\alpha}\right)^{n-1} \left(1 - \frac{2\alpha}{1+\alpha}\right) s.$$

It follows that (x_n) is Cauchy, so there exists $x^* \in X$ such that $x_n \to x^*$. As in the proof of (a)-(d), one can show that $x^* \in Tx^*$. Since $x_n \in \bar{B}(x_0, s)$ for each $n \geq 0$, we have $x^* \in \bar{B}(x_0, s) \subset B(x_0, r)$.

(i) We have to show that $Ty \subset \bar{B}(x_0, \frac{1+\alpha}{1-\alpha} r)$ for every $y \in \bar{B}(x_0, r)$. Let $z \in Ty$. Then, $d(z, x_0) \leq d(z, u) + d(u, x_0)$ for every $u \in Tx_0$. It follows that $d(z, x_0) \leq d(z, u) + \rho(x_0, Tx_0)$ for every $u \in Tx_0$, hence taking $\inf_{u \in Tx_0}$ we get

$$d(z, x_0) \leq D(z, Tx_0) + \rho(x_0, Tx_0) < H(Ty, Tx_0) + \frac{1-\alpha}{1+\alpha} r.$$

By hypothesis, $H(Ty, Tx_0) \leq \alpha P(y, x_0)$, where

$$P(y, x_0) = \max \{E_1(y, x_0), E_2(y, x_0)), E_3(y, x_0), E_4(y, x_0)\}.$$

We employ an analysis of the following cases:

If the maximum is $E_1(y, x_0) = d(y, x_0) + D(y, Ty) - D(x_0, Tx_0)$, then

$$\begin{aligned} d(z, x_0) &< \alpha d(y, x_0) + \alpha D(y, Ty) - \alpha D(x_0, Tx_0) + \frac{1-\alpha}{1+\alpha} r \\ &\leq \alpha r + \alpha d(y, z) + \frac{1-\alpha}{1+\alpha} r \\ &\leq \alpha r + \alpha [d(y, x_0) + d(x_0, Tx_0)] + \frac{1-\alpha}{1+\alpha} r \\ &= \alpha d(x_0, z) + (2\alpha + \frac{1-\alpha}{1+\alpha}) r. \end{aligned}$$

This means that

$$d(z, x_0) < (\frac{2\alpha}{1-\alpha} + \frac{1}{1+\alpha}) r = \frac{1+\alpha+2\alpha^2}{1-\alpha^2} r.$$

If the maximum is $E_2(y, x_0) = d(y, x_0) + D(x_0, Tx_0) - D(y, Ty)$, then

$$\begin{aligned} d(z, x_0) &< \alpha d(y, x_0) + \alpha D(x_0, Tx_0) - \alpha D(y, Ty) + \frac{1-\alpha}{1+\alpha} r \\ &\leq \alpha r + \alpha \frac{1-\alpha}{1+\alpha} r + \frac{1-\alpha}{1+\alpha} r \\ &= r. \end{aligned}$$

If the maximum is $E_3(y, x_0) = D(y, Ty) + D(x_0, Tx_0) - d(y, x_0)$, then

$$\begin{aligned} d(z, x_0) &< \alpha d(y, x_0) + \alpha D(y, Ty) - \alpha D(x_0, Tx_0) + \frac{1-\alpha}{1+\alpha} r \\ &\leq \alpha \frac{1-\alpha}{1+\alpha} r + \alpha d(y, z) + \frac{1-\alpha}{1+\alpha} r \\ &\leq (1-\alpha) r + \alpha [d(y, x_0) + d(x_0, z)] + \frac{1-\alpha}{1+\alpha} r \\ &\leq (1-\alpha) r + \alpha r + \alpha d(x_0, z). \end{aligned}$$

