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THE MULTI-DIMENSIONAL COAGULATION-FRAGMENTATION MODEL WITH UNBOUNDED KERNEL

D. GHOSH*, J. PAUL**, J. KUMAR*** AND J.-C. YAO****,§

*Debdulal Ghosh, School of Applied Sciences (Mathematics),
KIIT University, Campus-3 (Kathajori Campus), Bhubaneswar 751024, Odisha, India
E-mail: debdulal.email@gmail.com, debdulal.ghoshfma@kiit.ac.in

**Jayanta Paul, Department of Mathematics,
Indian Institute of Technology Kharagpur, Kharagpur-721302, India
E-mail: jayantapaul272@gmail.com

***Jitendra Kumar, Indian Institute of Technology Ropar,
Rupnagar, Punjab 140001, India
E-mail: jkumar@iitrpr.ac.in

****Jen-Chih Yao, Center for General Education,
China Medical University, Taichung, Taiwan
Academy of Romanian Scientists, 50044 Bucharest, Romania
§(Corresponding author) E-mail: yaojc@mail.cmu.edu.tw

Abstract. The study of the coagulation-fragmentation model has provided insights into various engineering and scientific disciplines. However, the characteristics of particle ensembles are determined by multiple parameters in a multidimensional parameter space, including mass, volume, porosity, binder content, enthalpy, mole number, and more. This work focuses on establishing the existence of a continuous solution for the higher dimensional model, subject to certain restrictions on the kernels. Additionally, the conservation of volume of the solution are investigated. The results are derived based on the compactness result of Arzelà-Ascoli and the Banach contraction mapping principle.
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1. Introduction

The population balance model incorporating coagulation and fragmentation phenomena in the particulate process is a well-established topic due to their generous impact in many scientific and engineering disciplines, namely, pharmaceuticals research, physical sciences, atmospheric engineering, astronomical disciplines etc. The population balance models are the mean-field model of desired properties for a population of particles subjected to different processes that result in the formation and disappearance of particles. In the one-dimensional population balance equation time evolution of the particle depend only on one property of the particle. To characterize the particle, though, additional distributed features (in a variety of physical situations like polymer degradation, combustion, grinding of minerals, fluidized bed agglomeration, or pharmaceutical processes) are frequently required. Thus, a multidimensional framework is prerequisite to describe the particle distribution function properly.

1.1. **Mathematical description of the problem.** In industrial plants and other sectors, particles (in dispersed media, batch crystallization, thermodynamic systems, etc.) are commonly characterized by multiple internal properties. These properties may include mass, volume, porosity, binder content, enthalpy, mole number, and more. [21, 16]. Denoting $\vec{x} = (x_1, x_2, \ldots, x_d) \in \mathbb{R}^d_+$ as the property characteristics vector, we define $\vec{x} \ominus \vec{y} = (x_1 - y_1, x_2 - y_2, \ldots, x_d - y_d)$, $\vec{x} \oplus \vec{y} = (x_1 + y_1, x_2 + y_2, \ldots, x_d + y_d)$ and $\vec{x} \otimes \vec{y} = (x_1y_1, x_2y_2, x_3y_3, ..., x_dy_d)$. Also, $\vec{x} < \vec{y}$ implies $x_i < y_i$ for all $i = 1, 2, ..., d$. With these notations, the mathematical form of the multidimension coagulation fragmentation model (multi-D CFM) is the following:

$$\begin{aligned}\frac{\partial f(t,\vec{x})}{\partial t} &= \frac{1}{2}\int_{\vec{0}}^{\vec{x}} K(\vec{x}\ominus\vec{y};\vec{y}) f(t,\vec{x}\ominus\vec{y}) f(t,\vec{y})\,\mathrm{d}\vec{y} - f(t,\vec{x})\int_{\vec{0}}^{\vec{\infty}} K(\vec{x};\vec{y}) f(t,\vec{y})\,\mathrm{d}\vec{y} \\ &\quad + \int_{\vec{x}}^{\vec{\infty}} b(\vec{x}|\vec{y}) S(\vec{y}) f(t,\vec{y})\,\mathrm{d}\vec{y} - S(\vec{x}) f(t,\vec{x}), \qquad (1.1)\end{aligned}$$

supported with the initial data

$$f(0,\vec{x}) = f_0(\vec{x}) \geq 0. \qquad (1.2)$$

In the preceding model,

- $f(t,\vec{x})$ is the particle number density function characterized by the property vector $\vec{x}$ at time $t \geq 0$.
- $K(\vec{x};\vec{x'})$ is the rate of formation of a particle identified by $\vec{x} \oplus \vec{x}'$ from two smaller particles characterized by $\vec{x}$, $\vec{x'}$ and is symmetric with respect to its respective characteristic argument.
- The selection function $S(\vec{x})$ describes the rate at which particles characterized by $\vec{x}$ are selected to fragment.
- The daughter distribution function $b(\vec{x}|\vec{y})$ describes the rate at which particles characterized by $\vec{x}$ are produced from a particle characterized by $\vec{y}$. Volume conservation requires that

$$\left(\prod_{i=1}^{d} y_i\right) = \int_0^{y_1}\int_0^{y_2}\cdots\int_0^{y_d}\left(\prod_{i=1}^{d} x_i\right) b(x_1, x_2, \cdots, x_d | y_1, y_2, \cdots, y_d)\mathrm{d}x_d \ldots \mathrm{d}x_2\mathrm{d}x_1.$$

Consequently, the average number of objects produced per fragmentation (linear) event is

$$N_l(y_1, y_2, \cdots, y_d) = \int_0^{y_1}\int_0^{y_2}\cdots\int_0^{y_d} b(x_1, x_2, \cdots, x_d | y_1, y_2, \cdots, y_d)\mathrm{d}x_d \ldots \mathrm{d}x_2\mathrm{d}x_1.$$

Similarly, the mass conservation along the axes indicates

$$\left(\sum_{i=1}^{d} y_i\right) = \int_0^{y_1}\int_0^{y_2}\dots\int_0^{y_d}\left(\sum_{i=1}^{d} x_i\right) b(x_1,x_2,\cdots,x_d|y_1,y_2,\cdots,y_d)\mathrm{d}x_d\dots\mathrm{d}x_2\mathrm{d}x_1.$$

$$M_{n_1,n_2,\dots,n_d}(t) = \int_0^{\infty}\int_0^{\infty}\dots\int_0^{\infty} x_1^{n_1}x_2^{n_2},\dots,x_d^{n_d} f(t,x_1,x_2,\dots,x_d)\mathrm{d}x_d\dots\mathrm{d}x_2\mathrm{d}x_1.$$

Here the zeroth moment, $M_{0,0,\dots,0}(t)$ is proportional to the total number of particles, whereas, the cross moment $\mu_1(t) = M_{1,1,\dots,1,\dots,1}(t)$ (1 in all the position) describes the total hyper-volume. Hyper-volume conservation indicates that

$$\frac{d}{dt}\mu_1(t) = 0.$$

1.2. **Literature review and motivation.** Over the last two decades, the general population balance model incorporating a one-dimensional coagulation fragmentation model has had a huge growth. In this regard, the mathematical study of existence-uniqueness result [15, 13, 10, 20], well-posedness of the solution [6], analytical solution by the developed Lie group analysis method [18], self-similar behavior of solution [4], stability and equilibrium criterion of the solution [14, 11, 7], large time asymptotic dynamics [12, 3], development of numerical techniques [19, 23], stochastic interpretation of solution [8] are of great interests. Recently, the numerical solution of multi-dimensional pure coagulation or pure fragmentation or combined problem have attracted many researchers [24, 23]. There is another circumstance where mother particle break into only two daughter particles in a linear breakage process [17]. In this case the corresponding model equation considered as

$$\frac{\partial f(t,\vec{x})}{\partial t} = 2^d\int_{\vec{x}}^{\vec{\infty}} F(\vec{y}-\vec{x};\vec{x})f(t,\vec{y})\,\mathrm{d}\vec{y} - f(t,\vec{x})\int_{\vec{0}}^{\vec{x}} F(\vec{x}-\vec{y};\vec{y})\mathrm{d}\vec{y}. \tag{1.3}$$

In the above model (1.3), the fragmentation kernel $F(\vec{x};\vec{y})$ is the rate of fragmentation of an object with volume characterized by $[(x_1+y_1),(x_2+y_2),\dots(x_d+y_d)]$ into two objects characterized by $(x_1,\dots,x_d)$, $(y_1,\dots,y_d)$. It is shown that the model (1.3) is a particular case of (1.1) for pure linear pure fragmentation case (coagulation kernel is considered to be zero there) by

$$b(x_1,\dots,x_d|y_1,\dots,y_d) = \frac{2^d F(x_1,\dots,x_d;y_1-x_1,\dots,y_d-x_d)}{S(y_1,\dots,y_d)}. \tag{1.4}$$

It is pertinent to mention that literature reveals only few mathematical investigations have been presented for the coagulation-fragmentation process with more than one degrees of freedom [1, 2] and also on collisional breakage problem [20]. These circumstances are motivated us to a brief mathematical study on multi-dimensional problem. The dynamics of coagulation-fragmentation model in the space inhomogeneous velocity fields has been a recent topic of interest to the science and engineering community, due to its enormous applications e.g., Chemistry (reacting polymers), Physics (aggregation of colloidal particles, growth of gas bubbles in solids), Industrial sectors (food processing, mineral processing, pharmaceutics, etc.), Astrophysics (formation of stars and planets), Meteorology (merging of drops in atmospheric clouds) etc.

