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EXISTENCE OF EQUILIBRIA FOR N-PERSON GAMES

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Abstract. Using recent fixed point theory of the author we establish some new maximal element results for general majorized type maps defined on Hausdorff topological vector spaces. Then by constructing appropriate majorized type maps we can then guarantee the existence of equilibria for N-person games.

Key Words and Phrases: Fixed point theory, maximal elements, majorized maps, equilibrium points.
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1. Introduction

Multivalued maps are point to set maps (i.e., their values are subsets) and we will denote multivalued maps by capital letters and the arrow $\to$ (i.e. for topological spaces X and Y we will denote the multivalued map F from X into the set of all subsets of Y (i.e. 2^Y) by $F : X \to Y$). In this paper we begin by presenting some new fixed point and maximal element results of the author and we will focus our results for majorized coercive, compact or condensing type maps. Establishing the existence of equilibria arises naturally when examining economic models and in this paper we present a new equilibrium theory for N-person games (abstract economies) Indeed it is also easy to remove $\{1, \dots, N\}$ with an index set I if one uses the results in [6] in their full generality. The maps considered (constraints, preferences) will be either of Ky Fan type [2, 3] or upper semicontinuous with compact convex values defined on Hausdorff topological vector spaces. The idea in our proof is to use a constraint and preference correspondence to construct a majorized type map related to a Ky Fan type map and this will then enable us to guarantee the existence of equilibria for N-person games. Our theory improves and complement results in the literature (see [3, 4, 5, 8, 9, 10] and the references therein) and in particular we note that in the literature constraints and preferences are defined on locally convex topological vector spaces

First we recall a result from [7] for majorized coercive maps.

Theorem 1.1. *Let $\{X_i\}_{i=1}^N$ be a family of convex sets each in a Hausdorff topological vector space with $X = \prod_{i=1}^N X_i$ paracompact. For each $i \in \{1, \dots, N\}$ suppose $H_i : X \to X_i$ and in addition there exists a map $T_i : X \to X_i$ with $H_i(w) \subseteq T_i(w)$ for $w \in X$, T_i has convex values, $T_i^{-1}(w)$ is open (in X) for each $w \in X_i$ and $w_i \notin T_i(w)$ for each $w \in X$ (here w_i denotes the projection of w on X_i). Also assume there is a compact subset K of X and for each $i \in \{1, .., N\}$ a convex compact subset Y_i of X_i such that for each $x \in X \backslash K$ there exists a $j \in \{1, .., N\}$ with $H_j(x) \cap Y_j \neq \emptyset$ (or, alternatively assume $T_j(x) \cap Y_j \neq \emptyset$). Then there exists a $x \in X$ with $H_i(x) = \emptyset$ for all $i \in \{1, \dots, N\}$.*

Remark 1.2. In Theorem 1.1 one could replace $\{X_i\}_{i=1}^N$ with $\{X_i\}_{i \in I}$ where I is an index set if we use a result in [6]. This is also true for other theorems in this paper but we will not refer to it again.

One could also obtain an analogue of Theorem 1.1 for majorized compact type maps if one uses our next result taken from [6, Theorem 2.7] (alternatively, one could consider Theorem 1.1).

Theorem 1.3. *Let $\{X_i\}_{i=1}^N$ be a family of convex sets each in a Hausdorff topological vector space E_i with $X = \prod_{i=1}^N X_i$ paracompact. For each $i \in \{1, \dots, N\}$ suppose $F_i : X \to X_i$ and there exists a map $T_i : X \to X_i$ with $T_i(w) \subseteq F_i(w)$ for $w \in X$, T_i has convex values and $T_i^{-1}(w)$ is open (in X) for each $w \in X_i$. Also suppose for each $i \in \{1, .., N\}$ there exists a convex compact set K_i with $F_i(X) \subseteq K_i \subseteq X_i$. Finally assume for each $x \in X$ there exists a $i \in \{1, \dots, N\}$ with $T_i(x) \neq \emptyset$. Then there exists a $x \in X$ and a $i \in \{1, .., N\}$ with $x_i \in F_i(x)$.*

Next note we can rewrite Theorem 1.3 as a maximal element type result.

