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A FIXED POINT PRINCIPLE FOR SOME NONSELF OPERATORS ON LARGE KASAHARA SPACES

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Abstract. In this paper we give a saturated fixed point principle for nonself operators on a large Kasahara space. Our results extend some results given by I.M Olaru and N.A. Secelean in *A New Approach of Some Contractive Mappings on Metric Spaces*, Mathematics, 2021 9(12).
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1. Introduction

There are several techniques in the fixed point theory for nonself operators on a complete metric space ([1], [7], [10], [13]). Throughout this paper we shall follow the standard terminologies and notations used in nonlinear analysis. For the convenience of the reader we shall recall some of them.

Let X be a nonempty set and $f : X \to X$ be an operator. We denote by $f^0 := 1_X$, $f^1 := f$, $f^{n+1} := f^n \circ f$, $n \in \mathbb{N}$ the iterate operators of the operator f. We also have

$$P(X) := \{Y \subset X \mid Y \neq \emptyset\}$$
$$F_f := \{x \in X \mid f(x) = x\}$$
$$P_b(X) := \{Y \in P(X) \mid Y \textit{ is bounded}\}$$

The notion of L-space was given by M. Frechet in 1906 (see [4])

Definition 1.1. Let X be a nonempty set and let us consider $s(X) := \{\{x_n\}_{n\in\mathbb{N}} \mid x_n \in X\}$, $c(X) \subset s(X)$ and $Lim : c(X) \to X$ be an operator.We say that $(X, c(X), Lim)$ is an $L-$space (denoted also by $(X, \xrightarrow{F})$) if the following conditions are satisfied:

(i) if $x_n = x$ for all $n \in \mathbb{N}$ then $\{x_n\}_{n\in\mathbb{N}} \in c(X)$ and $Lim\{x_n\}_{n\in\mathbb{N}} = x$
(ii) if $\{x_n\}_{n\in\mathbb{N}} \in c(X)$ and $Lim\{x_n\}_{n\in\mathbb{N}} = x$ then for all subsequences $\{x_{n_i}\}_{i\in\mathbb{N}}$ of $\{x_n\}_{n\in\mathbb{N}}$ we have $\{x_{n_i}\}_{i\in\mathbb{N}} \in c(X)$ and $Lim\{x_{n_i}\}_{i\in\mathbb{N}} = x$.

The notion of large Kasahara space which will be used in this paper, is the following one:

Definition 1.2. (see [2]) Let X be a nonempty set, $\xrightarrow{F}$ be an L-space structure on X, and $d : X \times X \to \mathbb{R}_+$ be a metric on X. The triple $(X, \xrightarrow{F}, d)$ is a large Kasahara space iff we have the following compatibility condition between $\xrightarrow{F}$ and d:

(i) $\{x_n\}_{n\in\mathbb{N}} \subset X$ a Cauchy sequence with respect to d implies that $\{x_n\}_{n\in\mathbb{N}}$ converges in $(X, \xrightarrow{F})$;
(ii) $x_n \xrightarrow{F} x$, $y_n \xrightarrow{F} y$ and $d(x_n, y_n) \to 0$ as $n \to \infty$ implies $x = y$.

Another useful notion used in the studies of fixed points for nonself operators is the comparison function.

Definition 1.3. (see [8]) A function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is a comparison function if the following conditions are satisfied:

(i) φ is increasing;
(ii) the sequence $\varphi^n(t) \to 0$ as $n \to \infty$, for every $t > 0$.

Definition 1.4. (see [8]) A comparison function is a strong comparison function if $\sum_{k\geq 0} \varphi^k(t) < \infty$, for any $t > 0$.

Lemma 1.1. *(see [8]) If $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is a strong comparison function then the function $s : \mathbb{R}_+ \to \mathbb{R}_+$ defined by*

$$s(t) = \sum_{k\geq 0} \varphi^k(t),$$

is increasing and continuous at 0.