In this context [5] considered corresponding representation by the following model

$$\begin{aligned}
&\frac{\partial f(t,\vec{x},z)}{\partial t} + \mathrm{div}_z(v(x,z)f(t,\vec{x},z)) \\
&= \frac{1}{2^d}\int_{\vec{0}}^{\vec{x}} K(\vec{x}\ominus\vec{y};\vec{y}) f(t,\vec{x}\ominus\vec{y},z) f(t,\vec{y},z)\,\mathrm{d}\vec{y} \\
&- f(t,\vec{x},z)\int_{\vec{0}}^{\vec{\infty}} K(\vec{x};\vec{y}) f(t,\vec{y},z)\,\mathrm{d}\vec{y} \\
&+ 2^d \int_{\vec{x}}^{\vec{\infty}} f(t,\vec{x'},z) F(x'\ominus x;x)\mathrm{d}\vec{x'} - f(t,\vec{x},z)\int_{\vec{0}}^{\vec{x}} F(\vec{x}\ominus\vec{x'};\vec{x'})\mathrm{d}\vec{x'},
\end{aligned} \tag{1.5}$$

supported with the initial data,

$$f(0,\vec{x},z) = f_0(\vec{x},z) \geq 0, \quad \text{for all} \quad x_i > 0. \tag{1.6}$$

Here $v(\vec{x},z) \in \mathbb{R}^3$ is a known velocity of transfer of particle with properties $\vec{x}$ in the space position $z \in \mathbb{R}^3$.

We examine the key findings of mathematical analysis i.e., the existence of a unique continuous solution for the multi-dimension population balance model. The article is constructed in the following pattern. In Section 2.1 the kernels truncation and assumptions on the kernels are considered. Section 2.2 deal with the existence of a solution in the truncated subspace or domain whereas Section 2.3 describes the existence of solution in the whole domain for the parameter space in a finite domain. An exquisite physical property of Hyper-volume conservation law is shown in Section 2.3. Finally, the conclusion of this research work is given in Section 3.

2. Step 1: Existence of solution

2.1. **The kernels truncation.** Using the truncation idea of [9, 22], we show that solution of (1.1)-(1.2) exists for the "cut-off" kernels $K_{\vec{n}},\ S_{\vec{n}},\ b_{\vec{n}}$ and $\beta_{\vec{n}}$ respectively, where

$$K_{\vec{n}}(\vec{x},\vec{y}) = \begin{cases} K(\vec{x},\vec{y}), & \text{when } 0 \leq x_i + y_i \leq n_i,\ \vec{n} = (n_1, n_2, \ldots, n_d) \in \mathbb{N}^d \\ 0, & \text{elsewhere.} \end{cases} \tag{2.1}$$

$$S_{\vec{n}}(\vec{x}) = \begin{cases} S(\vec{x}), & \text{when } 0 \leq x_i \leq n_i, \\ 0, & \text{elsewhere,} \end{cases} \tag{2.2}$$

and,

$$b_{\vec{n}}(\vec{x}|\vec{y}) = \begin{cases} b(\vec{x}|\vec{y}), & \text{when } 0 \leq x_i, y_i, z_i \leq n_i, \\ 0, & \text{elsewhere.} \end{cases} \tag{2.3}$$

For the truncated kernel, $f^{\vec{n}}$ is the solution of the following equation

$$\begin{aligned}\frac{\partial f^{\vec{n}}(t,\vec{x})}{\partial t} &= \frac{1}{2}\int_{\vec{0}}^{\vec{x}} K_n(\vec{x}\ominus\vec{y};\vec{y}) f^{\vec{n}}(t,\vec{x}\ominus\vec{y}) f^{\vec{n}}(t,\vec{y})\,\mathrm{d}\vec{y} \\ &- f^{\vec{n}}(t,\vec{x})\int_{\vec{0}}^{\vec{n}\ominus\vec{x}} K_n(\vec{x};\vec{y}) f^{\vec{n}}(t,\vec{y})\,\mathrm{d}\vec{y} \\ &+ \int_{\vec{x}}^{\vec{n}} b_n(\vec{x}|\vec{y}) S_n(\vec{y}) f_{\vec{n}}(t,\vec{y})\,\mathrm{d}\vec{y} - S_n(\vec{x}) f^{\vec{n}}(t,\vec{x}),\end{aligned} \tag{2.4}$$

corresponding truncated initial data is as follows

$$f^{\vec{n}}(0,\vec{x}) = f_0^{\vec{n}}(\vec{x}) = \begin{cases} f_0(\vec{x}), & 0\le x_i\le n_i, \\ 0, & x_i > n_i. \end{cases} \tag{2.5}$$

In the next, we introduce some spaces. To do this, let's define the strip

$$\mathcal{P}(T,\vec{X}) = \{(t,\vec{x}) : 0\le t\le T, 0\le x_i\le X_i\},$$

where T and X_i are finite real numbers. We denote $\mathcal{F}(T)$ as the set of continuous functions with the following norm

$$\|f\| = \sup_{0\le t\le T}\int_{\vec{0}}^{\vec{\infty}} \prod(1+x_i)|f(t,\vec{x})|\,\mathrm{d}\vec{x} < \infty.$$

Denote $\mathcal{F}^{\vec{n}}(T)$ as the set of all continuous functions f over the truncated domain with the following bounded norms

$$\|f\|^{\vec{n}} = \sup_{0\le t\le T}\int_{\vec{0}}^{\vec{n}} \prod(1+x_i)|f(t,\vec{x})|\,\mathrm{d}\vec{x} < \infty.$$

In the next, $\mathcal{F}^+(T)$ indicates the cone of the non-negative functions in $\mathcal{F}(T)$.

Assumptions: Let us assume the following restriction on the kernels to prove the desired result:

(A1) The coagulation kernel $K(\vec{x},\vec{y})$ is non-negative continuous functions on $\mathbb{R}_+^d \times \mathbb{R}_+^d$ and

$$K(\vec{x},\vec{y}) = K(\vec{y},\vec{x}).$$

(A2) The kernels $S(\vec{x})$ and $b(\vec{x}|\vec{y})$ are continuous, non-negative on $\mathbb{R}_d^+$ and $\mathbb{R}_d^+\times\mathbb{R}_d^+$, respectively.

(A3) The coagulation kernel satisfies the following,

$$K(\vec{x},\vec{y}) \le k\prod_{i=1}^{d}(1+x_i)(1+y_i),$$

where k is a positive constant.

(A4) For some positive constant S_1, selection function satisfies the following condition:

$$S(\vec{x}) \leq S_1 \prod_{i=1}^{d} x_i,$$

(A5) The breakage function satisfies the following condition

$$\lim_{\vec{y}\to\vec{\infty}} \sup_{\vec{x}\in[\vec{x}_1,\vec{x}_2]} b(\vec{x},\vec{y}) \leq \bar{b}, \forall\ \vec{0} < \vec{x}_1 < \vec{x}_2 < \vec{\infty},$$

and $\bar{b}$ is a constant.

2.2. **Local existence of solution.**

Theorem 2.1. *If the kernel conditions (A1-A5) and the initial condition $f_0(\vec{x}) \in \mathcal{F}^+(0)$, then for each n_i, the problem* (2.4), (2.5) *possesses a unique solution $f^{\vec{n}}$ in $\mathcal{F}^{\vec{n}}(T)$ with $f^n(t,\vec{x}) \geq 0$, for $0 \leq x_i \leq n_i$ and $t \in [0,T]$. Also, the volume conservation law holds, i.e.,*

$$\int_{\vec{0}}^{\vec{n}} \vec{x} f^{\vec{n}}(t,\vec{x})\,\mathrm{d}\vec{x} = \int_{\vec{0}}^{\vec{n}} \prod_i x_i f^{\vec{n}}(0,\vec{x})\,\mathrm{d}\vec{x},\ \textit{for } 0 \leq t \leq T. \tag{2.6}$$