Theorem 1.4. *Let $\{X_i\}_{i=1}^N$ be a family of convex sets each in a Hausdorff topological vector space with $X = \prod_{i=1}^N X_i$ paracompact. For each $i \in \{1, \dots, N\}$ suppose $F_i : X \to X_i$ and there exists a map $T_i : X \to X_i$ with $T_i(w) \subseteq F_i(w)$ for $w \in X$, T_i has convex values and $T_i^{-1}(w)$ is open (in X) for each $w \in X_i$. Also suppose for each $i \in \{1, .., N\}$ there exists a convex compact set K_i with $F_i(X) \subseteq K_i \subseteq X_i$. Finally assume for all $i \in \{1, .., N\}$ that $x_i \notin F_i(x)$ for each $x \in X$. Then there exists a $x \in X$ with $T_i(x) = \emptyset$ for all $i \in \{1, \dots, N\}$.*

Proof. Suppose the conclusion is false. Then for each $x \in X$ there exists a $i \in \{1, \dots, N\}$ with $T_i(x) \neq \emptyset$. Now Theorem 1.3 guarantees a $x \in X$ and a $i \in \{1, .., N\}$ with $x_i \in F_i(x)$, a contradiction. □

Theorem 1.5. *Let $\{X_i\}_{i=1}^N$ be a family of convex sets each in a Hausdorff topological vector space with $X = \prod_{i=1}^N X_i$ paracompact. For each $i \in \{1, \dots, N\}$ suppose $H_i : X \to X_i$ and there exists a map $T_i : X \to X_i$ with $H_i(w) \subseteq T_i(w)$ for $w \in X$, T_i has convex values, $T_i^{-1}(w)$ is open (in X) for each $w \in X_i$ and $w_i \notin T_i(w)$ for each $w \in X$. Also suppose for each $i \in \{1, .., N\}$ there exists a convex compact set K_i with $T_i(X) \subseteq K_i \subseteq X_i$. Then there exists a $x \in X$ with $H_i(x) = \emptyset$ for all $i \in \{1, \dots, N\}$.*

Proof. Apply Theorem 1.4 with $F_i = T_i$ so there exists a $x \in X$ with $T_i(x) = \emptyset$ for all $i \in \{1, \ldots, N\}$. Now since $H_j(w) \subseteq T_j(w)$ for $w \in X$ then $H_i(x) = \emptyset$ for all $i \in \{1, \ldots, N\}$. □

Finally we recall a result from [7] for majorized condensing type maps.

Theorem 1.6. *Let $\{X_i\}_{i=1}^N$ be a family of convex sets each in a Hausdorff topological vector space. For each $i \in \{1, \ldots, N\}$ suppose $H_i : X \equiv \prod_{i=1}^N X_i \to X_i$ and there exists a map $T_i : X \to X_i$ with $H_i(w) \subseteq T_i(w)$ for $w \in X$, T_i has convex values and $T_i^{-1}(w)$ is open (in X) for each $w \in X_i$. Also assume there is a compact convex subset K of X with $T(K) \subseteq K$ where $T(x) = \prod_{i=1}^N T_i(x)$ for $x \in X$. Finally suppose for each $w \in X$ there exists a $i \in \{1, \ldots, N\}$ with $w_i \notin T_i(w)$. Then there exists a $x \in X$ with $H_{i_0}(x) = \emptyset$ for some $i_0 \in \{1, \ldots, N\}$.*

Corollary 1.7. *Let X be a convex set in a Hausdorff topological vector space and $H : X \to X$. Also assume there exists a map $T : X \to X$ with $H(w) \subseteq T(w)$ for $w \in X$, T has convex values, $T^{-1}(w)$ is open (in X) for each $w \in X$ and $w \notin T(w)$ for $w \in X$. Also suppose there is a compact convex subset K of X with $T(K) \subseteq K$. Then there exists a $x \in X$ with $H(x) = \emptyset$.*

2. Generalized games

Now we present some results on generalized games (or abstract economies). A generalized game is given by $\Gamma = (X_i, A_i, B_i, P_i)_{i \in I}$ where I is a set of players (agents), X_i is a nonempty subset of a Hausdorff topological vector space, A_i, $B_i : X \equiv \prod_{i \in I} X_i \to X_i$ are constraint correspondences and $P_i : X \to E_i$ is a preference correspondence. An equilibrium of Γ is a point $x \in X$ such that for each $i \in I$ we have $x_i \in \overline{B_i}(x)$ and $A_i(x) \cap P_i(x) = \emptyset$.

