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EXISTENCE OF FIXED POINTS INVOLVING THE DIFFERENCE OF TWO NONLINEAR OPERATORS

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Abstract. Let X be a uniformly smooth Banach space and let T, A be hemicontinuous operators defined on X with values in X. Suppose, in addition, that T is strongly pseudo-contractive (with constant $k > 0$) while $cI + A$ is accretive for $0 < c < 1-k$. Then the operator $T - A$ has a unique fixed point, which represents a significant extension of the main theorem in [12]. Some surjectivity results are also discussed, which are related to the main result. In particular, we begin with a surjectivity result for monotone operators defined on reflexive Banach spaces.

Key Words and Phrases: Smooth Banach space, pseudo-contractive operator, fixed point, surjectivity.
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1. Introduction

Let X be a real Banach space with $\mathcal{D}(T)$ a subset of X. An operator $T : \mathcal{D}(T) \to X$ is said to be *pseudo-contractive* (see [8]) if for each $u,\ v \in \mathcal{D}(T)$ and $\lambda > 1$

$$(\lambda - 1)\|u - v\| \le \|(\lambda I - T)(u) - (\lambda I - T)(v)\|. \tag{1.1}$$

(with I denoting the identity mapping). Following Kato [7], we find an equivalent formulation of (1) that can be described as follows: A mapping $T : \mathcal{D}(T) \to X$ is pseudo-contractive if and only if for every $u,\ v \in \mathcal{D}(T)$, there exists $j \in J(u-v)$ such that

$$\langle Tu - Tv, j\rangle \le \|u - v\|^2$$

where $J : X \to 2^{X^*}$ is the normalized duality mapping which is defined by

$$J(u) = \{j \in X^* : \langle u, j\rangle = \|u\|^2,\ \|j\| = \|u\|\}$$

(see Browder [4] and Kato [7]). The symbol $\langle \cdot, \cdot\rangle$ denotes the generalized duality pairing. It is an immediate consequence of Hahn-Banach Theorem that $J(u)$ is nonempty for each $u \in X$. It is also known that J is single-valued if and only if X is smooth. This latter notion means that

$$\lim_{t\to 0} \frac{\|x + th\| - \|x\|}{t} \tag{1.2}$$

exists for each x and h on the unit sphere U of X. When this is the case, we say that the norm of X is *Gâteaux differentiable*. Moreover, if for each h in U the limit defined by (1.2) is uniformly attained for $x \in U$, we say that the norm of X is *uniformly Gâteaux differentiable*. It is also known that if X has a uniformly Gâteaux differentiable norm, then the mapping J is uniformly continuous on bounded subsets of X from the strong topology on X to the weak* topology on X^*. It is also known that J is norm-to-norm uniformly continuous on bounded sets is equivalent to the space X being *uniformly smooth.* Since some of the results will be given for reflexive spaces, we may assume that the reflexive Banach space X and its dual X^* are both locally uniformly convex after renorming (see Trojanski [13]). This very fact implies that the duality mapping J is single-valued and strictly monotone. In addition, we will present the main results for hemicontinuous operators, which are simply continuous mappings from the one dimensional strong topology into the weak topology.

It is known that if T is pseudo-contractive, then $A := I - T$ is accretive, which means that for every $u, v \in \mathcal{D}(A)$ there exists $j \in J(u-v)$ such that

$$\langle Au - Av, j\rangle \geq 0.$$

We should note that the family of pseudo-contractive mappings is quite ample, containing all nonexpansive mappings, and moreover, are not necessarily continuous. A related class is the so-called *k-pseudo-contractive operators*, which, for $k < 1$ are called strongly pseudo-contractive operators and are defined as follows: For every $u, v \in \mathcal{D}(T)$, there exists $j \in J(u-v)$ such that

$$\langle Tu - Tv, j\rangle \leq k\|u-v\|^2.$$

Accretive operators defined on Hilbert spaces are called monotone operators. We first study some surjectivity results for accretive, as well as, monotone operators, which happens to coincide under Hilbert spaces. However, monotone operators are also defined outside Hilbert spaces. In fact, we say that a mapping $A : D \to X^*$ is said to be monotone if

$$\langle Au - Av, u - v\rangle \geq 0 \;\text{ for all }\; u, v \in D,$$

where the notation $\langle Ax, z\rangle$ represents the action of the linear and bounded functional $Ax \in X^*$ into the element $z \in D$. Moreover, A is *maximal* if its graph is not properly contained in the graph of any other monotone operator. Similarly, we define a strongly monotone operator as follows: The operator $A : D \to X^*$ is said to be *strongly monotone* if there exists a constant $c > 0$ such that

$$\langle Au - Av, u - v\rangle \geq c\|u-v\|^2 \;\text{ for all }\; u, v \in \mathcal{D}(A).$$

