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# CERTAIN FIXED POINT RESULTS FOR ASYMPTOTICALLY $\mathcal{R}$-NONEXPANSIVE MAPPINGS

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**Abstract.** In this paper, we study the existence of fixed points for asymptotically $\mathcal{R}$-nonexpansive self-mappings defined on a subset $K$ of a uniformly convex Banach space equipped with a transitive binary relation $\mathcal{R}$.

**Key Words and Phrases**: Fixed point, asymptotically nonexpansive mapping, uniformly convex Banach space, binary relation.
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## 1. Introduction

It was a turning point in the literature of metric fixed point theory when Browder [12] and Göhde [16] independently but simultaneously proved a fixed point result (in 1965), which states that every nonexpansive mapping defined on a nonempty closed convex and bounded subset of a uniformly convex Banach space admits a fixed point. This is one of the most fundamental results in the theory of nonexpansive mappings. This result was further improved and enriched by Kirk [19] to a reflexive Banach space equipped with a normal structure which is not the point of discussion in the present paper. Thereafter, Goebel and Kirk [15] enlarged the class of nonexpansive mappings by introducing the class of asymptotically nonexpansive mappings and obtained a classical result for such mappings in a uniformly convex Banach space. In 2018, Alfuraidan and Khamsi [6] extended the Goebel and Kirk fixed point theorem for asymptotically nonexpansive mapping [15] to the case of monotone mappings in a Banach space $M$ endowed with the partial order '$\preceq$'. For an extensive study of fixed point theorems for various monotone nonexpansive mappings, one may consult [5, 8, 9, 11, 14, 18] and the references cited therein. With the same spirit as Alam and Imdad [3], very recently Alam et al. [2] proved a relation-theoretic analog of the fixed point theorems of Browder [12] and Göhde [16] for $\mathcal{R}$-nonexpansive mappings using a transitive binary relation which indeed generalize and extend the result of Bin Dehaish and Khamsi [11] for monotone nonexpansive mappings.

**Definition 1.1.** [7] let $(M, \|\cdot\|, \preceq)$ be a partially ordered Banach space. A mapping $S : D(S) \subseteq M \to M$ is called

- monotone if $S(r) \preceq S(s)$ for all $r, s \in D(S)$ with $r \preceq s$,
- monotone nonexpansive if $S$ is monotone and $\|S(r) - S(s)\| \leq \|r - s\|$ for all $r, s \in D(S)$ with $r \preceq s$,

where, $D(S)$ denotes the domain of the mapping $S$.

Clearly, a monotone nonexpansive mapping need not be continuous.

**Definition 1.2.** [7] Let $(M, \preceq)$ be a partial ordered set. For any $r, s \in M$, the subsets

$$[a, \rightarrow) = \{r \in M : a \preceq r\}$$

and

$$(\leftarrow, b] = \{r \in M : r \preceq b\}$$

are called order intervals with initial point $a$ and with end point $b$, respectively.

Also, the order intervals are closed and convex in a Banach lattice (cf. [22]).

**Definition 1.3.** [10] A Banach space $(M, \|\cdot\|)$ is called uniformly convex if for every $\varepsilon > 0$ there is a $\delta > 0$ such that for any $r, s \in M$ with

$$\|r\| \leq 1,\ \|s\| \leq 1,\ \|r - s\| \geq \varepsilon$$

$$\Longrightarrow \frac{1}{2}\|(r + s)\| \leq 1 - \delta.$$

**Example 1.1.** Euclidean space $\mathbb{R}^n$ is uniformly convex with Euclidean norm

$$\|r\| = \left(\sum_{i=1}^{n} r_i^2\right)^{\frac{1}{2}},$$

while, under the norm $\|r\| = \sum_{i=1}^{n} |r_i|$, it is not uniformly convex.

Now, we mention the following result of Alfuraidan and Khamsi [6], which is an improvement over a corresponding fixed point result of Goebel and Kirk [15] proved for monotone asymptotically nonexpansive mappings.

**Theorem 1.1.** [6] *Let $(M, \|\cdot\|, \preceq)$ be a partially ordered uniformly convex Banach space such that order intervals are convex and closed. Let $K$ be a nonempty bounded closed convex subset of $M$ not reducible to a single point. Assume there exists $r_0 \in K$ such that $r_0$ and $S(r_0)$ are comparable, where $S$ is a continuous monotone asymptotically nonexpansive self-mapping on $K$. Then $S$ admits a fixed point.*

Very recently, using the Baire category approach, Reich and Zaslavski [21] have shown that the fixed point problems for various types of monotone nonexpansive mappings are well-posed.