Theorem 2.1. *Let $\Gamma = (X_i, A_i, B_i, P_i)_{i=1}^N$ be a N-person game i.e. $\{X_i\}_{i=1}^N$ is a family of convex sets each in a Hausdorff topological vector space E_i with $X = \prod_{i=1}^N X_i$ paracompact, and for each $i \in \{1, \ldots, N\}$ the constraint correspondences A_i, $B_i : X \to X_i$ and the preference correspondence $P_i : X \to E_i$. Also for each $i \in \{1, \ldots, N\}$ suppose $cl\, B_i\, (\equiv \overline{B_i}) : X \to CK(X_i)$ is upper semicontinuous (here $CK(X_i)$ denotes the family of nonempty convex compact subsets of X_i) and assume the following conditions hold for each $i \in \{1, \ldots, N\}$:*

$$\begin{cases} A_i : X \to X_i \text{ has nonempty convex values and} \\ A_i^{-1}(x) \text{ is open (in } X\text{) for each } x \in X_i \end{cases} \tag{2.1}$$

$$A_i(x) \subseteq \overline{B_i(x)} \quad \text{for } x \in X \tag{2.2}$$

and

$$\begin{cases} \text{there exists a map } S_i : X \to X_i \text{ with } (A_i \cap P_i)(w) \subseteq S_i(w) \\ \text{for } w \in X,\ S_i \text{ has convex values, } S_i^{-1}(w) \text{ is open (in } X\text{)} \\ \text{for each } w \in X_i \text{ and } x_i \notin S_i(x) \text{ for } x \in X. \end{cases} \tag{2.3}$$

Finally suppose there exists a compact subset K of X and for each $i \in \{1,\dots,N\}$ a convex compact set Y_i of X_i such that for each $x \in X \backslash K$ there exists a $j \in \{1,\dots,N\}$ with $(A_j \cap P_j)(x) \cap Y_j \neq \emptyset$. Then there exists an equilibrium point of Γ i.e. there exists a $x \in X$ with $x_i \in \overline{B_i(x)}$ and $A_i(x) \cap P_i(x) = \emptyset$ for all $i \in \{1,\dots,N\}$.

Proof. For each $i \in \{1,\dots,N\}$ let

$$M_i = \{x \in X : \; x_i \notin \overline{B_i(x)}\}$$

and note M_i is open in X since $\overline{B_i} : X \to CK(X_i)$ is upper semicontinuous. Let $H_i : X \to X_i$ and $T_i : X \to X_i$ be given by

$$H_i(x) = \begin{cases} A_i(x) \cap P_i(x), & x \notin M_i \\ A_i(x), & x \in M_i \end{cases}$$

and

$$T_i(x) = \begin{cases} A_i(x) \cap S_i(x), & x \notin M_i \\ A_i(x), & x \in M_i. \end{cases}$$

First note for each $i \in \{1,\dots,N\}$ that $H_i(w) \subseteq T_i(w)$ for $w \in X$ and $T_i(x)$ has convex values for each $x \in X$. Now we show $T_i^{-1}(x)$ is open (in X) for each $x \in X_i$. To see see this let $x \in X_i$ and note

$$\begin{aligned} T_i^{-1}(x) &= \{w \in X : x \in T_i(w)\} \\ &= \{w \in M_i : x \in A_i(w)\} \cup \{w \in X \backslash M_i : x \in A_i(w) \cap S_i(w)\} \\ &= [M_i \cap \{w \in X : x \in A_i(w)\}] \cup [(X \backslash M_i) \cap \{w \in X : x \in A_i(w) \cap S_i(w)\}] \\ &= \big[M_i \cap A_i^{-1}(x)\big] \cup \big[(X \backslash M_i) \cap \big[A_i^{-1}(x) \cap S_i^{-1}(x)\big]\big] \\ &= \big[M_i \cup S_i^{-1}(x)\big] \cap A_i^{-1}(x) \end{aligned}$$

(note $A_i^{-1}(x) \cap S_i^{-1}(x) \subseteq A_i^{-1}(x)$) which is open in X. Next we claim for $i \in \{1,\dots,N\}$ that $x_i \notin T_i(x)$ for $x \in X$. Fix $i \in \{1,..,N\}$ and $x \in X$. First consider $x \in M_i$. Then $x_i \notin \overline{B_i(x)}$ so $x_i \notin A_i(x)$ from (2.2) i.e. $x_i \notin T_i(x)$ if $x \in M_i$. Next consider $x \notin M_i$. Then $x_i \notin (A_i \cap S_i)(x)$ since $(A_i \cap S_i)(x) \subseteq S_i(x)$ and $w_i \notin S_i(w)$ for $w \in X$ i.e. $x_i \notin T_i(x)$ if $x \notin M_i$. Consequently $x_i \notin T_i(x)$ for $x \in X$.