Proof. The truncated problem (2.4) can be reformulated in the following equivalent form

$$\frac{\partial}{\partial t}\big[\exp(J(f^{\vec{n}},t,\vec{x}))f^{\vec{n}}(t,\vec{x})\big] = \exp(J(f^{\vec{n}},t,\vec{x}))E(t,\vec{x},f^{\vec{n}}) \tag{2.7}$$

where

$$J(f^{\vec{n}},t,\vec{x}) = \int_0^t \left[\int_{\vec{0}}^{\vec{n}\ominus\vec{x}} K_{\vec{n}}(\vec{x},\vec{y})f^n(s,\vec{y})d\vec{y} + S_{\vec{n}}\right]ds \tag{2.8}$$

and

$$\begin{aligned} E(t,\vec{x},f^{\vec{n}}) = \frac{1}{2}\int_{\vec{0}}^{\vec{x}} K_n(\vec{x}\ominus\vec{y};\vec{y})f^{\vec{n}}(t,\vec{x}\ominus\vec{y})f^{\vec{n}}(t,\vec{y})\,\mathrm{d}\vec{y} \\ + \int_{\vec{x}}^{\vec{n}} b_n(\vec{x}|\vec{y})S_n(\vec{y})f_{\vec{n}}(t,\vec{y})\,\mathrm{d}\vec{y}. \end{aligned} \tag{2.9}$$

Integrating from 0 to t, we obtain

$$f^{\vec{n}}(t,\vec{x}) = C(f^{\vec{n}})(t,\vec{x}) \tag{2.10}$$

where

$$\begin{aligned} C(f^{\vec{n}})(t,\vec{x}) = f_0^{\vec{n}} \exp(-J(f^{\vec{n}},t,\vec{x})) \\ + \int_0^t \exp\{-[J(f^{\vec{n}},t,\vec{x}) - J(f^{\vec{n}},s,\vec{x})]\}E(s,\vec{x},f^{\vec{n}})ds. \end{aligned} \tag{2.11}$$

In order to prove the main result, we go through some Lemmas. To do this, we shall focus on the contraction mapping principle for the operator C on some interval $[0,t_0]$, for some $t_0 > 0$. Let $L = \|f_0^{\vec{n}}\|(1 + N_l T \prod_i (1+n_i)n_i)$ and choose $t', t'' > 0$ such that

$$\exp(2Mt(L+1))\left[1 + t\left(\frac{M}{2}L + N_l S_1 \prod_i (1+n_i)\right)\right] \leq 2 \text{ for } 0 \leq t \leq t',$$

and

$$\exp(Mt(L+1))\left[LMt + Mt^2\left(\frac{M}{2}L^2 + N_l S_1 L\prod_i(1+n_i)\right) + \ tS_1 N_l\prod_i(1+n_i)\right]$$
$$< 1, \ \text{for } 0 \le t \le t''. \tag{2.12}$$

Lemma 2.2. *For $f, g \in \mathcal{F}^{\vec{n}}(t_0)$ we have for $0 \le s \le t \le t_0$ and $0 \le x_i \le n_i$,*

$$|H(t,s,\vec{x})| \le M(t-s)\|f-g\|^{\vec{n}}\exp\{BM(t-s)\} \tag{2.13}$$

where $H(t,s,\vec{x}) = \exp\{-[J(f,t,\vec{x}) - J(f,s,\vec{x})]\} - \exp\{-[J(g,t,\vec{x}) - J(g,s,\vec{x})]\}$ and $B = \max\{\|f\|^n, \|g\|^n\}$.

Proof. For the operator D we assume, $J(g,t,\vec{x}) - J(g,s,\vec{x}) \le J(f,t,\vec{x}) - J(f,s,\vec{x})$. Therefore we have,

$$\begin{aligned}|H(t,s,\vec{x})| = -H(t,s,\vec{x}) &= \exp\{-[J(g,t,\vec{x}) - J(g,s,\vec{x})]\}\\ &\times\{1-\exp\{-[J(f,t,\vec{x}) - J(f,s,\vec{x}) - [J(g,t,\vec{x}) - J(g,s,\vec{x})]]\}\}.\end{aligned} \tag{2.14}$$

Since $x \ge 1-\exp(-x)$ in the range $x \ge 0$ we note that,

$$\begin{aligned}|H(t,s,\vec{x})| &\le \exp\{-[J(g,t,\vec{x}) - J(g,s,\vec{x})]\}\\ &\quad\times\{J(f,t,\vec{x}) - J(f,s,\vec{x}) - [J(g,t,\vec{x}) - J(g,s,\vec{x})]\}\\ &\le \exp\{-[J(g,t,\vec{x}) - J(g,s,\vec{x})]\}\times\int_s^t\int_{\vec{0}}^{\vec{n}\ominus\vec{x}}K(\vec{x};\vec{y})[f(\tau,\vec{y}) - g(\tau,\vec{y})]d\vec{y}d\tau\\ &\le \exp\left\{\int_s^t\int_{\vec{0}}^{\vec{n}\ominus\vec{x}}K(\vec{x};\vec{y})g(\tau,\vec{y})d\vec{y}d\tau\right\}\times M(t-s)\|f-g\|^{\vec{n}}\\ &\le M(t-s)\|f-g\|^{\vec{n}}\exp\{(t-s)BM\}.\end{aligned} \tag{2.15}$$

If $J(f,t,\vec{x}) - J(f,s,\vec{x}) \le J(g,t,\vec{x}) - J(g,s,\vec{x})$ then inequality (2.13) can be similarly derived. $\square$

Lemma 2.3. *The non-linear operator C maps Ω into itself, where Ω is the space*

$$\Omega := \{f \in C([0,t_0]\times[\vec{0},\vec{n}]) : \sup_{0\le t\le t_0}\int_{\vec{0}}^{\vec{n}}\prod_i(1+x_i)|f(t,\vec{x})|d\vec{x} < 2L\}$$

Proof. Considering $0 \le t \le t_0$ and $\|f\|^{\vec{n}} \le 2L$, we obtain

$$\int_{\vec{0}}^{\vec{n}}\prod_i(1+x_i)|C(f)(t,\vec{x})|d\vec{x} = \underbrace{\int_{\vec{0}}^{\vec{n}}\prod_i(1+x_i)f_0^{\vec{n}}\exp(-J(f,t,\vec{x}))d\vec{x}}_{A_1}$$
$$+\underbrace{\int_{\vec{0}}^{\vec{n}}\prod_i(1+x_i)\int_0^t\exp\{-[J(f,t,\vec{x}) - J(f,s,\vec{x})]\}E(s,\vec{x},f)dsd\vec{x}}_{A_2}. \tag{2.16}$$

Now,

$$A_1 = \int_{\vec{0}}^{\vec{n}} \prod_i (1+x_i) f_0^{\vec{n}} \exp(-J(f,t,\vec{x}))d\vec{x} \leq \int_{\vec{0}}^{\vec{n}} \prod_i (1+x_i) f_0^{\vec{n}} \exp(J(f,t,\vec{x}))d\vec{x}.$$

Considering,

$$M = \max\{\sup\{S(\vec{x}); \vec{x} \in [\vec{0},\vec{n}]\}, \sup\{K(\vec{x},\vec{y}); (\vec{x},\vec{y}) \in [\vec{0},\vec{n}], \sup\{b(\vec{x}|\vec{y})\};$$
$$(\vec{x},\vec{y}) \in [\vec{0},\vec{n}] \times [\vec{0},\vec{n}]\}\},$$

we obtain

$$A_1 \leq \|f_0\|^{\vec{n}} \exp(tM(\|f\|^{\vec{n}} + 1)).$$

Now, we estimate A_2 as,

$$\begin{aligned}
A_2 &= \int_{\vec{0}}^{\vec{n}} \prod_i (1+x_i) \int_0^t \exp\{-[J(f,t,\vec{x}) - J(f,s,\vec{x})]\} E(s,\vec{x},f) ds d\vec{x} \\
&\leq \exp(2Mt(\|f^{\vec{n}}\|+1)) \int_{\vec{0}}^{\vec{n}} \int_0^t \prod_i (1+x_i) E(s,\vec{x},f) ds d\vec{x} \\
&\leq \exp(2Mt(\|f^{\vec{n}}\|+1)) \int_{\vec{0}}^{\vec{n}} \int_0^t \prod_i (1+x_i) \bigg[\frac{1}{2}\int_{\vec{0}}^{\vec{x}} K_n(\vec{x}\ominus\vec{y};\vec{y}) f^{\vec{n}}(t,\vec{x}\ominus\vec{y}) f^{\vec{n}}(t,\vec{y})\, \mathrm{d}\vec{y} \\
&+ \int_{\vec{x}}^{\vec{n}} b_n(\vec{x}|\vec{y}) S_n(\vec{y}) f_{\vec{n}}(t,\vec{y})\, \mathrm{d}\vec{y}\bigg] ds d\vec{x}.
\end{aligned}$$

Now, changing the order of integration in the second term w.r.t. $\vec{x}$ and $\vec{y}$ we obtain

$$\begin{aligned}
A_2 \leq \exp(2Mt(\|f^{\vec{n}}\|+1)) \int_0^t \bigg[&\int_{\vec{0}}^{\vec{n}} \prod_i (1+x_i) \int_{\vec{0}}^{\vec{x}} \frac{1}{2} K_n(\vec{x}\ominus\vec{y};\vec{y}) f^{\vec{n}}(t,\vec{x}\ominus\vec{y}) f^{\vec{n}}(t,\vec{y})\, \mathrm{d}\vec{y} d\vec{x} \\
+ \prod_i (1+n_i) &\int_{\vec{0}}^{\vec{n}} \int_{\vec{0}}^{\vec{y}} b_n(\vec{x}|\vec{y}) S_n(\vec{y}) f_{\vec{n}}(t,\vec{y})\, d\vec{x} \mathrm{d}\vec{y}\bigg] ds.
\end{aligned}$$