Lemma 1.2. *(see [15]) Let $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ be a strong comparison function and $\{b_n\}_{n\in\mathbb{N}}$ be a sequence of positive numbers such that $b_n \to 0$ as $n \to \infty$. Then*

$$\lim_{n\to\infty} \sum_{k=0}^{n} \varphi^{n-k}(b_k) = 0.$$

Using the setup of large Kasahara spaces the author of [2] proved some results related to the existence, uniqueness and data dependence of fixed point for nonself operators, $f : Y \subset X \to X$, in the following two cases:

(A) there exists $l \in (0, 1)$ such that $d(f(x), f(y)) \leq ld(x, y)$, for all $x, y \in Y$
(B) there exists a strong comparisom function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ such that $d(f(x), f(y)) \leq \varphi(d(x, y))$, for all $x, y \in Y$

More results about fixed points of operators on Kasahara spaces can be found in [12], [16], [3].

2. Main results

In [5] the authors proved a result related to the existence and uniqueness of fixed point for an operator $T : X \to X$ which satisfies the metric condition

$$G(d(Tx, Ty)) \leq H(d(x, y)), \text{ for all } x, y \in X, \text{ with } Tx \neq Ty \tag{2.1}$$

where $G, H : (0,\infty) \to \mathbb{R}$ are some suitable control mappings. The above results extended the results provided earlier by Proinov in [6]. More results in regards to generalized contractions satisfying the metric condition (2.1) can be found in [17]. Next, by following the results of [11], our goal is to get a saturated fixed point principle for a nonself operator

$$f : Y \subset X \to X$$

on a large Kasahara space in the case when

$$G(d(f(x), f(y))) \leq H(d(x,y)), \text{ for all } x, y \in Y, \text{ with } f(x) \neq f(y),$$

where $G, H : (0,\infty) \to \mathbb{R}$ and Y will be defined further during our considerations. In order to prove our main result we need the following definitions and auxiliary results.

Definition 2.1. We say that a function $G : (0,\infty) \to \mathbb{R}$ satisfies property $\mathcal{P}$ if, for every sequence (t_k) of positive numbers one has $\lim\limits_{k\to\infty} G(t_k) = -\infty$, implies $\lim\limits_{k\to\infty} t_k = 0$.

Lemma 2.1. *Let us consider a sequence $\{t_j\}_{j\in\mathbb{N}}$ of positive numbers, and let $G, H : (0,\infty) \to \mathbb{R}$ be such that:*

(i) $G(t_j) \leq H(t_{j-1})$, $(\forall) j \geq 1$;
(ii) *for each $r \geq t > 0$ we have $G(r) > H(t)$;*
(iii) $\liminf\limits_{j\to\infty} \big(G(t_j) - H(t_j)\big) > 0$;
(iv) *G has the property $\mathcal{P}$.*

Then $\{t_j\}_{j\in\mathbb{N}}$ is a decreasing sequence and $t_j \searrow 0$ as $j \to \infty$.

Proof. The assumptions (i) and (ii) lead us to $t_k < t_{k-1}$ and

$$G(t_k) - G(t_{k-1}) \leq H(t_{k-1}) - G(t_{k-1}),$$

for any $k \geq 1$. Since $\{t_k\}_{k\in\mathbb{N}}$ is a decreasing sequence of positive numbers it follows that there exists $t \geq 0$ such that $t = \lim\limits_{j\to\infty} t_j$. Next, arguing by contradiction we suppose that $t > 0$. Then, via (iii) we get

$$G(t_j) \leq G(t_0) - \sum_{k=1}^{j} \Big(G(t_{k-1}) - H(t_{k-1})\Big) \to -\infty,$$

as $j \to \infty$.

Since G has the property $\mathcal{P}$ we conclude that $t_j \searrow 0$ as $j \to \infty$ which is a contradiction with the assumption $t > 0$. □

Lemma 2.2. *Let us consider the sequences $\{t_j\}_{j\in\mathbb{N}} \subset \mathbb{R}_+$, $\{s_j\}_{j\in\mathbb{N}}$ of positive numbers, $s_j \searrow 0$ as $j \to \infty$ and let $G, H : (0,\infty) \to \mathbb{R}$ be such that:*

(i) $G(t_j) \leq H(s_j + t_{j-1})$, $(\forall) j \geq 1$;
(ii) $\liminf\limits_{j\to\infty} \big(G(t_{j-1}) - H(s_j + t_{j-1})\big) > 0$
(iii) *G has the property $\mathcal{P}$.*

Then $t_j \to 0$ as $j \to \infty$.