Now let K and Y_i be as in the statement of Theorem 2.1. Note if $x \in X \backslash K$ then there exists a $j \in \{1,\dots,N\}$ with $(A_j \cap P_j)(x) \cap Y_j \neq \emptyset$, so if we let $x \in X \backslash K$ and $x \notin M_j$ then $H_j(x) \cap Y_j = (A_j \cap P_j)(x) \cap Y_j \neq \emptyset$ whereas if $x \in X \backslash K$ and $x \in M_j$ then $\emptyset \neq (A_j \cap P_j)(x) \cap Y_j \subseteq A_j(x) \cap Y_j = H_j(x) \cap Y_j$.

Then all the conditions in Theorem 1.1 are satisfied so there exists a $x \in X$ with $H_i(x) = \emptyset$ for all $i \in \{1,\dots,N\}$. Note for each $i \in \{1,\dots,N\}$ that A_i has nonempty values so as a result $x \notin M_i$. Thus $x \notin M_i$ with $H_i(x) = \emptyset$ for all $i \in \{1,\dots,N\}$ i.e. $x_i \in \overline{B_i(x)}$ and $A_i(x) \cap P_i(x) = \emptyset$ for all $i \in \{1,\dots,N\}$. □

Remark 2.2. (i). Suppose for each $i \in \{1,\dots,N\}$ and for each $x \in X$ there exists a map $\phi_{i,x} : X \to X_i$ and an open set $U_{i,x}$ containing x with $(A_i \cap P_i)(w) \subseteq \phi_{i,x}(w)$ for every $w \in U_{i,x}$, $\phi_{i,x}$ is convex valued, $(\phi_{i,x})^{-1}(w)$ is open (in X) for each $w \in X_i$ and $w_i \notin \phi_{i,x}(w)$ for each $w \in U_{i,x}$.

Then from [7] there exists a map $S_i : X \to X_i$ with $(A_i \cap P_i)(w) \subseteq S_i(w)$ for $w \in X$, S_i is convex valued, $(S_i)^{-1}(w)$ is open for each $w \in X_i$ and $w_i \notin S_i(w)$ for

$w \in X$ (i.e. (2.3) holds in this situation). More generality one could replace the map $A_i \cap P_i : X \to E_i$ with a map $\Psi_i : X \to E_i$.

(ii). From the proof of Theorem 2.1 note in (2.3) the assumption "$x_i \notin S_i(x)$ for $x \in X$" could be replaced by "$x_i \notin S_i(x)$ for $x \notin M_i$".

(iii). From the proof of Theorem 2.1 note in (2.3) the assumption "$(A_i \cap P_i)(w) \subseteq S_i(w)$ for $w \in X$" could be replaced by "'$(A_i \cap P_i)(w) \subseteq S_i(w)$ for $w \notin M_i$".

(iv). In (2.1) we assumed A_i has convex values but this can easily be removed if we replace A_i (in parts of the statement and proof) with $co\, A_i$.

This Remark is also true in the other results in this paper but we will not refer to it again.

Theorem 2.3. *Let* $\Gamma = (X_i, A_i, B_i, P_i)_{i=1}^N$ *be a* N*-person game i.e.* $\{X_i\}_{i=1}^N$ *is a family of convex sets each in a Hausdorff topological vector space* E_i *with* $X = \prod_{i=1}^N X_i$ *paracompact, and for each* $i \in \{1, \ldots, N\}$ *the constraint correspondences* A_i, $B_i : X \to X_i$, *the preference correspondence* $P_i : X \to E_i$ *and* $cl\, B_i\, (\equiv \overline{B_i}) : X \to CK(X_i)$ *is upper semicontinuous. Also for each* $i \in \{1, \ldots, N\}$ *suppose* (2.1), (2.2) *and* (2.3) *hold and in addition there is a compact convex subset* K_i *with* $A_i(X) \subseteq K_i \subseteq X_i$. *Then there exists an equilibrium point of* Γ *i.e. there exists a* $x \in X$ *with* $x_i \in \overline{B_i(x)}$ *and* $A_i(x) \cap P_i(x) = \emptyset$ *for all* $i \in \{1, \ldots, N\}$.

