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ON THE CONVERGENCE OF BROADCAST INCREMENTAL ALGORITHMS WITH APPLICATIONS

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Abstract. We consider a convex constrained optimization problem composed in part of finding fixed points of nonexpansive mappings and in part of solving a minimization problem. Two broadcast incremental algorithms are proposed to solve it, in the spirit of the steepest-descent method and Mann's iterative method. Under certain mild assumptions, the norm convergence of our suggested algorithms is established in the framework of real Hilbert spaces. Finally, numerical experiments on a peer to peer storage system are implemented to illustrate the performance of our algorithm.
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1. Introduction

Let $\mathcal{H}$ be a real Hilbert space associated with norm $\|\cdot\|$ and inner product $\langle\cdot,\cdot\rangle$. Let $\mathcal{C}$ be a convex and closed set in the space $\mathcal{H}$. The norm (strong) convergence of a vector sequence $\{x_n\}_{n=1}^{\infty}$ to a vector x is denoted by $x_n \to x$ as $n \to \infty$, while the weak convergence (convergence in the weak topology) of $\{x_n\}_{n=1}^{\infty}$ to x is denoted by $x_n \rightharpoonup x$ as $n \to \infty$ in $\mathcal{H}$. Let $\mathcal{F}^{(i)}, \mathcal{G}^{(i)} : \mathcal{H} \to \mathbb{R}$ be Fréchet differentiable and convex functions and let $\mathcal{S}^{(i)} : \mathcal{H} \to \mathcal{H}$ be a nonexpansive operator, i.e., $\|\mathcal{S}^{(i)}x - \mathcal{S}^{(i)}y\| \leq \|x - y\|$, $\forall x, y \in \mathcal{H}$, for each $i = 0, 1, 2, \cdots$. In this paper, we aim to present optimization algorithms for solving the following convex minimization problem

$$\text{minimize} \sum_{i\in\mathfrak{I}\cup\{0\}} (\mathcal{F}^{(i)}(x) + \mathcal{G}^{(i)}(x)) \text{ subject to } x \in \bigcap_{i\in\mathfrak{I}\cup\{0\}} Fix(\mathcal{S}^{(i)}), \tag{1.1}$$

where $\mathfrak{I} := \{1, 2, \cdots, I\}$ and $Fix(\mathcal{S}^{(i)})$ is the fixed-point set of mapping $\mathcal{S}^{(i)}$. Problem (1.1) includes some practical problems such as the signal processing, the storage allocation, the graph-based clustering, the power allocation and so on [22, 12, 30, 9, 26, 2, 28]. Problem (1.1) is one of the central convex optimization problems in the nonlinear functional analysis and the numerical optimization theory.

The variety of real-world applications stimulates the search of iterative methods for solving problem (1.1). In this sense, a well known iterative method named projection gradient method for solving the convex minimization problem, is stated as follows. Given $x_0 \in \mathcal{C}$, define $(x_n)_{n\in\mathbb{N}}$ by $x_{n+1} = Proj_{\mathcal{C}}(x_n - \lambda\nabla\mathcal{W}(x_n)), \forall n \geq 0$, where λ is some positive real number, $Proj_{\mathcal{C}} : \mathcal{H} \to \mathcal{C}$ is the metric projection from $\mathcal{H}$ onto $\mathcal{C}$, $\nabla\mathcal{W}$ is the gradient of a Fréchet differentiable and convex function $\mathcal{W}$. Recently, the projection gradient method has received much attention and been investigated extensively. However, this method requires the explicit expression of $Proj_C$, which is unfortunately always unknown. This might seriously affect the efficiency of the method. For avoiding the calculation of $Proj_{\mathcal{C}}$, Yamada [36] introduced a hybrid steepest descent method. It is of the form: given $x_0 \in \mathcal{C}$, define $(x_n)_{n\in\mathbb{N}}$ by $x_{n+1} = (I - \alpha\chi_n\mathcal{W})\mathcal{S}x_n, \forall n \geq 0$, where I denotes the identity operator on $\mathcal{H}$, the mapping $\mathcal{W}$ is κ-strongly monotone (i.e., $\kappa\|x-y\|^2 \leq \langle x-y, \mathcal{W}x - \mathcal{W}y\rangle$, $\forall x, y \in \mathcal{H}$) and ι-Lipschitz continuous (i.e., $\iota\|x-y\| \geq \|\mathcal{W}x - \mathcal{W}y\|$, $\forall x, y \in \mathcal{H}$) on $\mathcal{H}$ and the mapping $\mathcal{S} : \mathcal{C} \to \mathcal{H}$ is nonexpansive (i.e., $\|x-y\| \geq \|\mathcal{S}x - \mathcal{S}y\|$, $\forall x, y \in \mathcal{H}$), $\alpha \in (0, 2\kappa/\iota^2)$ and $(\chi_n)_{n\in\mathbb{N}} \subseteq (0,1)$ such that $\sum_{n=1}^{\infty}\chi_n = \infty$ and $\lim_{n\to\infty}\chi_n = 0$. Yamada established the convergent result of the above algorithm in Hilbert spaces, see [36]. A big advantage of the steepest descent method is that it avoids computing metric projections. Since then, this method has been extensively investigated and further extended to solve some optimization problems, see [16, 29, 18, 7, 11, 27, 33, 5]. Among which, Sahu and Yao [29] proposed a generalized hybrid steepest descent scheme, which stated as follows. Given $x_0 \in \mathcal{C}$, define $(x_n)_{n\in\mathbb{N}}$ by $y_n = x_n - \alpha\chi_n\mathcal{W}(x_n)$, $x_{n+1} = (1-\lambda_n)y_n + \lambda_n\mathcal{S}(y_n)$, $\forall n \geq 0$, where $\mathcal{W}$ is a κ-strongly monotone, ι-Lipschitz continuous mapping on $\mathcal{H}$, $\mathcal{S}$ is a nonexpansive mapping on $\mathcal{C}$ with $Fix(\mathcal{S}) \neq \emptyset$, $\alpha \in (0, 2\kappa/\iota^2)$, $\lambda_n \in (0,1)$, $\chi_n \in (0,1)$, $\lim_{n\to\infty}\chi_n = 0$ and $\sum_{n=1}^{\infty}\chi_n = \infty$. They proved a convergence result for solving strongly monotone variational inequality over $Fix(\mathcal{S})$.

There are two known optimization algorithms for solving problem (1.1).

(I) Broadcast optimization algorithm. We consider a system, in which user 0 can communicate with any other user i ($i \in \mathfrak{I}$), in the case that user 0 manages the whole system. Accordingly, the broadcast optimization algorithm is suitable to implement, see [13, 31, 19, 6] and references therein. In the iterative procedure, user 0 computes $x_n^{(0)} \in \mathcal{H}$ via x_n and its own private information $\mathcal{S}^{(0)}$ and $\mathcal{F}^{(0)}$, i.e., $x_n^{(0)} = x_n^{(0)}(x_n, \mathcal{S}^{(0)}, \mathcal{F}^{(0)})$. User $i (i \in \mathfrak{I})$ computes $x_n^{(i)} \in \mathfrak{I}$, via the transmitted information x_n from user 0 and its own private information $\mathcal{S}^{(i)}$ and $\mathcal{F}^{(i)}$, i.e., $x_n^{(i)} = x_n^{(i)}(x_n, \mathcal{S}^{(i)}, \mathcal{F}^{(i)})(i \in \mathfrak{I})$, and transmits $x_n^{(i)}$ to user 0. Then user 0 computes $x_{n+1} \in \mathcal{H}$ by using all $x_n^{(i)}(i \in \{0\} \cup \mathfrak{I})$, i.e., $x_{n+1} = x_{n+1}(x_n^{(0)}, x_n^{(1)}, \cdots, x_n^{(I)})(i \in \mathfrak{I})$. Assume that user 0 uses the transmitted information equally, $(x_{n+1})_{n\in\mathbb{N}}$ will be defined by

$$x_{n+1} := \frac{\sum_{i\in\{0\}\cup\mathfrak{I}} x_n^{(i)}}{I+1}.$$

(II) Incremental optimization algorithm. Assume that users can communicate with their neighbor users on the network. Thus, users can use their own private information and the transmitted information from their neighbor users. In this case, the incremental optimization algorithm is suitable to implement; see [21, 1, 20, 15] and references therein. Assume that user 0 has the information $y_n = y_n^{(0)} \in \mathcal{H}$ and user $i(i \in \mathfrak{I})$ has the information $y_n^{(i-1)}$ transmitted from user $(i-1)$, at the n-th iteration. Thus user i can compute the next iteration $y_n^{(i)}$ via $y_n^{(i-1)}$ and its own private information $\mathcal{S}^{(i)}$, $\mathcal{F}^{(i)}$, i.e., $y_n^{(i)} = y_n^{(i)}(y_n^{(i-1)}, \mathcal{S}^{(i)}, \mathcal{F}^{(i)})(i \in \mathfrak{I})$. User 0 computes y_{n+1} by using $y_n^{(I)}$ and its own private information $\mathcal{S}^{(0)}$, $\mathfrak{F}^{(0)}$, i.e., $y_{n+1} = y_{n+1}(y_n^{(I)}, \mathcal{S}^{(0)}, \mathcal{F}^{(0)})$.

From the above analysis of *(I)* and *(II)*, we note here that the broadcast optimization algorithm updates the next iteration x_{n+1}, only after user 0 has all the transmitted information $x_n^{(i)}(i \in \{0\} \cup \mathfrak{I})$. Therefore, when it comes to large-scale networked systems, this scheme will be time-consuming. The incremental optimization algorithm requires to go through all users to update the next iteration y_{n+1}. It seems to be physically difficult to go through all users, specially, in large-scale complex networked systems. Hence, the above two algorithms are not very efficient and convenient in practical situations. This requests a new method for avoiding the problems mentioned above. When each user $i(i \in \mathfrak{I})$ can communicate with its neighbors, we construct a subnetwork that consists of its neighbors and the user i itself. Hence, the network system can be divided into T subnetworks. In each subnetwork $t(t \in \mathfrak{T} := \{1, 2, \cdots, T\})$, users can implement the incremental optimization algorithm, by using their own private information and the transmitted information from their neighbor users. For each $t \in \mathfrak{T}$, we construct $x_n^{(t)}$ via $x_n = y_n^{(0)}$ and the incremental optimization algorithm. By using the broadcast distributed optimization algorithm and the information transmitted from subnetworks, user 0 can compute x_{n+1} via $x_n^{(t)}$ $(t \in \{0\} \cup \mathfrak{T})$, i.e., $x_{n+1} = x_{n+1}(x_n^{(0)}, x_n^{(1)}, \cdots, x_n^{(T)}) = \frac{x_n^{(0)} + \sum_{t \in \mathfrak{T}} x_n^{(t)}}{T+1}$.

In this paper, by combining steepest-descent method with Mann's iterative method [24], we investigate broadcast incremental algorithms, which acts as a useful tool in studying convex optimization problems (see [17, 14, 10]) for solving problem (1.1). This can be implemented in a multiuser storage system. Through all users cooperating in the system, which enables each user to obtain its own decision variable. This paper is organized as follows. Section 2 gives some necessary mathematical preliminaries. Section 3 gives the convergence analysis of our proposed iterative algorithms. Section 4 is devoted to a storage allocation problem and provides numerical experiences in a peer to peer data system.

2. Preliminaries

$\mathcal{W} : \mathcal{H} \to \mathbb{R}$ is an α-strongly convex function (see [3]), i.e., for any $x, y \in \mathcal{H}$,

$$\mu\mathcal{W}(x) + (1-\mu)\mathcal{W}(y) \geq \mathcal{W}(\mu x + (1-\mu)y) + \frac{1}{2}\alpha\mu(1-\mu)\|x-y\|^2, \ \mu \in [0,1].$$

Additionally assume that $\mathcal{W}$ is Fréchet differentiable. In such a case, we find that $\nabla\mathcal{W}$ is strongly monotone. A space $\mathcal{X}$ is said to have the Opial's condition if, for any

$\{x_n\} \subset \mathcal{X}$ with $x_n \rightharpoonup x$ $(n \to \infty)$, the following inequality holds

$$\liminf_{n\to\infty} \|x_n - y\| > \liminf_{n\to\infty} \|x_n - x\|,$$

for $y \in \mathcal{X}$ with $y \neq x$, see [25]. Notably, the above inequality is equivalent to

$$\limsup_{n\to\infty} \|x_n - y\| > \limsup_{n\to\infty} \|x_n - x\|,$$

for $y \in \mathcal{X}$ with $y \neq x$. The Opial's condition plays a significant role in the convergence analysis of various iterative algorithms. It is known that all Hilbert spaces satisfy the Opial's condition.

Lemma 1. [3] *Assume that $\mathcal{W} : \mathcal{H} \to \mathbb{R}$ is a convex and Gâteaux differentiable function. For any $x \in \mathcal{H}$, we have that $\mathcal{W}(x) + \langle y - x, \nabla\mathcal{W}(x)\rangle \leq \mathcal{W}(y)$, $\forall y \in \mathcal{H}$.*

Lemma 2. [36] *Assume that $\mathcal{W} : \mathcal{H} \to \mathbb{R}$ is Fréchet differentiable and $\nabla\mathcal{W} : \mathcal{H} \to \mathcal{H}$ is ι-Lipschitz continuous and κ-strongly monotone with respect to $\iota, \kappa > 0$. Assume that $\mathcal{S} := I - \alpha\chi\nabla\mathcal{W}$, where $\alpha \in (0, 2\kappa/\iota^2)$ and $\chi \in [0, 1]$. For any $x, y \in \mathcal{H}$, we have that $(1 - \vartheta\chi)\|x - y\| \geq \|\mathcal{S}(x) - \mathcal{S}(y)\|$, where $\vartheta := 1 - \sqrt{1 - \alpha(2\kappa - \alpha\iota^2)} \in (0, 1]$.*

Lemma 3. [35] *Let $\mathcal{C}$ be a convex and closed set of a Hilbert space $\mathcal{H}$. Assume that $\mathcal{W} : \mathcal{H} \to \mathbb{R}$ is Fréchet differentiable and $\nabla\mathcal{W} : \mathcal{H} \to \mathcal{H}$ is ι-Lipschitz continuous and κ-strongly monotone. Hence there exists a unique minimizer of $\mathcal{W}$ over $\mathcal{C}$.*

Lemma 4. [34] *Let $\{\mu_n\}$ and $\{\nu_n\}$ be sequences of real constants such that $\mu_n \in [0, 1]$, $\sum_{n=1}^{\infty} \mu_n = \infty$ and $\overline{\lim\limits_{n\to\infty}}\, \nu_n \leq 0$. Let $\{\chi_n\}$ be a sequence of nonnegative real constants such that $(1 - \mu_n)\chi_n + \mu_n\nu_n \geq \chi_{n+1}$. Then $\lim\limits_{n\to\infty} \chi_n = 0$.*

3. Algorithms and their convergence

We suppose that $\mathfrak{I} := \{1, 2, \cdots, I\}$ is a user set participating in a system Ξ. Let $\mathfrak{I}_t$ be a user set participating in a subnetwork $t \in \mathfrak{T}$ of the system Ξ, where $\mathfrak{T} = \{1, 2, \cdots, T\}$ is the set of subnetworks. Hence $\mathfrak{I} = \bigcup_{t\in\mathfrak{T}} \mathfrak{I}_t$. Let us denote $|\mathfrak{I}_t|$ by the element number of $\mathfrak{I}_t$ $(t \in \mathfrak{T})$. We set $I_t := |\mathfrak{I}_t|$. Doing so, we have that $I = \sum_{t\in\mathfrak{T}} I_t$. Let us choose $t \in \mathfrak{T}$ arbitrarily. Then we define $\mathfrak{I}_t^{(i)} (i = 1, 2, \cdots, I_t)$ as follows. Without ambiguity, we write $\mathfrak{I}_t^{(1)} := \mathfrak{I}_t$. First, we randomly choose $k_t^{(1)}$ in $\mathfrak{I}_t^{(1)}$. Let $\mathfrak{I}_t^{(2)} := \mathfrak{I}_t^{(1)} \backslash \{k_t^{(1)}\}$. Next, we randomly choose $k_t^{(2)}$ in $\mathfrak{I}_t^{(2)}$. Following a similar argument as above, one sets $\mathfrak{I}_t^{(i)} := \mathfrak{I}_t^{(i-1)} \backslash k_t^{(i-1)}$ $(i = 3, 4, \cdots, I_t)$. Then, we randomly choose $k_t^{(i)}$ in $\mathfrak{I}_t^{(i)}$ $(i = 3, 4, \cdots, I_t)$. Suppose that each user i has its own private constraint condition, denoted by a convex closed set $\mathcal{C}^{(i)} (\subset \mathcal{H})$ and its own private information, denoted by two convex, Fréchet differentiable objective functions, $\mathcal{F}^{(i)} : \mathcal{H} \to \mathbb{R}$ and $\mathcal{G}^{(i)} : \mathcal{H} \to \mathbb{R}$, for $i \in \mathfrak{I} \cup \{0\}$. From now on, we employ the following essential assumptions through the rest of the paper.