In this paper, we have obtained an illustrative example for asymptotically $\mathcal{R}$-nonexpansive mapping which is not an asymptotically nonexpansive in a Banach space. Thereafter, as a main result we have established a sharpened version of Theorem 1.1 in respect of the following observations:

- The partial ordering to endow on whole space $M$ is not necessary, it is enough to consider a transitive binary relation only on $K$.
- No need to use the assumption that all order intervals in $M$ are convex and closed. It suffices that order intervals in $K$ are convex and closed.
- Boundedness of $K$ must be replaced by a weaker assumption, boundedness of orbit of $r_0$.

An example is also provided to demonstrate the genuineness of our main result over corresponding relevant known results. Also, at the end, we have provided yet another fixed point result in order to replace the continuity condition of the involved mapping by an alternative suitable condition, namely $\mathcal{R}$-weak Opial condition.

## 2. Relation-theoretic notions

Throughout this manuscript, $\mathbb{N}$ and $\mathbb{N}_0$ denote the sets of natural numbers and whole numbers, respectively (*i.e.*, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$). In this section, to make our exposition self-contained, we recall some basic definitions, notions related to binary relations, which will be utilized to prove our main results.

A binary relation on a nonempty set $M$ is a subset $\mathcal{R}$ of $M^2$ (*i.e.*, $\mathcal{R} \subseteq M^2 := M \times M$). Trivially, $M^2$ and $\emptyset$ being subsets of $M^2$ are binary relations on $M$, often called the universal relation (or full relation) and empty relation, respectively. Throughout this paper, $\mathcal{R}$ stands for a nonempty binary relation. For the sake of simplicity, we often write 'binary relation' instead of 'nonempty binary relation'.

**Definition 2.1.** [17] Let $\mathcal{R}$ a binary relation on a nonempty set $M$.

($i$) The inverse (or transpose/dual relation) of $\mathcal{R}$, denoted by $\mathcal{R}^{-1}$, is defined by $\mathcal{R}^{-1} := \{(r, s) \in M^2 : (s, r) \in \mathcal{R}\}$.

($ii$) The symmetric closure of $\mathcal{R}$, denoted by $\mathcal{R}^s$, is defined to be the set $\mathcal{R} \cup \mathcal{R}^{-1}$ (*i.e.*, $\mathcal{R}^s := \mathcal{R} \cup \mathcal{R}^{-1}$). Clearly, $\mathcal{R}^s$ is the smallest symmetric relation on $M$ containing $\mathcal{R}$.

**Definition 2.2.** [3] Let $\mathcal{R}$ be a binary relation on a nonempty set $M$ and $r, s \in M$. We say that $r$ and $s$ are $\mathcal{R}$-comparable if either $(r, s) \in \mathcal{R}$ or $(s, r) \in \mathcal{R}$. We denote it by $[r, s] \in \mathcal{R}$.

**Proposition 2.1.** [3] *For a binary relation $\mathcal{R}$ on a nonempty set $M$,*

$$(r, s) \in \mathcal{R}^s \Longleftrightarrow [r, s] \in \mathcal{R}.$$

**Definition 2.3.** [3] Let $M$ be a nonempty set and $\mathcal{R}$ a binary relation on $M$. A sequence $\{r_n\} \subset M$ is called $\mathcal{R}$-preserving if

$$(r_n, r_{n+1}) \in \mathcal{R} \;\; \forall \; n \in \mathbb{N}_0.$$

**Definition 2.4.** [3] Let $M$ be a nonempty set and $S : M \to M$. A binary relation $\mathcal{R}$ on $M$ is called $S$-closed if for all $r, s \in M$,

$$(r, s) \in \mathcal{R} \Rightarrow (Sr, Ss) \in \mathcal{R}.$$

**Proposition 2.2.** [4] *Let $\mathcal{R}$ a binary relation on a nonempty set $M$ and $S$ a self-mapping on $M$. If $\mathcal{R}$ is $S$-closed, then, for all $n \in \mathbb{N}_0$, $\mathcal{R}$ is also $S^n$-closed, where $S^n$ denotes $n^{th}$-iterate of $S$.*

**Definition 2.5.** [1, 2] Let $\mathcal{R}$ be a binary relation on a nonempty set $K$, the image of an element $\mathbf{a} \in K$(associated with binary relation $\mathcal{R}$) is a subset of $K$ defined by

$$\text{Im}(\mathbf{a}, \mathcal{R}) = \{r \in K : \ (\mathbf{a}, r) \in \mathcal{R} \textit{ or } r = \mathbf{a}\}.$$

Similarly, the pre-image of $\mathbf{a} \in K$ is a subset of $K$ defined by

$$\text{PreIm}(\mathbf{a}, \mathcal{R}) = \{r \in K : \ (r, \mathbf{a}) \in \mathcal{R} \textit{ or } r = \mathbf{a}\}.$$

These subsets of $K$ based on point $\mathbf{a} \in K$ are called $\mathcal{R}$-intervals.