Proof. For $i \in \{1, \ldots, N\}$ let M_i, H_i and T_i be as in Theorem 2.1 and note $H_i(w) \subseteq T_i(w)$ for $w \in X$, T_i has convex values, $T_i^{-1}(w)$ is open (in X) for each $w \in X_i$ and $x_i \notin T_i(x)$ for $x \in X$. Let $i \in \{1, \ldots, N\}$ and let K_i be as in Theorem 2.3 and note $T_i(X) \subseteq A_i(X) \subseteq K_i \subseteq X_i$. Thus all the conditions in Theorem 1.5 are satisfied so there exists a $x \in X$ with $H_i(x) = \emptyset$ for all $i \in \{1, \ldots, N\}$. Note for $i \in \{1, \ldots, N\}$ that $x \notin M_i$ (since A_i has nonempty values) so for all $i \in \{1, \ldots, N\}$ we have $x \notin M_i$ with $H_i(x) = \emptyset$. □

Theorem 2.4. *Let* $\Gamma = (X_i, A_i, B_i, P_i)_{i=1}^N$ *be a* N*-person game i.e.* $\{X_i\}_{i=1}^N$ *is a family of convex sets each in a Hausdorff topological vector space* E_i *and for each* $i \in \{1, \ldots, N\}$ *the constraint correspondences* A_i, $B_i : X \to X_i$, *the preference correspondence* $P_i : X \to E_i$ *and* $cl\, B_i\, (\equiv \overline{B_i}) : X \to CK(X_i)$ *is upper semicontinuous. Also for each* $i \in \{1, \ldots, N\}$ *suppose* (2.1), (2.2) *and* (2.3) *hold and in addition there is a compact convex subset* K *of* X *with* $A(K) \subseteq K$ *where* $A(x) = \prod_{i=1}^N A_i(x)$ *for* $x \in X$. *Then there exists a* $x \in X$ *with* $x_{i_0} \in \overline{B_{i_0}(x)}$ *and* $A_{i_0}(x) \cap P_{i_0}(x) = \emptyset$ *for some* $i_0 \in \{1, \ldots, N\}$.

Proof. For $i \in \{1, \ldots, N\}$ let M_i, H_i and T_i be as in Theorem 2.1 and note $H_i(w) \subseteq T_i(w)$ for $w \in X$, T_i has convex values, $T_i^{-1}(w)$ is open (in X) for each $w \in X_i$ and $x_i \notin T_i(x)$ for $x \in X$. Let $T(x) = \prod_{i=1}^N T_i(x)$ for $x \in X$ and since $T_i(x) \subseteq A_i(x)$ for $x \in X$ then $T(K) \subseteq A(K) \subseteq K$. Thus all the conditions in Theorem 1.6 are satisfied so there exists a $x \in X$ with $H_{i_0}(x) = \emptyset$ for some $i_0 \in \{1, \ldots, N\}$. Note for each $i \in \{1, \ldots, N\}$ that A_i has nonempty values so as a result $x \notin M_{i_0}$ i.e. $x \notin M_{i_0}$ and $H_{i_0}(x) = \emptyset$ i.e. $x_{i_0} \in \overline{B_{i_0}(x)}$ and $A_{i_0}(x) \cap P_{i_0}(x) = \emptyset$. □

For Theorem 2.4 note in (2.3) we could replace "for each $i \in \{1, \ldots, N\}$ and each $x \in X$ we have $x_i \notin S_i(x)$" with "for each $x \in X$ there exists a $i \in \{1, \ldots, N\}$ with

$x_i \notin S_i(x)$" and this would guarantee that for each $x \in X$ there exists a $i \in \{1, \ldots, N\}$ with $x_i \notin T_i(x)$ which is all that is needed to apply Theorem 1.6.