Assumption 1. We suppose that, for any $i \in \mathfrak{I} \cup \{0\}$,

(i) $\nabla\mathcal{F}^{(i)} : \mathcal{H} \to \mathcal{H}$ is $\kappa^{(i)}$-Lipschitz continuous and $a^{(i)}$-strongly monotone for some $\kappa^{(i)}, a^{(i)} > 0$.

(ii) $\nabla\mathcal{G}^{(i)} : \mathcal{H} \to \mathcal{H}$ is $\iota^{(i)}$-Lipschitz continuous and $b^{(i)}$-strongly monotone for some $\iota^{(i)}, b^{(i)} > 0$.

(iii) $\mathcal{S}^{(i)} : \mathcal{H} \to \mathcal{H}$ is nonexpansive with $Fix(\mathcal{S}^{(i)}) = \mathcal{C}^{(i)}$.

Under Assumption 1, we consider problem (1.1) with $\bigcap_{i\in\mathfrak{I}\cup\{0\}} Fix(\mathcal{S}^{(i)}) \neq \emptyset$. Based on Assumption 1 (i), (ii), we have that

$$\nabla\left(\sum_{i\in\mathfrak{I}\cup\{0\}} \mathcal{F}^{(i)}\right) = \sum_{i\in\mathfrak{I}\cup\{0\}} \nabla\mathcal{F}^{(i)} \quad \text{and} \quad \nabla\left(\sum_{i\in\mathfrak{I}\cup\{0\}} \mathcal{G}^{(i)}\right) = \sum_{i\in\mathfrak{I}\cup\{0\}} \nabla\mathcal{G}^{(i)}$$

are Lipschitz continuous and strongly monotone. Thus

$$\nabla\left(\sum_{i\in\mathfrak{I}\cup\{0\}} \mathcal{F}^{(i)}(x)\right) + \nabla\left(\sum_{i\in\mathfrak{I}\cup\{0\}} \mathcal{G}^{(i)}(x)\right) = \nabla\left(\sum_{i\in\mathfrak{I}\cup\{0\}} \left(\mathcal{F}^{(i)}(x) + \mathcal{G}^{(i)}(x)\right)\right)$$

is Lipschitz continuous and strongly monotone. In addition, in view of Assumption 1 (iii), one sees that $\bigcap_{i\in\mathfrak{I}\cup\{0\}} Fix(\mathcal{S}^{(i)})$ is a convex and closed set. Therefore, it follows from Lemma 3 that problem (1.1) has a unique solution. Now, we are in the position to present our main algorithm (see Algorithm 1, below).

Algorithm 1 The broadcast incremental algorithm

procedure INPUT($(\alpha_n)_{n\in\mathbb{N}}$, $(\beta_n)_{n\in\mathbb{N}}$, μ, ν, $\mathfrak{T}$, T and $(I_t)_{t=1}^{T}$)
  User 0 chooses $x_0 \in \mathcal{H}$ arbitrarily
  Set $n \leftarrow 0$
  while not converged **do**
    User 0 computes $x_n^{(0)} \in \mathcal{H}$ as $x_n^{(0)} = \lambda_n(I - \mu\beta_n\nabla\mathcal{G}^{(0)})\mathcal{S}^{(0)}x_n + (1-\lambda_n)(I - \nu\alpha_n\nabla\mathcal{F}^{(0)})x_n$
    for t in $\mathfrak{T}$ **do**
      User 0 transmits x_n to user $k_t^{(1)}$
      User $k_t^{(1)}$ computes $x_n^{(k_t^{(0)})}$ as $x_n^{(k_t^{(0)})} = x_n$
    end for
    for t in $\mathfrak{T}$ **do**
      for $i = 1, 2, \cdots, I_t$ **do**
        User $k_t^{(i)}$ computes $x_n^{(k_t^{(i)})} \in \mathcal{H}$ as $x_n^{(k_t^{(i)})} = \lambda_n(I - \mu\beta_n\nabla\mathcal{G}^{(k_t^{(i)})})\mathcal{S}^{(k_t^{(i)})}x_n^{k_t^{(i-1)}} + (1-\lambda_n)(I - \nu\alpha_n\nabla\mathcal{F}^{(k_t^{(i)})})x_n^{k_t^{(i-1)}}$
      end for
      User $k_t^{(I_t)}$ transmits $x_n^{(k_t^{(I_t)})}$ to user $k_t^{(1)}$
      User $k_t^{(1)}$ computes $x_n^{(t)} \in \mathcal{H}$ as $x_n^{(t)} = x_n^{(k_t^{(I_t)})}$
      User $k_t^{(1)}$ transmits $x_n^{(t)}$ to user 0
    end for
    User 0 computes $x_{n+1} \in \mathcal{H}$ as $x_{n+1} = \frac{x_n^{(0)} + \sum_{t=1}^{T} x_n^{(t)}}{T+1}$
    Set $n \leftarrow n+1$
  end while
  return $x = x_n$
end procedure

Assumption 2. Assume that the following conditions are satisfied.

(i) The parameter sequences $(\alpha_n)_{n\in\mathbb{N}} \subset (0,1)$ and $(\beta_n)_{n\in\mathbb{N}} \subset (0,1)$ satisfy $\lim_{n\to\infty} |\alpha_n/\beta_n| < \infty$ and

$$\begin{cases} \sum_{n=0}^{\infty} \alpha_n = \infty, \\ \lim_{n\to\infty} \frac{1}{\alpha_{n+1}} \left| \frac{1}{\alpha_{n+1}} - \frac{1}{\alpha_n} \right| = 0, \\ \lim_{n\to\infty} \alpha_n = 0, \end{cases} \qquad \begin{cases} \sum_{n=0}^{\infty} \beta_n = \infty, \\ \lim_{n\to\infty} \frac{1}{\beta_{n+1}} \left| \frac{1}{\beta_{n+1}} - \frac{1}{\beta_n} \right| = 0, \\ \lim_{n\to\infty} \beta_n = 0. \end{cases}$$

(ii) Set $\mu \in \left(0, \min_{i\in\{0\}\cup\mathfrak{I}} \frac{2b^{(i)}}{\iota^{(i)2}}\right)$ and $\nu \in \left(0, \min_{i\in\{0\}\cup\mathfrak{I}} \frac{2a^{(i)}}{\kappa^{(i)2}}\right)$.

(iii) Let the sequence $(\lambda_n)_{n\in\mathbb{N}} \subset (0,1)$ satisfy
(a) $0 < e \le \lambda_n \le f < 1$;
(b) $\lim_{n\to\infty} \frac{|\lambda_{n-1}-\lambda_n|}{\alpha_n^2} = 0$; (c) $\lim_{n\to\infty} \frac{|\lambda_{n-1}-\lambda_n|}{\beta_n^2} = 0$.

Remark 1. Suppose that Assumptions 1, 2 are satisfied. All users participating in the network share the similar property depicted as follows. In view of Lemma 2 and Assumption 2 (i), (ii), one deduces that, for any $x, y \in \mathcal{H}$, $i \in \{0\} \cup \mathfrak{I}$,

$$\begin{aligned} &\|(I - \nu\alpha_n \nabla\mathcal{F}^{(i)})x - (I - \nu\alpha_n \nabla\mathcal{F}^{(i)})y\| \\ \le &\left(1 - \left(1 - \sqrt{1 - u(2a^{(i)} - u\kappa^{(i)2})}\right)\alpha_n\right)\|x - y\| \le (1 - \alpha_n\rho)\|x - y\|, \end{aligned}$$

and

$$\begin{aligned} &\|(I - \mu\beta_n \nabla\mathcal{G}^{(i)})x - (I - \mu\beta_n \nabla\mathcal{G}^{(i)})y\| \\ \le &\left(1 - \left(1 - \sqrt{1 - u(2b^{(i)} - u\iota^{(i)2})}\right)\beta_n\right)\|x - y\| \le (1 - \beta_n\sigma)\|x - y\|, \end{aligned}$$

where

$$\rho := \min_{i\in\{0\}\cup\mathfrak{I}} (1 - \sqrt{1 - (2a^{(i)} - u\kappa^{(i)2})u}),$$

and

$$\sigma := \min_{i\in\{0\}\cup\mathfrak{I}} (1 - \sqrt{1 - (2b^{(i)} - u\iota^{(i)2})u}).$$

Set $\phi_n = \min\{\varrho\alpha_n, \sigma\beta_n\}$. From Assumption 2, one has

$$(\phi_n)_{n\in\mathbb{N}} \in (0,1), \ \sum_{n=0}^{\infty} \phi_n = \infty$$

and

$$\lim_{n\to\infty} \phi_n = \lim_{n\to\infty} \frac{1}{\phi_{n+1}} \left| \frac{1}{\phi_{n+1}} - \frac{1}{\phi_n} \right| = \lim_{n\to\infty} \frac{|\lambda_{n-1} - \lambda_n|}{\phi_n^2} = 0.$$

Now, one is in a position to prove the main convergence result.

Theorem 1. *Assume that Assumptions 1 and 2 are satisfied. Let $(x_n)_{n\in\mathbb{N}}$ and $(x_n^{(k_t^{(i)})})_{n\in\mathbb{N}} (t \in \mathfrak{T}, i = 1, 2, \cdots, I_t)$ be sequences generated by Algorithm 1. Then $(x_n)_{n\in\mathbb{N}}$ strongly converges to the solution of problem (1.1).*

Proof. Let us now fix $z \in \Lambda$. Remark 1 yields

$$\begin{aligned}
&\|x_n^{(0)} - z\| \\
\leq&(1-\lambda_n)\|(I-\nu\alpha_n\nabla\mathcal{F}^{(0)})x_n - (I-\nu\alpha_n\nabla\mathcal{F}^{(0)})z\| + (1-\lambda_n)\nu\alpha_n\|\nabla\mathcal{F}^{(0)}z\| \\
&+ \lambda_n\|(I-\mu\beta_n\nabla\mathcal{G}^{(0)})\mathcal{S}^{(0)}x_n - (I-\mu\beta_n\nabla\mathcal{G}^{(0)})z\| + \mu\lambda_n\beta_n\|\nabla\mathcal{G}^{(0)}z\| \\
\leq&\mu\lambda_n\beta_n\|\nabla\mathcal{G}^{(0)}z\| + \lambda_n(1-\sigma\beta_n)\|x_n - z\| + (1-\lambda_n)(1-\varrho\alpha_n)\|x_n - z\| \\
&+ (1-\lambda_n)\nu\alpha_n\|\nabla\mathcal{F}^{(0)}z\| \\
\leq&\max\left\{\mu\beta_n\|\nabla\mathcal{G}^{(0)}z\|, \nu\alpha_n\|\nabla\mathcal{F}^{(0)}z\|\right\} + \max\left\{(1-\sigma\beta_n),(1-\varrho\alpha_n)\right\}\|x_n - z\|.
\end{aligned} \tag{3.1}$$

Set

$$\varphi_n = \max\left\{\mu\beta_n, \nu\alpha_n\right\}. \tag{3.2}$$

With the help of (3.1), (3.2), it ensues that

$$\|x_n^{(0)} - z\| \leq M_1\varphi_n + (1-\phi_n)\|x_n - z\|, \tag{3.3}$$

where

$$M_1 = \max\left\{\|\nabla\mathcal{G}^{(0)}z\|, \|\nabla\mathcal{F}^{(0)}z\|, \max_{t\in T, i=1,2,\cdots,I_t}\left\{\|\nabla\mathcal{G}^{(k_t^i)}z\|, \|\nabla\mathcal{F}^{(k_t^i)}z\|\right\}\right\}.$$

A similar argument gives that, for all $t \in \mathfrak{T}$, $k = 1, 2, \cdots, I_t$,

$$\left\|x_n^{(k_t^{(i)})} - z\right\| \leq M_1\varphi_n + (1-\phi_n)\left\|x_n^{(k_t^{(i)}-1)} - z\right\|.$$

This shows that, for all $t \in \mathfrak{T}$,

$$\begin{aligned}
\|x_n^{(t)} - z\| \leq& M_1\varphi_n + (1-\phi_n)\left\|x_n^{(k_t^{(I_t-1)})} - z\right\| \\
\leq& I_t M_1\varphi_n + (1-\phi_n)\left\|x_n^{k_t^{(0)}} - z\right\| \leq I_t M_1\varphi_n + (1-\phi_n)\|x_n - z\|.
\end{aligned} \tag{3.4}$$

By combining (3.3) with (3.4), one finds that

$$\begin{aligned}
\|x_{n+1} - z\| \leq& \frac{\|x_n^{(0)} - z\| + \sum_{t\in\mathfrak{T}}\|x_n^{(t)} - z\|}{T+1} \\
\leq& \frac{\left(T(1-\phi_n)\|x_n - z\| + \sum_{t\in\mathfrak{T}} I_t M_1\varphi_n\right) + (M_1\varphi_n + (1-\phi_n)\|x_n - z\|)}{T+1} \\
\leq& \phi_n\left(\sum_{t\in\mathfrak{T}} I_t + 1\right)\frac{M_1\psi_n}{T+1} + (1-\phi_n)\|x_n - z\| \\
\leq& \max\left\{\left(\sum_{t\in\mathfrak{T}} I_t + 1\right)\frac{M_1\psi_n}{T+1}, \|x_n - z\|\right\},
\end{aligned}$$

where $\psi_n = \frac{\varphi_n}{\phi_n}$. Due to Assumption 2 (i) and Remark 1, one concludes from (3.2) that $(\psi_n)_{n\in\mathbb{N}}$ is bounded. Thus one can easily check that there exists a real number

M_2 such that $\sup_{n\in\mathbb{N}}\psi_n \le M_2$. From this, one obtains that

$$\|x_n - z\| \le \max\left\{\left(\sum_{t\in\mathfrak{T}} I_t + 1\right)\frac{M_1M_2}{T+1}, \|x_0 - z\|\right\}.$$

Thus the sequence $(x_n)_{n\in\mathbb{N}}$ is bounded. By using (3.4), we find that

$$(x_n^{(k_t^{(i)})})_{n\in\mathbb{N}}(t\in\mathfrak{T},\ i=1,2,\cdots,I_t)$$

is also bounded. Now let us evaluate the term $\|x_{n+1}^{(0)} - x_n^{(0)}\|$.