Proof. The condition (i) leads us to $G(t_k) - G(t_{k-1}) \leq H(t_{k-1} + s_k) - G(t_{k-1})$ for any $k \geq 1$. Then, via (ii) we get

$$G(t_j) \leq G(t_0) - \sum_{k=1}^{j} \Big(G(t_{k-1}) - H(t_{k-1} + s_k) \Big) \to -\infty,$$

as $j \to \infty$. Since G satisfies the property $\mathcal{P}$ we conclude that $t_j \to 0$ as $j \to \infty$. □

Lemma 2.3. *Let us consider the sequence $\{t_j\}_{j \in \mathbb{N}} \subset \mathbb{R}_+$ converging to zero as $j \to \infty$, $g : \mathbb{R}_+ \to \mathbb{R}_+$ be a bijection with $g^{-1} : \mathbb{R}_+ \to \mathbb{R}_+$ an increasing function on $\mathbb{R}_+$ and $G, H : (0, \infty) \to \mathbb{R}$ be such that:*

(i) $G(t_j) \leq H(t_{j-1})$, $(\forall) j \geq 1$;
(ii) *there exists* $M_0 > 0$ *such that* $g(t_j)(G(t_0) - G(t_j)) \leq M_0$, $(\forall) j \geq 1$;
(iii) $\sum\limits_{j \geq 1} g^{-1}\left(\dfrac{M_0}{\sum\limits_{k=1}^{j} \Big(G(t_{k-1}) - H(t_{k-1}) \Big)} \right) < \infty$;
(iv) *for each* $r \geq t > 0$ *we have* $G(r) > H(t)$.

Then $\sum\limits_{j \geq 1} t_j < \infty$.

Proof. First of all we remark that the condition (i) implies

$$G(t_k) - G(t_{k-1}) \leq H(t_{k-1}) - G(t_{k-1}),$$

for any $k \geq 1$. Then by summing up for $k = \overline{1, j}$ we get

$$G(t_j) \leq G(t_0) - \sum_{k=1}^{j} \Big(G(t_{k-1}) - H(t_{k-1}) \Big) \Rightarrow$$

$$\sum_{k=1}^{j} \Big(G(t_{k-1}) - H(t_{k-1}) \Big) \leq G(t_0) - G(t_j) \Rightarrow$$

$$0 \leq g(t_j) \sum_{k=1}^{j} \Big(G(t_{k-1}) - H(t_{k-1}) \Big) \leq g(t_j)(G(t_0) - G(t_j)) \leq M_0 \Rightarrow$$

$$t_j \leq g^{-1}\left(\frac{M_0}{\sum\limits_{k=1}^{j} \Big(G(t_{k-1}) - H(t_{k-1}) \Big)} \right) \Rightarrow$$

$$\sum_{j \geq 1} t_j < \sum_{j \geq 1} g^{-1}\left(\frac{M_0}{\sum\limits_{k=1}^{j} \Big(G(t_{k-1}) - H(t_{k-1}) \Big)} \right) < \infty$$

□

Definition 2.2. We say that the function $g : \mathbb{R}_+ \to \mathbb{R}_+$ belongs to the class $\mathcal{R}$ iff

(g_0) g is a bijection with g^{-1} increasing on $\mathbb{R}_+$;
(g_1) $\lim\limits_{t \searrow 0} g(t) = 0$.

Example 2.1. Let us consider $k \in (0, 1)$ and $g : \mathbb{R}_+ \to \mathbb{R}_+$ defined by $g(t) = t^k$. Then $g \in \mathcal{R}$.

Definition 2.3. We say that the function $F : \mathbb{R}_+ \to \mathbb{R}$ is a regular function with respect to $g \in \mathcal{R}$ if and only if $\lim\limits_{t \searrow 0} g(t)F(t) = 0$.