Therefore,

$$A_2 \leq t\exp(2Mt(\|f^{\vec{n}}\|+1)) \left[\frac{M}{2}(\|f^{\vec{n}}\|)^2 + N_l S_1 \|f^{\vec{n}}\| \prod_i (1+n_i)\right].$$

Hence,

$$\begin{aligned}
&\int_{\vec{0}}^{\vec{n}} \prod_i (1+x_i) |C(f)(t,\vec{x})| d\vec{x} \\
&\leq \exp(2Mt(\|f^{\vec{n}}\|+1)) \left[\|f_0^{\vec{n}}\| + t\left(\frac{M}{2}(\|f^{\vec{n}}\|)^2 + N_l S_1 \|f^{\vec{n}}\| \prod_i (1+n_i)\right)\right] \\
&\leq L\exp(2Mt(L+1)) \left[1 + t\left(\frac{M}{2} L + N_l S_1 \prod_i (1+n_i)\right)\right].
\end{aligned}$$

The result follows by considering,

$$\exp(2Mt(L+1))\left[1+t\left(\frac{M}{2}L+N_lS_1\prod_i(1+n_i)\right)\right]\leq 2. \qquad \square$$

Lemma 2.4. *The non-linear operator C is a contraction on Ω, where Ω is the space*

$$\Omega := \{f\in C([0,t_0]\times[\vec{0},\vec{n}]) : \sup_{0\leq t\leq t_0}\int_{\vec{0}}^{\vec{n}}\prod_i(1+x_i)|f(t,\vec{x})|d\vec{x}<2L\}$$

Proof. The corresponding result follows in a similar way as of [20]. We note that

$$C(f^{\vec{n}})(t,\vec{x}) = f_0^{\vec{n}}\exp(-J(f^{\vec{n}},t,\vec{x})) + \int_0^t \exp\{-[J(f^{\vec{n}},t,\vec{x})-J(f^{\vec{n}},s,\vec{x})]\}E(s,\vec{x},f^{\vec{n}})ds, \tag{2.17}$$

where the notations $J(f^{\vec{n}},t,\vec{x})$ and $E(t,\vec{x},f^{\vec{n}})$ are used as following

$$J(f^{\vec{n}},t,\vec{x}) = \int_0^t\left[\int_{\vec{0}}^{\vec{n}}K_{\vec{n}}(\vec{x},\vec{y})f^{\vec{n}}(s,\vec{y})d\vec{y}+S_{\vec{n}}(\vec{x})\right]ds \tag{2.18}$$

and

$$E(t,\vec{x},f^{\vec{n}}) = \frac{1}{2}\int_{\vec{0}}^{\vec{x}}K_{\vec{n}}(\vec{x}\ominus\vec{y};\vec{y})f^{\vec{n}}(t,\vec{x}\ominus\vec{y})f^{\vec{n}}(t,\vec{y})d\vec{y}+\int_{\vec{x}}^{\vec{n}}b_{\vec{n}}(\vec{x}|\vec{y})S_{\vec{n}}(\vec{y})f_{\vec{n}}(t,\vec{y})d\vec{y}. \tag{2.19}$$

Here we recall that

$$H(t,s,\vec{x}) = \exp\{-[J(f,t,\vec{x})-J(f,s,\vec{x})]\}-\exp\{-[J(g,t,\vec{x})-J(g,s,\vec{x})]\},$$

therefore we obtain,

$$H(t,0,\vec{x}) = \exp[-J(f,t,\vec{x})]-\exp[-J(g,t,\vec{x})].$$

Hence from the definition of the operator C in (2.17), we get

$$\begin{aligned} C(f^n)(t,\vec{x})-C(g^n)(t,\vec{x}) &= f_0^{\vec{n}}H(t,0,\vec{x}) \\ &+\int_0^t F(t,s,\vec{x})\left[\frac{1}{2}\int_{\vec{0}}^{\vec{x}}K_{\vec{n}}(\vec{x}\ominus\vec{y};\vec{y})f^{\vec{n}}(t,\vec{x}\ominus\vec{y})f^{\vec{n}}(t,\vec{y})d\vec{y}+\int_{\vec{x}}^{\vec{n}}b_{\vec{n}}(\vec{x}|\vec{y})S_{\vec{n}}(\vec{y})f_{\vec{n}}(t,\vec{y})d\vec{y}\right] \\ &+\int_0^t\exp\{-[J(g,t,\vec{x})-J(g,s,\vec{x})]\}[\bar{B}(s,\vec{x},f)-\bar{B}(s,\vec{x},g)]ds, \end{aligned} \tag{2.20}$$

where the expression of $\bar{B}(s,\vec{x},f)$ is the following

$$\bar{B}(s,\vec{x},f) = \frac{1}{2}\int_{\vec{0}}^{\vec{x}}K_{\vec{n}}(\vec{x}\ominus\vec{y};\vec{y})f^{\vec{n}}(t,\vec{x}\ominus\vec{y})f^{\vec{n}}(t,\vec{y})d\vec{y}+\int_{\vec{x}}^{\vec{n}}b_{\vec{n}}(\vec{x}|\vec{y})S_{\vec{n}}(\vec{y})f_{\vec{n}}(t,\vec{y})d\vec{y}. \tag{2.21}$$

Now, to check the contraction of the operator C on Ω, we note that

$$\int_{\vec{0}}^{\vec{n}}\prod_i(1+x_i)^{\alpha_i}(C(f^{\vec{n}})-C(g^{\vec{n}}))(t,\vec{x})d\vec{x}\leq B_1+B_2+B_3, \tag{2.22}$$

where the expressions of B_1, B_2 and B_3 are given below

$$B_1 = \int_{\vec{0}}^{\vec{n}} \prod (1+x_i) f_0^{\vec{n}}(\vec{x}) H(t,0,\vec{x}) d\vec{x} \leq Mt\exp(tM(B+1))\|f_0^{\vec{n}}\|\|f^{\vec{n}} - g^{\vec{n}}\|, \quad (2.23)$$

and

$$B_2 = \int_{\vec{0}}^{\vec{n}} \prod (1+x_i) \int_0^t F(t,s,\vec{x}) \left[\frac{1}{2} \int_{\vec{0}}^{\vec{x}} K_{\vec{n}}(\vec{x} \ominus \vec{y}; \vec{y}) f^{\vec{n}}(t, \vec{x} \ominus \vec{y}) f^{\vec{n}}(t,\vec{y}) d\vec{y} \right.$$
$$\left. + \int_{\vec{x}}^{\vec{n}} b_{\vec{n}}(\vec{x}|\vec{y}) S_{\vec{n}}(\vec{y}) f_{\vec{n}}(t,\vec{y}) d\vec{y} \right] ds d\vec{x}, \quad (2.24)$$

at the end

$$B_3 = \int_{\vec{0}}^{\vec{n}} \prod (1+x_i) \int_0^t \exp\{-[J(g,t,\vec{x}) - J(g,s,\vec{x})]\}[\bar{B}(s,\vec{x},f) - \bar{B}(s,\vec{x},g)] ds d\vec{x}. \quad (2.25)$$

From Lemma 2.2, we note that

$$|H(t,s,\vec{x})| \leq M(t-s)\|f-g\|_{\vec{\alpha}}^{\vec{n}} \exp[(t-s)BM]. \quad (2.26)$$

Therefore, after simplification, we can obtain the following conclusion:

$$B_2 \leq Mt^2\|f^{\vec{n}} - g^{\vec{n}}\| \exp[tM(B+1)] \left[\frac{M}{2}(\|f\|^{\vec{n}})^2 + N_l S_1 \|f^{\vec{n}}\| \prod_i (1+n_i) \right]. \quad (2.27)$$

From the expression of (2.21), we obtain

$$\begin{aligned} &[\bar{B}(s,\vec{x},f) - \bar{B}(s,\vec{x},g)] \\ &= \left[\frac{1}{2} \int_{\vec{0}}^{\vec{x}} K_{\vec{n}}(\vec{x} \ominus \vec{y}; \vec{y}) [f^{\vec{n}}(t,\vec{x} \ominus \vec{y}) f^{\vec{n}}(t,\vec{y}) - g^{\vec{n}}(t,\vec{x} \ominus \vec{y}) g^{\vec{n}}(t,\vec{y})] d\vec{y} \right. \\ &\left. + \int_{\vec{x}}^{\vec{n}} b_{\vec{n}}(\vec{x}|\vec{y}) S_{\vec{n}}(\vec{y}) [f_{\vec{n}}(t,\vec{y}) - g_{\vec{n}}(t,\vec{y})] d\vec{y} \right]. \end{aligned} \quad (2.28)$$

