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THE BANACH SPACE c_0 AND ITS ROLE AMONG EXTREMAL SPACES

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Dedicated to the Memory of Professor Kazimierz Goebel

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Abstract. We present a unified approach to describe a possibly wide class of separable Banach spaces which are extremal with respect to the minimal displacement of k-Lipschitz self-maps of the closed unit ball. The prominent member of this class, which plays a central role in our considerations, is the Banach space c_0 of real sequences converging to 0, provided with the maximum norm. Indeed, we show that if a separable Banach space X contains an isomorphic (resp. isometric) copy of c_0, then X as well as all subspaces of X of finite codimension are extremal (resp. strictly extremal). Our result encompasses and significantly extends a collection of all known examples of separable Banach spaces which are extremal (resp. strictly extremal).
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1. Introduction

Suppose C is a bounded, closed and convex subset of an infinite-dimensional real Banach space X. If C is compact, then every continuous self-map T of C has a fixed point by the famous Schauder's theorem [20]. However, in noncompact setting it is always possible to construct a continuous map $T : C \to C$ without a fixed point. The general construction of such a map, which works in any space X, is due to Klee [15]. There are many various examples of uniformly continuous, k-Lipschitz, and even nonexpansive maps $T : C \to C$ without fixed points. It is usually the case that the *minimal displacement* of maps of the above type is zero, that is,

$$d(T) = \inf \{\|x - Tx\| : x \in C\} = 0.$$

This property is natural for certain classes of maps. For example, all nonexpansive self-maps of bounded, closed and convex sets have this feature. This is no longer the case if we consider a class of k-Lipschitz maps with $k > 1$.

Goebel [9] was probably the first to observe that there are k-Lipschitz maps $T : C \to C$ (in short, $T \in \mathcal{L}(k)$) having a positive minimal displacement. In the quoted paper, he proves the basic inequality

$$d(T) \leq \left(1 - \frac{1}{k}\right) r(C),$$

where $r(C)$ denotes the Chebyshev radius of C, i.e., $r(C) = \inf_{x \in C} \sup_{y \in C} \|x - y\|$, and gives examples of sets and maps for which the above estimate is exact. By following [9], we define the *minimal displacement characteristic* of the set C as

$$\varphi_C(k) = \sup \left\{ d(T) \mid T : C \to C,\ T \in \mathcal{L}(k) \right\}, \quad k \geq 1.$$

The *minimal displacement problem* involves finding or evaluating the function φ for concrete sets. One of the strongest result in this matter was obtained in 1985 by Lin and Sternfeld [16]: given $k > 1$ and a bounded, closed, convex but noncompact subset C of a Banach space X, one can always construct $T : C \to C$ such that $T \in \mathcal{L}(k)$ and $d(T) > 0$. In other words, $\varphi_C(k) > 0$ for every $k > 1$. However, there are still many quantitative aspects of the theory left.

Throughout we will be concerned in the most important case $C = B_X$, where B_X denotes the closed unit ball in X. In this special case, we will denote the minimal displacement characteristic of B_X by ψ_X, that is,

$$\psi_X(k) = \sup \left\{ d(T) \mid T : B_X \to B_X,\ T \in \mathcal{L}(k) \right\}, \quad k \geq 1.$$

Hence, for every space X,

$$\psi_X(k) \leq 1 - \frac{1}{k}.$$

For some spaces the above estimate is exact, that is, for every $k \geq 1$

$$\psi_X(k) = 1 - \frac{1}{k}.$$

We call such spaces *extremal*. Moreover, we say that X is *strictly extremal* if for every $k \geq 1$ there exists a map $T : B_X \to B_X$ such that $T \in \mathcal{L}(k)$ and $\|x - Tx\| > 1 - 1/k$ for every $x \in B_X$.

It is well-known that the space c_0 is strictly extremal (see Example 20.2 in [11] or Example 12.4 in [10]). Since this space will play a key role in our approach, we now recall the rationale for this.