$$\begin{aligned}
&\|x_{n+1}^{(0)} - x_n^{(0)}\| \\
\le&\Big\|(1-\lambda_{n+1})(I-\nu\alpha_{n+1}\nabla\mathcal{F}^{(0)})x_{n+1} - (1-\lambda_{n+1})(I-\nu\alpha_{n+1}\nabla\mathcal{F}^{(0)})x_n \\
&- (1-\lambda_{n+1})(I-\nu\alpha_n\nabla\mathcal{F}^{(0)})x_n + (1-\lambda_{n+1})(I-\nu\alpha_{n+1}\nabla\mathcal{F}^{(0)})x_n \\
&+ (1-\lambda_{n+1})(I-\nu\alpha_n\nabla\mathcal{F}^{(0)})x_n - (1-\lambda_n)(I-\nu\alpha_n\nabla\mathcal{F}^{(0)})x_n \\
&- \lambda_{n+1}(I-\mu\beta_{n+1}\nabla\mathcal{G}^{(0)})\mathcal{S}^{(0)}x_n + \lambda_{n+1}(I-\mu\beta_{n+1}\nabla\mathcal{G}^{(0)})\mathcal{S}^{(0)}x_{n+1} \\
&+ \lambda_{n+1}(I-\mu\beta_{n+1}\nabla\mathcal{G}^{(0)})\mathcal{S}^{(0)}x_n - \lambda_{n+1}(I-\mu\beta_n\nabla\mathcal{G}^{(0)})\mathcal{S}^{(0)}x_n \\
&+\lambda_{n+1}(I-\mu\beta_n\nabla\mathcal{G}^{(0)})\mathcal{S}^{(0)}x_n - \lambda_n(I-\mu\beta_n\nabla\mathcal{G}^{(0)})\mathcal{S}^{(0)}x_n\Big\| \qquad (3.5)\\
\le&(1-\lambda_{n+1})(I-\rho\alpha_{n+1})\|x_{n+1}-x_n\| + \nu(1-\lambda_{n+1})|\alpha_n-\alpha_{n+1}|\|\nabla\mathcal{F}^{(0)}x_n\| \\
&+ |\lambda_{n+1}-\lambda_n|\|(I-\nu\alpha_n\nabla\mathcal{F}^{(0)})x_n\| + \lambda_{n+1}(I-\sigma\beta_{n+1})\|x_{n+1}-x_n\| \\
&+ \mu\lambda_{n+1}|\beta_{n+1}-\beta_n|\|\nabla\mathcal{G}^{(0)}\mathcal{S}^{(0)}x_n\| + |\lambda_n-\lambda_{n+1}|\|(I-\mu\beta_n\nabla\mathcal{G}^{(0)})\mathcal{S}^{(0)}x_n\| \\
\le&(1-\phi_{n+1})\|x_n-x_{n+1}\| + \nu|\alpha_{n+1}-\alpha_n|\|\nabla\mathcal{F}^{(0)}x_n\| \\
&+ \mu\|\nabla\mathcal{G}^{(0)}\mathrm{S}^{(0)}x_n\||\beta_n-\beta_{n+1}| + \Big(\|(I-\mu\beta_n\nabla\mathcal{G}^{(0)})\mathcal{S}^{(0)}x_n\| \\
&+\|(I-\nu\alpha_n\nabla\mathcal{F}^{(0)})x_n\|\Big)|\lambda_n-\lambda_{n+1}|.
\end{aligned}$$

Recalling the fact that $(x_n)_{n\in\mathbb{N}}$ is bounded, together with the Lipschitz continuity of $\nabla\mathcal{F}^{(0)}$, $\nabla\mathcal{G}^{(0)}$, we deduce that

$$\begin{aligned}
&M_3 = \sup_{n\in\mathbb{N}}\left(\nu\|\nabla\mathcal{F}^{(0)}x_n\|\right) < \infty, \qquad M_4 = \sup_{n\in\mathbb{N}}\left(\mu\|\mathcal{G}^{(0)}\mathcal{S}^{(0)}x_n\|\right) < \infty, \\
&M_5 = \sup_{n\in\mathbb{N}}\left(\|(I-\nu\alpha_n\nabla\mathcal{F}^{(0)})x_n\| + \|(I-\mu\beta_n\nabla\mathcal{G}^{(0)})\mathcal{S}^{(0)}x_n\|\right) < \infty.
\end{aligned} \qquad (3.6)$$

According to (3.5), one concludes from (3.6) that

$$\begin{aligned}
&\|x_{n+1}^{(0)} - x_n^{(0)}\| \\
\le&|\alpha_n-\alpha_{n+1}|M_3 + |\beta_n-\beta_{n+1}|M_4 + |\lambda_n-\lambda_{n+1}|M_5 + (1-\phi_{n+1})\|x_{n+1}-x_n\|.
\end{aligned} \qquad (3.7)$$

By following a similar argument, one sees that, for all $t \in \mathfrak{T}$, $i = 1, 2, \cdots, I_t$,

$$\begin{aligned}
&\|x_{n+1}^{(k_t^i)} - x_n^{(k_t^i)}\| \\
\leq&\nu|\alpha_n - \alpha_{n+1}|\|\nabla \mathcal{F}^{(k_t^i)} x_n^{(k_t^{i-1})}\| + \mu\|\nabla \mathcal{G}^{(k_t^i)} \mathcal{S}^{(k_t^i)} x_n^{(k_t^{i-1})}\||\beta_n - \beta_{n+1}| \\
&+ (1 - \phi_{n+1})\|x_{n+1}^{(k_t^{i-1})} - x_n^{(k_t^{i-1})}\| \\
&+ |\lambda_n - \lambda_{n+1}| \left(\|(I - \mu\beta_n \nabla \mathcal{G}^{(k_t^i)})\mathcal{S}^{(k_t^i)} x_n^{(k_t^{i-1})}\| + \|(I - \nu\alpha_n \nabla \mathcal{F}^{(k_t^i)}) x_n^{(k_t^{i-1})}\|\right).
\end{aligned} \tag{3.8}$$

In view of the Lipschitz continuity of $\nabla F^{(0)}$, $\nabla G^{(0)}$ and Assumption 2 (i), one gets

$$\begin{aligned}
M_6 &= \max_{t\in\mathfrak{T},\ i=1,2,\cdots,I_t} \sup_{n\in\mathbb{N}} \left(\nu\|\nabla \mathcal{F}^{(k_t^i)} x_n^{(k_t^{i-1})}\|\right) < \infty, \\
M_7 &= \max_{t\in\mathfrak{T},\ i=1,2,\cdots,I_t} \sup_{n\in\mathbb{N}} \left(\mu\|\nabla \mathcal{G}^{(k_t^i)} \mathcal{S}^{(k_t^i)} x_n^{(k_t^{i-1})}\|\right) < \infty, \\
M_8 &= \max_{t\in\mathfrak{T},\ i=1,2,\cdots,I_t} \sup_{n\in\mathbb{N}} \left\{\|(I - \mu\beta_n \nabla \mathcal{G}^{(k_t^i)})\mathcal{S}^{(k_t^i)} x_n^{(k_t^{i-1})}\|\right. \\
&\quad \left. + \|(I - \nu\alpha_n \nabla \mathcal{F}^{(k_t^i)}) x_n^{(k_t^{i-1})}\|\right\} < \infty.
\end{aligned} \tag{3.9}$$

Successively using (3.8), (3.9), one has that, for all $t \in \mathfrak{T}$, $i = 1, 2, \cdots, I_t$,

$$\begin{aligned}
\|x_n^{(k_t^i)} - x_{n+1}^{(k_t^i)}\| \leq&|\alpha_n - \alpha_{n+1}|M_6 + |\beta_n - \beta_{n+1}|M_7 \\
&+ |\lambda_n - \lambda_{n+1}|M_8 + (1 - \phi_{n+1})\|x_{n+1}^{(k_t^{i-1})} - x_n^{(k_t^{i-1})}\|.
\end{aligned} \tag{3.10}$$

Hence, one arrives at

$$\begin{aligned}
\|x_{n+1}^{(t)} - x_n^{(t)}\| \leq&(1 - \phi_{n+1})\|x_{n+1}^{(k_t^{I_t-1})} - x_n^{(k_t^{I_t-1})}\| + |\alpha_n - \alpha_{n+1}|M_6 \\
&+ |\beta_n - \beta_{n+1}|M_7 + |\lambda_n - \lambda_{n+1}|M_8 \\
\leq&(1 - \phi_{n+1})\|x_{n+1} - x_n\| + I_t|\alpha_n - \alpha_{n+1}|M_6 \\
&+ I_t|\beta_n - \beta_{n+1}|M_7 + I_t|\lambda_n - \lambda_{n+1}|M_8.
\end{aligned} \tag{3.11}$$

Combining (3.10) with (3.11), one immediately obtains that

$$\begin{aligned}
&\|x_n - x_{n+1}\| \\
\leq&\frac{\sum_{t\in\mathfrak{T}} \|x_{n-1}^{(t)} - x_n^{(t)}\| + \|x_{n-1}^{(0)} - x_n^{(0)}\|}{T + 1} \\
\leq&\frac{1}{T + 1}\left\{(1 - \phi_n)\|x_{n-1} - x_n\| + |\alpha_{n-1} - \alpha_n|M_3 + |\beta_{n-1} - \beta_n|M_4\right. \\
&+ |\lambda_{n-1} - \lambda_n|M_5 + T(1 - \phi_n)\|x_{n-1} - x_n\| + \sum_{t\in\mathfrak{T}} I_t|\alpha_{n-1} - \alpha_n|M_6 \\
&\left.+ \sum_{t\in\mathfrak{T}} I_t|\beta_{n-1} - \beta_n|M_7 + \sum_{t\in\mathfrak{T}} I_t|\lambda_{n-1} - \lambda_n|M_8\right\} \\
\leq&|\alpha_{n-1} - \alpha_n|M_9 + |\beta_{n-1} - \beta_n|M_{10} + |\lambda_{n-1} - \lambda_n|M_{11} + (1 - \phi_n)\|x_{n-1} - x_n\|,
\end{aligned}$$

where

$$\begin{aligned} M_9 =& \frac{M_3 + \sum_{t \in \mathfrak{T}} I_t M_6}{T+1} < \infty, M_{10} = \frac{M_4 + \sum_{t \in \mathfrak{T}} I_t M_7}{T+1} < \infty, \\ M_{11} =& \frac{M_5 + \sum_{t \in \mathfrak{T}} I_t M_8}{T+1} < \infty. \end{aligned} \tag{3.12}$$

By a standard argument, one finds that

$$\begin{aligned} & \frac{\|x_n - x_{n+1}\|}{\phi_n} \\ \leq & (1-\phi_n)\frac{\|x_{n-1} - x_n\|}{\phi_{n-1}} + (1-\phi_n)\frac{\|x_{n-1} - x_n\|}{\phi_n} - (1-\phi_n)\frac{\|x_{n-1} - x_n\|}{\phi_{n-1}} \\ & + \frac{|\alpha_{n-1} - \alpha_n|}{\phi_n} M_9 + \frac{|\beta_{n-1} - \beta_n|}{\phi_n} M_{10} + \frac{|\lambda_{n-1} - \lambda_n|}{\phi_n} M_{11} \\ \leq & (1-\phi_n)\frac{\|x_{n-1} - x_n\|}{\phi_{n-1}} + \phi_n \left(\frac{1}{\phi_n} \left| \frac{1}{\phi_{n-1}} - \frac{1}{\phi_n} \right| M_{12} + \frac{1}{\alpha_n} \left| \frac{1}{\alpha_n} - \frac{1}{\alpha_{n-1}} \right| \gamma_n^2 M_9 \right. \\ & \left. + \frac{1}{\beta_n} \left| \frac{1}{\beta_n} - \frac{1}{\beta_{n-1}} \right| \tau_n^2 M_{10} + \frac{|\lambda_{n-1} - \lambda_n|}{\phi_n^2} M_{11} \right) \\ \leq & \phi_n X_n + \frac{1-\phi_n}{\phi_{n-1}} \|x_{n-1} - x_n\|, \end{aligned}$$

where

$$\gamma_n = \frac{\alpha_n}{\phi_n}, \ \tau_n = \frac{\beta_n}{\phi_n},$$

$$M_{12} = \sup_{n \in \mathbb{N}} \|x_n - x_{n+1}\|$$

and

$$\begin{aligned} X_n =& \frac{1}{\phi_n} \left| \frac{1}{\phi_{n-1}} - \frac{1}{\phi_n} \right| M_{12} + \frac{1}{\alpha_n} \left| \frac{1}{\alpha_n} - \frac{1}{\alpha_{n-1}} \right| \gamma_n^2 M_9 + \frac{1}{\beta_n} \left| \frac{1}{\beta_n} - \frac{1}{\beta_{n-1}} \right| \tau_n^2 M_{10} \\ & + \frac{|\lambda_{n-1} - \lambda_n|}{\phi_n^2} M_{11}. \end{aligned}$$

The boundedness of $(x_n)_{n \in \mathbb{N}}$ gives that $M_{12} < \infty$. Successively using Remark 1, Assumption 2 (i), (iii) and (3.12), one deduces that $\limsup_{n \to \infty} X_n \leq 0$. In light of Lemma 4, it suffices to prove that

$$\lim_{n \to \infty} \frac{\|x_n - x_{n+1}\|}{\phi_n} = 0. \tag{3.13}$$

In addition, one finds from Remark 1 that

$$\lim_{n \to \infty} \|x_n - x_{n+1}\| = 0. \tag{3.14}$$

In terms of Remark 1, one further asserts that

$$\begin{aligned}
&\|x_n^{(0)}-z\|^2\\
\le&\|x_n-z\|^2+2\alpha_n\nu(I-\rho\alpha_n)\|x_n-z\|\|\nabla\mathcal{F}^{(0)}z\|+\nu^2\alpha_n^2\|\nabla\mathcal{F}^{(0)}z\|^2\\
&+2\mu\beta_n(I-\beta_n\sigma)\|x_n-z\|\|\nabla\mathcal{G}^{(0)}z\|+\mu^2\beta_n^2\|\nabla\mathcal{G}^{(0)}z\|^2\\
&-\lambda_n(1-\lambda_n)\|x_n-\mathcal{S}^{(0)}x_n\|^2\\
&+2(1-\lambda_n)\lambda_n\|x_n-\mathcal{S}^{(0)}x_n\|\left(\nu\alpha_n\|\nabla\mathcal{F}^{(0)}x_n\|+\mu\beta_n\|\nabla\mathcal{G}^{(0)}\mathcal{S}^{(0)}x_n\|\right)\\
&+(1-\lambda_n)\lambda_n\|\alpha_n\nu\nabla\mathcal{F}^{(0)}x_n-\mu\beta_n\nabla\mathcal{G}^{(0)}\mathcal{S}^{(0)}x_n\|^2.
\end{aligned}$$

With the help of Assumption 2 (ii), the above inequality yields that

$$\|x_n^{(0)}-z\|^2\le\|x_n-z\|^2+\alpha_nM_{13}+\beta_nM_{14}-\lambda_n(1-\lambda_n)\|x_n-\mathcal{S}^{(0)}x_n\|^2, \tag{3.15}$$

where

$$\begin{aligned}
M_{13}=&\sup_{n\in\mathbb{N}}\,(2\nu\|x_n-z\|\|\nabla\mathcal{F}^{(0)}z\|+\nu^2\alpha_n\|\nabla\mathcal{F}^{(0)}z\|+2\nu\|x_n-\mathcal{S}^{(0)}x_n\|\|\nabla\mathcal{F}^{(0)}x_n\|\\
&+2\nu^2\alpha_n\|\nabla\mathcal{F}^{(0)}x_n\|^2)<\infty,\\
M_{14}=&\sup_{n\in\mathbb{N}}\,\Big(2\mu\|x_n-z\|\|\nabla\mathcal{G}^{(0)}z\|+\beta_n\mu^2\|\nabla\mathcal{G}^{(0)}z\|^2\\
&+2\beta_n\mu^2\|\nabla\mathcal{G}^{(0)}\mathcal{S}^{(0)}x_n\|^2+2\mu\|x_n-\mathcal{S}^{(0)}x_n\|\|\nabla\mathcal{G}^{(0)}\mathcal{S}^{(0)}x_n\|\Big)<\infty.
\end{aligned}\tag{3.16}$$