We can simplify the fist part of (2.28) in the following way

$$\begin{aligned} &\int_{\vec{0}}^{\vec{n}} \prod (1+x_i) \int_0^t \exp\{-[J(g,t,\vec{x}) - J(g,s,\vec{x})]\} \\ &\frac{1}{2} \int_{\vec{0}}^{\vec{x}} K_{\vec{n}}(\vec{x} \ominus \vec{y}; \vec{y}) [f^{\vec{n}}(t,\vec{x} \ominus \vec{y}) f^{\vec{n}}(t,\vec{y}) - g^{\vec{n}}(t,\vec{x} \ominus \vec{y}) g^{\vec{n}}(t,\vec{y})] d\vec{y} ds d\vec{x} \\ &\leq 2Bt\exp(tM(B+1))\|f^{\vec{n}} - g^{\vec{n}}\|. \end{aligned} \quad (2.29)$$

And for the last part of (2.28), we get

$$\int_{\vec{0}}^{\vec{n}} \prod (1+x_i) \int_0^t \exp\{-[J(g,t,\vec{x}) - J(g,s,\vec{x})]\}$$
$$\int_{\vec{x}}^{\vec{n}} b_{\vec{n}}(\vec{x}|\vec{y}) S_{\vec{n}}(\vec{y}) [f_{\vec{n}}(t,\vec{y}) - g_{\vec{n}}(t,\vec{y})] d\vec{y} ds d\vec{x}$$
$$\leq t S_1 N_l \exp(Mt(B+1)) \|f^{\vec{n}} - g^{\vec{n}}\| \prod_i (1+n_i). \tag{2.30}$$

Now we are in the final step to show the contraction property. We note that

$$\int_{\vec{0}}^{\vec{n}} \prod_i (1+x_i)^{\alpha_i} (C(f^{\vec{n}}) - C(g^{\vec{n}}))(t,\vec{x}) d\vec{x} \leq B_1 + B_2 + B_3$$
$$\leq \|f^{\vec{n}} - g^{\vec{n}}\| \Big[BMt \exp(Mt(B+1))$$
$$+ Mt^2 \exp(Mt(B+1)) \left(\frac{M}{2} B^2 + N_l S_1 B \prod_i (1+n_i) \right)$$
$$+ t S_1 N_l \exp(Mt(B+1)) \prod_i (1+n_i) \Big] \leq k_0 \|f-g\|^{\vec{n}},$$

where $0 \leq t \leq t_0$ and

$$k_0 = \Bigg[BMt \exp(Mt(B+1)) + Mt^2 \exp(Mt(B+1)) \left(\frac{M}{2} B^2 + N_l S_1 B \prod_i (1+n_i) \right)$$
$$+ t S_1 N_l \exp(Mt(B+1)) \prod_i (1+n_i) \Bigg]$$
$$= \exp(Mt(B+1)) \Bigg[BMt + Mt^2 \left(\frac{M}{2} B^2 + N_l S_1 B \prod_i (1+n_i) \right)$$
$$+ t S_1 N_l \prod_i (1+n_i) \Bigg] < 1. \tag{2.31}$$

This is the end of the proof of the Lemma 2.4. □

Now we predict the expression of L, to do this we have obtained the following:

$$\int_{\vec{0}}^{\vec{n}} \prod_i (1+x_i) f^{\vec{n}}(t,\vec{x}) d\vec{x} = \int_{\vec{0}}^{\vec{n}} \prod_i (1+x_i) f^{\vec{n}}(0,\vec{x}) d\vec{x}$$
$$+ \int_0^t \int_{\vec{0}}^{\vec{n}} \prod_i (1+x_i) \Bigg[\frac{1}{2} \int_{\vec{0}}^{\vec{x}} K_n(\vec{x} \ominus \vec{y}; \vec{y}) f^{\vec{n}}(t, \vec{x} \ominus \vec{y}) f^{\vec{n}}(t,\vec{y}) \, \mathrm{d}\vec{y}$$
$$- f^{\vec{n}}(t,\vec{x}) \int_{\vec{0}}^{\vec{n} \ominus \vec{x}} K_n(\vec{x}; \vec{y}) f^{\vec{n}} n(t,\vec{y}) \, \mathrm{d}\vec{y}$$
$$+ \int_{\vec{x}}^{\vec{n}} b_n(\vec{x}|\vec{y}) S_n(\vec{y}) f_{\vec{n}}(t,\vec{y}) \, \mathrm{d}\vec{y} - S_n(\vec{x}) f^{\vec{n}}(t,\vec{x}) \Bigg] d\vec{x} ds$$

$$
\begin{aligned}
&\le \|f_0^{\vec{n}}\| + \int_0^t \int_{\vec{0}}^{\vec{n}} \prod_i (1+x_i) \int_{\vec{x}}^{\vec{n}} b_n(\vec{x}|\vec{y}) S_n(\vec{y}) f^{\vec{n}}(t,\vec{y})\, \mathrm{d}\vec{y} d\vec{x} ds \\
&\le \|f_0^{\vec{n}}\| + N_l \prod_i (1+n_i) \int_0^t \int_{\vec{0}}^{\vec{n}} \left[\int_{\vec{0}}^{\vec{n}} \vec{y} f^{\vec{n}}(t,\vec{y}) d\vec{y} \right] d\vec{x} ds \\
&\le \|f_0^{\vec{n}}\| + N_l \prod_i (1+n_i) n_i \int_0^t \left[\int_{\vec{0}}^{\vec{n}} \vec{y} f^{\vec{n}}(0,\vec{y}) d\vec{y} \right] ds \\
&\le \|f_0^{\vec{n}}\| + N_l T \|f_0^{\vec{n}}\| \prod_i (1+n_i) n_i = L.
\end{aligned} \tag{2.32}
$$

It follows from Lemmas 2.3, 2.4 and the contraction mapping theorem that a unique solution $f^{\vec{n}}(t,\vec{x})$ exists for $0 \le t \le t_0$. □

2.3. Global existence of a solution in finite time.

Theorem 2.5. *Under the kernel conditions (A1)-(A5) and the initial condition satisfies $f_0(\vec{x}) \in \mathcal{F}^+(0)$. Then the problem (1.1), (1.2) possesses at least one solution in $\mathcal{F}^+(T)$.*

Denote the $\vec{j}^{\text{th}}$ order truncated central moment of $\hat{f}_{\vec{n}}(t,\vec{x})$ as

$$
N_{\vec{n},\vec{j}}(t) = \int_{\vec{0}}^{\vec{n}} \prod_{i=1}^{d} x_i^{j_i} \hat{f}_{\vec{n}}(t,\vec{x})\, \mathrm{d}\vec{x}, \;\; j_i \in \mathbb{R}, \;\; n \ge 1. \tag{2.33}
$$

We consider $f^{\vec{n}}$, the non-negative unique solution obtained from the last theorem. Let us consider the zero extension of the solution in full domain as

$$
\hat{f}^{\vec{n}}(t,\vec{x}) = \begin{cases} f^{\vec{n}}(t,\vec{x}), \;\; 0 \le x_i \le n_i, \;\; t \in [0,T], \\ 0, \;\; x_i > n_i. \end{cases} \tag{2.34}
$$

Lemma 2.6. *The moments $\{N_{\vec{n},\vec{p}}(t)\}_{n_i=1}^{\infty}$ are bounded uniformly, with respect to t and $\vec{n}$, for $0 \le t \le T$ and $0 \le p_i \le 2$, i.e.,*

$$
N_{\vec{n},\vec{p}}(t) \le \bar{N}_{\vec{p}} = \textit{ constant independent of } \vec{n}, \; t; \;\; n_i \ge 1, \;\; 0 \le t \le T \textit{ and } 0 \le p_i \le 2. \tag{2.35}
$$

Case 1: First cross-moment or hyper-volume conservation. For $t \in [0,T]$, Integrating equation (2.4) multiplied by $\prod_{i=1}^{d} x_i$, we obtain

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} \int_0^{\vec{n}} \prod_{i=1}^{d} x_i f^{\vec{n}}(t,\vec{x}) \mathrm{d}\vec{x} &= \frac{1}{2} \int_0^{\vec{n}} \int_{\vec{0}}^{\vec{x}} \prod_{i=1}^{d} x_i K_n(\vec{x} \ominus \vec{y}; \vec{y}) f^{\vec{n}}(t, \vec{x} \ominus \vec{y}) f^{\vec{n}}(t,\vec{y})\, \mathrm{d}\vec{y} \mathrm{d}\vec{x} \\
&- \int_0^{\vec{n}} \int_{\vec{0}}^{\vec{n} \ominus \vec{x}} \prod_{i=1}^{d} x_i f^{\vec{n}}(t,\vec{x}) K_n(\vec{x};\vec{y}) f^{\vec{n}} n(t,\vec{y})\, \mathrm{d}\vec{y} \mathrm{d}\vec{x} \\
+ \int_0^{\vec{n}} \int_{\vec{x}}^{\vec{n}} \prod_{i=1}^{d} x_i b_n(\vec{x}|\vec{y}) S_n(\vec{y}) f_{\vec{n}}(t,\vec{y})\, \mathrm{d}\vec{y} \mathrm{d}\vec{x} &- \int_0^{\vec{n}} \prod_{i=1}^{d} x_i S_n(\vec{x}) f^{\vec{n}}(t,\vec{x}) \mathrm{d}\vec{x}.
\end{aligned} \tag{2.36}
$$