The main purpose of this work is to explore the existence of fixed points for the difference of two nonlinear operators under weaker continuity assumptions. We begin discussing some surjectivity results, known in Hilbert spaces, defined for more general type of spaces such as reflexive Banach spaces.

Following Browder and Petryshyn [3], we say that an operator A defined on a normed space X is *α-inverse strongly accretive* ($\alpha > 0$) if for each $u, v \in X$ there exists $j \in J(u-v)$ so that

$$\langle Au - Av, j\rangle \geq \alpha \|Au - Av\|^2.$$

We may observe that this operator A is monotone and $\frac{1}{\alpha}$-Lipschitz. On the other hand, we say that A is *hemicontinuous at* $u \in \mathcal{D}(A)$ if for each $v \in X$, $t_n > 0$, $n \in \mathbb{N}$, with $t_n \to 0$ and $u + t_n v \in \mathcal{D}(A)$ imply $\{A(u + t_n v)\}$ converges weakly to Au as $n \to \infty$.

2. Surjectivity Results

Theorem 2.1. *Let X be a reflexive Banach space, and let $A : X \to X^*$ be a hemicontinuous monotone operator for which*

$$(\dagger) \qquad \lim_{\|x\| \to \infty} \|A(x)\| = +\infty.$$

Then A is surjective.

Proof. We first observe that the operator A is a maximal monotone operator defined on X (see for instance [5]). In addition, under the assumption that X is reflexive, the normalized duality mapping J is single valued and hemicontinuous. Therefore, by Theorem 1.2 of Barbu [1], the operator $A + \epsilon J$ (with $\epsilon > 0$) is surjective. Now, we show that A is surjective. To see this, let $f \in X^*$. Then, the shifting of A by f is also maximal monotone. Hence the equation

$$Ax + \epsilon J(x) = f$$

has a solution for every $\epsilon > 0$. This means, for every $k \in \mathbb{N}$, and $\epsilon_k > 0$ with $\epsilon_k \searrow 0$, there exists $x_k \in X$ such that

$$Ax_k - f + \epsilon_k J(x_k) = 0.$$

Next, we show that the sequence $\{x_k\}$ is bounded. For such a purpose, we use the fact that A is monotone, to derive

$$\langle f - \epsilon_k J(x_k) - A(0), x_k\rangle \geq 0.$$

Hence $\epsilon_k \|Jx_k\| \leq \|f - A(0)\|$, which implies that $\{Ax_k\}$ is bounded. Consequently, this boundedness and equation ($\dagger$) allow us to obtain that $\{x_k\}$ is bounded. Therefore, by Lemma 1 of [9], $x_k \to x$ for some $x \in X$, and thus $Ax = f$. Hence A is surjective. □

Corollary 2.1. *Let X be a reflexive Banach space and let $A : X \to X^*$ be a hemicontinuous strongly monotone operator. Then A is surjective.*

Proof. Since A is strongly monotone, then

$$\langle Au - A(0), u\rangle \geq c\|u\|^2 \quad \text{for all } u \in X.$$

Consequently, $\|Au\| \geq c\|u\| - \|A(0)\|$, which implies that $\lim_{\|u\|\to\infty} \|Au\| = +\infty$. Therefore, due to Theorem 2.1, A is surjective. □

The next theorem, will make reference to demicontinuous operators, meaning that they are simple continuous mappings from the strong topology into the weak topology in X. However, this surjectivity result is for accretive operators in Banach spaces that may be stated as follows:

Theorem 2.2. *Let X be a uniformly smooth Banach space and let $A : X \to X$ be a hemicontinuous accretive operator for which*

$$(^*) \qquad \liminf_{\|x\|\to\infty} \|A(x)\| = +\infty.$$

Then A is surjective.

Proof. We first observe that A holds a stronger continuity assumption, in fact, A is demicontinuous on X since X is uniformly smooth and its domain is an open set (see for instance Kato [6], Gajardo et al. [5]).