Example 2.2. Let us consider g as in Example 2.1, $0 < \alpha < k$ and $F : \mathbb{R}_+ \to \mathbb{R}$ defined by $F(t) = -\frac{1}{t^\alpha}$. Then F is a regular function with respect to g.

Lemma 2.4. *Let us consider $G, H : (0,\infty) \to \mathbb{R}$ such that:*

(a) *G is left lower semicontinuous and satisfies property $\mathcal{P}$;*
(b) *H is increasing;*
(c) *$\liminf\limits_{s \searrow t} \big(G(s) - H(s)\big) > 0$ for each $t > 0$;*
(d) *for each $r \geq t > 0$ we have $G(r) > H(t)$.*

Then the function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ defined by

$$\varphi(t) = \sup\{s \geq 0 \mid G(s) \leq H(t)\} \tag{2.2}$$

has the following properties:

(i) *φ is a comparison function;*
(ii) *if there exists $g \in \mathcal{R}$ such that G is regular with respect to g and*

$$\sum_{j \geq 1} g^{-1}\left(\frac{M}{\sum\limits_{k=1}^{j}\Big(G(t_{k-1}) - H(t_{k-1})\Big)}\right) < \infty$$

for any $M \in (0,\infty)$ and $\{t_j\}_{j\in\mathbb{N}}$ a decreasing sequence of positive numbers converging to zero then φ is a strong comparison function;

(iii) *if H is upper semicontinuous on $\mathbb{R}_+$ and $\{t_j\}_{j\in\mathbb{N}}$ is a bounded sequence of positive numbers such that $\lim\limits_{j\to\infty}(t_j - \varphi(t_j)) = 0$ then:*
  (α) *φ is upper semicontinuous on $\mathbb{R}_+$ and $\limsup\limits_{t\to\varepsilon} \varphi(t) < \varepsilon$ for any $\varepsilon > 0$;*
  (β) *$t_j \to 0$ as $j \to \infty$.*

Proof. (i) Let $t > 0$. Since G satisfies the property $\mathcal{P}$ it follows that the set

$$\{s \geq 0 \mid G(s) \leq H(t)\}$$

is not empty. On the other hand we remark that (d) implies

$$\{s \geq 0 \mid G(s) \leq H(t)\} \subseteq [0,t]$$

and thus there exists

$$\sup\{s \geq 0 \mid G(s) \leq H(t)\} =: \varphi(t) \leq t, \text{ for all } t > 0.$$

Further let us consider $0 \leq t_1 \leq t_2$. Since H is increasing it follows that

$$\varphi(t_1) = \sup\{s \geq 0 \mid G(s) \leq H(t_1)\} \leq \sup\{s \geq 0 \mid G(s) \leq H(t_2)\} = \varphi(t_2)$$

and from here one has that φ is increasing i.e condition (i) of Definition 1.3 is verified. Moreover, since G is left lower semicontinuous and taking into account the definition of φ we get that

$$G(s) \leq H(t),\ (\forall) s \in [0,\varphi(t)) \Rightarrow G(\varphi(t)) \leq \liminf_{s \nearrow \varphi(t)} G(s) \leq H(t). \tag{2.3}$$

In order to check the condition (ii) of Definition 1.3 let us consider $t > 0$ and the decreasing sequence $\{t_n\}_{n\in\mathbb{N}}$ defined by

$$t_0 = t,\ t_n = \varphi(t_{n-1}),\ n \geq 1.$$

The sequence $\{t_n\}_{n\in\mathbb{N}}$ being decreasing it follows that there exists $t \geq 0$ such that $t = \lim_{n\to\infty} t_n$. Next, arguing by contradiction we suppose that $t > 0$. Then, via relation (2.3) and summation over $k = \overline{1,n}$ we have

$$G(t_k) \leq H(t_{k-1}) \Rightarrow G(t_k) - G(t_{k-1}) \leq H(t_{k-1}) - G(t_{k-1}) \Rightarrow$$

$$G(t_n) \leq G(t_0) - \sum_{k=1}^{n} \Big(G(t_{k-1}) - H(t_{k-1}) \Big) \to -\infty,$$

as $n \to \infty$. Since G has the property $\mathcal{P}$ we conclude that $t_n \searrow 0$ as $n \to \infty$ which is a contradiction with the assumption $t > 0$.