All the integrals are finite as the corresponding kernels are finite in the compact domain of integration. We consider the first integral. Using Fubini's theorem and the symmetry of the coagulation kernel we obtain

$$\begin{aligned}
&\frac{1}{2}\int_{\vec{0}}^{\vec{n}}\int_{\vec{0}}^{\vec{x}}\prod_{i=1}^{d}x_iK_n(\vec{x}\ominus\vec{y};\vec{y})f^{\vec{n}}(t,\vec{x}\ominus\vec{y})f^{\vec{n}}(t,\vec{y})\,\mathrm{d}\vec{y}\mathrm{d}\vec{x}\\
=&\frac{1}{2}\int_{\vec{0}}^{\vec{n}}\int_{\vec{y}}^{\vec{n}}\prod_{i=1}^{d}x_iK_n(\vec{x}\ominus\vec{y};\vec{y})f^{\vec{n}}(t,\vec{x}\ominus\vec{y})f^{\vec{n}}(t,\vec{y})\,\mathrm{d}\vec{x}\mathrm{d}\vec{y}\\
=&\frac{1}{2}\int_{\vec{0}}^{\vec{n}}\int_{\vec{0}}^{\vec{n}\ominus\vec{y}}\prod_{i=1}^{d}(x_i+y_i)K_n(\vec{x};\vec{y})f^{\vec{n}}(t,\vec{x})f^{\vec{n}}(t,\vec{y})\,\mathrm{d}\vec{x}\mathrm{d}\vec{y}\\
=&\int_{\vec{0}}^{\vec{n}}\int_{\vec{0}}^{\vec{n}\ominus\vec{y}}\prod_{i=1}^{d}x_iK_n(\vec{x};\vec{y})f^{\vec{n}}(t,\vec{x})f^{\vec{n}}(t,\vec{y})\,\mathrm{d}\vec{x}\mathrm{d}\vec{y}\\
=&\text{ second integral of R.H.S.}
\end{aligned}$$

In a similar manner and using volume conservation properties of breakage kernels we can show that

$$\text{third integral of R.H.S.} = \text{fourth integral of R.H.S.}$$

Hence, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^{\vec{n}}\prod_{i=1}^{d}x_if^{\vec{n}}(t,\vec{x})\mathrm{d}\vec{x}=0,$$

which indicates

$$N_{\vec{n},\vec{1}}(t)=\int_0^{\vec{\infty}}\prod_{i=1}^{d}x_if_0(\vec{x})\mathrm{d}\vec{x}=\bar{N}_{\vec{1}},\ \text{say} <\infty. \tag{2.37}$$

Case 2: Second cross-moment or energy conservation. In this case, proceeding earlier fashion we obtain,

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_0^{\vec{n}}\prod_{i=1}^{d}x_i^2f^{\vec{n}}(t,\vec{x})\mathrm{d}\vec{x}\\
\le&\int_0^{\vec{n}}\int_{\vec{0}}^{\vec{n}\ominus\vec{y}}\prod_{i=1}^{d}x_iy_iK_n(\vec{x};\vec{y})f^{\vec{n}}(t,\vec{x})f^{\vec{n}}(t,\vec{y})\,\mathrm{d}\vec{y}\,\mathrm{d}\vec{x}\\
\le&\, k\int_0^{\vec{n}}\int_{\vec{0}}^{\vec{n}\ominus\vec{y}}\prod_{i=1}^{d}x_iy_i(1+x_i)(1+y_i)f^{\vec{n}}(t,\vec{x})f^{\vec{n}}(t,\vec{y})\,\mathrm{d}\vec{y}\,\mathrm{d}\vec{x}\\
\le&\, k[N_{\vec{n},\vec{1}}^2(t)+2N_{\vec{n},\vec{2}}(t)N_{\vec{n},\vec{1}}(t)+N_{\vec{n},\vec{2}}^2(t)].
\end{aligned}$$

Integrating the above relation from 0 to t, we have

$$N_{\vec{n},\vec{2}}(t)\le\bar{N}_{\vec{2}}<\infty. \tag{2.38}$$

Case 3: Following in the same way the above results holds for $N_{\vec{n},\vec{j}}(t)$ for $j = 3, 4, 5, \ldots$ independent of $\vec{n}$ and t.

Case 4: Zeroth moment or total number of particles. For zeroth order moment, the result reduces to

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} N_{\vec{n},\vec{0}}(t) &= \frac{\mathrm{d}}{\mathrm{d}t} \int_0^{\vec{n}} f^{\vec{n}}(t,\vec{x})\mathrm{d}\vec{x} \\
&\le \int_{\vec{0}}^{\vec{n}} \int_{\vec{x}}^{\vec{n}} b_{\vec{n}}(\vec{x}|\vec{y}) S_{\vec{n}}(\vec{y}) f_{\vec{n}}(t,\vec{y})\,\mathrm{d}\vec{y}\mathrm{d}\vec{x} - \int_0^{\vec{n}} S_n(\vec{x}) f^{\vec{n}}(t,\vec{x})\mathrm{d}\vec{x} \\
&\le \int_{\vec{0}}^{\vec{n}} \int_{\vec{x}}^{\vec{n}} b_{\vec{n}}(\vec{x}|\vec{y}) S_{\vec{n}}(\vec{y}) f_{\vec{n}}(t,\vec{y})\,\mathrm{d}\vec{y}\mathrm{d}\vec{x} \\
&\le N_l \int_{\vec{0}}^{\vec{n}} S_1 \prod_{i=1}^{d} y_i f_{\vec{n}}(t,\vec{y})\,\mathrm{d}\vec{y} \\
&\le N_l S_1 N_{\vec{n},\vec{1}}(t) \le N_l S_1 \bar{N}_{\vec{1}}.
\end{aligned}$$

Case 5: Moment of arbitrary order. Similarly we can show that

$$N_{\vec{n},\vec{p}}(t) \le \bar{N}_{\vec{p}} = \text{ constant independent of } \vec{n},\ t;\ n_i \ge 1,\ t \in [0,T] \text{ and } 0 \le p_i \le 2. \tag{2.39}$$

Combining all the case results the lemma follows consequently.

To obtain the main result, we state the following lemma:

Lemma 2.7. *The sequence $\{\hat{f}_{\vec{n}}\}_{n_i=1}^{\infty}$ is compact (relative) in the uniform-convergence topology of continuous functions on each rectangle $\mathcal{P}(T,\vec{X})$.*

Proof. We prove the Lemma 2.7 through three successive steps.

Step 1: In the first step, we show that $\{\hat{f}_{\vec{n}}\}_{n_i=1}^{\infty}$ is uniformly bounded on $\mathcal{P}(T,\vec{X})$. From (2.4) we have

$$\begin{aligned}
\frac{\partial f^{\vec{n}}(t,\vec{x})}{\partial t} &= \frac{1}{2}\int_{\vec{0}}^{\vec{x}} K_n(\vec{x}\ominus\vec{y};\vec{y}) f^{\vec{n}}(t,\vec{x}\ominus\vec{y}) f^{\vec{n}}(t,\vec{y})\,\mathrm{d}\vec{y} \\
&\quad - f^{\vec{n}}(t,\vec{x}) \int_{\vec{0}}^{\vec{n}\ominus\vec{x}} K_n(\vec{x};\vec{y}) f^{\vec{n}} n(t,\vec{y})\,\mathrm{d}\vec{y} \\
&\quad + \int_{\vec{x}}^{\vec{n}} b_n(\vec{x}|\vec{y}) S_n(\vec{y}) f_{\vec{n}}(t,\vec{y})\,\mathrm{d}\vec{y} - S_n(\vec{x}) f^{\vec{n}}(t,\vec{x}) \\
&\le \frac{1}{2}\int_{\vec{0}}^{\vec{x}} K_n(\vec{x}\ominus\vec{y};\vec{y}) f^{\vec{n}}(t,\vec{x}\ominus\vec{y}) f^{\vec{n}}(t,\vec{y})\,\mathrm{d}\vec{y} + \int_{\vec{x}}^{\vec{n}} b_n(\vec{x}|\vec{y}) S_n(\vec{y}) f_{\vec{n}}(t,\vec{y})\,\mathrm{d}\vec{y} \\
&\le \frac{1}{2}\int_{\vec{0}}^{\vec{x}} k \prod_i (1+X_i)^2 f^{\vec{n}}(t,\vec{x}\ominus\vec{y}) f^{\vec{n}}(t,\vec{y})\,\mathrm{d}\vec{y} + S_1\bar{b}\int_{\vec{x}}^{\vec{n}} \vec{y} f^{\vec{n}}(t,\vec{y})\,\mathrm{d}\vec{y} \\
&= \frac{1}{2} k \prod_i (1+X_i)^2 (f^{\vec{n}} * f^{\vec{n}}) + S_1\bar{b}\bar{N}_{\vec{1}}.
\end{aligned} \tag{2.40}$$

