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FIXED POINT THEOREMS WITHOUT CONTINUITY IN METRIC VECTOR SPACES

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Abstract. In this paper, we prove some fixed point theorems in metric vector spaces, in which the continuity is not required for the considered mappings and the underlying set is not necessarily compact. We provide some concrete examples to demonstrate these theorems. We also give some counterexamples to show that every condition in the theorems is necessary for the considered mapping to have a fixed point.
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1. Introduction

In fixed point theory, a topological vector space X is said to have the fixed point property if any continuous self-mapping on an arbitrary given nonempty compact and convex subset of X has a fixed point. In the concept of fixed point property, a certain type of continuity of the considered mapping is absolutely required (see [4], [6], [7]). In this paper, we prove a fixed point theorem in metric vector spaces. In this theorem, the continuity is not a necessary condition for the considered mapping to satisfy, which is replaced by a certain type of mapping convexity.

After K. Fan proved the so-called Fan-KKM theorem in 1961, it has been widely applied in fixed point theory, optimization theory, variational inequalities and other fields (see [5]). The theorems of this paper are also proved by using Fan-KKM theorem in two different ways. Since Fan-KKM Theorem has more than one different version, for easy reference, we briefly review the version of Fan-KKM Theorem used in this paper (see [1], [2], [3], [5]).

In [2], Aliprantis and Border introduced more general definitions of KKM mappings and they extended the Fan-KKM theorem to more general underlying spaces, which will help to generalize the results proved in this paper to more general cases. For example, the considered underlying set C in every theorem in this paper is assumed to be closed and convex. If we apply the results in [2], the closedness and convexity of C in the obtained results is not needed. By applying the results in [2], one may consider to generalize the results in this paper by assuming that the considered function f

takes values in the whole space X and the surjectivity condition is replaced with the assumption that C should be a subset of $f(C)$.

Let C be a nonempty convex subset of a vector space X. A set-valued mapping $G : C \to 2^X \setminus \{\emptyset\}$ is called a KKM mapping if, for any finite subset $\{x_1, x_2, \ldots, x_n\}$ of C, we have

$$co\{x_1, x_2, \ldots, x_n\} \subseteq \bigcup_{1\le i\le n} G(x_i),$$

where $co\{x_1, x_2, \ldots, x_n\}$ denotes the convex hull of $\{x_1, x_2, \ldots, x_n\}$.

Fan-KKM Theorem. *Let C be a nonempty closed convex subset of a Hausdorff topological vector space X and let $G : C \to 2^X \setminus \{\emptyset\}$ be a KKM mapping with closed values. If there exists a point $x^* \in C$ such that $G(x^*)$ is a compact subset of C, then*

$$\bigcap_{x\in C} G(x) \neq \emptyset.$$

2. The first fixed point theorem in this paper

2.1. The main theorem.

Theorem 2.1. *Let (X, d) be a metric vector space and let C be a nonempty closed and convex subset of X. Let $f : C \to C$ be a single-valued mapping. Suppose that f satisfies the following conditions:*

(a) $f : C \to C$ is onto.

(b_1) For each finite subset $\{x_1, x_2, \ldots, x_n\} \subseteq C$ and for any point u from its convex hull, we have

$$\max\{d(f(x_i), u) - d(x_i, u) : i = 1, 2, \ldots, n\} \geq 0.$$

(c_1) There is $x^* \in C$ such that the following subset of C is compact

$$\{y \in C : d(x^*, y) \leq d(f(x^*), y)\}.$$

Then f has a fixed point.

Proof. We define a set-valued mapping $G : C \to 2^C$ by

$$G(x) = \{y \in C : d(x, y) \leq d(f(x), y)\}, \text{ for every } x \in C.$$

For every $x \in C$ $G(x)$ is nonempty because $x \in G(x)$. $G(x)$ is a closed subset of C since, for every fixed $x \in C$, $d(f(x), \cdot) - d(x, \cdot)$ is a continuous function on C (on X).