By a similar argument, we have that, for all $t\in\mathfrak{T}$, $i=1,2,\cdots,I_t$,

$$\|x_n^{(k_t^i)}-z\|^2\le\|x_n^{(k_t^i-1)}-z\|^2+\alpha_nM_{15}+\beta_nM_{16}-\lambda_n(1-\lambda_n)\|x_n^{(k_t^i-1)}-\mathcal{S}^{(k_t^i)}x_n^{(k_t^i-1)}\|^2,$$

where

$$\begin{aligned}
M_{15}=&\max_{t\in\mathfrak{T},i=1,2,\cdots,I_t}\sup_{n\in\mathbb{N}}\Big(2\nu\left\|x_n^{(k_t^i-1)}-z\right\|\left\|\nabla\mathcal{F}^{(k_t^i)}z\right\|+\nu^2\alpha_n\left\|\nabla\mathcal{F}^{(k_t^i)}z\right\|\\
&+2\nu\left\|x_n^{(k_t^i-1)}-\mathcal{S}^{(k_t^i)}x_n^{(k_t^i-1)}\right\|\left\|\nabla\mathcal{F}^{(k_t^i)}x_n^{(k_t^i-1)}\right\|\\
&+2\nu^2\alpha_n\left\|\nabla\mathcal{F}^{(k_t^i)}x_n^{(k_t^i-1)}\right\|^2\Big)<\infty,\\
M_{16}=&\max_{t\in\mathfrak{T},i=1,2,\cdots,I_t}\sup_{n\in\mathbb{N}}\Big(2\mu\left\|x_n^{(k_t^i-1)}-z\right\|\left\|\nabla\mathcal{G}^{(k_t^i)}z\right\|+\mu^2\beta_n\left\|\nabla\mathcal{G}^{(k_t^i)}z\right\|^2\\
&+2\mu\left\|x_n^{(k_t^i-1)}-\mathcal{S}^{(k_t^i)}x_n^{(k_t^i-1)}\right\|\left\|\nabla\mathcal{G}^{(k_t^i)}\mathcal{S}^{(k_t^i)}x_n^{(k_t^i-1)}\right\|\\
&+2\mu^2\beta_n\left\|\nabla\mathcal{G}^{(k_t^i)}\mathcal{S}^{(k_t^i)}x_n^{(k_t^i-1)}\right\|^2\Big)<\infty.
\end{aligned}\tag{3.17}$$

An elementary calculation gives that, for all $t\in\mathfrak{T}$, $i=1,2,\cdots,I_t$,

$$\begin{aligned}
\|x_n^{(t)}-z\|^2\le&\|x_n-z\|^2+\alpha_nI_tM_{15}+\beta_nI_tM_{16}\\
&-\lambda_n(1-\lambda_n)\sum_{i=1}^{I_t}\left\|x_n^{(k_t^{i-1})}-\mathcal{S}^{(k_t^{I_t})}x_n^{(k_t^{i-1})}\right\|^2.
\end{aligned}\tag{3.18}$$

By putting (3.15) and (3.18) together, one concludes that

$$\begin{aligned}\|x_{n+1}-z\|^2 \leq & \frac{\|x_n^{(0)}-z\|^2+\sum_{t\in\mathfrak{T}}\|x_n^{(t)}-z\|^2}{T+1} \\ \leq & \|x_n-z\|^2+\frac{1}{T+1}\left(\alpha_n\left(M_{13}+\sum_{t\in\mathfrak{T}}I_tM_{15}\right)\right. \\ & +\beta_n\left(M_{14}+\sum_{t\in\mathfrak{T}}I_tM_{16}\right)-\lambda_n(1-\lambda_n)\|x_n-\mathcal{S}^{(0)}x_n\|^2 \\ & \left.-(1-\lambda_n)\lambda_n\sum_{i=1}^{I_t}\|x_n^{(k_t^{i-1})}-\mathcal{S}^{(k_t^i)}x_n^{(k_t^{i-1})}\|^2\right).\end{aligned}$$

Let us reformulate the above expression as

$$\begin{aligned}0\leq & \frac{\lambda_n(1-\lambda_n)}{T+1}\left(\|\mathcal{S}^{(0)}x_n-x_n\|^2+\sum_{i=1}^{I_t}\left\|x_n^{(k_t^{i-1})}-\mathcal{S}^{(k_t^i)}x_n^{(k_t^{i-1})}\right\|^2\right) \\ \leq & (\|x_n-z\|+\|x_{n+1}-z\|)\|x_n-x_{n+1}\| \\ & +\frac{1}{T+1}\left(\alpha_n\left(M_{13}+\sum_{t\in\mathbb{T}}I_tM_{15}\right)+\beta_n\left(M_{14}+\sum_{t\in\mathbb{T}}I_tM_{16}\right)\right).\end{aligned} \tag{3.19}$$

By successively using (3.14), (3.16), (3.17) and Assumption 1 (i), one observes that the right-hand side of (3.19) converges to 0 as n goes to infinity. Accordingly,

$$\lim_{n\to\infty}\|x_n-\mathcal{S}^{(0)}x_n\|=0, \tag{3.20}$$

and

$$\lim_{n\to\infty}\left\|x_n^{(k_t^{i-1})}-\mathcal{S}^{((k_t^i)}x_n^{(k_t^{i-1})}\right\|=0\ (t\in\mathfrak{T}, i=1,2,\cdots,I_t). \tag{3.21}$$

Recalling the definition of $x_n^{(0)}$, one further has

$$\left\|x_n^{(0)}-x_n\right\|\leq\lambda_n\left\|x_n-\mathcal{S}^{(0)}x_n\right\|+(1-\lambda_n)\nu\alpha_n\left\|\nabla\mathcal{F}^{(0)}x_n\right\|+\lambda_n\mu\beta_n\left\|\nabla\mathcal{G}^{(0)}\mathcal{S}^{(0)}x_n\right\|.$$

By passing to the limit in the above inequality, one concludes that

$$\lim_{n\to\infty}\|x_n^{(0)}-x_n\|=0. \tag{3.22}$$

In addition, one also has

$$\lim_{n\to\infty}\left\|x_n^{(k_t^{i-1})}-x_n^{(k_t^i)}\right\|=0, \tag{3.23}$$

and

$$\|x_n^{(0)}-x_n^{(k_t^i)}\|\leq\sum_{j=1}^{i}\left\|x_n^{(k_t^{j-1})}-x_n^{(k_t^j)}\right\|\quad(t\in\mathfrak{T},\ i=1,2,\cdots,I_t,\ \ n\in\mathbb{N}). \tag{3.24}$$

From (3.23) and (3.24), one arrives at

$$\lim_{n\to\infty}\left\|x_n^{(0)}-x_n^{(k_t^i)}\right\|=0\quad(t\in\mathfrak{T}, i=1,2,\cdots,I_t). \tag{3.25}$$

By collecting (3.20), (3.22) and (3.25), one concludes that

$$\lim_{n\to\infty}\left\|x_n - x_n^{(k_t^i)}\right\| = 0 \quad (t \in \mathfrak{T}, i = 1, 2, \cdots, I_t). \tag{3.26}$$

The boundedness of the sequence $(x_n)_{n\in\mathbb{N}}$ yields that there is a subsequence $(x_{n_j})_{j\in\mathbb{N}}(\subset (x_n)_{n\in\mathbb{N}})$ such that $(x_{n_j})_{j\in\mathbb{N}}$ converges weakly to $\hat{x} \in \mathcal{H}$. Let $t \in \mathfrak{T}$ $(i \in \{1, 2, \cdots, I_t\})$ be chosen arbitrarily. If $\hat{x} \notin Fix(\mathcal{S}^{(k_t^i)})$, then the Opial's condition and the nonexpansivity of $\mathcal{S}^{(k_t^i)}$ guarantee that

$$\begin{aligned}\limsup_{j\to\infty}\|x_{n_j} - \hat{x}\| &\geq \limsup_{j\to\infty}\left\|\mathcal{S}^{(k_t^i)}x_{n_j} - \mathcal{S}^{(k_t^i)}\hat{x}\right\| \\ &= \limsup_{j\to\infty}\left\|(x_{n_j} - \mathcal{S}^{(k_t^i)}x_{n_j}) + (T^{(k_t^i)}x_{n_j} - \mathcal{S}^{(k_t^i)}\hat{x})\right\| \\ &= \limsup_{k\to\infty}\left\|x_{n_j} - \mathcal{S}^{(k_t^i)}\hat{x}\right\| \\ &> \limsup_{j\to\infty}\left\|x_{n_j} - \hat{x}\right\|.\end{aligned}$$

This yields a contradiction. Hence, $\hat{x} \in Fix(\mathcal{S}^{(k_t^i)})$, i.e., $\hat{x} \in \bigcap_{t\in\mathfrak{T}}\bigcap_{i=1}^{I_t} Fix(\mathcal{S}^{(k_t^i)}) = \bigcap_{i\in\mathfrak{I}} Fix(\mathcal{S}^{(i)})$. If $\hat{x} \notin Fix(\mathcal{S}^{(0)})$, then

$$\limsup_{k\to\infty}\|x_0 - \hat{x}\| \geq \limsup_{n\to\infty}\|x_{n_k} - \mathcal{S}^{(0)}\hat{x}\| > \limsup_{k\to\infty}\|x_{n_k} - \hat{x}\|.$$

This shows $\hat{x} \in Fix(\mathcal{S}^{(0)})$. So $\hat{x} \in \cap_{i\in\{0\}\cup\mathfrak{I}}Fix(\mathcal{S}^{(i)})$. In view of $\nabla\mathcal{F}^{(0)}(x_n) = \partial\mathcal{F}^{(0)}(x_n)$, we have $\mathcal{F}^{(0)}(x) \geq \mathcal{F}^{(0)}(x_n) + \left\langle z - x_n, \nabla\mathcal{F}^{(0)}x_n\right\rangle$. An application of the nonexpansivity of $\mathcal{S}^{(0)}$ gives that

$$\begin{aligned}&\|x_n^{(0)} - z\|^2 \\ \leq&\lambda_n\left\|\mathcal{S}^{(0)}x_n - \mu\beta_n\nabla\mathcal{G}^{(0)}\mathcal{S}^{(0)}x_n - z\right\|^2 + (1-\lambda_n)\left\|x_n - \nu\alpha_n\nabla\mathcal{F}^{(0)}x_n - z\right\|^2 \\ =&(1-\lambda_n)\left[\|x_n - z\|^2 - 2\nu\alpha_n\left\langle x_n - z, \nabla\mathcal{F}^{(0)}x_n\right\rangle + \nu^2\alpha_n^2\left\|\nabla\mathcal{F}^{(0)}x_n\right\|^2\right] \\ &+\lambda_n\left[\left\|\mathcal{S}^{(0)}x_n - z\right\|^2 - 2\mu\beta_n\left\langle\mathcal{S}^{(0)}x_n - z, \nabla\mathcal{G}^{(0)}\mathcal{S}^{(0)}x_n\right\rangle + \mu^2\beta_n^2\left\|\nabla\mathcal{G}^{(0)}\mathcal{S}^{(0)}x_n\right\|^2\right] \\ \leq&\|x_n - z\|^2 - 2(1-\lambda_n)\nu\alpha_n\left\langle x_n - z, \nabla\mathcal{F}^{(0)}x_n\right\rangle + \nu^2\alpha_n^2\left\|\nabla\mathcal{F}^{(0)}x_n\right\|^2 \\ &- 2\mu\lambda_n\beta_n\left\langle\mathcal{S}^{(0)}x_n - z, \nabla\mathcal{G}^{(0)}\mathcal{S}^{(0)}x_n\right\rangle + \mu^2\beta_n^2\left\|\nabla\mathcal{G}^{(0)}\mathcal{S}^{(0)}x_n\right\|^2 \\ \leq&\|x_n - z\|^2 + 2\nu\alpha_n(1-\lambda_n)(\mathcal{F}^{(0)}(z) - \mathcal{F}^{(0)}(x_n))) + \nu^2\alpha_n^2\left\|\nabla\mathcal{F}^{(0)}x_n\right\|^2 \\ &+ \mu^2\beta_n^2\left\|\nabla\mathcal{G}^{(0)}\mathcal{S}^{(0)}x_n\right\|^2 + 2\mu\lambda_n\beta_n\left(\mathcal{G}^{(0)}(z) - \mathcal{G}^{(0)}(\mathcal{S}^{(0)}x_n)\right).\end{aligned} \tag{3.27}$$

Following a similar argument as above, one sees that

$$\begin{aligned}\|x_n^{(k_t^i)} - z\|^2 \le &\|x_n^{(k_t^{i-1})} - z\|^2 + 2(1-\lambda_n)\nu\alpha_n \left(\mathcal{F}^{(k_t^i)}(z) - \mathcal{F}^{(k_t^i)}(x_n^{(k_t^{i-1})}))\right) \\ &+ \nu^2\alpha_n^2\|\nabla\mathcal{F}^{(k_t^i)}x_n^{(k_t^{i-1})}\|^2 + 2\mu\lambda_n\beta_n\left(\mathcal{G}^{(k_t^i)}(z)\mathcal{G}^{(k_t^i)}(\mathcal{S}^{(k_t^i)}x_n^{(k_t^{i-1})})\right) \\ &+ \mu^2\beta_n^2\|\nabla\mathcal{G}^{(k_t^i)}\mathcal{S}^{(k_t^i)}x_n^{(k_t^{i-1})}\|^2,\end{aligned} \tag{3.28}$$

which in turn implies that

$$\begin{aligned}&\|x_n^{(t)} - z\|^2 \\ \le&\|x_n^{(k_t^{I_t-1})} - z\|^2 + 2\nu\alpha_n(1-\lambda_n)\left(\mathcal{F}^{(k_t^{I_t})}(z) - \mathcal{F}^{(k_t^{I_t})}(x_n^{(k_t^{I_t-1})})\right) \\ &+ \nu^2\alpha_n^2\|\nabla\mathcal{F}^{(k_t^{I_t})}x_n^{(k_t^{I_t-1})}\|^2 + 2\mu\lambda_n\beta_n(\mathcal{G}^{(k_t^{I_t})}(z) - \mathcal{G}^{(k_t^{I_t})}(\mathcal{S}^{(k_t^{I_t})}x_n^{(k_t^{I_t-1})})) \\ &+ \mu^2\beta_n^2\|\nabla\mathcal{G}^{(k_t^{I_t})}\mathcal{S}^{(k_t^{I_t})}x_n^{(k_t^{I_t-1})}\|^2 \\ \le&\|x_n - z\|^2 + 2(1-\lambda_n)\nu\alpha_n\sum_{i=1}^{I_t}\left(\mathcal{F}^{(k_t^i)}(x) - \mathcal{F}^{(k_t^i)}(x_n^{(k_t^{i-1})})\right) \\ &+ \beta_n^2\mu^2\sum_{i=1}^{I_t}\|\nabla\mathcal{G}^{(k_t^i)}\mathcal{S}^{(k_t^i)}x_n^{(k_t^{i-1})}\|^2 + 2\mu\beta_n\lambda_n\sum_{i=1}^{I_t}\Big(\mathcal{G}^{(k_t^i)}(z) \\ &-\mathcal{G}^{(k_t^i)}(\mathcal{S}^{(k_t^i)}x_n^{(k_t^{i-1})})\Big) + \alpha_n^2\nu^2\sum_{i=1}^{I_t}\|\nabla\mathcal{F}^{(k_t^i)}x_n^{(k_t^{i-1})}\|^2.\end{aligned}$$

Summing up the above inequality over all t leads to

$$\begin{aligned}&\sum_{t\in\mathfrak{T}}\|x_n^{(t)} - z\|^2 \\ \le&T\|x_n - z\|^2 + 2(1-\lambda_n)\nu\alpha_n\sum_{t\in\mathfrak{T}}\sum_{i=1}^{I_t}\left(\mathcal{F}^{(k_t^i)}(z) - \mathcal{F}^{(k_t^i)}(x_n^{(k_t^{i-1})})\right) \\ &+ \beta_n^2\mu^2\sum_{i=1}^{I_t}\|\nabla\mathcal{G}^{(k_t^i)}\mathcal{S}^{(k_t^i)}x_n^{(k_t^{i-1})}\|^2 \\ &+ 2\mu\beta_n\lambda_n\sum_{t\in\mathfrak{T}}\sum_{i=1}^{I_t}\left(\mathcal{G}^{(k_t^i)}(x) - \mathcal{G}^{(k_t^i)}(\mathcal{S}^{(k_t^i)}x_n^{(k_t^{i-1})})\right) \\ &+ \alpha_n^2\nu^2\sum_{t\in\mathfrak{T}}\sum_{i=1}^{I_t}\|\nabla\mathcal{F}^{(k_t^i)}x_n^{(k_t^{i-1})}\|^2.\end{aligned} \tag{3.29}$$