Let $z \in X$ and let $A_z(x) = Ax - z$. Select $\eta > \|A(0)\| + 3\|z\|$. Then by $(^*)$, we may choose $r > 0$ such that

$$\inf\{\|Ax\| : \|x\| \geq r\} \geq \eta.$$

This implies

$$\|A(0)\| + 3\|z\| < \|Ax\| \quad \text{for} \quad x \in \partial B(0;r),$$

so that

$$\|A_z(0)\| < \inf_{x\in\partial B} \|A_z x\|.$$

Consequently, by Proposition 2 of [10], there exists a nonexpansive mapping g from $\overline{B}(0;r)$ into $B(0;r)$ whose fixed points determine zeros for A_z. Since X is uniformly smooth, g has a fixed point $x_0 \in B(0;r)$, a result due to J. Baillon that was published in [14]. Hence $z = Ax_0$. Therefore A is surjective.

We should observe that Theorem 2.2 also holds true for super reflexive Banach spaces, since these spaces admit an equivalent renorming to become uniformly smooth. □

Corollary 2.2. *Let X be a uniformly smooth Banach space and let $A : X \to X$ be a hemicontinuous strongly accretive operator. Then A is surjective.*

3. Main Result

Theorem 3.1. *Let X be a uniformly smooth Banach space and $T : X \to X$ be a hemicontinuous strongly pseudo-contractive mapping. If $A : X \to X$ is a hemicontinuous operator such that $cI + A$ is accretive (with $0 < c < 1 - k$), then $T - A$ has a unique fixed point in X.*

Proof. Define $F = I - T + A$. Then for $x, y \in X$ there exists $j \in J(x - y)$ such that

$$\begin{aligned} \langle Fx - Fy, j\rangle &= \langle x - Tx + Ax - (y - Ty + Ay), j\rangle \\ &= \langle x - Tx - (y - Ty), j\rangle + \langle Ax - Ay, j\rangle \\ &\geq (1 - k - c)\|x - y\|^2. \end{aligned}$$

Consequently, by Corollary 2.2, F is surjective, which means, there exists $x \in X$ such that

$$x - Tx + Ax = 0.$$

Therefore $T - A$ has a unique fixed point in X. □

Corollary 3.1. *Let X be a uniformly smooth Banach space and $T : X \to X$ be a hemicontinuous strongly pseudo-contractive mapping. If $A : X \to X$ is a α-inverse strongly accretive operator, then $T - A$ has a unique fixed point in X.*

Proof. Since an α-inverse strongly accretiveness of A trivially implies that $cI + A$ is accretive for any $c > 0$, the conclusion follows from Theorem 3.1. □

Corollary 3.2. *Let H be a Hilbert space and $T : H \to H$ be a hemicontinuous strongly pseudo-contractive mapping. If $A : H \to H$ is a hemicontinuous monotone operator, then $T - A$ has a unique fixed point in H.*

We should mention that Corollary 3.2 represents a significant extension of Theorem 2.2 of [12]. In fact, we have a weaker continuity assumption, the strong pseudo-contractiveness is much more general than contraction mappings, and mostly, the monotonicity on the second operator is significantly much weaker that being *inverse strongly monotone.*

We close, with a more general setting of Corollary 2.2 under weaker assumptions on the operator A. To see this, we need the following definition:

Let $\phi : [0, \infty) \to [0, \infty)$ be a function that is continuos at 0, $\phi(0) = 0$ and $\phi(r) > 0$ for every $r > 0$. Let X be a (real) normed space. A mapping $A : D(A) \subset X \to X$ is said to be ϕ-*expansive* if

$$||Ax - Ay|| \geq \phi(||x - y||).$$

Additionally, A is said to be ϕ-*accretive* if for every $x, y \in D(A)$ there exists $j \in J(x - y)$ such that

$$\big\langle Ax - Ay, j \big\rangle \geq \phi(||x - y||). \tag{3.1}$$

Theorem 3.2. *Let X be a uniformly smooth Banach space and let $A : X \to X$ be a hemicontinuous ϕ-accretive operator. Then A is surjective.*