Consider upper function $g^{\vec{n}}$ that satisfies

$$g^{\vec{n}}(t,\vec{x}) = g_0^{\vec{n}} + \int_0^t \left[\frac{1}{2}k\prod_i(1+X_i)^2(g^{\vec{n}} * g^{\vec{n}}) + g^{\vec{n}}(s,\vec{x})\right]ds, \tag{2.41}$$

where

$$g_0^{\vec{n}} = \max\{f_0^{\vec{n}}, S_1\bar{b}\bar{N}_{\vec{1}}\}.$$

The next, we used the Laplace transform, and obtain

$$g^{\vec{n}}(t,\vec{x}) = g_0 \exp\left[\frac{1}{2}g_0 k\prod_i X_i(1+X_i)^2(\exp(t)-1)+t\right].$$

Now we show $f^{\vec{n}}(t,\vec{x}) \le g^{\vec{n}}(t,\vec{x})$ for $(t,\vec{x}) \in \mathcal{P}(T,\vec{X})$.

Using similar analysis of [22], we can prove that

$$0 \le f^{\vec{n}}(t,\vec{x}) \le g_0\left[\frac{1}{2}g_0 k\prod_i X_i(1+X_i)^2(\exp(T)-1)+T\right] = L_1, \text{ a constant.} \tag{2.42}$$

Step 2: In this part, we prove the equicontinuity property of $\{f^{\vec{n}}\}_{n_i=1}^{\infty}$ for the variable t. Using (2.4), for $t \in [0,T], t \le t',\ x \in [0,X_i],\ n_i \ge 1$, we observe that

$$\begin{aligned}|f^{\vec{n}}(t',\vec{x}) - f^{\vec{n}}(t,\vec{x})| \le \int_t^{t'} \Bigg\{ &\frac{1}{2}\int_{\vec{0}}^{\vec{x}} K_{\vec{n}}(\vec{x}\ominus\vec{y};\vec{y})f^{\vec{n}}(s,\vec{x}\ominus\vec{y})f^{\vec{n}}(s,\vec{y})\,\mathrm{d}\vec{y} \\ &- f^{\vec{n}}(s,\vec{x})\int_{\vec{0}}^{\vec{\infty}} K_n(\vec{x};\vec{y})f^{\vec{n}}(s,\vec{y})\,\mathrm{d}\vec{y} \\ &+ \int_{\vec{x}}^{\vec{\infty}} b_{\vec{n}}(\vec{x}|\vec{y})S_{\vec{n}}(\vec{y})f_{\vec{n}}(s,\vec{y})\,\mathrm{d}\vec{y} - S_{\vec{n}}(\vec{x})f^{\vec{n}}(s,\vec{x})\Bigg\}ds.\end{aligned} \tag{2.43}$$

It follows form (2.1), (2.2), (2.3) and (2.42) that the first and the fourth terms of (2.43) are uniformly bounded. For the second integral of (2.43), we have

$$\begin{aligned}\int_{\vec{0}}^{\vec{\infty}} K_{\vec{n}}(\vec{x};\vec{y})f^{\vec{n}}(s,\vec{y})\,\mathrm{d}\vec{y} &\le k\int_{\vec{0}}^{\vec{\infty}}\prod_i(1+X_i)(1+y_i)f^{\vec{n}}(s,\vec{y})d\vec{y} \\ &\le k\prod_i(1+X_i)(\bar{N}_{\vec{0}}+\bar{N}_{\vec{1}}),\end{aligned} \tag{2.44}$$

$$\text{and} \quad \int_{\vec{x}}^{\vec{\infty}} b_{\vec{n}}(\vec{x}|\vec{y})S_{\vec{n}}(\vec{y})f^{\vec{n}}(t,\vec{y})\,\mathrm{d}\vec{y} \le S_1\bar{b}\bar{N}_{\vec{1}}. \tag{2.45}$$

We finally obtain

$$\sup_{0\le x_i\le X_i}|f^{\vec{n}}(t',\vec{x}) - f^{\vec{n}}(t,\vec{x})| \le M_2|t-t'|,\ 0\le t\le t'\le T,\ n_i \ge 1. \tag{2.46}$$

The constant M_2 is independent of $\vec{n}$ and hence $\{f^{\vec{n}}\}_{n_i=1}^{\infty}$ is equicontinuous w.r.t. the variable t on $\mathcal{P}(T,\vec{X})$.

Step 3: In this step, we justify the equicontinuity property of $\{f^{\vec{n}}\}_{n_i=1}^{\infty}$ for the variable $\vec{x}$. Let, $\vec{x} \in [0, \vec{X}], \vec{x} \leq \vec{x}' \leq \vec{X}$; therefore for $n_i \geq 1$ we note that

$$\begin{aligned}
|f^{\vec{n}}(\vec{x'}, t) - f^{\vec{n}}(\vec{x}, t)| \leq\ & |f_0(\vec{x}') - f_0(\vec{x})| \\
& + \int_0^t \Bigg[\frac{1}{2} \int_{\vec{x}}^{\vec{x'}} K_{\vec{n}}(\vec{x'} - \vec{y}, \vec{y}) f^{\vec{n}}(\vec{x'} - \vec{y}, s) f^{\vec{n}}(\vec{y}, s) d\vec{y} \\
& + \frac{1}{2} \int_{\vec{0}}^{\vec{x}} |K_{\vec{n}}(\vec{x'} - \vec{y}, \vec{y}) - K_{\vec{n}}(\vec{x} - \vec{y}, \vec{y})| f^{\vec{n}}(\vec{x'} - \vec{y}, s) f^{\vec{n}}(\vec{y}, s) d\vec{y} \\
& + \frac{1}{2} \int_{\vec{0}}^{\vec{x}} K_{\vec{n}}(\vec{x} - \vec{y}, \vec{y}) |f^{\vec{n}}(\vec{x'} - \vec{y}, s) - f^{\vec{n}}(\vec{x} - \vec{y}, s)| f^{\vec{n}}(\vec{y}, s) d\vec{y} \\
& + |f^{\vec{n}}(\vec{x'}, s) - f^{\vec{n}}(\vec{x}, s)| \int_{\vec{0}}^{\vec{\infty}} K_{\vec{n}}(\vec{x'}, \vec{y}) f^{\vec{n}}(\vec{y}, s) d\vec{y} \\
& + f^{\vec{n}}(\vec{x}, s) \int_{\vec{0}}^{\vec{\infty}} |K_{\vec{n}}(\vec{x'}, \vec{y}) - K_{\vec{n}}(\vec{x}, \vec{y})| f^{\vec{n}}(\vec{y}, s) d\vec{y} \qquad (2.47) \\
& + \int_{\vec{x}}^{\vec{x'}} b_{\vec{n}}(\vec{x}|\vec{y}) S_{\vec{n}}(\vec{y}) f^{\vec{n}}(t, \vec{y}) d\vec{y} \\
& + \int_{\vec{x'}}^{\vec{\infty}} |b_{\vec{n}}(\vec{x'}|\vec{y}) - b_{\vec{n}}(\vec{x}|\vec{y})| S_{\vec{n}}(\vec{y}) f^{\vec{n}}(t, \vec{y}) d\vec{y} \qquad (2.48) \\
& + |S_{\vec{n}}(\vec{x'}) - S_{\vec{n}}(\vec{x})| f^{\vec{n}}(\vec{x'}, s) + S_{\vec{n}}(\vec{x}) |f^{\vec{n}}(\vec{x'}, s) - f^{\vec{n}}(\vec{x}, s)| \Bigg] ds. \qquad (2.49)
\end{aligned}$$

Next, we show that when $\|\vec{x'} - \vec{x}\| = \max\{|x_i' - x_i| : i = 1, 2, ..., d\}$ is small enough, then the L.H.S. of (2.49) is also small. AS the kernels $K_{\vec{n}}, S_{\vec{n}}, b_{\vec{n}}$ and IC $f_0(\vec{x})$ are continuous, therefore for $\epsilon > 0$, there exists $\delta(\epsilon)$ such that

$$\begin{aligned}
&\sup_{\|\vec{x'} - \vec{x}\| < \delta} |f_0(\vec{x'}) - f_0(\vec{x})| < \epsilon, \\
&\sup_{\|\vec{x'} - \vec{x}\| < \delta} |K_{\vec{n}}(\vec{x'}, \vec{y}) - K_{\vec{n}}(\vec{x}, \vec{y})| < \epsilon, \\
&\sup_{\|\vec{x'} - \vec{x}\| < \delta} |S_{\vec{n}}(\vec{x'}) - S_{\vec{n}}(\vec{x})| < \epsilon \text{ and} \\
&\sup_{\|\vec{x'} - \vec{x}\| < \delta} |b_{\vec{n}}(\vec{x'}, \vec{y}) - b_{\vec{n}}(\vec{x}, \vec{y})| < \epsilon. \qquad (2.50)
\end{aligned}$$