(ii) Let us consider $t > 0$ and the decreasing sequence $\{t_n\}_{n\in\mathbb{N}}$ defined by

$$t_0 = t,\ t_n = \varphi(t_{n-1}),\ n \geq 1.$$

By considering the same arguments as in the proof of (i) we get that $t_n \to 0$ as $n \to \infty$ and $G(t_j) \leq H(t_{j-1})$ for any $j \geq 1$. On the other hand, since G is regular with respect to $g \in \mathcal{R}$ we have that $\lim_{t\searrow 0} g(t)(G(t_0) - G(t)) = 0$. Consequently, there exists $M_0 > 0$ such that $g(t_j)(G(t_0) - G(t_j)) \leq M_0$, $(\forall) j \geq 1$. Now, by applying Lemma 2.3 to the sequence $\{t_j\}_{j\in\mathbb{N}}$ defined above, we get that φ is a strong comparison function.

(iii) (α) Let us consider $a \in \mathbb{R}_+$, $\mathcal{U}_a(\varphi) = \{t \in \mathbb{R}_+ \mid \varphi(t) \geq a\}$ and $\{t_j\}_{j\in\mathbb{N}}$ such that $t_j \to t$ as $j \to \infty$. Since $t_j \in \mathcal{U}_a(\varphi)$ we get that

$$\begin{aligned} a \leq \varphi(t_j) &= \sup\{s \geq 0 \mid G(s) \leq H(t_j)\} \\ &\leq \sup\{s \geq 0 \mid G(s) \leq \limsup_{j\to\infty} H(t_j)\} \\ &\leq \sup\{s \geq 0 \mid G(s) \leq H(t)\} = \varphi(t). \end{aligned}$$

From here we get that $t \in \mathcal{U}_a(\varphi)$ and therefore $\mathcal{U}_a(\varphi)$ is closed i.e φ is upper semicontinuous. Moreover for any $\varepsilon > 0$ we have

$$\limsup_{t\to\varepsilon} \varphi(t) \leq \varphi(\varepsilon) < \varepsilon.$$

(β) Arguing by contradiction we suppose that the sequence $\{t_j\}_{j\in\mathbb{N}}$ does not converge to zero. Since it is bounded it follows that there exists $\varepsilon > 0$ and a subsequence t_{n_k} such that $t_{n_k} \to \varepsilon$ as $k \to \infty$.Then

$$\varphi(\epsilon) < \varepsilon = \lim_{k\to\infty} t_{n_k} = \lim_{k\to\infty} \varphi(t_{n_k}) \leq \limsup_{t\to\varepsilon} \varphi(t) \leq \varphi(\epsilon),$$

which is a contradiction. □

The diameter functional $\delta : \mathrm{P(X)} \to \mathbb{R}_+ \cup \{\infty\}$ is defined by:

$$\delta(A) := \sup\{d(a,b) \mid a, b \in A\}, \text{ for all } A \in P(X)$$

Remark 2.1. Let us consider $G, H : (0, \infty) \to \mathbb{R}$ and $f : Y \to X$ be an operator such that

$$G(\delta(f(B))) \leq H(\delta(B)),$$

for all $B \in P_b(Y)$, with $\delta(f(B)) \neq 0$. Then

$$G(d(f(x), f(y))) \leq H(d(x, y)),$$

for all $x, y \in Y$, with $f(x) \neq f(y)$;

Proof. Let us consider $x, y \in Y$ such that $f(x) \neq f(y)$. Then by using hypothesis for $B = \{x, y\}$ we get the conclusion. □

Remark 2.2. Let us consider $G, H : (0, \infty) \to \mathbb{R}$ increasing and left continuous and let $f : Y \to X$ be an operator such that

$$G(d(f(x), f(y))) \leq H(d(x, y)),$$

for all $x, y \in Y$, with $f(x) \neq f(y)$ Then

$$G(\delta(f(B))) \leq H(\delta(B)),$$

for all $B \in P_b(Y)$, with $\delta(f(B)) \neq 0$.