Coming back to (3.27) and (3.29), one sees that

$$\begin{aligned}
&\|x_{n+1}-z\|^2\\
=&\frac{\sum_{t\in\mathfrak{T}}\|x_n^{(t)}-z\|^2+\|x_n^{(0)}-z\|^2}{T+1}\\
\leq&\frac{1}{T+1}\Big\{T\|x_n-z\|^2+\|x_n-z\|^2+2(1-\lambda_n)\nu\alpha_n(\mathcal{F}^{(0)}(z)-\mathcal{F}^{(0)}(x_n))\\
&+2(1-\lambda_n)\nu\alpha_n\sum_{t\in\mathfrak{T}}\sum_{i=1}^{I_t}\Big(\mathcal{F}^{(k_t^i)}(z)-\mathcal{F}^{(k_t^i)}(x_n^{(k_t^{i-1})})\Big)\\
&+2\mu\lambda_n\beta_n\Big(\mathcal{G}^{(0)}(z)-\mathcal{G}^{(0)}\Big(\mathcal{S}^{(0)}x_n\Big)\Big)\\
&+2\mu\lambda_n\beta_n\sum_{t\in\mathfrak{T}}\sum_{i=1}^{I_t}\Big(\mathcal{G}^{(k_t^i)}(z)-\mathcal{G}^{(k_t^i)}\Big(\mathcal{S}^{(k_t^i)}x_n^{(k_t^{i-1})}\Big)\Big)+\nu^2\alpha_n^2\Big\|\nabla\mathcal{F}^{(0)}x_n\Big\|^2\\
&+\nu^2\alpha_n^2\sum_{t\in\mathfrak{T}}\sum_{i=1}^{I_t}\Big\|\nabla\mathcal{F}^{(k_t^i)}x_n^{(k_t^{i-1})}\Big\|^2+\mu^2\beta_n^2\Big\|\nabla\mathcal{G}^{(0)}\mathcal{S}^{(0)}x_n\Big\|^2\\
&+\mu^2\beta_n^2\sum_{i=1}^{I_t}\Big\|\nabla\mathcal{G}^{(k_t^i)}\mathcal{S}^{(k_t^i)}x_n^{(k_t^{i-1})}\Big\|^2\Big\}.
\end{aligned}$$

Putting $\delta_n=\frac{\beta_n}{\alpha_n}$, one concludes from the above inequality that

$$\begin{aligned}
&2(1-\lambda_n)\nu\Big(\mathcal{F}^{(0)}(x_n)-\mathcal{F}^{(0)}(z)\Big)\\
&+2(1-\lambda_n)\nu\sum_{t\in\mathfrak{T}}\sum_{i=1}^{I_t}\Big(\mathcal{F}^{(k_t^i)}\Big(x_n^{(k_t^{i-1})}\Big)-\mathcal{F}^{(k_t^i)}(z)\Big)\\
&+2\mu\lambda_n\delta_n\Big(\mathcal{G}^{(0)}(\mathcal{S}^{(0)}x_n)-\mathcal{G}^{(0)}(z)\Big)\\
&+2\mu\lambda_n\delta_n\sum_{t\in\mathfrak{T}}\sum_{i=1}^{I_t}\Big(\mathcal{G}^{(k_t^i)}\Big(\mathcal{S}^{(k_t^i)}x_n^{(k_t^{i-1})}\Big)-\mathcal{G}^{(k_t^i)}(z)\Big)\\
\leq&\frac{\|x_n-x_{n+1}\|(\|x_n-z\|+\|x_{n+1}-z\|)}{\alpha_n}\\
&+\alpha_n(T+1)\Big(\nu^2\Big\|\nabla\mathcal{F}^{(0)}x_n\Big\|^2+\nu^2\sum_{t\in\mathfrak{T}}\sum_{i=1}^{I_t}\Big\|\nabla\mathcal{F}^{(k_t^i)}x_n^{(k_t^{i-1})}\Big\|^2\Big)\\
&+\beta_n(T+1)\Big(\mu^2\delta_n\Big\|\nabla\mathcal{G}^{(0)}\mathcal{S}^{0)}x_n\Big\|^2+\mu^2\delta_n\sum_{i=1}^{I_t}\Big\|\nabla\mathcal{G}^{(k_t^i)}\mathcal{S}^{(k_t^i)}x_n^{(k_t^{i-1})}\Big\|^2\Big)\\
\leq&\frac{M_{14}\|x_n-x_{n+1}\|}{\alpha_n}+\alpha_nM_{15}+\beta_nM_{16},
\end{aligned}\tag{3.30}$$

where

$$
\begin{aligned}
M_{14} =& \sup_{n\in\mathbb{N}} \left(\|x_n - z\| + \|x_{n+1} - z\|\right) < \infty, \\
M_{15} =& \max_{t\in\mathfrak{T}, i=1,2,\cdots,I_t} \sup_{n\in\mathbb{N}} \Bigg\{ (T+1) \left(\nu^2 \left\| \nabla \mathcal{F}^{(0)} x_n \right\|^2 \right. \\
& \left. + \nu^2 \sum_{t\in\mathfrak{T}} \sum_{i=1}^{I_t} \left\| \nabla \mathcal{F}^{(k_t^i)} x_n^{(k_t^{i-1})} \right\|^2 \right) \Bigg\} < \infty, \\
M_{16} =& \max_{t\in\mathfrak{T}, i=1,2,\cdots,I_t} \sup_{n\in\mathbb{N}} \Bigg\{ (T+1) \left(\mu^2 \delta_n \left\| \nabla \mathcal{G}^{(0)} \mathcal{S}^{0)} x_n \right\|^2 \right. \\
& \left. + \mu^2 \delta_n \sum_{i=1}^{I_t} \left\| \nabla \mathcal{G}^{(k_t^i)} \mathcal{S}^{(k_t^i)} x_n^{(k_t^{i-1})} \right\|^2 \right) \Bigg\} < \infty.
\end{aligned}
\tag{3.31}
$$

Set

$$
\begin{aligned}
\mathcal{F} &:= \sum_{t\in\mathfrak{T}} \sum_{i=1}^{I_t} \mathcal{F}^{(k_t^i)} + \mathcal{F}^{(0)} = \sum_{i=1}^{\mathfrak{I}} \mathcal{F}^{(i)} + \mathcal{F}^{(0)}, \\
\mathcal{G} &:= \sum_{t\in\mathfrak{T}} \sum_{i=1}^{I_t} \mathcal{G}^{(k_t^i)} + \mathcal{G}^{(0)} = \sum_{i=1}^{\mathfrak{I}} \mathcal{G}^{(i)} + \mathcal{G}^{(0)}.
\end{aligned}
\tag{3.32}
$$

By rearranging the terms of the left-hand side of (3.30), we infer that

$$
\begin{aligned}
& 2(1-\lambda_n)\nu \left(\mathcal{F}^{(0)}(x_n) - \mathcal{F}^{(0)}(z) \right) \\
& + 2(1-\lambda_n)\nu \sum_{t\in\mathfrak{T}} \sum_{i=1}^{I_t} \left(\mathcal{F}^{(k_t^i)} \left(x_n^{(k_t^{i-1})} \right) - \mathcal{F}^{(k_t^i)}(z) \right) \\
& + 2\mu\lambda_n\delta_n \left(\mathcal{G}^{(0)} \left(\mathcal{S}^{(0)} x_n \right) - \mathcal{G}^{(0)}(z) \right) \\
& + 2\mu\lambda_n\delta_n \sum_{t\in\mathfrak{T}} \sum_{i=1}^{I_t} \left(\mathcal{G}^{(k_t^i)} \left(\mathcal{S}^{(k_t^i)} x_n^{(k_t^{i-1})} \right) - \mathcal{G}^{(k_t^i)}(z) \right) \\
=& 2\nu(1-\lambda_n) \left[\mathcal{F}(x_n) - \mathcal{F}(z) + \sum_{t\in\mathfrak{T}} \sum_{i=1}^{I_t} \left(\mathcal{F}^{(k_t^i)} \left(x_n^{(k_t^{i-1})} \right) - F^{(k_t^i)}(x_n) \right) \right] \\
& + 2\mu\lambda_n\delta_n \Bigg\{ \mathcal{G}(x_n) - \mathcal{G}(z) + \left[\mathcal{G}^{(0)} \left(\mathcal{S}^{(0)} x_n \right) - \mathcal{G}^{(0)}(x_n) \right] \\
& + \sum_{t\in\mathfrak{T}} \sum_{i=1}^{I_t} \left(\mathcal{G}^{(k_t^i)} \left(\mathcal{S}^{(k_t^i)} x_n^{(k_t^{i-1})} \right) - \mathcal{G}^{(k_t^i)} \left(x_n^{(k_t^{i-1})} \right) \right) \\
& + \sum_{t\in\mathfrak{T}} \sum_{i=1}^{I_t} \left(\mathcal{G}^{(k_t^i)} (x_n^{(k_t^{i-1})}) - \mathcal{G}^{(k_t^i)}(x_n) \right) \Bigg\}.
\end{aligned}
\tag{3.33}
$$

Set $\kappa = \min\{2(1-\lambda_n)\nu, 2\mu\lambda_n\delta_n\} > 0$. The condition $\kappa > 0$ is guaranteed by Assumption 2 (iii). Hence, it follows immediately from (3.30) and (3.33) that

$$\begin{aligned}
&\kappa[\mathcal{F}(x_n)+\mathcal{G}(x_n)-(\mathcal{F}(z)+\mathcal{G}(z))]\\
\leq &\frac{\|x_n-x_{n+1}\|M_{14}}{\alpha_n}+\alpha_n M_{15}+\beta_n M_{16}\\
&+2(1-\lambda_n)\nu\left[\sum_{t\in\mathfrak{T}}\sum_{i=1}^{I_t}\left(\mathcal{F}^{(k_t^i)}(x_n)-\mathcal{F}^{(k_t^i)}\left(x_n^{(k_t^{i-1})}\right)\right)\right]\\
&+2\mu\lambda_n\delta_n\{\left[\mathcal{G}^{(0)}(x_n)-\mathcal{G}^{(0)}(\mathcal{S}^{(0)}x_n)\right]\\
&+\sum_{t\in\mathfrak{T}}\sum_{i=1}^{I_t}\left(\mathcal{G}^{(k_t^i)}\left(x_n^{(k_t^{i-1})}\right)-\mathcal{G}^{(k_t^i)}\left(T^{(k_t^i)}x_n^{(k_t^{i-1})}\right)\right)\\
&+\sum_{t\in\mathfrak{T}}\sum_{i=1}^{I_t}\left(\mathcal{G}^{(k_t^i)}(x_n)-\mathcal{G}^{(k_t^i)}\left(x_n^{(k_t^{i-1})}\right)\right)\}.
\end{aligned}\tag{3.34}$$

Due to Assumption 2 (i) and (3.13), one concludes that

$$\limsup_{n\to\infty}\frac{\|x_n-x_{n+1}\|}{\alpha_n}=0. \tag{3.35}$$

By applying Proposition 1 , the Cauchy-Schwarz inequality and (3.26), one concludes that, for all $t\in\mathfrak{T}$,

$$\begin{aligned}
0\leq &\limsup_{n\to\infty}\sum_{t\in\mathfrak{T}}\sum_{i=1}^{I_t}\left[\mathcal{F}^{(k_t^i)}(x_n)-\mathcal{F}^{(k_t^i)}\left(x_n^{(k_t^{i-1})}\right)\right]\\
\leq &\sum_{t\in\mathfrak{T}}\sum_{i=1}^{I_t}\left(\limsup_{n\to\infty}\left[\mathcal{F}^{(k_t^i)}(x_n)-\mathcal{F}^{(k_t^i)}\left(x_n^{(k_t^{i-1})}\right)\right]\right)\\
\leq &\sum_{t\in\mathfrak{T}}\sum_{i=1}^{I_t}\left(\limsup_{n\to\infty}\left\langle x_n-x_n^{(k_t^{i-1})},\nabla\mathcal{F}^{(k_t^i)}(x_n)\right\rangle\right)\\
\leq &\sum_{t\in\mathfrak{T}}\sum_{i=1}^{I_t}\left(\limsup_{n\to\infty}\left\|\nabla\mathcal{F}^{(k_t^i)}(x_n)\right\|\left\|x_n-x_n^{(k_t^{i-1})}\right\|\right)=0.
\end{aligned}\tag{3.36}$$

On the other hand, Lemma 1, Assumption 1 (ii), (3.20), the boundedness of $(x_n)_{n\in\mathbb{N}}$ and the Cauchy-Schwarz inequality infer that

$$\begin{aligned}
0=&\limsup_{n\to\infty}\|x_n-\mathcal{S}^{(0)}x_n\|\|\nabla\mathcal{G}^{(0)}(x_n)\|\\
\geq&\limsup_{n\to\infty}\langle x_n-\mathcal{S}^{(0)}x_n,\nabla\mathcal{G}^{(0)}(x_n)\rangle\\
\geq&\limsup_{n\to\infty}(\mathcal{G}^{(0)}(x_n)-\mathcal{G}^{(0)}(\mathcal{S}^{(0)}x_n)).
\end{aligned}\tag{3.37}$$

By applying Lemma 1, Assumption 1 (ii), the Cauchy-Schwarz inequality, together with (3.21), the boundedness of $(x_n^{(k_t^{(i)})})_{n\in\mathbb{N}}$ $(t\in\mathfrak{T},\ i=1,2,\cdots,I_t)$, we conclude that

$$\begin{aligned}
&\limsup_{n\to\infty}\sum_{t\in\mathfrak{T}}\sum_{i=1}^{I_t}\left[\mathcal{G}^{(k_t^i)}\left(x_n^{(k_t^{i-1})}\right)-\mathcal{G}^{(k_t^i)}\left(\mathcal{S}^{(k_t^i)}x_n^{(k_t^{i-1})}\right)\right]\\
\le&\sum_{t\in\mathfrak{T}}\sum_{i=1}^{I_t}\left(\limsup_{n\to\infty}\left[\mathcal{G}^{(k_t^i)}\left(x_n^{(k_t^{i-1})}\right)-\mathcal{G}^{(k_t^i)}\left(\mathcal{S}^{(k_t^i)}x_n^{(k_t^{i-1})}\right)\right]\right)\\
\le&\sum_{t\in\mathfrak{T}}\sum_{i=1}^{I_t}\left(\limsup_{n\to\infty}\left\langle x_n^{(k_t^{i-1})}-\mathcal{S}^{(k_t^i)}x_n^{(k_t^{i-1})},\nabla\mathcal{G}^{(k_t^i)}\left(x_n^{(k_t^{i-1})}\right)\right\rangle\right)\\
\le&\sum_{t\in\mathfrak{T}}\sum_{i=1}^{I_t}\left(\limsup_{n\to\infty}\left\|\nabla\mathcal{G}^{(k_t^i)}\left(x_n^{(k_t^{i-1})}\right)\right\|\left\|x_n^{(k_t^{i-1})}-\mathcal{S}^{(k_t^i)}\left(x_n^{(k_t^{i-1})}\right)\right\|\right)=0.
\end{aligned}\tag{3.38}$$

Similarly, one has

$$\begin{aligned}
&\limsup_{n\to\infty}\sum_{t\in\mathfrak{T}}\sum_{i=1}^{I_t}\left[\mathcal{G}^{(k_t^i)}(x_n)-\mathcal{G}^{(k_t^i)}\left(x_n^{(k_t^{i-1})}\right)\right]\\
\le&\sum_{t\in\mathfrak{T}}\sum_{i=1}^{I_t}\left(\limsup_{n\to\infty}\left[\mathcal{G}^{(k_t^i)}(x_n)-\mathcal{G}^{(k_t^i)}\left(x_n^{(k_t^{i-1})}\right)\right]\right)\\
\le&\sum_{t\in\mathfrak{T}}\sum_{i=1}^{I_t}\left(\limsup_{n\to\infty}\left\langle x_n-x_n^{(k_t^{i-1})},\nabla\mathcal{G}^{(k_t^i)}(x_n)\right\rangle\right)\\
\le&\sum_{t\in\mathfrak{T}}\sum_{i=1}^{I_t}\left(\limsup_{n\to\infty}\left\|x_n-x_n^{(k_t^{i-1})}\right\|\left\|\nabla\mathcal{G}^{(k_t^i)}(x_n)\right\|\right)=0.
\end{aligned}\tag{3.39}$$

Thanks to (3.32)-(3.39), one reaches

$$\limsup_{n\to\infty}(\mathcal{F}(x_n)+\mathcal{G}(x_n)-(\mathcal{F}(z)+\mathcal{G}(z)))\le 0.\tag{3.40}$$