Hence, $d(z,x_0) \le \frac{r}{1-\alpha}$.
Finally, if the maximum is

$$E_4(y,x_0) = [D(y,Tx_0) + D(x_0,Ty) + |D(y,Ty) - D(x_0,Tx_0)|]/2,$$

then for $D(y,Ty) > D(x_0,Tx_0)$ we obtained

$$\begin{aligned} d(z,x_0) &< (\alpha/2)[D(y,Tx_0) + D(x_0,Ty) + D(y,Ty)] + \frac{1-\alpha}{1+\alpha}r \\ &\le (\alpha/2)[d(y,x_0) + D(x_0,Tx_0) + d(x_0,z) + D(y,Ty)] + \frac{1-\alpha}{1+\alpha}r \\ &\le (\alpha/2)[r + \frac{1-\alpha}{1+\alpha}r + d(x_0,z) + d(y,z)] + \frac{1-\alpha}{1+\alpha}r \\ &\le (\alpha/2)[\frac{2}{1+\alpha}r + d(x_0,z) + d(y,x_0) + d(x_0,z)] + \frac{1-\alpha}{1+\alpha}r \\ &= \left(\frac{1}{1+\alpha} + \frac{\alpha}{2}\right) r + \alpha d(x_0,z). \end{aligned}$$

It follows that

$$d(z,x_0) \le \frac{\alpha^2+\alpha+2}{2(1-\alpha^2)} \le \frac{1+\alpha+2\alpha^2}{1-\alpha^2}r.$$

Also, for $D(y,Ty) \le D(x_0,Tx_0)$ we have

$$\begin{aligned} d(z,x_0) &< (\alpha/2)[D(y,Tx_0) + D(x_0,Ty) + D(x_0,Tx_0)] + \frac{1-\alpha}{1+\alpha}r \\ &\le (\alpha/2)[d(y,x_0) + D(x_0,Tx_0) + d(x_0,z) + D(x_0,Tx_0)] + \frac{1-\alpha}{1+\alpha}r \\ &\le (\alpha/2)[r + 2\frac{1-\alpha}{1+\alpha}r + d(x_0,z)] + \frac{1-\alpha}{1+\alpha}r \\ &= (\alpha/2)d(x_0,z) + (1-\alpha/2)r. \end{aligned}$$

by where, $d(z,x_0) \le r$. Since $\frac{1}{1-\alpha} \le \frac{1+\alpha+2\alpha^2}{1-\alpha^2}$, in all cases we have

$$d(z,x_0) \le \frac{1+\alpha+2\alpha^2}{1-\alpha^2}r.$$

This means that

$$T(\bar{B}(x_0,r)) \subset \bar{B}(x_0, \frac{1+\alpha+2\alpha^2}{1-\alpha^2}r).$$

Since $D(x_0,Tx_0) \le \rho(x_0,Tx_0) < \frac{1-\alpha}{1+\alpha}r$, by (h) it follows that there exists $x^* \in F_T \cap B(x_0,r)$.

(j) Let V be an open subset of U. We shall prove that $G(V)$ is open in X. This means that for $x_0 \in U$ and $r > 0$ such that $B(x_0,r) \subset U$, we have $V(Gx_0, \frac{1-\alpha}{1+\alpha}r) \subset G(B(x_0,r))$. Taking $y \in V(Gx_0, \frac{1-\alpha}{1+\alpha}r)$, i.e. $D(y,Gx_0) < \frac{1-\alpha}{1+\alpha}r$, we shall prove that there exists $x^* \in B(x_0,r)$ such that $y \in Gx^*$. Let us consider the multi-valued operator $F : B(x_0,r) \to P_{cl}(X)$, defined by $F(x) := y + Tx$. Then, for $x,z \in B(x_0,r)$ we have that:

$$H(Fx,Fz) = H(y+Tx, y+Tz) = H(Tx,Tz) \le \alpha P(x,z).$$

Moreover,

$$D(x_0, Fx_0) = D(x_0, y + Tx_0) = D(y, x_0 - Tx_0) = D(y, Gx_0) < \frac{1-\alpha}{1+\alpha} r.$$

It follows from (h) that F has a fixed point $x^* \in B(x_0, r)$, i.e. $x^* \in Fx^*$. Hence, $x^* \in y + Tx^*$ or $y \in x^* - Tx^* = Gx^*$. Therefore, $y \in V(Gx_0, \frac{1-\alpha}{1+\alpha} r)$, and then, G is open.