Proof. Let us consider $B \in P_b(Y)$ such that $\delta(f(B)) \neq 0$. Since G, H are increasing on $(0, \infty)$ it follows that

$$\begin{aligned} G(\delta(f(B))) &= G(\sup_{x,y \in B} d(f(x), f(y))) = \sup_{x,y \in B} G(d(f(x), f(y))) \\ &\leq \sup_{x,y \in B} H(d(x, y)) = H(\sup_{x,y \in B} d(x, y)) = H(\delta(B)). \end{aligned}$$

□

Next we shall provide our main results in which is proved a saturated fixed point principle for nonself operators $f : Y \subseteq X \to X$ on large Kasahara spaces under generalized contractive conditions.

Theorem 2.1. *Let $(X, \stackrel{F}{\to}, d)$ be a large Kasahara space, $Y \subset X$ be a closed subset of $(X, \stackrel{F}{\to})$ and $f : Y \to X$ be an operator. We suppose that:*

- (a_1) *there exists the bounded sequence $(y_n)_{n \in \mathbb{N}^*} \in Y$ such that $f^i(y_n)$ is defined for all $i = \overline{1, n}$, $n \in \mathbb{N}^*$;*
- (a_2) *f is continuous in $(X, \stackrel{F}{\to})$;*
- (a_3) *there exists $G, H : (0, \infty) \to \mathbb{R}$ such that:*
  - (i) *$G(\delta(f(B))) \leq H(\delta(B))$, for all $B \in P_b(Y)$, $\delta(f(B)) \neq 0$;*
  - (ii) *for each $r \geq t > 0$ we have $G(r) > H(t)$;*
  - (iii) *there exists $g \in \mathcal{R}$ such that G is regular with respect to g and*

$$\sum_{j \geq 1} g^{-1}\left(\frac{M}{\sum_{k=1}^{j} \left(G(t_{k-1}) - H(t_{k-1}) \right)} \right) < \infty$$

for any $M \in (0, \infty)$ and $\{t_j\}_{j \in \mathbb{N}}$ a decreasing sequence of positive numbers converging to zero;

  - (iv) *G has the property $\mathcal{P}$ and H is increasing;*

(v) $\liminf_{j\to\infty}(G(t_j) - H(s_{j+1} + t_j)) > 0$ *for any sequences* $\{t_j\}_{j\in\mathbb{N}}$, $\{s_j\}_{j\in\mathbb{N}}$ *of positive numbers,* $s_j \to 0$ *and* $t_j \to t > 0$ *as* $j \to \infty$.

Then

(b_1) *there exists* $x^* \in X$ *such that* $f^n(y_n) \xrightarrow{F} x^*$, *as* $n \to \infty$;
(b_2) $F_f = \{x^*\}$;
(b_3) $f^n(y_n) \xrightarrow{d} x^*$, *as* $n \to \infty$;
(b_4) *if the function* φ *defined by (2.2) is subadditive,* G *is left lower semicontinuous and* $\{z_n\}_{n\in\mathbb{N}} \subset Y$ *such that* $d(z_{n+1}, f(z_n))$ *converges to 0 as* $n \to \infty$ *then* $z_n \xrightarrow{d} x^*$, *as* $n \to \infty$.

Proof. (b_1) First of all we remark that the set $\{f^i(y_n) \mid i = \overline{0, n-1}, n \in \mathbb{N}^*\}$ is bounded. Indeed, we remark that for a given $y_0 \in Y$ there exists $R > 0$ such that $d(y_0, y_n) \leq R$ for each $n \in \mathbb{N}^*$. Then by considering (a_3), (ii) we have

$$\begin{aligned} d(y_0, f(y_n)) &\leq d(y_0, f(y_0)) + d(f(y_0), f(y_n)) \\ &\leq d(y_0, f(y_0)) + d(y_0, y_n) \\ &\leq d(y_0, f(y_0)) + R. \end{aligned}$$

Let us consider the sequence $\{t_j\}_{j\geq1}$ defined by $t_j = d(f^j(y_n), f^{j+1}(y_n)) > 0$. From ($a_3$) ($i$) and Remark 2.1 we get that $G(t_j) \leq H(t_{j-1})$. Since H is increasing and considering (a_3) (v) one has

$$\liminf_{j\to\infty}(G(t_j) - H(t_j)) \geq \liminf_{j\to\infty}(G(t_j) - H(t_j + s_{j+1})) > 0.$$