Next, we show that G is a KKM mapping. To this end, for any finite subset $\{x_1, x_2, \ldots, x_n\} \subseteq C$, let u by an arbitrary point from its convex hull. That is, u is an arbitrary (linear) convex combination of $x_1, x_2, \ldots, x_n$. We can suppose that there are positive numbers $\alpha_1, \alpha_2, \ldots, \alpha_n$ with $\sum_{i=1}^{n} \alpha_i = 1$ such that

$$u = \sum_{i=1}^{n} \alpha_i x_i.$$

By condition (b_1) in this theorem, we have

$$\max\{d(f(x_j), u) - d(x_j, u) : j = 1, 2, \ldots, n\} \geq 0.$$

It follows that there must be an integer k with $1 \leq k \leq n$ such that

$$d(x_k, u) \leq d(f(x_k), u).$$

That is,

$$\sum_{i=1}^{n} \alpha_i x_i = u \in G(x_k) \subseteq \bigcup_{1 \leq i \leq j} G(x_j).$$

This implies that $G : C \to 2^C$ is a KKM mapping with nonempty closed values in C. By condition (c_1) and by using Fan-KKM Theorem, we obtain

$$\bigcap_{x \in C} G(x) \neq \emptyset.$$

Then, taking any $y_0 \in \bigcap_{x \in C} G(x)$, we have

$$d(x, y_0) \leq d(f(x), y_0), \text{ for every } x \in C. \tag{2.1}$$

By condition (a) in this theorem, for the arbitrarily selected $y_0 \in \bigcap_{x \in C} G(x) \subseteq C$ satisfying (2.1), there is $x_0 \in C$ such that $f(x_0) = y_0$. Substituting x_0 for x în (2.1) gives

$$d(x_0, f(x_0)) = d(x_0, y_0) \leq d(f(x_0), y_0) = d(y_0, y_0) = 0.$$

This implies that x_0 is a fixed point of f, which proves this theorem. □

Corollary 2.2. *Let (X, d) be a metric vector space and let C be a nonempty compact and convex subset of X. Let $f : C \to C$ be a single-valued mapping. If f satisfies conditions (a) and (b_1) in Theorem 2.1, then f has a fixed point.*

2.2. **Examples regarding to Theorem 2.1.** In the following examples, we always take $(X, d) = (\mathbb{R}, |\cdot|)$, let C be a closed interval of $\mathbb{R}$, and f to be a single-valued self-mapping on C.

Example 2.3. Let $C = [0, \infty)$ and the self-mapping f satisfies

(i) $f(0) = 12$;
(ii) $0 \leq f(x) \leq x$, for $0 < x \leq 6$ and $f(0, 6] = [0, 6]$;
(iii) $f(x) \geq x$, for $6 \leq x < \infty$ and $f[6, \infty) = [6, \infty)$.

Then

(I) f satisfies all conditions (a, b_1, c_1) in Theorem 2.1;
(II) f has at least one fixed point, $x = 6$.

The following graph shows an example of such a function.

Proof. Conditions (i-iii) in this example show that f satisfies condition (a) in Theorem 2.1. We next show that f satisfies conditions (b_1). For any finite subset $\{x_1, x_2, \ldots, x_n\} \subseteq C$, let u be a convex combination of $x_1, x_2, \ldots, x_n$ with $u = \sum_{i=1}^{n} \alpha_i x_i$, for some positive numbers $\alpha_1, \alpha_2, \ldots, \alpha_n$ with $\sum_{i=1}^{n} \alpha_i = 1$. Suppose $0 \leq x_1 < x_2 < \ldots < x_n < \infty$. It follows that $0 \leq x_1 < u < x_n < \infty$. Then the proof is divided into two cases:

Case 1. $0 \leq x_1 < u \leq 6$. By condition (ii), we have

$$0 < u - x_1 \leq u - f(x_1), \text{ for any } u \text{ with } 0 < x_1 < u \leq 6$$

and

$$0 < u - 0 = u \leq 12 - u = f(0) - u, \text{ for any } u \text{ with } 0 = x_1 < u \leq 6.$$