(k) Let $\epsilon := \frac{1-\alpha}{2(1+\alpha)}, \beta := \frac{2\alpha}{1+\alpha}$ and $\varphi(x) := (1/\epsilon) D(x, Tx)$. Then, obviously, we have $\epsilon + \beta = \frac{1+3\alpha}{2+2\alpha} < 1$ and φ is bounded below by 0. Since $1/(\epsilon+\beta) > 1$, for each $x \in X$ we can choose $tx \in Tx$ such that $d(x, tx) \leq 1/(\epsilon+\beta) D(x, Tx)$. Since $D(tx, Tx) = 0$ and $D(x, Tx) \leq d(x, tx)$, we get for $D(tx, Ttx) \geq D(x, Tx)$ that

$$d(x,tx) + D(x,Tx) - D(tx,Ttx) \leq d(x,tx) + D(tx,Ttx) - D(x,Tx),$$

$$D(x,Tx) + D(tx,Ttx) - d(x,tx) \leq d(x,tx) + D(tx,Ttx) - D(x,Tx),$$

$$\begin{aligned}
&(1/2)[D(x,Ttx) + D(tx,Tx) + |D(x,Tx) - D(tx,Ttx)|] \\
&\leq (1/2)[d(x,tx) + D(tx,Ttx) + D(tx,Ttx) - D(x,Tx) \\
&\leq D(tx,Ttx) + d(x,tx) - D(x,Tx),
\end{aligned}$$

so $E_1(x,tx) \leq E_2(x,tx)$, $E_3(x,tx) \leq E_2(x,tx)$ and $E_4(x,tx) \leq E_2(x,tx)$. Hence, $P(x,tx) = d(x,tx) + D(tx,Ttx) - D(x,Tx)$. Then, by hypothesis, we have:

$$D(tx,Ttx) \leq H(Tx,Ttx) \leq \alpha P(x,tx),$$

so

$$D(tx,Ttx) \leq \alpha[d(x,tx) + D(tx,Ttx) - D(x,Tx)].$$

Hence, we get

$$D(x,Tx) \leq D(tx,Ttx) \leq \frac{\alpha}{1-\alpha}[d(x,tx) - D(x,Tx)],$$

by where

$$D(x,Tx) \leq \alpha d(x,tx) \leq \frac{\alpha}{\epsilon+\beta} D(x,Tx).$$

Since $\frac{\alpha}{\epsilon+\beta} < 1$, it follows that $D(x,Tx) = 0$. This implies $d(x,tx) = 0$, so we have $D(x,Tx) \leq \beta d(x,tx)$. If $D(tx,Ttx) < D(x,Tx)$, then

$$d(x,tx) + D(x,Tx) - D(tx,Ttx) > d(x,tx) + D(tx,Ttx) - D(x,Tx),$$

$$D(x,Tx) + D(tx,Ttx) - d(x,tx) \leq d(x,tx) + D(tx,Ttx) - D(x,Tx),$$

$$\begin{aligned}
&(1/2)[D(x,Ttx) + D(tx,Tx) + |D(x,Tx) - D(tx,Ttx)|] \\
&\leq (1/2)[d(x,tx) + D(tx,Ttx) + D(x,Tx) - D(tx,Ttx) \\
&= (1/2)[d(x,tx) + D(x,Tx)] \\
&\leq d(x,tx) + D(x,Tx) - D(tx,Ttx),
\end{aligned}$$

so $E_1(x,tx) > E_2(x,tx)$, $E_3(x,tx) \leq E_2(x,tx)$ and $E_4(x,tx) \leq E_1(x,tx)$. Hence, $P(x,tx) = d(x,tx) + D(x,Tx) - D(tx,Ttx)$. Then, by hypothesis, we have:

$$D(tx,Ttx) \leq H(Tx,Ttx) \leq \alpha P(x,tx),$$

so

$$D(tx,Ttx) \leq \alpha[d(x,tx)+D(x,Tx)-D(tx,Ttx)].$$

Hence, $D(tx,Ttx) \leq \frac{\alpha}{1+\alpha}[d(x,tx)+D(x,Tx)] \leq \frac{2\alpha}{1+\alpha}d(x,tx)$.
Therefore, in all cases we have for each $x \in X$ that

$$D(tx,Ttx) \leq \frac{2\alpha}{1+\alpha}d(x,tx) = \beta d(x,tx).$$

Now, we will prove that t is a Caristi type operator. Indeed, for each $x \in X$ we have:

$$\begin{aligned} d(x,tx) &= (1/\epsilon)[(\epsilon+\beta)d(x,tx) - \beta d(x,tx)] \\ &\leq (1/\epsilon)[D(x,Tx) - D(tx,Ttx)] \\ &= \varphi(x) - \varphi(tx). \end{aligned}$$