We use Assumption 1 (i), (ii) to deduce that

$$\mathcal{F}:=\sum_{i\in\{0\}\cup\mathfrak{I}}\mathcal{F}^{(i)}$$

and

$$\mathcal{G}:=\sum_{i\in\{0\}\cup\mathfrak{I}}\mathcal{F}^{(i)}$$

are convex and continuous. Then there exists a subsequence $(x_{n_j})_{j\in\mathbb{N}}$ of $(x_n)_{n\in\mathbb{N}}$ such that $(x_{n_j})_{j\in\mathbb{N}}$ converges weakly to $\hat{x}\in\bigcap_{i\in\{0\}\cup\mathfrak{I}}Fix(\mathcal{S}^{(i)})$. As we shall see, this fact implies that

$$\liminf_{j\to\infty}\mathcal{G}(x_{n_j})\ge\mathcal{G}(\hat{x}),\ \ \liminf_{j\to\infty}\mathcal{F}(x_{n_j})\ge\mathcal{F}(\hat{x}).\tag{3.41}$$

As a classical result, (3.32), (3.40) and (3.41) guarantee that, for all $z \in \bigcap_{i\in\{0\}\cup\mathbb{I}} Fix(\mathcal{S}^{(i)})$,

$$\begin{aligned}
&\sum_{i\in\{0\}\cup\mathfrak{I}}\left[\mathcal{F}^{(i)}(\hat{x})+\mathcal{G}^{(i)}(\hat{x})\right]\\
=&\mathcal{F}(\hat{x})+\mathcal{G}(\hat{x})\le\liminf_{j\to\infty}\mathcal{F}(x_{n_j})+\liminf_{j\to\infty}\mathcal{G}(x_{n_j})\\
\le&\liminf_{j\to\infty}\left[\mathcal{F}(x_{n_j})+\mathcal{G}(x_{n_j})\right]\le\limsup_{j\to\infty}\left[\mathcal{F}(x_{n_j})+\mathcal{G}(x_{n_j})\right]\\
\le&\mathcal{F}(z)+\mathcal{G}(z)=\sum_{i\in\{0\}\cup\mathfrak{I}}\left[\mathcal{F}^{(i)}(z)+\mathcal{G}^{(i)}(z)\right].
\end{aligned} \tag{3.42}$$

Accordingly, one concludes that $\hat{x}\in\Lambda$ is the solution of problem (1.1). In view of Lemma 3, one finds that $\hat{x}$ is the unique solution of problem (1.1). Suppose that there exists another weakly convergent subsequence, $(x_{n_l})_{l\in\mathbb{N}}$ of $(x_n)_{n\in\mathbb{N}}$. Proceeding as in the proof above, we can derive that $(x_{n_l})_{l\in\mathbb{N}}$ also weakly converges to the unique solution $\hat{x}\in\Lambda$. Hence, we can conclude that $(x_n)_{n\in\mathbb{N}}$ weakly converges to $\hat{x}\in\Lambda$. Using (3.26), one obtains that $x_n^{k_t^{(i)}}$ also weakly converges to $\hat{x}\in\Lambda$, for all $t\in\mathfrak{T}$, $i=1,2,\cdots,I_t$. With the help of the elementary inequality $\|a+b\|^2\le\|a\|^2-2\langle a,b\rangle$, $a,b\in\mathcal{H}$ and Remark 1, one has that

$$\begin{aligned}
&\|x_n^{(0)}-\hat{x}\|^2\\
\le&(1-\lambda_n)\|(I-\nu\alpha_n\nabla\mathcal{F}^{(0)})x_n-(I-\nu\alpha_n\nabla\mathcal{F}^{(0)})\hat{x}-\nu\alpha_n\nabla\mathcal{F}^{(0)}\hat{x}\|^2\\
&+\lambda_n\|(I-\mu\beta_n\nabla\mathcal{G}^{(0)})\mathcal{S}^{(0)}x_n-(I-\mu\beta_n\nabla\mathcal{G}^{(0)})\mathcal{S}^{(0)}\hat{x}-\mu\beta_n\nabla\mathcal{G}^{(0)}\mathcal{S}^{(0)}\hat{x}\|^2\\
\le&(1-\lambda_n)\left[(I-\rho\alpha_n)^2\|x_n-\hat{x}\|^2-2\nu\alpha_n\langle\nabla\mathcal{F}^{(0)}\hat{x},(I-\nu\alpha_n\nabla\mathcal{F}^{(0)})x_n-\hat{x}\rangle\right]\\
&+\lambda_n\left[(I-\sigma\beta_n)^2\|x_n-\hat{x}\|^2-2\mu\beta_n\langle\nabla\mathcal{G}^{(0)}\mathcal{S}^{(0)}\hat{x},(I-\mu\beta_n\nabla\mathcal{G}^{(0)})\mathcal{S}^{(0)}x_n-\hat{x}\rangle\right]\\
\le&(1-\phi_n)\|x_n-\hat{x}\|^2+\phi_n\Big\{2\nu\gamma_n(1-\lambda_n)\langle\nabla\mathcal{F}^{(0)}\hat{x},\hat{x}-x_n\rangle\\
&+2\mu\tau_n\lambda_n\langle\nabla\mathcal{G}^{(0)}\mathcal{S}^{(0)}\hat{x},\hat{x}-\mathcal{S}^{(0)}x_n\rangle+\alpha_nM_{17}+\beta_nM_{18}\Big\},
\end{aligned} \tag{3.43}$$

where

$$\begin{aligned}
&\gamma_n=\frac{\alpha_n}{\phi_n},\tau_n=\frac{\beta_n}{\phi_n},M_{17}=\sup_{n\in\mathbb{N}}\left(2\nu^2\gamma_n\|\nabla\mathcal{F}^{(0)}\hat{x}\|\|\nabla\mathcal{F}^{(0)}x_n\|\right),\\
&M_{18}=\sup_{n\in\mathbb{N}}\left(2\mu^2\tau_n\|\nabla\mathcal{G}^{(0)}\mathcal{S}^{(0)}\hat{x}\|\|\nabla\mathcal{G}^{(0)}\mathcal{S}^{(0)}x_n\|\right).
\end{aligned} \tag{3.44}$$

A calculation similar to (3.43) guarantees that, for all $t\in\mathfrak{T}$, $i=1,2,\cdots,I_t$,

$$\begin{aligned}
&\|x_n^{(k_t^i)}-\hat{x}\|^2\\
\le&(1-\phi_n)\left\|x_n^{(k_t^{i-1})}-\hat{x}\right\|^2+\phi_n\Big\{2\nu\gamma_n(1-\lambda_n)\left\langle\nabla\mathcal{F}^{(k_t^i)}\hat{x},\hat{x}-x_n^{(k_t^{i-1})}\right\rangle\\
&+2\mu\tau_n\lambda_n\left\langle\nabla\mathcal{G}^{(k_t^i)}\mathcal{S}^{(k_t^i)}\hat{x},\hat{x}-\mathcal{S}^{(k_t^i)}x_n^{(k_t^{i-1})}\right\rangle+\alpha_nM_{19}+\beta_nM_{20}\Big\},
\end{aligned}$$

where

$$\begin{aligned} M_{19} &= \max_{t\in\mathfrak{T}, i=1,2,\cdots,I_t} \sup_{n\in\mathbb{N}} \left(2\nu^2\gamma_n \left\|\nabla\mathcal{F}^{(k_t^i)}\hat{x}\right\| \left\|\nabla\mathcal{F}^{(k_t^i)}x_n^{(k_t^{i-1})}\right\|\right), \\ M_{20} &= \max_{t\in\mathfrak{T}, i=1,2,\cdots,I_t} \sup_{n\in\mathbb{N}} \left(2\mu^2\tau_n \left\|\nabla\mathcal{G}^{(k_t^i)}\mathcal{S}^{(k_t^i)}\hat{x}\right\| \left\|\nabla\mathcal{G}^{(k_t^i)}\mathcal{S}^{(k_t^i)}x_n^{(k_t^{i-1})}\right\|\right). \end{aligned} \tag{3.45}$$

Accordingly, one has that, for all $t \in \mathfrak{T}$

$$\begin{aligned} &\|x_n^{(t)} - \hat{x}\|^2 \\ \leq &(1-\phi_n)\|x_n - \hat{x}\|^2 + \phi_n \left\{2\nu\gamma_n(1-\lambda_n)\sum_{i=1}^{I_t}\left\langle\nabla\mathcal{F}^{(k_t^i)}\hat{x}, \hat{x} - x_n^{(k_t^{i-1})}\right\rangle\right. \\ &\left. + 2\mu\tau_n\lambda_n \sum_{i=1}^{I_t}\left\langle\nabla\mathcal{G}^{(k_t^i)}\mathcal{S}^{(k_t^i)}\hat{x}, \hat{x} - \mathcal{S}^{(k_t^i)}x_n^{(k_t^{i-1})}\right\rangle + \alpha_n\sum_{i=1}^{I_t} M_{19} + \beta_n \sum_{i=1}^{I_t} M_{20}\right\}. \end{aligned}$$

Summing up the above inequality over all $t \in \mathfrak{T}$ shows that

$$\begin{aligned} &\sum_{t\in\mathfrak{T}}\|x_n^{(t)} - \hat{x}\|^2 \\ \leq &T(1-\phi_n)\|x_n - \hat{x}\|^2 + \phi_n \left\{2\nu\gamma_n(1-\lambda_n)\sum_{t\in\mathfrak{T}}\sum_{i=1}^{I_t}\left\langle\nabla\mathcal{F}^{(k_t^i)}\hat{x}, \hat{x} - x_n^{(k_t^{i-1})}\right\rangle\right. \\ &+ 2\mu\tau_n\lambda_n \sum_{t\in\mathfrak{T}}\sum_{i=1}^{I_t}\left\langle\nabla\mathcal{G}^{(k_t^i)}\mathcal{S}^{(k_t^i)}\hat{x}, \hat{x} - \mathcal{S}^{(k_t^i)}x_n^{(k_t^{i-1})}\right\rangle \\ &\left. + \alpha_n\sum_{t\in\mathfrak{T}}\sum_{i=1}^{I_t} M_{19} + \beta_n \sum_{t\in\mathfrak{T}}\sum_{i=1}^{I_t} M_{20}\right\}. \end{aligned} \tag{3.46}$$

Hence, by combining (3.43) with (3.46), we derive that

$$\begin{aligned} \|x_{n+1} - \hat{x}\|^2 &\leq \frac{\sum_{t\in\mathfrak{T}}\|x_n^{(t)} - \hat{x}\|^2 + \|\hat{x} - x_n^{(0)}\|^2}{T+1} \\ &\leq (1-\phi_n)\|x_n - \hat{x}\|^2 \\ &+ \phi_n\left\{\frac{2\nu\gamma_n(1-\lambda_n)}{T+1}\left[\langle\nabla\mathcal{F}^{(0)}\hat{x}, \hat{x} - x_n\rangle + \sum_{t\in\mathfrak{T}}\sum_{i=1}^{I_t}\left\langle\nabla\mathcal{F}^{(k_t^i)}\hat{x}, \hat{x} - x_n^{(k_t^{i-1})}\right\rangle\right]\right. \\ &+ \frac{2\mu\tau_n\lambda_n}{T+1}\left[\left\langle\nabla\mathcal{G}^{(0)}\mathcal{S}^{(0)}\hat{x}, \hat{x} - \mathcal{S}^{(0)}x_n\right\rangle\right. \\ &\left. + \sum_{t\in\mathfrak{T}}\sum_{i=1}^{I_t}\left\langle\nabla\mathcal{G}^{(k_t^i)}\mathcal{S}^{(k_t^i)}\hat{x}, \hat{x} - \mathcal{S}^{(k_t^i)}x_n^{(k_t^{i-1})}\right\rangle\right] \\ &\left. + \alpha_n\left[\frac{M_{17} + \sum_{t\in\mathfrak{T}}\sum_{i=1}^{I_t} M_{19}}{T+1}\right] + \beta_n\left[\frac{M_{18} + \sum_{t\in\mathfrak{T}}\sum_{i=1}^{I_t} M_{20}}{T+1}\right]\right\} \\ &= (1-\phi_n)\|x_n - \hat{x}\|^2 + \phi_n Y_n, \end{aligned} \tag{3.47}$$

where

$$\begin{aligned} Y_n =&\frac{2\nu\gamma_n(1-\lambda_n)}{T+1}\left[\langle\nabla\mathcal{F}^{(0)}\hat{x},\hat{x}-x_n\rangle+\sum_{t\in\mathfrak{T}}\sum_{i=1}^{I_t}\left\langle\nabla\mathcal{F}^{(k_t^i)}\hat{x},\hat{x}-x_n^{(k_t^{i-1})}\right\rangle\right]\\ &+\frac{2\mu\tau_n\lambda_n}{T+1}\Big[\langle\nabla\mathcal{G}^{(0)}\mathcal{S}^{(0)}\hat{x},\hat{x}-\mathcal{S}^{(0)}x_n\rangle\\ &+\sum_{t\in\mathfrak{T}}\sum_{i=1}^{I_t}\left\langle\nabla\mathcal{G}^{(k_t^i)}\mathcal{S}^{(k_t^i)}\hat{x},\hat{x}-\mathcal{S}^{(k_t^i)}x_n^{(k_t^{i-1})}\right\rangle\Big]\\ &+\alpha_n\left[\frac{M_{17}+\sum_{t\in\mathfrak{T}}\sum_{i=1}^{I_t}M_{19}}{T+1}\right]+\beta_n\left[\frac{M_{18}+\sum_{t\in\mathfrak{T}}\sum_{i=1}^{I_t}M_{20}}{T+1}\right]. \end{aligned} \tag{3.48}$$

Since $x_n \rightharpoonup \hat{x}$ and $x_n^{(k_t^{(i)})} \rightharpoonup \hat{x}$ $(t\in\mathfrak{T}, i=1,2,\cdots,I_t)$, we find from (3.20) and (3.21) that $\mathcal{S}^{(0)}x_n \rightharpoonup \hat{x}$ and $\mathcal{S}^{(k_t^i)}x_n^{(k_t^{(i-1)})} \rightharpoonup \hat{x}$ $(t\in\mathfrak{T}, i=1,2,\cdots,I_t)$. Therefore, by applying Assumption 2, Remark 1, (3.44), (3.45), (3.48), and the boundeness of (x_n) and $(x_n^{(k_t^{(i)})})_{n\in\mathbb{N}}$ $(t\in\mathfrak{T},\ i=1,2,\cdots,I_t)$, one concludes that $\lim_{n\to\infty}Y_n=0$. Hence, Lemma 4, Remark 1, (3.47) and (3.48) send us to

$$\lim_{n\to\infty}\|x_n-\hat{x}\|=0. \tag{3.49}$$

This implies that $(x_n)_{n\in\mathbb{N}}$ converges strongly to $\hat{x}$. In addition,

$$\|x_n^{(k_t^i)}-\hat{x}\|\le\|x_n^{(k_t^i)}-x_n\|+\|x_n-\hat{x}\|\ \ (t\in\mathfrak{T}, i=1,2,\cdots,I_t). \tag{3.50}$$

By combining (3.26), (3.49) with (3.50), we find that

$$\lim_{n\to\infty}\|x_n^{(k_t^i)}-\hat{x}\|=0\ \ (t\in\mathfrak{T}, i=1,2,\cdots,I_t),$$

that is, $(x_n^{(k_t^i})})_{n\in\mathbb{N}}\,(t\in\mathfrak{T}, i=1,2,\cdots,I_t)$ also converges strongly to $\hat{x}$. This proves Theorem 1. □

Remark 2. Theorem 1 is more general in nature. It is an improvement upon corresponding results of Sahu and Yao [29], Ceng and Ansari et al. [4], and Maingé [23].