Case 2. $0 < u < x_n < \infty$. By condition (iii), we have

$$0 < x_n - u \leq f(x_n) - u, \text{ for any } u \text{ with } 6 < u < x_n < \infty.$$

Finally, we show that f satisfies condition (c_1) in Theorem 2.1. Take any point x^* with $x^* > 6$. Since $f(x^*) \geq x^*$, we have

$$\{y \in C : |x^* - y| \leq |f(x^*) - y|\} = \left[0, \frac{x^* + f(x^*)}{2}\right].$$

It is a compact subset in C. □

Example 2.4. Let $C = [0, \infty)$. Define two continuous functions φ, ψ on C as follows:

$$\varphi(x) = \begin{cases} \frac{1}{2}x, & \text{for} \quad x \in [0, 4]; \\ 3x - 10, & \text{for} \quad x \in (4, 6]; \\ \frac{3}{2}x - 1, & \text{for} \quad x \in (6, \infty), \end{cases}$$

and

$$\psi(x) = \begin{cases} \frac{3}{4}x, & \text{for} \quad x \in [0, 4]; \\ 2x - 5, & \text{for} \quad x \in (4, 6]; \\ \frac{5}{4}x - \frac{1}{2}, & \text{for} \quad x \in (6, \infty). \end{cases}$$

φ and ψ satisfy the following conditions:

(i) $0 < \varphi(x) < \psi(x) < x$, for $0 < x < 5$;

(ii) $\varphi(x) > \psi(x) > x$, for $5 < x < \infty$.

Then, we define $f : C \to C$ by

$$f(x) = \begin{cases} \varphi(x), & \text{for } 0 \leq x < \infty \text{ and } x \text{ is rational;} \\ \psi(x), & \text{for } 0 \leq x < \infty \text{ and } x \text{ is irrational.} \end{cases}$$

Then, we have

(I) f satisfies all conditions (a, b_1, c_1) in Theorem 2.1.

(II) f has two fixed points, $x = 0$ and $x = 5$.

(III) f is discontinuous at every point in $(0, \infty) \setminus \{5\}$.

Proof. Verifying of condition (a) in Theorem 2.1. $f(C)$ is the range of f. We have $\{0, 5\} \subseteq f(C)$. For any $b \in (0, \infty) \setminus \{5\}$, the horizontal line $y = b$ intersects each of the curves $y = \varphi(x)$ and $y = \psi(x)$ at exactly one point. Suppose that (x_1, b) and (x_2, b) are the intersections of the line $y = b$ and the curves $y = \varphi(x)$ and $y = \psi(x)$, respectively. Then if b is rational, both x_1 and x_2 are rational. It follows that $f(x_1) = \varphi(x_1) = b$. If b is irrational, then both x_1 and x_2 are irrational. It follows that $f(x_2) = \psi(x_2) = b$. This proves (a). Similarly to the proof of Example 2.3, we can show that f satisfies condition (b_1) in Theorem 2.1.

Finally we show that f satisfies condition (c_1) in Theorem 2.1. Take point $x^* = 6$. By $f(6) = 8$, we have

$$\{y \in C : |6 - y| \leq |f(6) - y|\} = \{y \in C : |6 - y| \leq |8 - y|\} = [0, 7].$$

It is a compact subset in C. □

2.3. **Counter examples regarding to Theorem 2.1.** Next, we give three counter examples to show that every condition in Theorem 2.1 is necessary for the considered mapping to have a fixed point.

Example 2.5. Let $C = [0, \infty)$ and let $f : C \to C$ be the function given in Example 2.4. Based on f, we define a function $g : C \to C$ by

$$g(x) = \begin{cases} f(x), & \text{for} \quad x \notin \{0, 5\}; \\ 10, & \text{for} \quad x = 0; \\ 8, & \text{for} \quad x = 5. \end{cases}$$

Then g satisfies condition (b_1, c_1) but not (a) in Theorem 2.1 and g has no fixed point.