(l) Let $\epsilon > 0$ and consider $y^* \in X$ that satifies $D(y^*,Ty^*) \leq \epsilon$. Then, for each $x_0 \in X$ and $x_1 \in Tx_0$ with $d(x_0,x_1) \leq (1/\alpha)D(x_0.Tx_0)$ there exists $x^* \in F_T$ such that

$$d(x_0,x^*) \leq \frac{1+\alpha}{1-\alpha}d(x0,x_1).$$

Taking $x_0 = y^*$, we obtain

$$d(y^*,x^*) \leq \frac{1+\alpha}{1-\alpha}d(y^*,x_1),$$

where $x_1 \in Ty^*$ with $d(y^*,x_1) \leq (1/\alpha)D(y^*,Ty^*)$. Since $Ty^* \in P_{cp}(X)$, there exists $x_1 \in Ty^*$ such that $d(y^*,x_1) = D(y^*,Ty^*)$. It follows that

$$d(y^*,x^*) \leq \frac{1+\alpha}{1-\alpha}D(y^*,Ty^*) \leq \frac{1+\alpha}{1-\alpha}\epsilon = \psi(\epsilon),$$

where $\psi(t) = \frac{1+\alpha}{1-\alpha}t$.
(m) Since there exists $x^* \in F_T$, we have $D(x^*,Tx^*) = 0$, so $\inf_{x\in X} D(x,Tx) = 0$, i.e. T has the approximate fixed point property.
(n) Let $\epsilon > 0$ and $E_\epsilon(T) := \{x \in X : \sup_{z\in Tx} d(x,z) \leq \epsilon\}$. If $x,y \in E_\epsilon(T)$, then $H(x,Tx) \leq \epsilon$ and $H(y,Ty) \leq \epsilon$. Since

$$d(x,y) \leq H(x,Tx) + H(Tx,Ty) + H(y,Ty),$$

we get by hypothesis that $d(x,y) \leq \alpha P(x,y) + 2\epsilon$. Hence, we have the following cases:
If $P(x,y) = d(x,y) + D(x,Tx) - D(y,Ty)$, then

$$\begin{aligned} d(x,y) &\leq \alpha d(x,y) + \alpha D(x,Tx) + 2\epsilon \\ &\leq \alpha d(x,y) + \alpha H(x,Tx) + 2\epsilon \\ &\leq \alpha d(x,y) + (\alpha+2)\epsilon. \end{aligned}$$

It follows that

$$d(x,y) \leq \frac{\alpha+2}{1-\alpha}\epsilon.$$

Similarly, if $P(x,y) = d(x,y) + D(y,Ty) - D(x,Tx)$, then

$$d(x,y) \leq \frac{\alpha+2}{1-\alpha}\epsilon.$$

If $P(x,y) = D(x,Tx) + D(y,Ty) - d(x,y)$, then

$$\begin{aligned} d(x,y) &\leq \alpha D(x,Tx) + \alpha D(y,Ty) + 2\epsilon \\ &\leq \alpha H(x,Tx) + \alpha H(y,Ty) + 2\epsilon \\ &\leq 2(\alpha+1)\epsilon. \end{aligned}$$

Finally, if $P(x,y) = [D(x,Ty) + D(y,Tx) + |D(x,Tx) - D(y,Ty)|]/2$, then

$$\begin{aligned} P(x,y) &\leq [d(x,y) + D(y,Ty) + d(x,y) + D(x,Tx) + D(x,Tx) + D(y,Ty)]/2 \\ &= d(x,y) + D(x,Tx) + D(y,Ty) \\ &\leq d(x,y) + 2\epsilon. \end{aligned}$$

Hence, we get that $d(x,y) \leq \alpha d(x,y) + 2\alpha\epsilon + 2\epsilon$, by where

$$d(x,y) \leq \frac{2(1+\alpha)\epsilon}{1-\alpha}.$$

Therefore, it follows that

$$d(x,y) \leq \epsilon \max\left\{\frac{\alpha+2}{1-\alpha}, 2(\alpha+1), \frac{2(1+\alpha)}{1-\alpha}\right\} = \frac{2(1+\alpha)}{1-\alpha}\epsilon.$$