**Remark 2.1.** [2, 1] The following conclusions are obvious.

- $\text{Im}(\mathbf{a}, \mathcal{R}) = \text{PreIm}(\mathbf{a}, \mathcal{R}^{-1})$,
- $\text{PreIm}(\mathbf{a}, \mathcal{R}) = \text{Im}(\mathbf{a}, \mathcal{R}^{-1})$,
- $\text{Im}(\mathbf{a}, \mathcal{R}^s) = \text{PreIm}(\mathbf{a}, \mathcal{R}^s)$.

**Remark 2.2.** The following implications also hold. The proofs are straightforward in view of Definition 2.5.

($i$) If $x \in \text{Im}(\mathbf{a}, \mathcal{R}) \iff \mathbf{a} \in \text{Im}(x, \mathcal{R}^{-1})$,
($ii$) If $x \in \text{PreIm}(\mathbf{a}, \mathcal{R}) \iff \mathbf{a} \in \text{PreIm}(x, \mathcal{R}^{-1})$,
($iii$) If $x \in \text{Im}(\mathbf{a}, \mathcal{R}^s) \iff \mathbf{a} \in \text{Im}(x, \mathcal{R}^s)$.

**Remark 2.3.** Under $\mathcal{R} :=\preceq$ (partial ordering), $\text{Im}(\mathbf{a}, \mathcal{R})$ and $\text{PreIm}(\mathbf{a}, \mathcal{R})$ coincide with order intervals $[\mathbf{a}, \to)$ and $(\leftarrow, \mathbf{a}]$, respectively.

It is a well-established fact that the Lipschitzian mappings play a significant role in the existence and uniqueness of solutions for ordinary differential equation and have nice topological behavior (such as: uniform continuity). Here, we introduce the notion of $\mathcal{R}$-Lipschitzian mappings in a metric space.

**Definition 2.6.** Let $(M, d)$ be a metric space and $\mathcal{R}$ a binary relation on $M$. A mapping $S : M \to M$ is called $\mathcal{R}$-Lipschitzian if

($i$) $\mathcal{R}$ is $S$-closed,
($ii$) there exists $\alpha \geq 0$ such that

$$d(Sr, Ss) \leq \alpha d(r, s).$$

for all $r, s \in M$ with $(r, s) \in \mathcal{R}$.

**Remark 2.4.** The following conclusions are obvious.

($i$) Under universal relation $\mathcal{R} = M^2$, the notion of $\mathcal{R}$-Lipschitzian mapping reduces to Lipschitzian mapping.
($ii$) $\mathcal{R}$-Lipschitzian mapping need not necessarily be continuous.

(*iii*) If we choose $\mathcal{R}$ to be partial ordering (*i.e.*, $\mathcal{R} :=\preceq$) on $M$, then we obtain the notion of monotone Lipschitzian mappings (cf. [6]).

Now, we give the definition of asymptotically $\mathcal{R}$-nonexpansive mappings in metric space.

**Definition 2.7.** Let $(M, d)$ be a metric space and $\mathcal{R}$ a binary relation on $M$. A mapping $S : M \to M$ is said to be asymptotically $\mathcal{R}$-nonexpansive if

(*i*) $\mathcal{R}$ is $S$-closed,

(*ii*) there exists a sequence of positive numbers $\{\alpha_n\}_{n\in\mathbb{N}}$ such that $\lim\limits_{n\to\infty} \alpha_n = 1$ and

$$d(S^n(r), S^n(s)) \leq \alpha_n d(r, s)$$

for all $r, s \in M$ with $(r, s) \in \mathcal{R}$.

**Remark 2.5.** The following are some immediate conclusions.

(*i*) If we choose $\mathcal{R}$ to be partial ordering (*i.e.*, $\mathcal{R} :=\preceq$) on $M$ in Definition 2.7, then we obtain the notion of monotone asymptotically nonexpansive mappings (cf. [6]).

(*ii*) Trivially, every asymptotically nonexpansive mapping is asymptotically $\mathcal{R}$-nonexpansive, but the converse need not be true in general. The below example in a Banach space consideration substantiates this fact.