Example 1.1. Let $\alpha : \mathbb{R} \to [-1, 1]$ be defined as

$$\alpha(t) = \begin{cases} -1 & \text{if } t \leq -1, \\ t & \text{if } t \in [-1, 1], \\ 1 & \text{if } t \geq 1. \end{cases}$$

Fix $k \geq 1$. Let $T : B_{c_0} \to B_{c_0}$ be defined for every $x = (x(1), x(2), \dots) \in B_{c_0}$ by

$$T(x(1), x(2), \dots) = (1, \alpha(kx(1)), \alpha(kx(2)), \dots).$$

Since $\alpha \in \mathcal{L}(1)$, it follows that $T \in \mathcal{L}(k)$, that is,

$$\|Tx - Ty\| \leq k \|x - y\|$$

for any $x, y \in B_{c_0}$. Moreover,

$$\|x - Tx\| > 1 - \frac{1}{k}$$

for any $x \in B_{c_0}$. Indeed, the opposite inequality $\|x - Tx\| \leq 1 - 1/k$ implies that $x(i) \geq 1/k$ for every $i \in \mathbb{N}$, contradicting the fact that $x \in c_0$. Consequently, the space c_0 is strictly extremal.

In what follows, by a subspace of a given Banach space X we always mean a closed linear subspace.

Below we provide currently known examples of strictly extremal spaces, pointing out that they are only a special case of our results. Let us start with the following:

- the Banach space $C([0,1])$ of all continuous real-valued functions on $[0,1]$, endowed with the maximum norm (see Example 20.1 in [11] or Example 12.3 in [10]);
- the Banach space c of convergent sequences, provided with the supremum norm (see Claim 1 in [17]);
- the space of differentiable functions $C^{(n)}([0,1])$ provided with the norm $\| f \| = \sum_{i=0}^{n-1} |f^{(i)}(0)| + \max_{t\in[0,1]} |f^{(n)}(t)|$ (see [2]);
- every separable infinite-dimensional L_1-predual (see Theorem 2.5 in [4]), in particular, every separable space $C_0(K)$, where K is a locally compact Hausdorff space and $C_0(K)$ denotes the Banach space of all continuous real-valued functions vanishing at infinity on K, furnished with the maximum norm; we recall that a function $f : K \to \mathbb{R}$ is said to vanish at infinity on K if for every $\varepsilon > 0$ the set $\{x \in K : |f(x)| \geq \varepsilon\}$ is compact.

It is well-known that each of the above mentioned space contains an isometric copy of c_0; here we only recall an old result by Zippin [21] stating that every infinite-dimensional L_1-predual contains an isometric copy of c_0. Therefore, all the results mentioned above follow from Theorem 3.1.

It is also known that

- every subspace of $C([0,1])$ of finite codimension is strictly extremal (see [3]).

This result, in turn, is a consequence of Corollary 3.4.

Finally,

- the spaces of differentiable functions $C^{(n)}([0,1])$ provided with the norm $\| f \| = \sum_{i=0}^{n} \max_{t\in[0,1]} |f^{(n)}(t)|$ are extremal (see [2]);
- the space $C([0,1])$ renormed to be uniformly convex in every direction is extremal (see [1]).

Observe that the above space (in fact, with any equivalent norm) contains an isomorphic copy of c_0. Therefore, its extremality follows directly from Theorem 2.2.

To present the landscape surrounding the issue under discussion, we recall that a Hilbert space H is not extremal and we have:

$$1 - \frac{2\sqrt{\sqrt{2}(k+1)}}{k} \leq \psi_H(k) \leq \left(1 - \frac{1}{k}\right)\sqrt{\frac{k}{k+1}}$$

(see [9] and [5]). More generally, uniformly convex spaces are not extremal (see Theorem 20.2 in [11]). It should be emphasized here that the space ℓ_1 is also not extremal and we have (see Example 12.5 in [10]):

$$\psi_{\ell_1}(k) \leq \begin{cases} \frac{2+\sqrt{3}}{4}\left(1-\frac{1}{k}\right) & \text{for } 1 \leq k \leq 3+2\sqrt{3}, \\ \frac{k+1}{k+3} & \text{for } k > 3+2\sqrt{3}. \end{cases}$$

The reader interested in the minimal displacement and related problems is referred to the books [10], [11], [18] and papers [6], [12], [13].

We use the standard notation and terminology of Banach space theory. In particular, if $A \subset X$, then $[A]$ stands for the closed linear span of A. The dual of X is denoted by X^*.