Then, we get that

$$\delta(E_\epsilon(T)) \leq \frac{2(1+\alpha)}{1-\alpha}\epsilon,$$

where $\delta(A) := \sup_{a,b\in A} d(a,b)$ means the diameter of the set A.
Let $x_n \in E_\epsilon(T)$ such that $x_n \to x$ as $n \to \infty$ and $z \in Tx$. Since T is lower semi-continuous, then there exists $z_n \in Tx_n$ with $z_n \to z$. Since $x_n \in E_\epsilon(T)$ we have $\sup_{y\in Tx_n} d(x_n, y) \leq \epsilon$, so $d(x_n, z_n) \leq \epsilon$, for every $n \geq 1$. Taking the limit as $n \to \infty$, it follows that $d(x,z) \leq \epsilon$. Then, we have $\sup_{z\in Tx} d(x,z) \leq \epsilon$. Hence $x \in E_\epsilon(T)$. Therefore, $E_\epsilon(T)$ is closed.
Now, suppose that T has the approximate endpoint property and define $C_n := E_{1/n}(T)$. Then, by our hypothesis, C_n is nonempty for each n, and it is obvious that $C_{n+1} \subset C_n$ for all $n \geq 1$. Also, C_n is closed and

$$\delta(C_n) = \delta(E_{1/n}(T)) \leq \frac{2(1+\alpha)}{1-\alpha}\epsilon.$$

Since $\lim_{n\to\infty} \delta(C_n) = 0$, by Cantor's intersection Theorem, it follows that $\cap_{n\in\mathbb{N}} C_n = \{x^*\}$. As $x^* \in C_n$ for each $n \geq 1$, we obtain that $\sup_{y\in Tx^*} d(x^*, y) \leq 1/n$, so $d(x^*, y) = 0$ for each $y \in Tx^*$. Hence, $Tx^* = \{x^*\}$, i.e. T has a strict fixed point. If y^* is another strict fixed point of T, then by hypothesis, we have

$$d(x^*, y^*) = D(x^*, Ty^*) \leq H(TX^*, Ty^*) \leq \alpha P(x^*, y^*).$$

Since $D(x^*, Tx^*) = D(y^*, Ty^*) = 0$, and $D(x^*, Ty^*) = D(y^*, Tx^*) = d(x^*, y^*)$, it follows that $P(x^*, y^*) = d(x^*, y^*)$. Hence, we get that $d(x^*, y^*) \leq \alpha d(x^*, y^*)$, by where $d(x^*, y^*) = 0$. Therefore, T has a unique strict fixed point.
Reciprocally, if T has a unique strict fixed point x^*, then $Tx^* = \{x^*\}$. It follows that $\sup_{y\in Tx^*} d(x^*, y) = 0$, so $\inf_{x\in X} \sup_{y\in Tx} d(x,y) = 0$, i.e. T has the approximate endpoint property.

(o) By (d), we have that F_T is closed in (X, d). Let $x^*, y^* \in F_T$. By hypothesis, we have

$$D(x^*, Ty^*) \leq H(Tx^*, Ty^*) \leq \alpha P(x^*, y^*).$$

Since $D(x^*, Tx^*) = D(y^*, Ty^*) = 0$, and $D(x^*, Ty^*) \leq d(x^*, y^*), D(y^*, Tx^*) \leq d(x^*, y^*)$, we get that $P(x^*, y^*) = d(x^*, y^*)$. Hence, $D(x^*, Ty^*) \leq \alpha d(x^*, y^*)$. As Ty^* is compact, there exists $z \in Ty^*$ such that $d(x^*, z) = D(x^*, Ty^*) \leq \alpha d(x^*, y^*)$. It follows that $d(x^*, y^*) - d(z, y^*) \leq \alpha d(x^*, y^*)$, so

$$d(x^*, y^*) \leq \frac{1}{1-\alpha} d(z, y^*) \leq \frac{1}{1-\alpha} \delta(Ty^*).$$

Since Ty^* is compact, then Ty^* is bounded and $\delta(Ty^*) < \infty$. Hence, $F_T \subset \bar{B}(y^*, \frac{1}{1-\alpha}\delta(Ty^*))$, i.e. F_T is bounded.

Obviously, F_T is complete with respect to d. Let us suppose that F_T is not compact. Then, F_T is not precompact, i.e. there exist $\delta > 0$ and $(x_k)_{k\in\mathbb{N}} \subset F_T$ such that $d(x_i, x_j) \geq \delta$ for all $i \neq j$.