These inequalities hold uniformly w.r.t. $\vec{n} \geq \vec{1}$ and $\vec{y} \leq \vec{z}$. The restrictions of $\vec{z}$ are given below. Let,

$$w_n(t) = \sup_{\|\vec{x'} - \vec{x}\| < \delta} |f^{\vec{n}}(\vec{x'}, t) - f^{\vec{n}}(\vec{x}, t)|, \ \vec{0} \leq \vec{x} \leq \vec{X}. \qquad (2.51)$$

From (2.47), we get

$$\begin{aligned}
&\int_{\vec{0}}^{\vec{\infty}} |K_{\vec{n}}(\vec{x'},\vec{y}) - K_{\vec{n}}(\vec{x},\vec{y})| f^{\vec{n}}(\vec{y},s)d\vec{y} \\
&= \int_{\vec{0}}^{\vec{z}} |K_{\vec{n}}(\vec{x'},\vec{y}) - K_{\vec{n}}(\vec{x},\vec{y})| f^{\vec{n}}(\vec{y},s)d\vec{y} + \int_{\vec{z}}^{\vec{\infty}} |K_{\vec{n}}(\vec{x'},\vec{y}) - K_{\vec{n}}(\vec{x},\vec{y})| f^{\vec{n}}(\vec{y},s)d\vec{y} \\
&\leq \epsilon \bar{N}_{\vec{0}} + k \int_{\vec{z}}^{\vec{\infty}} \prod_i [(1+x_i') - (1+x_i)](1+y_i) f^{\vec{n}}(\vec{y},s)d\vec{y} \\
&\leq \epsilon \bar{N}_{\vec{0}} + k \prod_i \frac{(1+X_i)}{1+z_i} \int_{\vec{z}}^{\vec{\infty}} \prod_i (1+y_i)^2 f^{\vec{n}}(\vec{y},s)d\vec{y} \\
&\leq \epsilon \bar{N}_{\vec{0}} + k \prod_i (1+X_i) \prod_i \frac{(\bar{N}_{\vec{0}}^2 + 2\bar{N}_{\vec{1}} + \bar{N}_{\vec{2}}^2)}{1+z_i}, \\
&\leq \epsilon (\bar{N}_0 + k \prod_i (1+X_i)), \qquad (2.52)
\end{aligned}$$

choose z_i such that $\prod_i \frac{(\bar{N}_{\vec{0}}^2 + 2\bar{N}_{\vec{1}} + \bar{N}_{\vec{2}}^2)}{1+z_i} < \epsilon$. From (2.48), we get

$$\begin{aligned}
&\int_{\vec{x'}}^{\vec{\infty}} |b_{\vec{n}}(\vec{x'}|\vec{y}) - b_{\vec{n}}(\vec{x}|\vec{y})| S_{\vec{n}}(\vec{y}) f^{\vec{n}}(t,\vec{y})d\vec{y} \\
&= \int_{\vec{x'}}^{\vec{z}} |b_{\vec{n}}(\vec{x'}|\vec{y}) - b_{\vec{n}}(\vec{x}|\vec{y})| S_{\vec{n}}(\vec{y}) f^{\vec{n}}(t,\vec{y})d\vec{y} + \int_{\vec{z}}^{\vec{\infty}} |b_{\vec{n}}(\vec{x'}|\vec{y}) - b_{\vec{n}}(\vec{x}|\vec{y})| S_{\vec{n}}(\vec{y}) f^{\vec{n}}(t,\vec{y})d\vec{y} \\
&\leq \epsilon \int_{\vec{x'}}^{\vec{z}} \prod_i S_1 y_i f^{\vec{n}}(t,\vec{y})d\vec{y} + 2S_1\bar{b} \int_{\vec{z}}^{\vec{\infty}} \prod_i y_i f^{\vec{n}}(t,\vec{y})d\vec{y} \\
&\leq \epsilon S_1 \bar{N}_{\vec{1}} + 2S_1\bar{b} \prod_i \frac{1}{z_i} \int_{\vec{z}}^{\vec{\infty}} \prod_i y_i^2 f^{\vec{n}}(t,\vec{y})d\vec{y} \\
&\leq \epsilon [S_1 \bar{N}_{\vec{1}} + 2S_1\bar{b}], \qquad (2.53)
\end{aligned}$$

choose $\vec{z}$ such a way that $\prod_i \frac{1}{z_i} \bar{N}_{\vec{2}} < \epsilon$. From (2.49), we get

$$\int_0^t |S_{\vec{n}}(\vec{x'}) - S_{\vec{n}}(\vec{x})| f^{\vec{n}}(\vec{x'},s)ds < \epsilon L_1 T, \text{ by using (2.42).} \qquad (2.54)$$

From (2.49), we have

$$\int_0^t S_{\vec{n}}(\vec{x}) |f^{\vec{n}}(\vec{x'},s) - f^{\vec{n}}(\vec{x},s)| ds < S_1 \prod_i X_i \int_0^t w_n(s)ds. \qquad (2.55)$$

Finally, using (2.42), all inequality in (2.50) (2.52), (2.53), (2.54), (2.55) and (2.49), we get:

$$w_{\vec{n}}(t) \leq M_3 \epsilon + M_4 \int_0^t w_{\vec{n}}(s)ds, \ 0 \leq t \leq T,$$

where the constants M_3 and M_4 do not depend on $\vec{n}$ and ϵ. With the help of Gronwall's inequality, we note that

$$w_{\vec{n}}(t) \leq M_3\epsilon \exp(M_4 T) = M_5\epsilon \text{ (say)}. \tag{2.56}$$

We conclude from (2.46) and (2.56) that

$$\sup_{\|\vec{x}'-\vec{x}\|<\delta, 0\leq t\leq t'\leq T} |f^{\vec{n}}(\vec{x}', t') - f^{\vec{n}}(x, t)| \leq (M_2 + M_5)\epsilon. \tag{2.57}$$

Hence the Lemma 2.7 is proved. □

Proof of theorem 2.5

Using the diagonal method we choose a subsequence $\{f^{\vec{p}}\}_{\vec{p}=\vec{1}}^{\vec{\infty}}$ from $\{f^{\vec{n}}\}_{\vec{n}=\vec{1}}^{\vec{\infty}}$ which converges uniformly to a continuous non-negative function f on each compact set in $\mathcal{P}$. In the next, we focus on the following integral

$$\int_{\vec{0}}^{\vec{z}} \vec{x}^k f(\vec{x}, t) d\vec{x}, \text{ where } \vec{0} \leq \vec{k} \leq \vec{2}.$$

We note that

$$\forall\ \epsilon > 0\ \exists\ \vec{r} \geq \vec{1} \text{ such that} \int_{\vec{0}}^{\vec{z}} \vec{x}^{\vec{k}} f(\vec{x}, t) d\vec{x} \leq \int_{\vec{0}}^{\vec{z}} \vec{x}^{\vec{k}} f^{\vec{r}}(\vec{x}, t) d\vec{x} \leq \bar{N}_{\vec{k}} + \epsilon. \tag{2.58}$$

Here, for $\vec{x} = (x_1, x_2, x_3, ..., x_d)$ and $\vec{k} = (k_1, k, ..., k_d)$, we define $\vec{x}^{\vec{k}} = \prod_{i=1}^{d} x_i^{k_i}$. Then

$$\int_{\vec{0}}^{\vec{z}} \vec{x}^{\vec{k}} f(\vec{x}, t) d\vec{x} \leq \bar{N}_{\vec{k}},$$

because in (2.58) both $\vec{z}$ and ϵ are arbitrary. Similar to [22] we can show that the function $f(\vec{x}, t)$ is a solution to (1.1), (1.2).

3. Concluding remarks

In this article, we establish the existence of solutions for the higher-dimensional coagulation-fragmentation equation. The regulations pertaining to the kernels are applicable to a wide class of practically relevant kernels. We demonstrate the existence of a continuous solution that conserves volume by considering the continuity of the initial condition, the boundedness of the zeroth and first initial moments, and additional restrictions on regularity of the kernels. To prove the result, we have defined a new norm and examined the equicontinuity with respect to time and the property characteristics vector. By the boundedness of the moments and contraction mapping theorem, we obtained the required result. The form of the coagulation kernel is the following:

$$K(\vec{x}, \vec{y}) \leq k \prod_{i=1}^{d} (1 + x_i)_i^{\alpha} (1 + y_i)_i^{\alpha}$$

where $\alpha_i = 1$. In the future, we shall focus on the mentioned coagulation kernel for $\alpha_i > 1$.

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