**Example 2.1.** Let $M = (l^2, \|\cdot\|_2)$, the space of all absolute square summable sequences and $K := \{r \in l^2 : \|r\|_2 \leq \frac{1}{3}\} \cup \{e_1\} \subset l^2$, wherein $e_1 := (1, 0, \cdots, 0, \cdots)$. Consider $S : K \to K$ such that

$$S(r_1, r_2, r_3, \cdots) = (0, r_1^2, c_2 r_2, c_3 r_3, \cdots),$$

where $0 < c_i < 1$ such that $\prod_{i=2}^{\infty} c_i = 1$. Now if we consider a binary relation $\mathcal{R}$ on $M$ as follows:

$$\mathcal{R} := \{(r, s) \in \mathcal{R} \iff \|r - s\|_2 \leq \frac{1}{3}, \ \forall\, r, s \in M\}.$$

Then for any $(r, s) \in \mathcal{R}$ we have $\|S(r) - S(s)\|_2 \leq \|r - s\|_2 \leq \frac{1}{3}$, implying that $\mathcal{R}$ is $S$-closed. Also, for any $(r, s) \in \mathcal{R}$ it is easy to observe that

$$\|S^j(r) - S^j(s)\|_2 \leq \|r + s\|_2 \prod_{i=2}^{j} c_i \|r - s\|_2 \leq \alpha_i \|r - s\|_2,$$

wherein $\alpha_i = \prod_{i=2}^{j} c_i$ and $\lim_{i\to\infty} \alpha_i = 1$. Hence, $S$ is asymptotically $\mathcal{R}$-nonexpansive mapping. However, for any $r \in K$ with $\|r\|_2 \leq \frac{1}{3}$ and $s = e_1$, we get

$$\|S^j(r) - S^j(e_1)\|_2 \leq \|r + e_1\|_2 \prod_{i=2}^{j} c_i \|r - e_1\|_2 \leq \beta_i \|r - e_1\|_2,$$

where $\beta_i = \frac{4}{3} \prod_{i=2}^{j} c_i \nrightarrow 1(\text{as } i \to \infty)$. This shows that $S$ is not asymptotically nonexpansive.

For the sake of simplicity, we record a well-known results due to Smulian [23], which characterizes reflexivity of the underlying Banach space.

**Lemma 2.1.** [23] *A Banach space $M$ is reflexive iff every decreasing sequence $\{A_n\}$ of nonempty bounded, closed and convex subsets of $M$ have nonempty intersection, i.e., $\bigcap_{n=0}^{\infty} A_n \neq \emptyset$.*

**Definition 2.8.** [11] Let $(M, \|\cdot\|)$ be a Banach space, $K \subseteq M$ and $\{r_n\}$ a bounded sequence in $K$. Then a function $\tau : K \to [0, \infty)$ defined by

$$\tau(r) = \limsup_{n\to\infty} \|r_n - r\| \quad \forall r \in K,$$

is called a type function generated by $\{r_n\}$.

The following lemma is crucial in our main result. Although the root of this lemma can be traced out in the work of Edelstein [13]. However, detailed and systematic proof can be found in the recent work of Alfuraidan and Khamsi [6].

**Lemma 2.2.** *Let $K$ be a nonempty, closed and convex subset of a uniformly convex Banach space $(M, \|\cdot\|)$ and $\tau : K \to [0, \infty)$ a type function. Then $\tau$ has a unique minimum point $x \in K$ such that*

$$\tau(x) = \inf\{\tau(r) : r \in K\} = \tau_0.$$

*Moreover, if $\{x_n\}$ is a minimizing sequence in $K$, (i.e., $\lim_{n\to\infty} \tau(x_n) = \tau_0$) then $\{x_n\}$ converges (strongly) to $x$.*

## 3. Main result

Since the main result of this paper is aimed to obtain in a Banach space, we mention analogous relation-theoretic variant of asymptotically nonexpansive mappings in a Banach space $(M, \|\cdot\|)$ for the sake of convenience.

**Definition 3.1.** Let $K$ be a subset of Banach space $(M, \|\cdot\|)$ and $\mathcal{R}$ a binary relation on $K$. A mapping $S : K \to K$ is said to be asymptotically $\mathcal{R}$-nonexpansive if

$(i)$ $\mathcal{R}$ is $S$-closed,

$(ii)$ there exists a sequence of positive numbers $\{\alpha_n\}_{n\in\mathbb{N}}$ such that $\lim_{n\to\infty} \alpha_n = 1$ and

$$\|S^n(r) - S^n(s)\| \leq \alpha_n \|r - s\|$$

for all $r, s \in M$ with $(r, s) \in \mathcal{R}$.

Now, we present relation-theoretic variant of Goebel and Kirk fixed point theorem for asymptotically nonexpansive mappings (see [15]) under a transitive binary relation. We essentially adopt similar approaches which appeared in the works of Alam et al. [2] and Alfuraidan-Khamsi [6].