Assumption 3. The sequences $(\alpha_n)_{n\in\mathbb{N}}(\subset(0,1))$ and $(\beta_n)_{n\in\mathbb{N}}(\subset(0,1))$ satisfy

$$\begin{cases}\sum_{n=0}^{\infty}\alpha_n=\infty,\\ \lim_{n\to\infty}\frac{1}{\alpha_{n+1}}\left|\frac{1}{\alpha_{n+1}}-\frac{1}{\alpha_n}\right|=0,\\ \lim_{n\to\infty}\alpha_n=0,\end{cases}\qquad\begin{cases}\sum_{n=0}^{\infty}\beta_n=\infty,\\ \lim_{n\to\infty}\frac{1}{\beta_{n+1}}\left|\frac{1}{\beta_{n+1}}-\frac{1}{\beta_n}\right|=0,\\ \lim_{n\to\infty}\beta_n=0.\end{cases}$$

Remark 3. Assume that $(\alpha_n)_{n\in\mathbb{N}}\subset(0,p)$ and $(\beta_n)_{n\in\mathbb{N}}\subset(0,q)$, where

$$p:=\min_{i\in\{0\}\cup\mathfrak{I}}2a_i/\kappa_i^2,\ q:=\min_{i\in\{0\}\cup\mathfrak{I}}2b_i/\iota_i^2.$$

With the help of Assumption 3, one concludes from Lemma 1 that

$$\begin{aligned}\|(I-\alpha_n\nabla F^{(i)})x-(I-\alpha_n\nabla F^{(i)})y\| \leq& (1-\alpha_n\vartheta^{(i)})\|x-y\| \leq (1-\alpha_n\vartheta)\|x-y\|,\\ \|(I-\beta_n\nabla G^{(i)})x-(I-\beta_n\nabla G^{(i)})y\| \leq& (1-\beta_n\varsigma^{(i)})\|x-y\| \leq (1-\beta_n\varsigma)\|x-y\|,\end{aligned} \tag{3.51}$$

where

$$\vartheta := \min_{i\in\{0\}\cup\mathfrak{I}} 1-\sqrt{1-p(2a^{(i)}-p\kappa^{(i)2})} \quad \text{and} \quad \varsigma := \min_{i\in\{0\}\cup\mathfrak{I}} 1-\sqrt{1-q(2b^{(i)}-q\iota^{(i)2})}.$$

Remark 4. Assumption 3 guarantees that there exists $n_0 \in \mathbb{N}$ such that $(\alpha_n)_{n\geq n_0} \subset (0,p)$, $(\beta_n)_{n\geq n_0} \subset (0,q)$, (3.51) holds. Without loss of generality, we may assume that (3.51) is always true for all $n \in \mathbb{N}$.

Assumption 3 and Remark 3 naturally lead us to construct the following simple algorithm.

Algorithm 2 The broadcast incremental algorithm

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procedure INPUT($(\alpha_n)_{n\in\mathbb{N}}$, $(\beta_n)_{n\in\mathbb{N}}$, T and $(I_t)_{t=1}^T$)
  User 0 chooses $x_0 \in \mathcal{H}$ arbitrarily
  Set $n \leftarrow 0$
  while not converged **do**
    User 0 computes $x_n^{(0)} \in \mathcal{H}$ as $x_n^{(0)} = (1-\lambda_n)(I-\alpha_n\nabla\mathcal{F}^{(0)})x_n + \lambda_n(I-\beta_n\nabla\mathcal{G}^{(0)})\mathcal{S}^{(0)}x_n$
    for $t = 1, 2, \cdots, T$ **do**
      User 0 transmits x_n to user $k_t^{(1)}$
      User $k_t^{(1)}$ computes $x_n^{(k_t^{(0)})}$ as $x_n^{(k_t^{(0)})} = x_n$
    end for
    for $t = 1, 2, \cdots, T$ **do**
      for $i = 1, 2, \cdots, I_t$ **do**
        User $k_t^{(i)}$ computes $x_n^{(k_t^{(i)})} \in \mathcal{H}$ as $x_n^{(k_t^{(i)})} = (1-\lambda_n)(I-\alpha_n\nabla\mathcal{F}^{(k_t^{(i)})})x_n^{k_t^{(i-1)}} + \lambda_n(I-\beta_n\nabla\mathcal{G}^{(k_t^{(i)})})\mathcal{S}^{(k_t^{(i)})}x_n^{k_t^{(i-1)}}$
      end for
      User $k_t^{(I_t)}$ computes $x_n^{(t)}$ as $x_n^{(t)} = x_n^{(k_t^{(I_t)})}$
      User $k_t^{(I_t)}$ transmits $x_n^{(t)}$ to user 0
    end for
    User 0 computes x_{n+1} as $x_{n+1} = \frac{x_n^{(0)}+\sum_{t=1}^T x_n^{(t)}}{T+1}$
    Set $n \leftarrow n+1$
  end while
  return $x = x_n$
end procedure

Corollary 1. *Under Assumption 3, the sequences* $(x_n)_{n\in\mathbb{N}}$ *and* $(x_n^{i_s^{(j)}})_{n\in\mathbb{N}}$ ($t \in \mathfrak{T}$, $i = 1, 2, \cdots, I_t$) *generated by Algorithm 2 strongly converge to the solution of problem (1.1).*

Remark 5. Inspired and motivated by problem (3.1), in the next step, we will consider a more general problem wherein user i ($i \in \mathfrak{I} \cup \{0\}$) has more private information such as $\mathcal{S}^{(i)}, \mathcal{F}_1^i, \mathcal{F}_2^i, \cdots, \mathcal{F}_m^i$ ($m \in \mathbb{N}$). The main objective of this problem is to

$$\text{minimize} \sum_{i \in \mathfrak{I} \cup \{0\}} \left(\sum_{q=1}^{m} \mathcal{F}_q^{(i)}(x) \right) \text{ subject to } x \in \bigcap_{i \in \mathfrak{I} \cup \{0\}} Fix(\mathcal{S}^{(i)}) \ (m \in \mathbb{N})$$

where $\mathcal{F}_q^{(i)} : H \to \mathbb{R}$ ($i \in \mathfrak{I} \cup \{0\}$, $q \in \{1, 2, \cdots, m\}$) is Fréchet differentiable, $\nabla \mathcal{F}_q^{(i)} : \mathcal{H} \to \mathcal{H}$ ($i \in \mathfrak{I} \cup \{0\}$, $q \in \{1, 2, \cdots, m\}$) is strongly monotone and Lipschitz continuous, $\mathcal{S}^{(i)} : \mathcal{H} \to \mathcal{H}$ ($i \in \mathfrak{I} \cup \{0\}$) is nonexpansive with $\bigcap_{i \in \mathfrak{I} \cup \{0\}} Fix(\mathcal{S}^{(i)}) \neq \emptyset$.

4. Applications

In a peer-to-peer (P2P) storage network, participants act as not only the service users but also the service providers. In other word, each participant enjoys the service by storing her own online data in the network and at the same time provides the service by offering some memory space to the others. To work properly, an economic model developed for P2P file sharing systems has already been applied to the P2P storage service allocation. This leads to a mechanism where the contribution of each peer should equal her use in the storage system. Such a mechanism is called a symmetric scheme, see [8, 32]. Based on the monetary transaction, users can 'sell' their own memory space to the system at a fixed unit price and 'buy' the storage space from the system at another unit price. Thus, the P2P data storage system can be managed by a simple payment-based mechanism. Recalling that, we can fix such prices so as to maximize the revenue in a profit-maximizing entity.

Let us consider a P2P data storage system network with its user set denoted by $\mathfrak{I}$. Assume that a peer i ($i \in \mathfrak{I}$) benefits from the service by demanding a storage capacity $\mathcal{C}_i^s$ that is to be used for storing its own data and offers a storage capacity $\mathcal{C}_i^o$ that is to be shared with other users. Based on a simple payment-based mechanism, users can 'sell' some of their own memory space for a unit price (denoted by p^o) and 'buy' storage space in the system for a unit price (denoted by p^s). Meanwhile, the two price thresholds $p_i^{\max}$ and $p_i^{\min}$ correspond to p^s, the maximum value of the unit price such that user i buys some storage space and p^o, the minimum value of the unit price such that she sells some of her memory space. As depicted in Figure 1, two price sensitivities a_i and b_i respectively represent the increase of the sold capacity when the unit price satisfies $p_i^{\min} \leq p^o$ and the decrease of the bought storage space when the unit price satisfies $p^s \leq p_i^{\max}$. Let us give $p (\geq 0)$. Assume that $d_i(p)$ (resp., $s_i(p)$) is the amount of storage capacity that peer i would choose to buy (resp., sell), if user i was charged (resp., paid) a unit price p for it (also see, Figure 1).

Assumption 4. Suppose that the supply function and demand function of user i ($i \in \mathfrak{I}$) are affine. There exist some values a_i, b_i, $p_i^{\min}$, $p_i^{\max} \geq 0$ such that

$$d_i(p) := b^{(i)}[p_i^{\max} - p]^+, \quad s_i(p) := a^{(i)}[p - p_i^{\min}]^+,$$

where $\min_i p_i^{\max} > \max_i p_i^{\min}$ and $x^+ = \max\{0, x\}$.

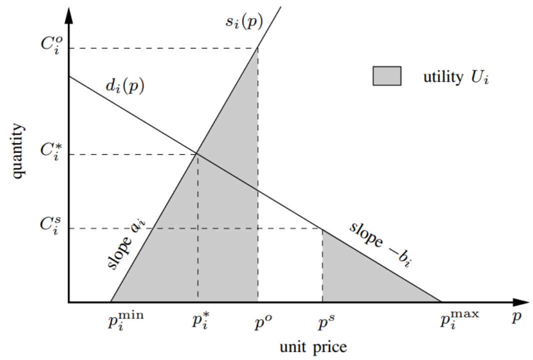

FIGURE 1. Reactions to prices and the utility of user $i \in \mathfrak{I}$ under Assumption 4.

In the profit-oriented pricing scheme, the operator (peer 0) plays a role of the manager in the P2P data storage system. Here, it is used to maximize the profit out of the business in the system by selling/buying the total amount of storage space. Suppose that peer 0 knows that peer $i(i \in \mathfrak{I})$ will buy $d^{(i)}(p^s)$ and sell $s^{(i)}(p^o)$. In this case, the utility function is defined as follows

$$\mathcal{U}^{(0)}(p^s, p^o) := \sum_{i \in \mathfrak{I}} \left[p^s d^{(i)}(p^s) - p^o s^{(i)}(p^o) \right]. \tag{4.1}$$

It strives to choose optimal prices p^s and p^o so as to maximize its profit $\mathcal{U}^{(0)}(p^s, p^o)$. As a matter of fact, $\sum_{i \in \mathfrak{I}} \mathcal{C}_i^s = \sum_{i \in \mathfrak{I}} d_i(p^s)$, which is used for storing the data, must not exceed $\sum_{i \in \mathfrak{I}} \mathcal{C}_i^o = \sum_{i \in \mathfrak{I}} s_i(p^o)$ offered by the peers. Accordingly, we define the objective function and the constrained set of peer 0 by, for all $(p^s, p^o) \in \mathbb{R}^2$,

$$\begin{aligned} &\mathcal{F}^{(0)}(p^s, p^o) = -\sum_{i \in \mathfrak{I}} p^s d^{(i)}(p^s), \;\; \mathcal{G}^{(0)}(p^s, p^o) = \sum_{i \in \mathfrak{I}} p^o s^{(i)}(p^o), \\ &\mathcal{C}^{(0)} = \mathbb{R}_+^2 \cap \left\{ (p^s, p^o) \in \mathbb{R}^2 : \sum_{i \in \mathfrak{I}} d_i(p^s) \le \sum_{i \in \mathfrak{I}} s_i(p^o) \right\}. \end{aligned} \tag{4.2}$$

Taking account of the fact that $d^{(i)}$ and $s^{(i)}$ are affine, both $\mathcal{F}^{(0)}$ and $\mathcal{G}^{(0)}$ in (4.2) are strongly convex. Furthermore, we find that both $\nabla \mathcal{F}^{(0)}$ and $\nabla \mathcal{G}^{(0)}$ are Lipschitz continuous and strongly monotone. Let $\mathcal{S}^{(0)} : \mathbb{R}^2 \to \mathbb{R}^2$ be a function defined by,

$$\mathcal{S}^{(0)}(p^s, p^o) := \frac{Proj_{\mathbb{R}_+^2} \{ Proj_{\hat{\mathcal{C}}^{(0)}}(p^s, p^o) \} + (p^s, p^o)}{2}, \;\; \forall (p^s, p^o) \in \mathbb{R}^2, \tag{4.3}$$

where $\hat{\mathcal{C}}^{(0)} := \left\{ (p^s, p^o) \in \mathbb{R}^2 : \sum_{i \in \mathfrak{I}} d_i(p^s) \le \sum_{i \in \mathfrak{I}} s_i(p^o) \right\}$. Thus, we find that $\mathcal{S}^{(0)}$ is nonexpansive. Furthermore, since $Fix(\mathcal{S}^{(0)}) = Fix(Proj_{\mathbb{R}_+^2} Proj_{\hat{\mathcal{C}}^{(0)}}) = \mathbb{R}_+^2 \cap \hat{\mathcal{C}}^{(0)} = \mathcal{C}^{(0)}$, it leads to $Fix(\mathcal{S}^{(0)}) := \mathcal{C}^{(0)}$.

The utility function $\mathcal{U}^{(i)}$ of peer i ($i \in \mathfrak{I}$) can be represented by

$$\mathcal{U}^{(i)}(\mathcal{C}_i^s, \mathcal{C}_i^o, \varepsilon^{(i)}) := \mathcal{V}^{(i)}(\mathcal{C}_i^s) - \varepsilon^{(i)} - \mathcal{P}^{(i)}(\mathcal{C}_i^o). \tag{4.4}$$

(i) $\mathcal{V}^{(i)}(\mathcal{C}_i^s)$ is the storage service valuation of user i. It can be expressed as a quadratic function for the opportunity and valuation cost ($\wedge$ denotes the min),

$$\mathcal{V}^{(i)}(\mathcal{C}_i^s) := \frac{b_i p_i^{\max}(\mathcal{C}_i^s \wedge b_i p_i^{\max}) - \frac{(\mathcal{C}_i^s \wedge b_i p_i^{\max})^2}{2}}{b_i}.$$

(ii) $\mathcal{P}^{(i)}(\mathcal{C}_i^o)$ is the overall non-monetary cost of user i for offering capacity $\mathcal{C}_i^o$ to the system, which is of the form

$$\mathcal{P}^{(i)}(\mathcal{C}_i^o) := p_i^{\min}\mathcal{C}_i^o + \mathcal{O}^{(i)}(\mathcal{C}_i^o),$$

(a) $O^{(i)}(\mathcal{C}_i^o)$ is an opportunity cost of offering capacity $\mathcal{C}_i^o$ for other peers without using $\mathcal{C}_i^o$ for itself. This actually corresponds to

$$\mathcal{O}^{(i)}(\mathcal{C}_i^o) := (1/a_i)(\mathcal{C}_i^o)^2/2.$$

(b) $p_i^{\min}\mathcal{C}_i^o$ is the data transfer cost used for the data protection mechanism implemented by the system.

(iii) The monetary price (possible negative) paid by peer i is $\varepsilon^{(i)} := p^s\mathcal{C}_i^s - p^o\mathcal{C}_i^o$, where $\mathcal{C}_i^o = s_i(p^o)$ and $\mathcal{C}_i^s = d_i(p^s)$.