Example 2.6. Let $C = [0, 10]$. Let $f : C \to C$ be given by

$$f(x) = \begin{cases} 0, & \text{for} x = 10; \\ \frac{4}{5}x, & \text{for} x \in (0, 5); \\ \frac{6}{5}x - 2, & \text{for} x \in [5, 10); \\ 10, & \text{for} x = 0. \end{cases}$$

Then f satisfies condition (a, c_1) but not (b_1) in Theorem 2.1 and f has no fixed point.

Proof. We only prove that f does not satisfy condition (b_1) in Theorem 2.1. Take $n = 2$ with $x_1 = 0$ and $x_2 > 5$. We can take x_2 very close to 10 such that there is u with $5 < u < x_2$ satisfying $u < f(x_2)$. Then we have

$$0 < u - x_1 = u > 10 - u = (0) - u$$

and

$$x_2 - u > f(x_2) - u > 0.$$

It shows that f does not satisfy condition (b_1) in Theorem 2.1. □

Example 2.7. Let $C = (-\infty, \infty)$ and define a linear function $f(x) = x - 1$. Then f satisfies conditions (a, b_1) but not (c_1) in Theorem 2.1 and f has no fixed point.

Proof. It is clear that f satisfies condition (a). Similar to the proof of Example 2.3, we can show that f satisfies condition (b_1) in Theorem 2.1. From the function $f(x) = x - 1$, we have

$$\begin{aligned} \{y \in C : |x - y| \leq |f(x) - y|\} &= \{y \in C : |x - y| \leq |x - 1 - y|\} \\ &= \left[x - \frac{1}{2}, \infty\right), \text{ for any } x \in (-\infty, \infty). \end{aligned}$$

This proves that f does not satisfy condition (c_1). □

3. The second fixed point theorem in this paper

The proof of the following theorem is similar to the proof of Theorem 2.1. Hence, we only give a proof sketch.

Theorem 3.1. *Let (X,d) be a metric vector space and let C be a nonempty closed and convex subset of X. Let $f : C \to C$ be a single-valued mapping. Suppose that f satisfies the following conditions:*

(a) $f : C \to C$ is onto;

(b_2) *For each finite subset $\{x_1, x_2, \ldots, x_n\} \subseteq C$ and for any point u from its convex hull, we have*

$$\max\{d(f(x_i),u) - d(f(x_i),x_i) : i = 1,2,\ldots,n\} \geq 0;$$

(c_2) *There is $x^* \in C$ such that the following subset of C is compact*

$$\{y \in C : d(f(x^*),x^*) \leq d(f(x^*),y)\}.$$

Then f has a fixed point.

Proof. We define a set-valued mapping $G : C \to 2^C$ by

$$G(x) = \{y \in C : d(f(x),x) \leq d(f(x),y)\}, \text{ for every } x \in C.$$

Similarly to the proof of Theorem 2.1, by condition (b_2), we can show that $G : C \to 2^C$ is a KKM mapping with nonempty closed values on C. By condition (c_2) and by using Fan-KKM Theorem, we obtain $\bigcap_{x\in C} G(x) \neq \emptyset$. Then, taking any $y_0 \in \bigcap_{x\in C} G(x)$, we have

$$d(f(x),x) \leq d(f(x),y_0), \text{ for every } x \in C. \tag{3.1}$$

By condition (a) in this theorem, for the arbitrarily selected $y_0 \in \bigcap_{x\in C} G(x) \subseteq C$ satisfying (3.1), there is $x_0 \in C$ such that $f(x_0) = y_0$. Substituting x_0 for x in (3.1) gives

$$d(f(x_0),x_0) \leq d(f(x_0),y_0) = d(y_0,y_0) = 0.$$

This implies that x_0 is a fixed point of f, which proves this theorem. □

Corollary 3.2. *Let (X,d) be a metric vector space and let C be a nonempty compact and convex subset of X. Let $f : C \to C$ be a single-valued mapping. If f satisfies conditions (a) and (b_2) in Theorem 3.1, then f has a fixed point.*

Similar to the examples regarding to Theorem 2.1, we give three examples below for Corollary 3.2. The readers interested in these topics can construct more examples and counter examples to demonstrate Theorem 3.1 and Corollary 3.2.