**Theorem 3.1.** *Let $(M, \|\cdot\|)$ be a uniformly convex Banach space and $K$ a nonempty, closed and convex subset of $M$ not reducible to a single point. Let $\mathcal{R}$ be a transitive binary relation on $K$ and $S : K \to K$ an asymptotically $\mathcal{R}$-nonexpansive mapping. If there exists $r_0 \in K$ such that*

$(i)$ *$S(r_0) \in Im(r_0, \mathcal{R})$,*

$(ii)$ *$\mathrm{Im}(S^n(r_0), \mathcal{R})$ is nonempty, closed and convex for each $n \in \mathbb{N}_0$,*

$(iii)$ *$\{S^n(r_0)\}$ is bounded,*

$(iv)$ *$S$ is continuous,*

*then $S$ admits a fixed point.*

*Proof.* By assumption, choose $r_0 \in K$ such that $S(r_0) \in Im(r_0, \mathcal{R})$ which implies that $(r_0, S(r_0)) \in \mathcal{R}$. Hence, using $S$-closedness of $\mathcal{R}$ and Proposition 2.2, we obtain

$$(S^n(r_0), S^{n+1}(r_0)) \in \mathcal{R},\ \forall n \in \mathbb{N}_0 \tag{1}$$

*i.e.*,

$$S^{n+1}(r_0) \in Im(S^n(r_0), \mathcal{R}).$$

In view of assumption $(ii)$, for each $n \in \mathbb{N}_0$, $A_n := \mathrm{Im}(S^n(r_0), \mathcal{R})$ is a closed and convex subset of $M$. Then $\{A_n\}$ is bounded as $\{S^n(r_0)\}$ is bounded (by assumption $(iii)$). Clearly, $\{A_n\}$ is decreasing *i.e.*, $A_n \supset A_{n+1}$, for all $n$. Hence, $\{A_n\}$ is a bounded decreasing sequence of closed and convex subsets of $M$. Also, being uniformly convex, the Banach space $M$ is reflexive. Therefore, using Lemma 2.1, we get

$$A := \bigcap_{n=0}^{\infty} \mathrm{Im}(S^n(r_0), \mathcal{R}) \neq \emptyset.$$

Take $r \in A$, then $(S^n(r_0), r) \in \mathcal{R}$ for each $n \in \mathbb{N}_0$. Using $S$-closedness of $\mathcal{R}$, we obtain

$$(S^{n+1}(r_0), S(r)) \in \mathcal{R}. \tag{2}$$

Using (1), (2) and transitivity of $\mathcal{R}$, we deduce

$$(S^n(r_0), S(r)) \in \mathcal{R},\ \ \forall n \in \mathbb{N}_0,$$

which yielding hereby

$$S(A) \subseteq A. \tag{3}$$

By assumption $(iii)$, $\{S^n(r_0)\}$ is bounded. Now, consider the type function $\tau : M \to [0, \infty)$ generated by $\{S^n(r_0)\}, i.e.$,

$$\tau(r) = \limsup_{n\to\infty} \|S^n(r_0) - r\|. \tag{4}$$

Then Lemma 2.2 ensures that there exists a unique minimum point $x \in A, i.e.$,

$$\tau(x) = \inf_{r \in M} \tau(r) = \tau_0.$$

Also, as $x \in A$, from (3), we have $S^l(x) \in A$, for any $l \in \mathbb{N}$. Now, using (4), we obtain

$$\tau(S^l(x)) = \limsup_{n\to\infty} \|S^n(r_0) - S^l(x)\| = \limsup_{n\to\infty} \|S^{n+l}(r_0) - S^l(x)\|. \tag{5}$$

Since $x \in A$, we get $(S^n(r_0), x) \in \mathcal{R}$. Hence, by $S$-closedness of $\mathcal{R}$ and Proposition 2.2, we deduce

$$(S^{n+l}(r_0), S^l(x)) \in \mathcal{R}. \tag{6}$$

Using (6) and asymptotically $\mathcal{R}$-nonexpansiveness of $S$, we obtain

$$\|S^{n+l}(r_0) - S^l(x)\| \leq \alpha_l \|S^n(r_0) - x\|, \tag{7}$$

where, $\{\alpha_l\}_{l\in\mathbb{N}}$ is given by condition $(ii)$ of Definition 3.1 such that $\lim_{l\to\infty} \alpha_l = 1$. Therefore, Equations (5) and (7) yielding hereby

$$\tau(S^l(x)) \leq \alpha_l \limsup_{n\to\infty} \|S^n(r_0) - x\| = \alpha_l \cdot \tau_0,$$

implies that $\tau_0 \leq \tau(S^l(x)) \leq \alpha_l \cdot \tau_0$, for every $l \in \mathbb{N}$. Consequently, by the squeeze (or sandwich) theorem of limits, we obtain

$$\lim_{l\to\infty} \tau(S^l(x)) = \tau_0.$$

Which shows that $\{S^l(x)\}_{l\in\mathbb{N}}$ is a minimizing sequence of $\tau$. Again, by using Lemma 2.2 we conclude that $\{S^l(x)\}_{l\in\mathbb{N}}$ converges to $x$.
Now, if $S$ is continuous then

$$\lim_{l\to\infty} S(S^l(x)) = \lim_{l\to\infty} S^{l+1}(x) = S(x).$$

Owing to the uniqueness of limit, we obtain $S(x) = x$, *i.e.*, $x$ is a fixed point of $S$. □

We adopt the following example to demonstrate the genuineness of our main result over corresponding relevant known results.