Let consider a performance measure, called the social welfare. It can be viewed as the sum of the utility functions of all peers in the whole system. Because user 0 is a member of the society, all money it exchanges with other users stays within the system and therefore does not influence the social welfare. It follows from (4.1) and (4.2) that the social welfare $\mathcal{W} : \mathbb{R}^2 \to \mathbb{R}$ can be expressed as follows. For all $\mathcal{C}_s := (\mathcal{C}_1^s, \mathcal{C}_2^s, \cdots, \mathcal{C}_I^s)^T$, $\mathcal{C}_o := (\mathcal{C}_1^o, \mathcal{C}_2^o, \cdots, \mathcal{C}_I^o)^T \in \mathbb{R}^I$,

$$\mathcal{W}(\mathcal{C}_s, \mathcal{C}_o) := \sum_{i \in \mathfrak{I}} \mathcal{U}^{(i)}(\mathcal{C}_i^s, \mathcal{C}_i^o, \varepsilon^{(i)}) + \mathcal{U}^{(0)}(p^s, p^o) = \sum_{i \in \mathfrak{I}} \left[\mathcal{V}^{(i)}(\mathcal{C}_i^s) - \mathcal{P}^{(i)}(\mathcal{C}_i^o)\right]. \tag{4.5}$$

Since the social welfare makes the system reliable and stable, we would like to maximize $\mathcal{W}$. Through all users cooperating in the system, it is desirable to enable each user i ($i \in \mathfrak{I}$) to decide the optimal $(\mathcal{C}_i^s, \mathcal{C}_i^o) \in \mathbb{R}^2$, by maximizing its own private welfare. We call $\mathcal{W}^{(i)} : \mathbb{R}^2 \to \mathbb{R}$ the welfare of peer i ($i \in \mathfrak{I}$), which is defined by, for all $(\mathcal{C}_i^s, \mathcal{C}_i^o) \in \mathbb{R}^2$ (resp.$(d_i(p^s), s_i(p^o)) \in \mathbb{R}^2$),

$$\mathcal{W}^{(i)}(\mathcal{C}_i^s, \mathcal{C}_i^o) := \mathcal{V}^{(i)}(\mathcal{C}_i^s) - \mathcal{P}^{(i)}(\mathcal{C}_i^o).$$

Taking account of the fact that peer $i (i \in \mathfrak{I})$ will behave selfishly in order to choose a strategy that maximize its own welfare $\mathcal{W}^{(i)}$. Therefore, the objective function and the box constrained set of peer $i (i \in \mathfrak{I})$ can be defined as, for all $(p^s, p^o) \in \mathbb{R}^2$,

$$\begin{gathered} \mathcal{F}^{(i)}(p^s, p^o) = -\mathcal{V}^{(i)}(d_i(p^s)), \;\; \mathcal{G}^{(i)}(p^s, p^o) = \mathcal{P}^{(i)}(s_i(p^o)), \\ \mathcal{C}^{(i)} = [p_i^{\min}, p_i^{\max}] \times [p_i^{\min}, p_i^{\max}] \; (i \in \mathfrak{I}). \end{gathered} \tag{4.6}$$

Because d_i and s_i are affine, one sees that $\mathcal{F}^{(i)}, \mathcal{G}^{(i)} (i \in \mathfrak{I})$ in (4.6) satisfy the strong convexity condition. As a consequence, one has that both $\nabla\mathcal{F}^{(i)}$ and $\nabla\mathcal{G}^{(i)} (i \in \mathfrak{I})$

are strongly monotone and Lipschitz continuous. Let us define mappings $\mathcal{S}^{(i)}:\mathbb{R}^2\to\mathbb{R}^2$ $(i\in\mathfrak{I})$ by, for all $(p^s,p^o)\in\mathbb{R}^2$,

$$\mathcal{S}^{(i)}(p^s,p^o)=Proj_{\mathcal{C}^{(i)}}(p^s,p^o). \tag{4.7}$$

It can be easily checked that $\mathcal{S}^{(i)}$ $(i\in\mathfrak{I})$, defined in (4.7), satisfies the nonexpansive condition with

$$Fix(\mathcal{S}^{(i)})=Fix(Proj_{\mathcal{C}^{(i)}})=\left\{(p^s,p^o)\in\mathbb{R}^2:\mathcal{S}^{(i)}(p^s,p^o)=(p^s,p^o)\right\}=\mathcal{C}^{(i)}.$$

The main aim of the profit-oriented pricing scheme is to determine optimal prices p^o and p^s, in order to maximize $\mathcal{U}^{(0)}$, the profit of operator (user 0). To make our system reliable and stable, one may maximize the social welfare $\mathcal{W}$. Accordingly, one can construct the storage allocation problem of the form

Problem 1. (The storage allocation problem)

$$\begin{aligned}\text{Maximize } &(1-\varpi)\mathcal{W}(p^s,p^o)+\varpi\mathcal{U}^{(0)}(p^s,p^o)\\ &=-\left[(1-\varpi)\sum_{i\in\mathfrak{I}}(\mathcal{F}^{(i)}+\mathcal{G}^{(i)})+\varpi(\mathcal{G}^{(0)}+\mathcal{F}^{(0)})\right](p^s,p^o)\end{aligned}$$

subject to

$$\begin{aligned}(p^s,p^o)&\in\bigcap_{i\in\mathfrak{I}}[p_i^{\min},p_i^{\max}]\times[p_i^{\min},p_i^{\max}]\cap\left\{(p^s,p^o)\in\mathbb{R}_+^2:\sum_{i\in\mathfrak{I}}d_i(p^s)\le\sum_{i\in\mathfrak{I}}s_i(p^o)\right\}\\ &=\bigcap_{i\in\{0\}\cup\mathfrak{I}}Fix(\mathcal{S}^{(i)}),\end{aligned}$$

where $\varpi\in(0,1)$, $\mathcal{F}^{(i)}:\mathbb{R}^2\to\mathbb{R}$, $\mathcal{G}^{(i)}:\mathbb{R}^2\to\mathbb{R}$ and $\mathcal{S}^{(i)}:\mathbb{R}^2\to\mathbb{R}(i\in\{0\}\cup\mathfrak{I})$ are respectively defined as in (4.2), (4.3), (4.6) and (4.7).

Let us consider two computational experiments for solving Problem 1 with $\varpi=\frac{1}{2}$. Experiment 1 is generated with Matlab version 5.0. Experiment 2 is generated with Python 3.7. All calculations are performed on a personal computer Intel(R) Core(TM) i5-8250U CPU @ 1.60GHz.

Experiment 1. Consider a storage networked system with $I=120$, $T=1,2,6,12,60,120$. Thus the element number of each subnetwork is $I_t=I/T$. One randomly chooses the values in the range of $(0,10]$ for $a^{(i)}$ and $b^{(i)}$. One randomly chooses $p_i^{\min}$ in the range of $[20,30]$ and $p_i^{\max}$ in the range of $[120,130]$ $(i=1,2,\cdots,120)$. One sets $\alpha_n:=\frac{10^{-3}}{(n+1)^{0.35}}$, $\beta_n:=0.5\times\frac{10^{-3}}{(n+1)^{0.35}}$ and $\lambda_n:=0.5$ $(\forall n\in\mathbb{N})$. It ensures that $(\alpha_n)_{n\in\mathbb{N}}$ and $(\beta_n)_{n\in\mathbb{N}}$ satisfy Assumption 3 . Algorithm 2 is initialized with the staring point $x_0=(p_0^s,p_0^o)^T$ randomly chosen in the range of $[0,100]\times[0,100]$. We set the number of iterations $N=10000,200000,500000$ as the stoping criterion. Now we will compare the computational performance with different number of subnetworks.

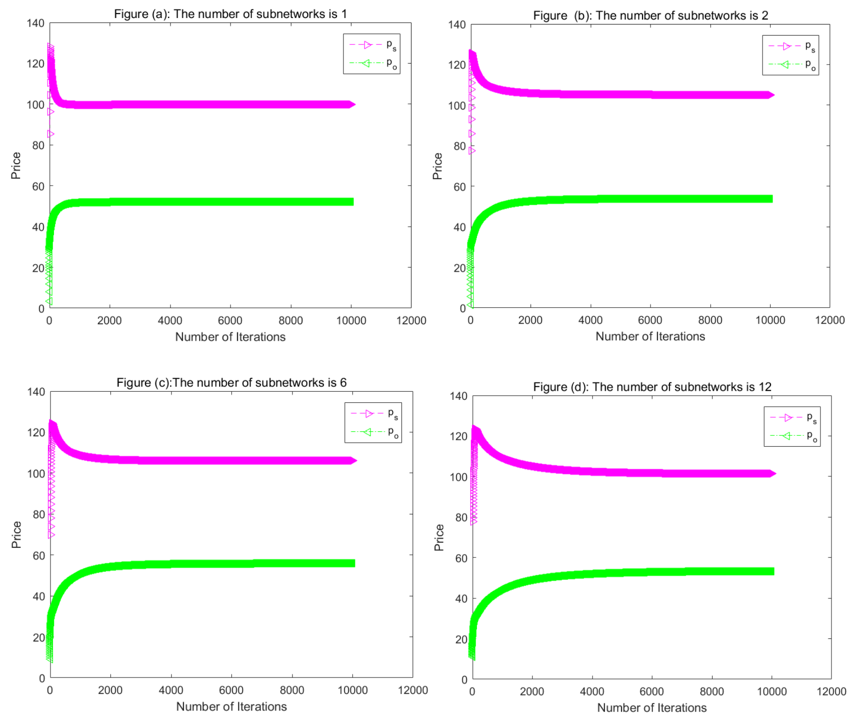

FIGURE 2. (a)-(b): Behaviors of p^s and p^o with the number of subnetworks $T = 1$, 2, 6, 12. The number of iterations is 10000.

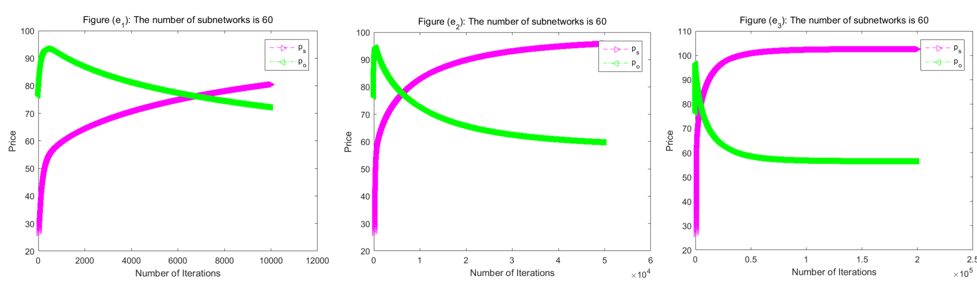

FIGURE 3. (e_1)-(e_3): Behaviors of p^s and p^o with the number of subnetworks $T = 60$. The number of iterations is 10000, 50000, 200000

Figures 2, 3, 4 describe the changing processes of p^o and p^s in the case of $T = 1, 2, 6, 12, 60, 120$, respectively. Those results indicate that Algorithm 2 with $T = 1, 2, 6, 12, 20, 60$ dramatically reduces the required number of iterations and enjoys a better rate of the convergence than Algorithm 2 with $T = 120$.

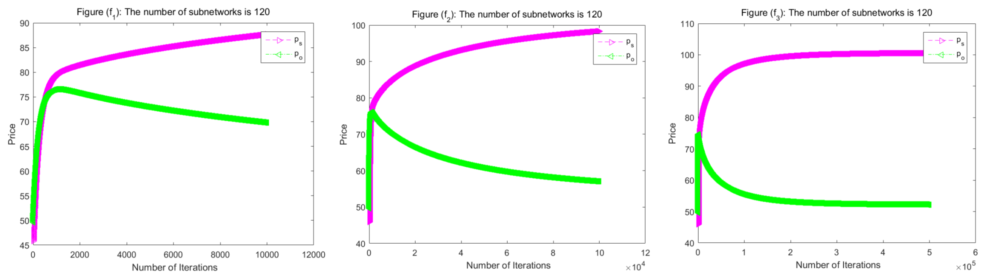

FIGURE 4. (f_1)-(f_3): Behaviors of p^s and p^o with the number of subnetworks $T = 120$. The number of iterations is 10000, 100000, 500000.

Hence, one concludes that Algorithm 2 has a better behavior with $T < 120$ than the conventional broadcast optimization algorithm (i.e., Algorithm 2 with $I = T = 120$). Meanwhile, these figures indicate the relation between the number of iterations and the number of subnetworks. As many peers as possible participate in subnetworks, the operator can quickly find the optimization solution to Problem 1 due to full cooperation from many peers. Note that it is not easy for all peers to implement the incremental optimization algorithm (i.e., Algorithm 2 with $I = 120, T = 1$), especially when the real networked system is composed of a number of subnetworks. As a consequence, our proposed algorithm is efficient to solve convex optimization problems in a large scale networked system.

Experiment 2. Consider a storage networked system with $I = 20$, $T = 10, 2$, $I_t = I/T$. Let us randomly choose the values in the range of $(0, 10]$ for $a^{(i)}$ and $b^{(i)}$. The lower bound $p_i^{\min}$ is a random value chosen in the range of $[10, 20]$. Meanwhile, the upper bound $p_i^{\max}$ is a random value chosen in the range of $[90, 100]$ $(i = 1, 2, \cdots, 20)$. We set $\alpha_n := \frac{10^{-4}}{(n+1)^{10^{-3}}}$, $\beta_n := \frac{10^{-4}}{(n+1)^{10^{-3}}}$ and $\lambda_n := 0.5$ $(\forall n \in \mathbb{N})$ for Algorithm 2. The initial points are generated randomly in the range of $[0, 100] \times [0, 100]$. Let us take the number of iterations $N = 10000$ as our stopping criterion.

We depict the changing process of (p_s, p_o) for Algorithm 2. As shown in Figure 5 (g_1), when the number of subnetworks is 2, the optimal $(p^s, p^o)^T$ is convergent to $(65.57, 42.20)^T$, user 0's revenue $\mathcal{U}^{(0)}(p^s, p^o)$ is approximately 85398.44. On the other hand, from Figure 5 (g_2), we have that when the number of subnetworks is 10, the optimal $(p^s, p^o)^T$ is convergent to $(55.70, 41.12)^T$. In this case, user 0's revenue $\mathcal{U}^{(0)}(p^s, p^o)$ has a approximate value 84228.42. Note that all the subfigures in Figure 5 in this experiment plot that the optimal p^o (denoted by $\hat{p^o}$) is smaller than the optimal p^s (denoted by $\hat{p^s}$). In this case, user 0's revenue $\mathcal{U}^{(0)}(\hat{p^s}, \hat{p^o})$ is nonnegative. In other words, user 0 makes a profit in this system.

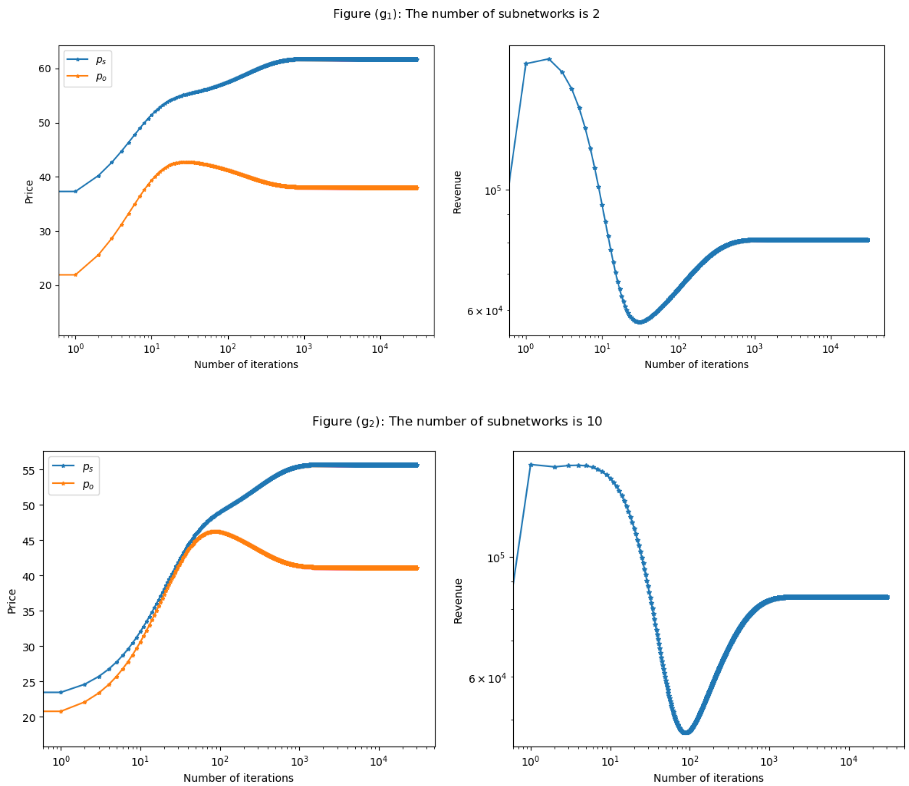

FIGURE 5. Behaviors of p^s and p^o (Left). Behaviors of $\mathcal{U}^{(0)}(p^s, p^o)$ (Right). The number of iterations is 30000.

5. Conclusion

In this paper, we discussed the minimization problem for the sum of objective functions over the intersection of fixed point sets of a family of nonexpansive mappings in a real Hilbert space. Our proposed iterative algorithms are devised by combining the broadcast distributed idea, the incremental optimization idea with the steepest descent method. We obtained the corresponding norm convergence results. Finally, we gave the numerical examples for the data storage allocation in a peer to peer data system, which demonstrates the computational effectiveness and convergence performance of our proposed algorithm.

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