Example 3.3. Let $C = [0,10]$. Let $f : C \to C$ be defined with $f(C) = C$ and

$$f(x) \leq x, \text{ for all } x \in C.$$

f satisfies conditions (a, b_2) in Corollary 3.2 has at least two fixed points, $x = 0$ and $x = 10$.

Proof. $f(C) = C$ is assumed, which proves (a). Next, we prove that f satisfies condition (b_2). For any finite subset $\{x_1, x_2, \dots, x_n\} \subseteq C$, let u be a convex combination of $x_1, x_2, \dots, x_n$ with $u = \sum_{i=1}^{n} \alpha_i x_i$, for some positive numbers $\alpha_1, \alpha_2, \dots, \alpha_n$ with $\sum_{i=1}^{n} \alpha_i = 1$. Suppose $0 \le x_1 < u < x_n < 10$. Then, we have

$$0 \le x_1 - f(x_1) < u - f(x_1), \text{ for any } y \text{ with } 0 \le x_1 < u < 10. \qquad \square$$

Similar to Example 3.3, we have

Example 3.4. Let $C = [0, 10]$. Let $f : C \to C$ be defined with $f(C) = C$ and

$$f(x) \ge x, \text{ for all } x \in C.$$

Then, f satisfies conditions (a) and (b_2) in Corollary 3.2 and f has at least two fixed points, $x = 0$ and $x = 10$.

Proof. From the proof of Example 3.3, we have

$$0 \le f(x_n) - x_n < (x_n) - u, \text{ for any } u \text{ with } 0 < u < x_n \le 10. \qquad \square$$

As a special case of the above Example 3.3, we have the following example.

Example 3.5. Let $C = [0, 10]$. Define two continuous functions φ, ψ on [0, 10] as follows:

$$\varphi(x) = \begin{cases} \frac{3}{5}x, & \text{for} \quad x \in [0,5]; \\ \frac{7}{5}x - 4, & \text{for} \quad x \in [5,10] \end{cases}$$

and

$$\psi(x) = \begin{cases} \frac{2}{5}x, & \text{for} \quad x \in [0,5]; \\ \frac{8}{5}x - 6, & \text{for} \quad x \in [5,10]. \end{cases}$$

Based on the above given continuous functions φ, ψ, we define a function $f : C \to C$ by

$$f(x) = \begin{cases} \varphi(x), & \text{for } 0 \le x \le 10 \text{ and } x \text{ is rational}; \\ \psi(x), & \text{for } 0 < x < 10 \text{ and } x \text{ is irrational}. \end{cases}$$

Then, we have
(I) f satisfies conditions (a, b_2) in Corollary 3.2;
(II) f has two fixed points, $x = 0$ and $x = 10$;
(III) f is discontinuous at every point in $(0, 10)$.

Proof. The proofs of (a) and (b_2) in this example are similar to the proofs of (a) in Example 2.4 and the proof of (b_1) in Example 3.3, respectively. $\square$

4. The third fixed point theorem in this paper

4.1. **The theorem.** In this subsection, we use the Fan-KKM theorem to prove the third fixed point theorem in this paper. The way, the proof is different from the proofs of Theorems 2.1 and 3.1, is said to be "a proof by indirectly applying the Fan-KKM theorem". For a self-mapping f on a given set, let $\mathcal{F}(f)$ denote the collection of fixed point(s) of f.