**Example 3.1.** Let $M = (l^2, \|\cdot\|_2)$, the space of all absolute square summable sequences and $K := \{r \in l^2 : \|r\|_2 \leq \frac{1}{3}\} \cup \{e_1\} \subset l^2$, wherein $e_1 := (1, 0, \cdots, 0, \cdots)$. Define $S : K \to K$ by

$$S(r) = S(r_1, r_2, r_3, \cdots, r_n, \cdots) = (c_1 r_1, r_1^2, c_2 r_2, c_3 r_3, \cdots, c_n r_n, \cdots),$$

where $0 < c_i < 1$ such that $\prod_{i=1}^{\infty} c_i = 1$. Now, we consider a binary relation $\mathcal{R}$ on $M$ as follows:

$$\mathcal{R} := \big\{(r,s) \in \mathcal{R} \iff \text{“}\|r - s\|_2 \leq \frac{1}{3} \text{ and } r \prec s\text{”},\ \forall\, r, s \in M\big\}, \tag{8}$$

where, $r = (r_1, r_2, r_3, \cdots, r_n, \cdots)$, $s = (s_1, s_2, s_3, \cdots, s_n, \cdots)$.
Define $r \prec s$ by "$r \prec s \iff r_i < s_i$, for all $i$". Then, for any $(r, s) \in \mathcal{R}$ we have $\|S(r) - S(s)\|_2 \leq \|r - s\|_2 \leq \frac{1}{3}$, and $r \prec s$ i.e., $r_i < s_i$ for all $i$. Now, by definition of $S$, we get $S(r) \prec S(s)$ so that $\mathcal{R}$ is $S$-closed. Also, for any $(r, s) \in \mathcal{R}$ it is easy to observe that

$$\|S^j(r) - S^j(s)\|_2 \leq \|r + s\|_2 \prod_{i=1}^{j} c_i \|r - s\|_2 \leq \alpha_i \|r - s\|_2,$$

wherein $\alpha_i = \prod_{i=1}^{j} c_i$ and $\lim_{i\to\infty} \alpha_i = 1$. Hence, $S$ is asymptotically $\mathcal{R}$-nonexpansive mapping. Observe that the binary relation $\mathcal{R}$ defined by (8) is not a partial order on $M$ due to the absence of reflexivity and hence, the given mapping $S$ is not a monotone asymptotically nonexpansive mapping. Therefore, one can not use the results of [6] in the context of present example whereas our main result is applicable which substantiates the utility of our newly proved result. On the other hand, observe that for any $r \in K$ with $\|r\|_2 \leq \frac{1}{3}$ and $s = e_1$, we get

$$\|S^j(r) - S^j(e_1)\|_2 \leq \|r + e_1\|_2 \prod_{i=1}^{j} c_i \|r - e_1\|_2 \leq \beta_i \|r - e_1\|_2,$$

where $\beta_i = \frac{4}{3} \prod_{i=1}^{j} c_i \nrightarrow 1$(as $i \to \infty$). This shows that $S$ is not asymptotically nonexpansive as well.

## 4. Result involving $\tau$-Opial condition and binary relation

Throughout this section we assume that $K$ is nonempty bounded and convex subset of $M$ not reducible to one point. Let $S : K \to K$ be an asymptotically $\mathcal{R}$-nonexpansive mapping. Now, we recall the definition of $\tau$-Opial condition [20] and also define relation-theoretic analog of the same, which play a crucial role in the forthcoming result.

**Definition 4.1.** [20] A Banach space $M$ is said to satisfy the $\tau$-Opial condition if for any sequence $\{r_n\} \subset M$ $\tau$-converges to $r$, we have

$$\limsup_{n\to\infty} \|r_n - r\| < \limsup_{n\to\infty} \|r_n - s\|$$

for any $s \in M$ such that $s \neq r$.

**Definition 4.2.** Let $\mathcal{R}$ be a binary relation on $(M, \|\cdot\|)$, then $M$ is said to be satisfied $\mathcal{R}$-$\tau$-Opial condition if whenever any $\mathcal{R}$-preserving sequence $\{r_n\}$ in $M$ $\tau$-converges to $r$, we have

$$\limsup_{n\to\infty} \|r_n - r\| \leq \limsup_{n\to\infty} \|r_n - s\|$$

for any $s \in M$ such that $(r, s) \in \mathcal{R}$.