Proof. In view of Theorem 2.2 It is suffices to show that

$$(^*) \qquad \liminf_{\|x\| \to \infty} \|A(x)\| = +\infty.$$

To see this, we first observe that due to Theorem 1 of [11], there exists a strictly increasing function $\psi : [0, \infty) \to [0, \infty)$ with $\psi(0) = 0$ such that $\psi(t) \to \infty$ as $t \to \infty$ and for every $x, y \in X$, there exists $j \in J(x - y)$ such that

$$\big\langle Ax - Ay, j \big\rangle \geq \psi(||x - y||)||x - y||.$$

However, this implies that $\psi(||x||) \leq \|Ax - A(0)\|$.
Consequently, $\lim\limits_{\|x\| \to \infty} \|A(x)\| = +\infty$. Hence, by Theorem 2.2, we conclude that A is surjective. □

Corollary 3.3. *Let X be a reflexive Banach space, and let $A : X \to X^*$ be a hemicontinuous monotone operator such that*

$$\big\langle Ax - Ay, x - y \big\rangle \geq \phi(\|x - y\|) \quad \textit{for all} \quad x, y \in X. \tag{3.2}$$

Then A is surjective.

Proof. Observe that the proof of Theorem 1 of [11], holds easily for monotone operators, and the proof follows Theorem 3.2. □

Remark 3.1. In [2], the authors introduce the notion of *uniform monotonicity* which appears to be strongly than our assumption (3.2) in the above corollary. They also define *supercoercivity* as:

$$\lim_{\|x\| \to \infty} \frac{\|A(x)\|}{\|x\|} = +\infty.$$

It turns out that Corollary 3.3 represents a significant extension of Proposition 22.8 of [2]. In fact, no supercoercivity assumption is needed on the operator A to prove surjectivity.

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References

[1] V. Barbu, *Nonlinear Semigroups and Differential Equations in Banach Spaces*, Editura Academiei Republicii Socialiste Romania, Bucharest, Noordhoff International Publishing, Leiden, 1976. Translated from the Romanian.

[2] H.H. Bauschke, P.L. Combettes, *Convex Analysis and Monotone Operator Theory in Hilbert Spaces*, CMS Books in Mathematics/Ouvrages de Mathématiques de la SMC. Springer, New York, 2011. With a foreword by Hédy Attouch.

[3] F.E. Browder, W.V. Petryshyn, *Construction of fixed points of nonlinear mappings in Hilbert space*, J. Math. Anal. Appl., **20**(1967), 197-228.

[4] F.E. Browder, *Nonlinear mappings of nonexpansive and accretive type in Banach spaces*, Bull. Amer. Math. Soc., **73**(1967), 875-882.

[5] P. Gajardo, C.H. Morales, J. Peypouquet, *Monotonicity beyond Minty and Kato on locally convex spaces*, J. Math. Anal. Appl., **435**(2016), no. 2, 1701-1709.

[6] T. Kato, *Demicontinuity, hemicontinuity and monotonicity*, Bull. Amer. Math. Soc., **70**(1964), 548-550.

[7] T. Kato, *Nonlinear semigroups and evolution equations*, J. Math. Soc. Japan, **19**(1967), 508-520.

[8] C.H. Morales, *Pseudocontractive mappings and the Leray-Schauder boundary condition.*, Comment. Math. Univ. Carolin., **20**(1979), no. 4, 745-756.

[9] C.H. Morales, *A Bolzano's theorem in the new millennium*, Nonlinear Anal., Ser. A: Theory Methods, **51**(2002), no. 4, 679-691.

[10] C.H. Morales, *Existence theorems for demicontinuous accretive operators in Banach spaces*, Houston J. Math., **10**(1984), no. 4, 535-543.

[11] C.H. Morales, *Variational inequalities for Φ-pseudo-contractive mappings*, Nonlinear Anal., **75**(2012), no. 2, 477-484.

[12] D. Teodorescu, *On the difference of a contraction and an inverse strongly monotone operator*, Oper. Matrices, **11**(2017), no. 3, 635-638.

[13] S.L. Troyanski, *On locally uniformly convex and differentiable norms in certain non-separable Banach spaces*, Studia Math., **37**(1970/71), 173-180.

[14] B. Turett, *A dual view of a theorem of Baillon*, In: Nonlinear Analysis and Applications (St. Johns, Nfld., 1981), volume 80, Lecture Notes in Pure and Appl. Math., Dekker, New York, 1982, 279-286.

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