Theorem 4.1. *Let (X, d) be a metric vector space and let C be a nonempty compact and convex subset of X. Let $f : C \to C$ be a single-valued mapping. Suppose that f satisfies the following conditions:*

(a) $f : C \to C$ is onto;

(b_3) For each finite subset $\{x_1, x_2, \ldots, x_n\} \subseteq C$ and for any point u from its convex hull, we have

$$d(f(u), u) \leq \max\{d(f(x_i), u) : i = 1, 2, \ldots, n\};$$

(c_3) $\{x \in C : d(f(x), x) \leq \beta\}$ is closed, for any $\beta > 0$.

Then $\mathcal{F}(f)$ is a nonempty closed subset of C.

Proof. $\mathcal{F}(f)$ is precisely defined by $\mathcal{F}(f) = \{x \in C : d(f(x), x) = 0\}$. Assume, by contradiction, that

$$\mathcal{F}(f) = \emptyset. \tag{4.1}$$

Under the assumption (3), we show that there is $\delta > 0$ such that

$$\{x \in C : d(f(x), x) \leq \delta\} = \emptyset. \tag{4.2}$$

Assume again, by contradiction, that (4.2) does not hold for any $\delta > 0$. That is,

$$\{x \in C : d(f(x), x) \leq \delta\} \neq \emptyset, \text{ for any } \delta > 0. \tag{4.3}$$

Let

$$E_m = \left\{x \in C : d(f(x), x) \leq \frac{1}{m}\right\}, \text{ for } m = 1, 2, \ldots$$

By the assumption (4.3) and condition (c_3) in this theorem, E_m is a nonempty closed subset of C, for $m = 1, 2, \ldots$. Then $\{E_m\}$ is a decreasing (with respect to the inclusion ordering) sequence of nonempty closed subsets of the given compact subset C of X. It follows that

$$\bigcap_{m=1}^{\infty} E_m \neq \emptyset.$$

Since

$$\mathcal{F}(f) = \bigcap_{m=1}^{\infty} E_m, \tag{4.4}$$

it yields a contradiction to the assumption (4.1). Then (4.2) is proved. Hence, there is $\delta > 0$ that satisfies (4.2). Then we take $\frac{\delta}{2} > 0$ such that

$$d(f(x), x) \geq \frac{\delta}{2}, \text{ for every } x \in C. \tag{4.5}$$

With respect to this fixed $\frac{\delta}{2} > 0$ given in (4.5), we define a set-valued mapping $G : C \to 2^C$ by

$$G(x) = \left\{ y \in C : d(f(x), y) \geq \frac{\delta}{2} \right\}, \text{ for every } x \in C.$$

By the assumption (4.5), we have $x \in G(x)$, for every $x \in C$. It follows that

$$C \supseteq G(x) \neq \emptyset, \text{ for every } x \in C.$$

Since, for every fixed $x \in C$, $d(f(x), \cdot)$ is a continuous function on C (on X), it follows that, for every $x \in C$, $G(x)$ is a nonempty closed subset of C.

Next, we show that G is a KKM mapping. To this end, for any finite subset $\{x_1, x_2, \ldots, x_n\} \subseteq C$, let u by an arbitrary point from its convex hull. That is, u is an arbitrary (linear) convex combination of $x_1, x_2, \ldots, x_n$. We can suppose that there are positive numbers $\alpha_1, \alpha_2, \ldots, \alpha_n$ with $\sum_{i=1}^{n} \alpha_i = 1$ such that $u = \sum_{i=1}^{n} \alpha_i x_i$. By condition (b_3) in this theorem, we have

$$\begin{aligned} d(f(u), u) &= d\left(f\left(\sum_{i=1}^{n} \alpha_i x_i \right), \sum_{i=1}^{n} \alpha_i x_i \right) \\ &\leq \max\left\{ d(f(x_j), \sum_{i=1}^{n} \alpha_i x_i) : j = 1, 2, \ldots, n \right\}. \end{aligned}$$

This implies that there must be an integer k with $1 \leq k \leq n$ such that

$$d(f(u), u) = d\left(f\left(\sum_{i=1}^{n} \alpha_i x_i \right), \sum_{i=1}^{n} \alpha_i x_i \right) \leq d\left(f(x_k), \sum_{i=1}^{n} \alpha_i x_i \right).$$