2. Spaces containing an isomorphic copy of c_0

A classical James's Distortion Theorem for c_0 states that whenever a Banach space contains a subspace isomorphic to c_0, then, for every $\varepsilon > 0$, it contains a subspace that is $(1+\varepsilon)$-isomorphic to c_0 (see [14]). In some cases one can even find $(1+\varepsilon)$-isomorphic copies of c_0 which are $(1+\varepsilon)$-complemented. Indeed, Dowling, Randrianantoanina, and Turett [8] proved the following result:

Theorem 2.1 (Theorem 6 in [8])**.** *Let X be a Banach space whose closed unit ball in X^* is weak* sequentially compact and let $\varepsilon > 0$. If X contains a subspace isomorphic to c_0, then there exists a subspace Z of X and a projection P from X onto Z such that Z is $(1+\varepsilon)$-isomorphic to c_0 and $\|P\| \leqslant 1+\varepsilon$. Moreover, if X contains a subspace isometric to c_0, then there exists a subspace Z of X such that Z is isometric to c_0 and $\|P\| = 1$.*

It is worth noting that the above theorem can be also concluded from the proof of 1969 Zippin's result. Recall that Zippin [21] proved that every infinite-dimensional L_1-predual X contains an isometric copy of c_0. Moreover, he showed that if we additionally assume that X is separable, then it contains an isometric copy of c_0 which is 1-complemented in X. This part of Zippin's proof can be generalized in a natural way to the case of spaces satisfying the assumptions of Theorem 2.1. As a result, we obtain exactly the same construction as in [8]. For completeness, we include the proof of the case when X contains an isometric copy of c_0.

Proof. Since X contains an isometric copy of c_0, there exists a sequence $(x_n)_{n\in\mathbb{N}}$ in X such that

$$\left\|\sum_{n=1}^{\infty} \lambda_n x_n\right\| = \max_{n\in\mathbb{N}} |\lambda_n|$$

for all $(\lambda_n) \in c_0$. Put $Y = [\{x_n : n \in \mathbb{N}\}]$ and define $x_n^* : Y \to \mathbb{R}$ by

$$x_n^*\left(\sum_{i=1}^{\infty} \lambda_i x_i\right) = \lambda_n.$$

Then $x_n^* \in Y^*$ and $\|x_n^*\| = 1$ for every $n \in \mathbb{N}$. Let y_n^*'s be norm one extensions of x_n^*'s to the whole space X. By our assumption, we can choose a weak* convergent subsequence $\left(y^*_{n_k}\right)_{k\in\mathbb{N}}$ of $(y^*_n)_{n\in\mathbb{N}}$. Let $Z = \left[\left\{x_{n_{2k}} - x_{n_{2k-1}} : k \in \mathbb{N}\right\}\right]$. Then Z is isometric to c_0. Moreover, it is 1-complemented in X. Indeed, the formula

$$Px = \frac{1}{2}\sum_{k=1}^{\infty}\left(\left(y^*_{n_{2k}}(x) - y^*_{n_{2k-1}}(x)\right)x_{n_{2k}} + \left(y^*_{n_{2k-1}}(x) - y^*_{n_{2k}}(x)\right)x_{n_{2k-1}}\right)$$

defines a linear projection from X onto Z such that $\|P\| = 1$. □

By following the above reasoning in tandem with the aforementioned James's Distortion Theorem for c_0, we get the first part of Theorem 2.1.

We are now in a position to prove the following

Theorem 2.2. *Let X be a Banach space whose closed unit ball in X^* is weak* sequentially compact. If X contains an isomorphic copy of c_0, then X is extremal, that is,*

$$\psi_X(k) = 1 - \frac{1}{k}$$

for every $k \geq 1$.