Lemma 2.1 implies that $\{t_j\}_{j\in\mathbb{N}}$ is a decreasing sequence and $t_j \searrow 0$ as $j \to \infty$. On the other hand, the regularity of G implies that there exists $g \in \mathcal{R}$ such that

$$\lim_{t\searrow 0} g(t)(G(t_0) - G(t)) = 0.$$

Consequently, there exists $M_0 > 0$ such that $g(t_j)(G(t_0) - G(t_j)) \leq M_0$, $(\forall) j \geq 1$. Now by using (a_3) (iii) and applying Lemma 2.3 we get that $\sum_{j\geq1} t_j < \infty$ and therefore there exists $M_1 > 0$ such that $\sum_{j=1}^{i-1} t_j < M_1$ for any $i \geq 2$. Then for each $i \geq 2$ we have

$$d(y_0, f^i(y_n)) \leq d(y_0, f(y_0)) + d(f(y_0), f^i(y_n)))$$

$$\leq d(y_0, f(y_0)) + d(f(y_0), f(y_n)) + d(f(y_n), f^i(y_n))$$

$$\leq d(y_0, f(y_0)) + R + \sum_{j=1}^{i-1} t_j \leq d(y_0, f(y_0)) + R + M_1.$$

It follows that there exists $A \in P_b(Y)$ such that

$$\{f^i(y_n) \mid i = \overline{0, n-1}, n \in \mathbb{N}^*\} \subseteq A.$$

Let $A_1 =: f(A), A_2 = f(A_1 \cap A), \cdots, A_{n+1} = f(A_n \cap A)$. From the above construction we get that $A_{n+1} \subseteq A_n$ and $f^n(y_n) \in A_n$ for each $n \geq 1$. Further we state that the condition (a_3) (i) leads us to $G(d_{n+1}) \leq H(d_n)$, where $d_n := \delta(A_n)$. Indeed, the fact that H is increasing it easily implies that

$$G(d_{n+1}) = G(\delta(A_{n+1})) = G(\delta(f(A_n \cap A)))$$
$$\leq H(\delta(A_n \cap A)) \leq H(\delta(A_n)) = H(d_n).$$

By applying Lemma 2.1 for the sequence $\{d_n\}_{n\in\mathbb{N}}$ we get that $d_n \to 0$ as $n \to \infty$. Since $f^n(y_n) \in A_n$ and $f^{n-1}(y_n) \in A_{n-1} \cap A \subseteq A_{n-1}$ we get $\{f^n(y_n)\}_{n\geq 1}$ and $\{f^{n-1}(y_n)\}_{n\geq 1}$ are fundamental sequences in (X,d). Therefore by the condition (i) of Definition 1.2 one has $f^n(y_n) \overset{F}{\to} u^*$ and $f^{n-1}(y_n) \overset{F}{\to} v^*$. On the other hand, $d(f^{n-1}(y_n), f^n(y_n))) \overset{n\to\infty}{\longrightarrow} 0$ and consequently the condition (ii) of Definition 1.2 implies that $u^* = v^* =: x^*$.

(b_2) Since f is continuous in $(X, \overset{F}{\to})$ we get

$$f^n(y_n) = f(f^{n-1}(y_n)) \overset{F}{\to} f(x^*).$$

So, $\{x^*\} \subseteq F_f$. From Remark 2.1 and (a_3) (ii) it follows that f is a contractive operator and thus $F_f = \{x^*\}$.