By (4.5), it follows that

$$d\left(f(x_k), \sum_{i=1}^{n} \alpha_i x_i \right) \geq \frac{\delta}{2}.$$

That is,

$$u = \sum_{i=1}^{n} \alpha_i x_i \in G(x_k) \subseteq \bigcup_{1 \leq j \leq n} G(x_j).$$

This implies that $G : C \to 2^C$ is a KKM mapping with nonempty closed values in C. Since C is compact, by Fan-KKM Theorem, we obtain $\bigcap_{x \in C} G(x) \neq \emptyset$. Taking any $y_0 \in \bigcap_{x \in C} G(x)$, we have

$$d(f(x), y_0) \geq \frac{\delta}{2}, \text{ for every } x \in C. \tag{4.6}$$

By condition (a) in this theorem, for the selected $y_0 \in C$ satisfying (4.6), there is $x_0 in C$ such that $f(x_0) = y_0$. Substituting x_0 for x in (4.6) gives

$$d(y_0, y_0) = d(f(x_0), y_0) \geq \frac{\delta}{2} > 0.$$

It is a contradiction. Hence (4.1) does not hold. That is, we must have $\mathcal{F}(f) \neq \emptyset$. By (4.4), $\mathcal{F}(f)$ is the intersection of a decreasing sequence of nonempty closed subsets of C. It follows that $\mathcal{F}(f)$ is closed. This proves this theorem. □

Remark 4.2. The following (stronger) condition implies condition (b_3) in Theorem 4.1

$$d(f(u), u) \leq \sum_{j=1}^{n} \alpha_j d(f(x_j), u).$$

4.2. **Examples regarding to Theorem 4.1.** In the following examples, we take $C = [0, 10]$. Let f be a single-valued self-mapping on C.

Example 4.3.

$$f(x) = \begin{cases} -\frac{5}{3}x + 10, & \text{for} \quad 0 \leq x < 3; \\ 1, & \text{for} \quad x = 3; \\ 5, & \text{for} \quad 3 < x < 7; \\ 9, & \text{for} \quad x = 7; \\ -\frac{5}{3}x + \frac{50}{3}, & \text{for} \quad 7 < x \leq 10. \end{cases}$$

Then we have
(i) f satisfies all conditions in Theorem 4.1;
(ii) f has one fixed point, $x = 5$;
(iii) f is neither lower semi-continuous, nor upper semi-continuous.

Proof. It is straightforward to check that f satisfies conditions (a, c_3) in Theorem 4.1, and therefore it is omitted here. We only show that f satisfies conditions (b_3). For any finite subset $\{x_1, x_2, \ldots, x_n\} \subseteq C$, let u be a convex combination of $x_1, x_2, \ldots, x_n$ with $u = \sum_{i=1}^{n} \alpha_i x_i$ for positive numbers $\alpha_1, \alpha_2, \ldots, \alpha_n$ with $\sum_{i=1}^{n} \alpha_i = 1$. Suppose $0 \leq x_1 < x_2 < \ldots < x_n \leq 10$. So we have $0 \leq x_1 < u < x_n \leq 10$. Then, we have

$$|f(u) - u| \leq |f(x_1) - u|, \text{ for } x_1 < u \leq 5$$

and

$$|f(u) - u| \leq |f(x_n) - u|, \text{ for } 5 < u < x_n.$$ □

Similar to Example 4.3, we construct more complicated examples. The proofs of the following two examples are similar to the proof of Example 4.3 and are omitted.

Example 4.4. Example 10.

$$f(x) = \begin{cases} -\frac{5}{3}x + 10, & \text{for } 0 \leq x < 3; \\ 2x - 5, & \text{for } 3 \leq x \leq 7 \text{ and } x \text{ is rational;} \\ 5, & \text{for } 3 < x < 7 \text{ and } x \text{ is irrationa;} \\ -\frac{5}{3}x + \frac{50}{3}, & \text{for } 7 < x \leq 10. \end{cases}$$

Then
(i) f satisfies all conditions in Theorem 4.1 and f has one fixed point, $x = 5$;
(ii) f is continuous on $[0, 3) \cup (7, 10] \cup \{5\}$;
(iii) f is discontinuous at every point in $[3, 7] \setminus \{5\}$;
(iv) f is neither lower semi-continuous, nor upper semi-continuous on C.