Corollary 2.5. *Let X be a convex set in a Hausdorff topological vector space E and let A, $B. P : X \to E$ with $cl\, B\, (\equiv \overline{B}) : X \to CK(X)$ upper semicontinuous. Suppose the following hold:*

$$\text{(2.4)} \qquad \begin{cases} A : X \to X \ \ \text{has nonempty convex values and} \\ A^{-1}(x) \ \ \text{is open (in } X\text{) for each } \ x \in X \end{cases}$$

$$\text{(2.5)} \qquad A(x) \subseteq \overline{B(x)} \ \ \text{for} \ \ x \in X$$

and

$$\text{(2.6)} \qquad \begin{cases} \text{there exists a map } \ S : X \to X \ \ \text{with} \ \ (A \cap P)(w) \subseteq S(w) \\ \text{for } \ w \in X, \ S \ \ \text{has convex values, } \ S^{-1}(w) \ \ \text{is open (in } \ X\text{)} \\ \text{for each } \ \ w \in X \ \ \text{and} \ \ x \notin S(x) \ \ \text{for} \ \ x \in X. \end{cases}$$

Also assume there is a compact convex subset K of X with $A(K) \subseteq K$. Then there exists a $x \in X$ with $x \notin \overline{B(x)}$ and $A(x) \cap P(x) = \emptyset$.

Proof. The result follows from Theorem 2.4 with $N = 1$. □

Remark 2.6. We note here that Corollary 2.5 could supply a proof of Theorem 2.4 (so if one wishes one could prove Corollary 2.5 first). To do this let

$$A(x) = \prod_{i=1}^{N} A_i(x), \ \ B(x) = \prod_{i=1}^{N} B_i(x) \ \ \text{ and } \ \ P(x) = \prod_{i=1}^{N} P_i(x) \ \ \text{for} \ \ x \in X.$$

Note [1 pp. 472] guarantees that $\overline{B} : X \to CK(X)$ is upper semicontinuous. Also note (2.5) is immediate since $A_i(x) \subseteq \overline{B_i(x)}$ for $x \in X$ and $i \in \{1, \ldots, N\}$. To show (2.4) holds we just need to show that $A^{-1}(x)$ is open (in X) for each $x \in X$. Let $\pi_i : X \to X_i$ be the i^{th} projection and let $A_i^\star : X \to X$ be given by $A_i^\star = \pi_i^{-1}\, A_i$. Let $x \in X$ and note

$$\begin{aligned} A^{-1}(x) &= \{w \in X : \ x \in A(w)\} = \{w : \ \pi_1\, x \in A_1(w)\} \cap \ldots \cap \{w : \ \pi_N\, x \in A_N(w)\} \\ &= \{w \in X : \ x \in A_1^\star(w)\} \cap \ldots \cap \{w \in X : \ x \in A_N^\star(w)\} \end{aligned}$$

so

$$A^{-1}(x) = \cap_{i=1}^{N}\, (A_i^\star)^{-1}(x)$$

which is open in X (note $(A_i^\star)^{-1}(x) = A_i^{-1}\, \pi_i(x)$). Thus (2.4) holds. To apply Corollary 2.5 it remains to show (2.6) holds. Let

$$S(x) = \prod_{i=1}^{N} S_i(x) \ \ \text{for} \ \ x \in X \ \ \text{ and } \ \ S_i^\star = \pi_i^{-1}\, S_i.$$

Now $(A \cap P)(w) \subseteq S(w)$ for $w \in X$ since $(A_i \cap P_i)(w) \subseteq S_i(w)$ for $w \in X$ and $i \in \{1, \ldots, N\}$. Also $S : X \to X$ and S is convex valued. In addition for $x \in X$ then $S^{-1}(x) = \cap_{i=1}^{N}\, (S_i^\star)^{-1}(x)$ which is open in X. Finally for $x \in X$ we have $x_i \notin S_i(x)$ for each $i \in \{1, \ldots, N\}$ so $x \notin S(x)$. Thus (2.6) holds. Now Corollary 2.5 guarantees

$x \in X$ with $x \in \overline{B(x)}$ (so $x_i \in \overline{B_i(x)}$ for $i \in \{1, \dots, N\}$) and $A(x) \cap P(x) = \emptyset$ (so there exists a $i \in \{1, \dots, N\}$ with $A_i(x) \cap P_i(x) = \emptyset$).

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