**Remark 4.1.** An analogous definition can also be obtained for $\mathcal{R}$-reversing sequence (*i.e.*, $\{r_n\}$ is a sequence such that $(r_{n+1}, r_n) \in \mathcal{R}, \ \forall n \in \mathbb{N}_0$).

If we consider $\tau$ to be the 'weak topology' then we have the following immediate consequence.

**Definition 4.3.** Let $\mathcal{R}$ be a binary relation on $(M, \|\cdot\|)$, then $M$ is said to satisfy $\mathcal{R}$-weak Opial condition if whenever any $\mathcal{R}$-preserving sequence $\{r_n\}$ in $M$ weakly converges to $r$, we have

$$\limsup_{n\to\infty} \|r_n - r\| \leq \limsup_{n\to\infty} \|r_n - s\|$$

for any $s \in M$ such that $(r, s) \in \mathcal{R}$.

Now, we utilize the notion of $\mathcal{R}$-weak Opial condition in order to relax the continuity condition of the involved mapping $S$ in Theorem 3.1.

**Theorem 4.1.** *Let $(M, \|\cdot\|)$ be a uniformly convex Banach space and $K$ a nonempty, closed and convex subset of $M$ not reducible to a single point. Let $\mathcal{R}$ be a transitive binary relation on $K$ and $S : K \to K$ an asymptotically $\mathcal{R}$-nonexpansive mapping. If there exists $r_0 \in K$ such that*

(*i*) *$S(r_0) \in Im(r_0, \mathcal{R})$,*
(*ii*) *$\mathcal{R}$-intervals (nonempty) are closed and convex,*
(*iii*) *$\{S^n(r_0)\}$ is bounded,*
(*iv*) *$M$ satisfies $\mathcal{R}$-weak Opial condition,*

*then $S$ has a fixed point.*

*Proof.* As proceed in Theorem 3.1, by our assumption we get an $\mathcal{R}$-preserving bounded sequence $\{S^n(r_0)\} \subset K$. As the underlying space $M$ is reflexive, the sequence $\{S^n(r_0)\}$ converges weakly to some point (say $r^*$) in $K$, *i.e.*, $S^n(r_0) \rightharpoonup r^*$ such that $r^* \in \mathrm{Im}(S^n(r_0), \mathcal{R})$, for any $n \in \mathbb{N}$. Now, since $M$ enjoys $\mathcal{R}$-weak Opial condition, we have

$$\limsup_{n\to\infty} \|S^n(r_0) - r^*\| \leq \limsup_{n\to\infty} \|S^n(r_0) - z\|,$$

for any $z \in \widehat{K} := K \cap Im(r^*, \mathcal{R})$. Clearly, $\widehat{K}$ is a nonempty closed and convex subset of $K$ due to assumption $(ii)$. Therefore, if we consider the type function generated by $\{S^n(r_0)\}$, then Lemma 2.2 ensures that $r^*$ is the desired unique minimum point in $\widehat{K}$. In a similar fashion, as we have already done in the proof of Theorem 3.1, we also obtain $\{S^l(r^*)\}_{l\in\mathbb{N}}$ converges strongly to $r^*$. Next, our aim is to show $(r^*, S(r^*)) \in \mathcal{R}$. As $S^n(r_0) \in \mathrm{Im}(r^*, \mathcal{R}^{-1}) \cap K$ for every $n \in N$, the $S$-closedness of $\mathcal{R}$ implies that $S^{n+1}(r_0) \in \mathrm{Im}(S(r^*), \mathcal{R}^{-1}) \cap K$. Now the weak-limit $r^*$ of $\{S^{n+1}(r_0)\}$ also lies in the set $\mathrm{Im}(S(r^*), \mathcal{R}^{-1}) \cap K$, due to its convexity and closedness property. That is $r^* \in \mathrm{Im}(\mathrm{S}(\mathrm{r}^*), \mathcal{R}^{-1}) \cap K \implies (r^*, S(r^*)) \in \mathcal{R}$. Thus, $\{S^n(r^*)\}$ is an $\mathcal{R}$-preserving sequence that converges to $r^*$ such that $(r^*, S(r^*)) \in \mathcal{R}$. Since $\mathcal{R}$ is transitive, $S$-closed and $S$ is asymptotically $\mathcal{R}$-nonexpansive mapping, we assert that

$$0 \leq \|S(r^*) - r^*\| \leq \alpha_1 \|r^* - S^l(r^*)\| + \|S^{l+1}(r^*) - r^*\|,$$

for any $l \in \mathbb{N}$. Therefore, taking the limit (as $l \to \infty$), we get $S(r^*) = r^*$. This shows that $S$ admits a fixed point. □

## 5. Conclusion

In this paper we introduced the notion of asymptotically $\mathcal{R}$-nonexpansive mappings and proved some fixed point results for such mappings on a subset $K$ of a uniformly convex Banach space equipped with a transitive binary relation $\mathcal{R}$. Our main result (*i.e.*, Theorem 3.1) is indeed a relation-theoretic variant of Goebel and Kirk fixed point theorem for asymptotically nonexpansive mappings (see [15]). We also have provided an example of asymptotically $\mathcal{R}$-nonexpansive mapping which is not monotone asymptotically nonexpansive in a Banach space. Since, a directed graph or a partial order '$\preceq$' can be realized as a specific binary relation, we can easily deduce corollaries for such consideration from our main results (*i.e.*, Theorems 3.1 and 4.1).