Example 4.5.

$$f(x) = \begin{cases} -\frac{5}{3}x + 10, & \text{for } 0 \leq x < 3; \\ 2x - 5, & \text{for } 3 \leq x \leq 7; \\ -\frac{5}{3}x + \frac{50}{3}, & \text{for } 7 < x \leq 10. \end{cases}$$

Then
(i) f satisfies all conditions in Theorem 4.1 and f has one fixed point, $x = 5$;
(ii) f is continuous on $[0, 10] \setminus \{3, 7\}$'
(iii) f is neither lower semi-continuous, nor upper semi-continuous on C.

4.3. **Counter examples regarding to Theorem 4.1.** Now we provide three counterexamples to show that every condition in Theorem 4.1 is necessary for the considered mapping to have a fixed point.

Example 4.6.

$$f(x) = \begin{cases} 6, & \text{for } \quad x \in [0, 5); \\ 4, & \text{for } \quad x \in [5, 10). \end{cases}$$

Then f satisfies conditions (b_3) and (c_3) but not (a) in Theorem 4.1 and f has no fixed point.

Proof. The proof is straightforward and it is omitted here. □

Example 4.7.

$$f(x) = \begin{cases} 0, & \text{for } \quad x = 10; \\ \frac{4}{5}x, & \text{for } \quad x \in (0, 5); \\ \frac{6}{5}x - 2, & \text{for } \quad x \in [5, 10); \\ 10, & \text{for } \quad x = 0. \end{cases}$$

Then f satisfies conditions (a) and (b_3) but not (c_3) in Theorem 4.1 and f has no fixed point.

Proof. It is clear to see that f satisfies condition (a). We next show that f satisfies conditions (c). For any finite subset $\{x_1, x_2, \ldots, x_n)\} \subseteq C$, let u be an arbitrary convex combination of $x_1, x_2, \ldots, x_n$ with $u = \sum_{i=1}^{n} \alpha_i x_i$, for some positive numbers $\alpha_1, \alpha_2, \ldots, \alpha_n$ with $\sum_{i=1}^{n} \alpha_i = 1$. Suppose $0 \le x_1 < x_2 < \ldots < x_n \le 10$. It implies that $0 \le x_1 < u < x_n \le 10$. Then, we calculate

$$0 < u - f(u) < u - f(x_1), \text{ for any } 0 < x_1 < u < x_n \le 10$$

and

$$0 < u - f(u) < 10 - u, \text{ for any } 0 = x_1 < u < x_n \le 10.$$

It shows that f satisfies condition (b_3). Since $d(f(0), 0) = 10$, it implies that $\{x \in C : d(f(x), x) \le \beta\}$ is not closed, for any β with $0 < \beta < 1$. condition (c_3) is not satisfied. f does not have a fixed point. □

Example 4.8.

$$f(x) = \begin{cases} -\frac{5}{3}x + 10, & \text{for} \quad x \in [0, 3]; \\ x + 2, & \text{for} \quad x \in (3, 5); \\ x - 2, & \text{for} \quad x \in [5, 7]; \\ -\frac{5}{3}x + \frac{50}{3}, & \text{for} \quad x \in (7, 10]. \end{cases}$$

Then f satisfies conditions (a) and (c_3) but not (b_3) in Theorem 4.1 and f has no fixed point.

Proof. It is clear to see that f satisfies conditions (a) and (c_3). Take $x_1 = 3$, $x_2 = 7$ and let $u = 5$. We have $f(x_1) - 5 = f(x_2) - 5 = 0$, and $f(u) - 5 = 2$. So, condition (b_3) is not satisfied. □

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