(b_3) Since

$$G(d(f^n(y_n), x^*)) \leq H(d(f^{n-1}(y_n), x^*)) \ , \ (\forall)n \geq 1$$

we get, via (a_3) (ii), that

$$d(f^n(y_n), x^*) < d(f^{n-1}(y_n), x^*) < \cdots < d(y_n, x^*), \ (\forall)n \geq 1$$

and consequently the sequence $\{d(f^n(y_n), x^*)\}_{n\in\mathbb{N}}$ is bounded. Let us consider

$$t_n := d(f^n(y_n), x^*) \text{ and } s_n := d(f^{n-1}(y_n), f^{n-1}(y_{n-1})) \geq 0.$$

We remark that

$$f^{n-1}(y_n) \in A_{n-1}, \ (\forall)n \geq 1$$
$$f^{n-1}(y_{n-1}) \in A_{n-1} \ (\forall)n \geq 1$$

and therefore $s_n \to 0$ as $n \to \infty$. Moreover,

$$G(t_n) = G(d(f^n(y_n), x^*)) \leq H(d(f^{n-1}(y_n), x^*))$$
$$\leq H(d(f^{n-1}(y_n), f^{n-1}(y_{n-1})) + d(f^{n-1}(y_{n-1}), x^*)) = H(s_n + t_{n-1}),$$

for each $j \geq 1$. Hence the hypothesis of Lemma 2.2 are fulfilled and thus $t_n \to 0$ as $n \to \infty$. Therefore $d(f^n(y_n), x^*) \to 0$ as $n \to \infty$.

(b_4) Let us consider $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ defined as in Lemma 2.4. Then we remark that

$$d(f(x), f(y)) \leq \varphi(d(x,y)),$$

for each $x, y \in Y$. On the other hand for $n \geq 0$ we have

$$d(z_{n+1}, x^*) \leq d(z_{n+1}, f(z_n)) + d(f(z_n), f(x^*)) \leq d(z_{n+1}, f(z_n)) + \varphi(d(z_n, x^*)). \quad (2.4)$$

In the same way we get

$$d(z_n, x^*) \leq d(z_n, f(z_{n-1})) + \varphi(d(z_{n-1}, x^*))$$

which applied back to (2.4) and considering the subadditivity of φ yield

$$d(z_{n+1}, x^*) \leq d(z_{n+1}, f(z_n)) + \varphi(d(z_n, f(z_{n-1}))) + \varphi^2(d(z_{n-1}, x^*)).$$

By induction we get

$$d(z_{n+1},x^*) \le d(z_{n+1},f(z_n))+\varphi(d(z_n,f(z_{n-1})))+\cdots+\varphi^n(z_1,f(z_0))+\varphi^{n+1}(d(x_0,x^*)).$$

Now the conclusion follows by taking into account that φ is a comparison function and by applying Lemma 1.2 for φ and the sequence $b_n = d(z_{n+1}, f(z_n))$, for all $n \in \mathbb{N}^*$. $\square$

Example 2.3. Let us consider g as in Example 2.1, F as in Example 2.2, $\tau > 0$ and $G, H : (0,\infty) \to \mathbb{R}$ defined by

$$G(t) = \tau + F(t)$$
$$H(t) = F(t).$$

Then G, H verify the hypothesis $(a_3)(ii), (iii), (iv), (v)$ from Theorem 2.1.

Proof. (ii) Let us consider $r \ge t > 0$. Then

$$G(r) = \tau - \frac{1}{r^\alpha} \ge -\frac{1}{t^\alpha} \ge -\frac{1}{t^\alpha} = H(t).$$

(iii) Let us consider $M \in (0,\infty)$ and $\{t_j\}_{j\in\mathbb{N}}$ a decreasing sequence of positive numbers converging to zero and let us denote $p = \frac{1}{k} > 1$ and $q = \left(\frac{M}{\tau}\right)^p$. Then

$$\sum_{j\ge1} g^{-1}\left(\frac{M}{\sum\limits_{k=1}^{j}\left(G(t_{k-1}) - H(t_{k-1})\right)}\right) = q\sum_{j\ge1}\frac{1}{j^p} < \infty.$$

(iv) Obviously.

(v)Let us consider the sequences $\{t_j\}_{j\in\mathbb{N}}$, $\{s_j\}_{j\in\mathbb{N}}$ of positive numbers such that $s_j \to 0$ and $t_j \to t > 0$ as $j \to \infty$. Then $\liminf\limits_{j\to\infty}(G(t_j) - H(s_{j+1} + t_j)) = \tau > 0$. $\square$

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