Denote

$$\rho := \inf \{R : \exists a \in X \text{ such that } B(a, R) \text{ contains an infinity of } x_k\text{'s}\}.$$

Since F_T is bounded, we have $\rho < \infty$. Moreover, $\rho \geq \delta/2$ because for each $a \in X, B(a, \delta/2)$ contains at most one x_k. Furthermore, consider $0 < \epsilon < \frac{1-3\alpha}{1+\alpha}\rho$ and take $a \in X$ such that the set $J := \{k : x_k \in B(a, \rho + \epsilon)\}$ is infinite. Then, for each $k \in J$, we have

$$D(x_k, Ta) \leq H(Tx_k, Ta) \leq \alpha P(x_k, a).$$

Since $x_k \in F_T$, we have $D(x_k, Tx_k) = 0$, so

$$\begin{aligned} E_4(x_k, a) &= [D(x_k, Ta + D(a, Tx_k) + D(a, Ta)]/2 \\ &\leq [d(x_k, a) + D(a, Ta) + d(a, x_k) + D(x_k, Tx_k) + D(a, Ta)]/2 \\ &= d(x_k, a) + D(a, Ta). \end{aligned}$$

It follows that $P(x_k, a) = d(x_k, a) + D(a, Ta)$, and then

$$\begin{aligned} D(x_k, Ta) &\leq \alpha d(x_k, a) + \alpha D(a, Ta) \\ &\leq \alpha d(x_k, a) + \alpha d(a, x_k) + \alpha D(x_k, Ta). \end{aligned}$$

Hence,

$$D(x_k, Ta) \leq \frac{2\alpha}{1-\alpha} d(x_k, a) \leq \frac{2\alpha}{1-\alpha}(\rho + \epsilon),$$

for each $k \in J$. Since Ta is compact, then there exists $y_k \in Ta$ such that $d(x_k, y_k) \leq \frac{2\alpha}{1-\alpha}(\rho + \epsilon)$ for each $k \in J$. Moreover, since Ta is compact, then there exists $b \in Ta$, for which the set $J' = \{k \in J : d(y_k, b) < \epsilon\}$ is infinite. Then, we have for each $k \in J'$ that

$$d(x_k, b) \leq d(x_k, y_k) + d(y_k, b) < \frac{2\alpha}{1-\alpha}(\rho + \epsilon) + \epsilon < \rho.$$

Hence, the ball $B(b, R)$ contains an infinite number of elements x_k'-s, where

$$R = \frac{2\alpha}{1-\alpha}\rho + \frac{1+\alpha}{1-\alpha}\epsilon < \rho.$$

This contradicts the choice of ρ. Therefore, F_T is compact.
(p) Let $F_p^* := \{x \in X : D(x, Tx) < p\}$ for each $p > 0$. If $x \in F_T$ then $D(x, Tx) = 0$, so $x \in F_p^*$. Hence $F_T \subset F_p^*$ for each $p > 0$. This implies that

$$H(F_p^*, F_T) = \rho(F_p^*, F_T) = \sup_{x \in F_p^*} D(x, F_T),$$

for all $p > 0$, where ρ denotes the excess functional. Now, let $\epsilon > 0$ arbitrarly chosen and $x \in F_p^*$. Then, $D(x, Tx) < p$ and there exists $x_1 \in Tx$ such that $d(x, x_1) < \min\{(1+\epsilon)p, (1/\alpha)D(x, Tx)\}$. Following (b) there exists a s.s.a. (x_n) starting from $x_0 = x \in X$, such that $d(x_0, x^*) \leq \frac{1+\alpha}{1-\alpha} d(x_0, x_1)$, where $x_n \to x^* \in F_T$ as $n \to \infty$. Then, we have $d(x_0, x^*) \leq \frac{1+\alpha}{1-\alpha}(1+\epsilon)p$. Taking $\epsilon \downarrow 0$, it follows that $d(x_0, x^*) \leq \frac{1+\alpha}{1-\alpha}p$, for each $x \in F_p^*$, i.e. $H(F_p^*, F_T) \leq \frac{1+\alpha}{1-\alpha}p$.

The following result is an extended version of the strict fixed point principle for multi-valued $\mathcal{P}$-operators.