Proof. Fix $\varepsilon > 0$. By Theorem 2.1, there exists a subspace Z of X which is $(1+\varepsilon)$-isomorphic to c_0 and $(1+\varepsilon)$-complemented in X. Let P denote a projection from X onto Z with $\|P\| \leq 1+\varepsilon$ and let $\phi : Z \to c_0$ be an isomorphism such that for every $x \in Z$ we have

$$\|x\| \leq \|\phi x\| \leq (1+\varepsilon)\|x\|.$$

Choose any $k \geq 1$. There exists a map $T : B_{c_0} \to B_{c_0}$ such that $T \in \mathcal{L}(k)$ and $\|x - Tx\| > 1 - 1/k$ for every $x \in B_{c_0}$ (see Example 1.1). Let $\hat{T} : (1+\varepsilon)B_{c_0} \to (1+\varepsilon)B_{c_0}$ be given by

$$\hat{T}x = (1+\varepsilon)T\left(\frac{1}{1+\varepsilon}x\right).$$

Obviously, $\hat{T} \in \mathcal{L}(k)$. Moreover, for any $x \in (1+\varepsilon)B_{c_0}$,

$$\left\|\hat{T}x - x\right\| = (1+\varepsilon)\left\|T\left(\frac{1}{1+\varepsilon}x\right) - \frac{1}{1+\varepsilon}x\right\| > (1+\varepsilon)\left(1 - \frac{1}{k}\right).$$

Consider now a map $S : (1+\varepsilon)B_Z \to (1+\varepsilon)B_Z$ defined by

$$Sx = \left(\phi^{-1} \circ \hat{T} \circ \phi\right)\left(\frac{1}{1+\varepsilon}x\right).$$

Then, $S \in \mathcal{L}(k)$ and for any $x \in (1+\varepsilon)B_Z$ we have

$$\begin{aligned}
\|Sx - x\| &= \left\| \phi^{-1}\hat{T}\phi\left(\frac{1}{1+\varepsilon}x\right) - \frac{1}{1+\varepsilon}x - \frac{\varepsilon}{1+\varepsilon}x \right\| \\
&\geq \left\| \phi^{-1}\hat{T}\phi\left(\frac{1}{1+\varepsilon}x\right) - \frac{1}{1+\varepsilon}x \right\| - \varepsilon \\
&\geq \frac{1}{1+\varepsilon}\left\| \hat{T}\phi\left(\frac{1}{1+\varepsilon}x\right) - \phi\left(\frac{1}{1+\varepsilon}x\right) \right\| - \varepsilon \\
&> 1 - \frac{1}{k} - \varepsilon.
\end{aligned}$$

Finally, let $\hat{S} : B_X \to B_X$ be given by

$$\hat{S}x = \frac{1}{1+\varepsilon}SPx.$$

Clearly, $\hat{S} \in \mathcal{L}(k)$. Moreover, for any $x \in B_X$,

$$\begin{aligned}
\left\| \hat{S}x - x \right\| &= \left\| \frac{1}{1+\varepsilon}SPx - \frac{1}{1+\varepsilon}x - \frac{\varepsilon}{1+\varepsilon}x \right\| \\
&\geq \frac{1}{1+\varepsilon}\|SPx - x\| - \frac{\varepsilon}{1+\varepsilon} \\
&\geq \frac{1}{(1+\varepsilon)^2}\|SPx - Px\| - \frac{\varepsilon}{1+\varepsilon} \\
&> \frac{1}{(1+\varepsilon)^2}\left(1 - \frac{1}{k} - \varepsilon\right) - \frac{\varepsilon}{1+\varepsilon}.
\end{aligned}$$

Letting $\varepsilon \to 0$ we conclude that X is an extremal space. □

Remark 2.3. The above result significantly improves Proposition 2.4 in [4], which states that

$$\psi_X(k) \geq \frac{1}{2} - \frac{1}{k}, \quad k \geq 1,$$

for every separable Banach space X containing an isometric copy of c_0.

It is a known fact that any infinite-dimensional subspace of c_0 contains a further subspace that is isomorphic to c_0 (see e.g. Theorem 6 in [7], where a stronger result is proved). Hence, by applying Theorem 2.2, we obtain the following

Corollary 2.4. *Let X be a Banach space whose closed unit ball in X^* is weak* sequentially compact. If X contains an isomorphic copy of c_0, then every subspace of X of finite codimension is extremal.*

3. Spaces containing an isometric copy of c_0

In this section we investigate in more details the case when a Banach space contains an isometric copy of c_0.