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## References

[1] A. Aftab, R. George, M. Imdad, M. Hasanuzzaman, *Fixed point theorems for nonexpansive mappings under binary relations*, Mathematics, **9**:2059, (2021), 10 pp.

[2] A. Aftab, M. Imdad, M. Asim, S. Sessa, *A Relation-Theoretic Formulation of Browder–Göhde fixed point theorem*, Axioms, **10**:285, (2021), 9 pp.

[3] A. Alam, M. Imdad, *Relation-theoretic contraction principle*, J. Fixed Point Theory Appl., **17**(2015), 693-702.

[4] A. Alam, M. Imdad, *Relation-theoretic metrical coincidence theorems*, Filomat, **31**(2017), 4421-4439.

[5] M.R. Alfuraidan, *Fixed points of monotone nonexpansive mappings with a graph*, Fixed Point Theory Appl., **2015**:49, (2015), 6 pp.

[6] M.R. Alfuraidan, M.A. Khamsi, *A fixed point theorem for monotone asymptotically nonexpansive mappings*, Proc. Amer. Math. Soc., **146**(2018), 2451-2456.

[7] M. Bachar, M.A. Khamsi, *Fixed points of monotone mappings and application to integral equations*, Fixed Point Theory Appl., **2015**:110, (2015), 7 pp.

[8] M. Bachar, M.A. Khamsi, *Properties of fixed point sets of monotone nonexpansive mappings in Banach spaces*, Numer. Func. Anal. Opt., **37**(2016), 277-283.

[9] M. Bachar, M.A. Khamsi, *Recent contributions to fixed point theory of monotone mappings*, J. Fixed Point Theory Appl., **19**(2017), 1953-1976.

[10] B. Beauzamy, *Introduction to Banach Spaces and Their Geometry*, North-Holland, Amsterdam, 1985.

[11] B.A. Bin Dehaish, M.A. Khamsi, *Browder and Göhde fixed point theorem for monotone nonexpansive mappings*, Fixed Point Theory Appl., **2016**:20, (2016), 9 pp.

[12] F.E. Browder, *Nonexpansive nonlinear operators in a Banach space*, Proc. Natl. Acad. Sci. USA, **54**(1965), 1041-1044.

[13] M. Edelstein, *The construction of an asymptotic center with a fixed-point property*, Bull. Amer. Math. Soc., **78**(1972), 206-208.

[14] R. Espínola, A. Wiśnicki, *The Knaster-Tarski theorem versus monotone nonexpansive mappings*, Bull. Polish Acad. Sci. Math., **66**(2018), 1-7.

[15] K. Goebel, W.A. Kirk, *A fixed point theorem for asymptotically nonexpansive mappings*, Proc. Amer. Math. Soc., **35**(1972), 171-174.

[16] D. Göhde, *Zum Prinzip der kontraktiven Abbildung*, Math. Nachr., **30**(1965), 251-258.

[17] S. Lipschutz, *Schaum's Outlines of Theory and Problems of Set Theory and Related Topics*, McGraw-Hill, New York, 1964.

[18] M.A. Khamsi, *Notes on Knaster-Tarski theorem versus monotone nonexpansive mappings*, Moroc. J. Pure Appl. Anal., **4**(2018), 1-8.

[19] W.A. Kirk, *A fixed point theorem for mappings which do not increase distances*, Amer. Math. Monthly, **72**(1965), 1004-1006.

[20] Z. Opial, *Weak convergence of the sequence of successive approximations for nonexpansive mappings*, Bull. Amer. Math. Soc., **73**(1967), 591-597.

[21] S. Reich, A. Zaslavski, *Generic Well-posedness of the Fixed Point Problem for Monotone Nonexpansive Mappings*, Chapter 9, Mathematics Almost Everywhere, 2018, 169-179.

[22] H.H. Schaefer, *Banach Lattices and Positive Operators*, Die Grundlehren Der Mathematischen Wissenschaften, Band 215, Springer-Verlag, New York, 1974.

[23] V. Smulian, *On the principle of inclusion in the space of the type (B)*, Matematicheskii Sbornik, **47**(1939), 317-328.

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