Theorem 2.5 *Let (X, d) be a complete metric space and $T : X \to P_{cl}(X)$ be a multi-valued $\mathcal{P}$-type operator. Suppose that $(SF)_T \neq \emptyset$. Then, the following conclusions hold:*
(a) $(SF)_T = F_T = \{x^*\}$.
(b) if $\alpha < \frac{1}{3}$, then T has the Ostrowski property;
(c) if $\alpha < \frac{1}{3}$, then $H(Tx, x^*) \leq \frac{2\alpha}{1-\alpha} d(x, x^*)$, for each $x \in X$;
(d) if $\alpha < \frac{1}{3}$, then $d(x, x^*) \leq \frac{1-\alpha}{1-3\alpha} H(x, Tx)$, for each $x \in X$;
(e) the fixed point inclusion $x \in Tx$ is generalized Ulam-Hyers stable;
(f) the strict fixed point inclusion $\{x\} = Tx$ is generalized Ulam-Hyers stable;
(g) the fixed point problem is well-posed for T, with respect to D and, respectively, with respect to H;
(h) if $G : X \to P(X)$ is a multi-valued operator with $F_G \neq \emptyset$, and there exists $\eta > 0$ such that $H(Tx, Gx) \leq \eta$ for all $x \in X$, then $H(F_T, F_G) \leq \frac{1+\alpha}{1-\alpha}\eta$.

Proof. (a) Since $(SF)_T \neq \emptyset$, then there exists $x^* \in (SF)_T \subset F_T$. If $y^* \in F_T$, then by hypothesis, we have

$$d(x^*, y^*) = D(Tx^*, y^*) \leq H(Tx^*, Ty^*) \leq \alpha P(x^*, y^*).$$

Since $D(x^*, Tx^*) = D(y^*, Ty^*) = 0, D(x^*, Ty^*) \leq d(x^*, y^*)$ and $D(y^*, Tx^*) = d(x^*, y^*)$, it follows that $P(x^*, y^*) = d(x^*, y^*)$. Hence, $d(x^*, y^*) \leq \alpha d(x^*, y^*)$, by where $d(x^*, y^*) = 0$, i.e. $x^* = y^*$. Therefore, $F_T = (SF)_T = \{x^*\}$.
(b) Let (y_n) be a sequence such that $D(y_{n+1}, Ty_n) \to 0$ as $n \to \infty$. Then, by hypothesis, we have for each $n \geq 1$ that $H(Tx^*, Ty_n) \leq \alpha P(x^*, y_n)$. Since $D(x^*, Tx^*) = 0$, we have

$$\begin{aligned} E_4(x^*, y_n) &= [D(x^*, Ty_n) + D(y_n, Tx^*) + D(y_n, Ty_n)]/2 \\ &\leq [d(x^*, y_n) + D(y_n, Ty_n) + d(x^*, y_n) + D(y_n, Ty_n)]/2 \\ &= d(x^*, y_n) + D(y_n, Ty_n). \end{aligned}$$

It follows that $P(x^*, y_n) = d(x^*, y_n) + D(y_n, Ty_n)$. Hence, we get that for each $n \geq 1$:

$$H(Tx^*, Ty_n) \leq \alpha[d(x^*, y_n) + D(y_n, Ty_n)].$$

Then, we obtain that:

$$\begin{aligned} H(Tx^*, Ty_n) &\leq \alpha d(x^*, y_n) + \alpha[d(x^*, y_n) + D(x^*, Ty_n)] \\ &= 2\alpha d(x^*, y_n) + \alpha D(x^*, Ty_n) \\ &\leq 2\alpha d(x^*, y_n) + \alpha H(Tx^*, Ty_n). \end{aligned}$$

Hence, we have that $H(Tx^*, Ty_n) \leq \frac{2\alpha}{1-\alpha} d(x^*, y_n)$. Since $d(x^*, y_{n+1}) \leq H(Tx^*, Ty_n) + D(y_{n+1}, Ty_n)$, we obtain

$$d(x^*, y_{n+1}) \leq \frac{2\alpha}{1-\alpha} d(x^*, y_n) + D(y_{n+1}, Ty_n) = kd(x^*, y_n) + D(y_{n+1}, Ty_n),$$

where $k = \frac{2\alpha}{1-\alpha} < 1$. By Lemma 1.3, we get $\lim_{n\to\infty} d(x^*, y_n) = 0$, so $y_n \to x^*$ as $n \to \infty$.