Theorem 3.1. *Let X be a Banach space whose closed unit ball in X^* is weak* sequentially compact. If X contains an isometric copy of c_0, then X is strictly extremal.*

Proof. It is enough to follow the proof of Theorem 2.2 with $\varepsilon = 0$. □

Perhaps the following lemma is known, but we were not able to find a suitable reference.

Lemma 3.2. *Every subspace of c_0 of finite codimension contains an isometric copy of c_0.*

Proof. For $x = (x(1), x(2), \dots) \in c_0$ we put

$$\operatorname{supp} x = \{i \in \mathbb{N} : x(i) \neq 0\}\,.$$

Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in c_0 such that:

(1) $\operatorname{supp} x_n$ is finite for every $n \in \mathbb{N}$,
(2) $\max \operatorname{supp} x_n < \min \operatorname{supp} x_{n+1}$ for every $n \in \mathbb{N}$,
(3) $\|x_n\| = 1$ for every $n \in \mathbb{N}$.

Obviously, $[\{x_n : n \in \mathbb{N}\}]$ is isometric to c_0. Take any non-zero functional

$$f = (f(1), f(2), \dots) \in \ell_1 = c_0^*.$$

Let

$$A = \left\{ n \in \mathbb{N} : f(x_n) = \sum_{i=1}^{\infty} f(i) x_n(i) = 0 \right\}.$$

We have two cases.

Case 1. The set A is infinite. Then $[\{x_n : n \in A\}] \subset \ker f$ is isometric to c_0.

Case 2. The set A is finite. W.L.O.G. we can assume that A is empty. Since $\lim_{n\to\infty} f(x_n) = 0$, there exist a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ of $(x_n)_{n\in\mathbb{N}}$ and a sequence $(\lambda_k)_{k\in\mathbb{N}} \subset [-1, 1]$ such that

$$f(x_{n_{2k}}) = \lambda_k f(x_{n_{2k-1}})$$

for $k \in \mathbb{N}$. Observe that the sequence $\left(x_{n_{2k}} - \lambda_k x_{n_{2k-1}}\right)_{k\in\mathbb{N}}$ satisfies (1), (2), (3), and $\left[\left\{x_{n_{2k}} - \lambda_k x_{n_{2k-1}} : k \in \mathbb{N}\right\}\right] \subset \ker f$ is isometric to c_0.

Now, if Y is a subspace of c_0 of codimension m, we begin with a sequence $(x_n)_{n\in\mathbb{N}}$ defined as $x_n = e_n$ for $n \in \mathbb{N}$, where $e_n = (0, \dots, 0, 1, 0, 0, \dots)$ and 1 appears in the n-th position. Then, by applying the above procedure m times, we conclude that Y has a subspace isometric to c_0. □

As noted on page 387 in [7], there is an infinite-dimensional subspace of c_0 which does not contain an isometric copy of c_0. Thus, Lemma 3.2 can be considered as the optimal result of this type.

Corollary 3.3. *If a Banach space X contains an isometric copy of c_0, then every subspace of X of finite codimension contains an isometric copy of c_0.*

Proof. Let Z be a subspace of X isometric to c_0 and let Y be a subspace of X of finite codimension. Since $Z \cap Y$ is a subspace of Z of finite codimension, it contains an isometric copy of c_0 by Lemma 3.2. □

Corollary 3.4. *Let X be a Banach space whose closed unit ball in X^* is weak* sequentially compact. If X contains an isometric copy of c_0, then every subspace of X of finite codimension is strictly extremal.*

Proof. It follows from Corollary 3.3 and Theorem 3.1. □

4. Questions

We finish our paper with two open problems.

1. In [19], it is proved that there exists a bounded, closed and convex set $C \subset c_0$ with the Chebyshev radius $r(C) = 1$ such that for every $k \geq 1$ there is a k-contractive map $T : C \to C$ such that $\|x - Tx\| > 1 - 1/k$ for any $x \in C$; recall that T is called *k-contractive* if $\|Tx - Ty\| < k\|x - y\|$ for all $x \neq y$. A natural question that still remains unanswered is whether B_{c_0} has this property.
2. We do not know if the assumption of weak* sequential compactness of the closed unit ball in X^* in Theorem 2.2 can be removed. In particular, it is unknown whether the space ℓ_∞ is extremal.

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