(c) By hypothesis, we have for each $x \in X$ that $H(Tx, Tx^*) \leq \alpha P(x, x^*)$. Since $D(x^*, Tx^*) = 0$, we obtain that $P(x, x^*) = d(x, x^*) + D(x, Tx)$. Hence

$$\begin{aligned} H(Tx, Tx^*) &\leq \alpha d(x, x^*) + \alpha D(x, Tx) \\ &\leq \alpha d(x, x^*) + \alpha[d(x, x^*) + H(Tx, Tx^*)] \\ &= 2\alpha d(x, x^*) + \alpha H(Tx, Tx^*), \end{aligned}$$

by where we get $H(Tx, Tx^*) \leq \frac{2\alpha}{1-\alpha} d(x, x^*)$.

(d) We have

$$d(x, x^*) \leq H(x, Tx) + H(Tx, x^*) \leq H(x, Tx) + \frac{2\alpha}{1-\alpha} d(x, x^*).$$

Therefore, we get $d(x, x^*) \leq \frac{1-\alpha}{1-3\alpha} H(x, Tx)$.

(e) Now, let us consider $y \in X$ and x$x \in Ty$. Then, we have:

$$\begin{aligned} d(y, x^*) &\leq d(y, x) + H(Ty, Tx^*) \leq d(x, y) + \alpha P(y, x^*) \\ &= d(x, y) + \alpha[d(y, x^*) + D(y, Ty)] \\ &\leq (1 + \alpha)d(x, y) + \alpha d(y, x^*). \end{aligned}$$

It follows that $d(y, x^*) \leq \frac{1+\alpha}{1-\alpha} d(x, y)$ for each $x \in Ty$. Taking $\inf_{x\in Ty}$ we get $d(y, x^*) \leq \frac{1+\alpha}{1-\alpha} D(y, Ty) = \psi(D(y, Ty))$, where $\psi(t) := \frac{1+\alpha}{1-\alpha} t$. It is obvious that ψ is continuous in 0, increasing and $\psi(0) = 0$. Let $\epsilon > 0$ and consider $y^* \in X$ that satisfies $D(y^*, Ty^*) \leq \epsilon$. Then, we have

$$d(y^*, x^*) \leq \psi(\epsilon) = \frac{1+\alpha}{1-\alpha}\epsilon.$$

(f) Since $D(y^*, Ty^*) \leq H(y^*, Ty^*)$ for every $y^* \in X$, the conclusion follows from (e).

(g) Let $x_n \in X$ with $D(x_n, Tx_n) \to 0$ as $n \to \infty$. Then,

$$d(x_n, x^*) \leq D(x_n, Tx_n) + H(Tx_n, Tx^*) \leq D(x_n, Tx_n) + \alpha P(x_n, x^*).$$

Since $P(x_n, x^*) = d(x_n, x^*) + D(x_n, Tx_n)$, we get that $d(x_n, x^*) \leq \alpha d(x_n, x^*) + (1 + \alpha)D(x_n, Tx_n)$. Hence, $d(x_n, x^*) \leq \frac{1+\alpha}{1-\alpha} D(x_n, Tx_n)$. Letting $n \to \infty$ we obtain $d(x_n, x^*) \to 0$, so $x_n \to x^*$.

(h) Let $x^* \in (SF)_T$ and $y^* \in F_G$. Then, we have

$$d(x^*, y^*) \leq H(Gy^*, x^*) \leq H(Gy^*, Ty^*) + H(Ty^*, x^*) \leq \eta + \alpha P(y^*, x^*).$$

Since $P(y^*, x^*) = d(y^*, x^*) + D(y^*, Ty^*)$, we get

$$\begin{aligned} d(x^*, y^*) &\leq \eta + \alpha d(x^*, y^*) + \alpha D(y^*, Ty^*) \\ &\leq \eta + \alpha d(x^*, y^*) + \alpha H(y^*, Ty^*) \\ &\leq \eta(1+\alpha) + \alpha d(x^*, y^*). \end{aligned}$$

Hence, we obtain that $d(x^*, y^*) \leq \frac{1+\alpha}{1-\alpha}\eta$ for each $y^* \in F_G$. It follows that

$$\sup_{y^* \in F_G} d(x^*, y^*) \leq \frac{1+\alpha}{1-\alpha}\eta,$$

by where

$$H(F_T, F_G) \leq \frac{1+\alpha}{1-\alpha}\eta.$$

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