Fixed Point Theory, 25(2024), No. 2, 545-568
DOI: 10.24193/fpt-ro.2024.2.07
http://www.math.ubbcluj.ro/~nodeacj/sfptcj.html

HR-TYPE INTERPOLATIVE AND P-CONTRACTIONS VIA MAIA TYPE RESULT IN B-METRIC SPACES WITH APPLICATIONS

SUDIPTA KUMAR GHOSH*, C. NAHAK** AND RAVI P. AGARWAL***

Dedicated to Academician Professor Ram N. Mohapatra, Department of Mathematical Sciences, University of Central Florida USA, on the occasion of his 80^{th} *birthday.*

*Department of Mathematics, School of Applied Sciences,
Kalinga Institute of Industrial Technology (KIIT) Deemed to be University,
Bhubaneswar 751024, Odisha, India
E-mail: ghosh.sudipta516@gmail.com

**Department of Mathematics, IIT Kharagpur, India
E-mail: cnahak@maths.iitkgp.ac.in

***Department of Mathematics, Texas A & M University-Kingsville,
700 University Blvd., MSC 172, Kingsville, TX, USA
Emeritus Research Professor,
Department of Mathematics and Systems Engineering,
Florida Institute of Technology,
Melbourne, FL 32901, USA
E-mail: Ravi.Agarwal@tamuk.edu

Abstract. The main aim of this work is to study multivalued HR-type interpolative contraction and multivalued P-contraction through $(\mathcal{H}, \mathcal{F})$-contraction and C-class function, respectively via Maia type result in b-metric spaces setting. Moreover, by using the notion of multi-valued triangular α-admissible mappings of type Λ, we have investigated our new findings. Some related fixed point results for such mappings are also proved in this set-up. Our results extend, generalize and modify many famous results that exist in the literature. We furnish an example to justify our new findings. As an application, we implement our results in data dependence and stability of fixed point.

Key Words and Phrases: Multivalued mapping, admissible mapping, interpolative contraction, P-contraction, b-metric spaces, Maia-type result, $(\mathcal{H}, \mathcal{F})$-contraction, C-class function, generalized data dependence, generalized stability problem.
2020 Mathematics Subject Classification: 47H09, 47H10, 54H25.

1. Introduction and preliminaries

The notion of b-metric spaces was initiated by Bakhtin [26]. After that, some new fixed point results along with the well known Banach's contraction principle (in brief BCP), were investigated on this set-up by Czerwik [30]. Czerwik's [30] work attracted many researchers to work on this set up. Many famous fixed point results have been studied on this set-up (for example, readers can see [9], [16], [43], [48], [50], [51] and the references therein). By using Pompeiu-Hausdorff metric, fixed point results were extended from single valued to multi-valued mappings by Nadler [46]. The result of Nadler became very popular in fixed point theory, and consequently, many researchers started to work on multi-valued mappings into various direction. For results on multi-valued mappings, readers can look into [1],[4],[5], [7], [8], [12], [15], [19], [20], [22], [25], [16], [24], [28], [29] [52] [54] and the references therein. In 2012, Samet et al. [53] introduced the notion of α-admissible mappings to study some fixed point results. In fixed point theory, the concept of α-admissible mappings is a very powerful tool since it covers many important structures like the structure of standard metric spaces, the structure of a metric space endowed with a graph, the structure of a metric space endowed with a partial order, and the structure of cyclic mappings via closed subsets of a metric space etc. Due to the reasons mentioned just, researchers have started to work on α-admissible mappings in large scale. To study some recent fixed point results on α-admissible mappings, readers can view [2], [3], [16], [32], [33], [34], [42] and the references therein.

Very recently, Karapinar [38] introduced the notion of interpolative Kannan contraction (in brief IKC) by modifying the famous Kannan contraction [37]. The interpolative Kannan contraction condition is given by

IKC: For a metric space (X, δ), a self mapping $\mathcal{J}$ from X into X is said to satisfy IKC if $\exists\ \beta \in [0, 1)$ and $\varrho \in (0, 1)$, such that the following holds

$$\delta(\mathcal{J}u, \mathcal{J}v) \leq \beta\big[\delta(u, \mathcal{J}u)\big]^{\varrho} \cdot \big[\delta(v, \mathcal{J}v)\big]^{1-\varrho},$$

$\forall\ u, v \in X$ with $u \neq \mathcal{J}u$.

Theorem 1.1. [38] *Let (X, δ) be a complete metric space. Suppose that $\mathcal{J}$ is a self mapping from X into X which satisfies IKC. Then $\mathcal{J}$ has a unique fixed point.*

But immediately after the work of Karapinar [38], Karapinar et al. [40] observed a small error about the supposition of fixed point set being unique. After the work of Karapinar et al. [40], different types of results on interpolative contraction have been investigated by many researchers. For example, one can look into [21], [23], [39], [41].

Next, we recollect another important fixed point result due to Maia which states as:

Theorem 1.2. [45] *Let (X, δ, ρ) be a bimetric space and $\mathcal{J}$ be a self mapping from X into X. Suppose that the following assertions hold:*

(1) $\delta(u, v) \leq \rho(u, v), \forall\ u, v \in X$;
(2) $\mathcal{J}$ *is continuous w.r.t* δ;
(3) $\mathcal{J}$ *is complete w.r.t* δ;
(4) $\exists$ *a* $\beta \in [0, 1)$ *such that*

$$\rho(\mathcal{J}u, \mathcal{J}v) \leq \beta\rho(u, v), \forall\ u, v \in X.$$

Then $\mathcal{J}$ has a unique fixed point in X.

The result of Maia [45] influenced many authors to work on different types of fixed point results. For example, reader can look into [18], [6] [27] [44], [47] and the references therein.

Next, we state another important fixed point result due to Popescu [49], which greatly extends the famous BCP.

P-contraction: Let (X, δ) be a metric space. Suppose that $\mathcal{J}$ is a self mapping from X into X. Then $\mathcal{J}$ is said to be a P-contraction if $\exists$ a $\beta \in [0,1)$, such that the following holds:

$$\delta(\mathcal{J}u, \mathcal{J}v) \leq \beta\big[\delta(u,v) + \mid \delta(u, \mathcal{J}u) - \delta(v, \mathcal{J}v) \mid\big], \forall\ u, v \in X.$$

Some researchers call the above contraction as "E-contraction" in stead of "P-contraction". But, we prefer to call the contraction as " P-contraction".

Theorem 1.3. [49] *Let (X, δ) be a complete metric space and $\mathcal{J}$ be a self mapping from X into X which satisfies P-contraction. Then $\mathcal{J}$ has a unique fixed point. Moreover, every Picard iteration converges to the fixed point.*

Many authors put their attention on P-contraction and established different kinds of fixed point results. For example, reader can view [13], [14], [31], [35].

For the sake of completeness, we now recall some basic definitions, propositions, lemmas, and necessary results from the existing literature for our investigations.

From now, we will write $P(X), CL(X), CB(X), K(X)$ to denote the collection of non-empty subsets, non empty closed subsets, non empty closed-bounded subsets and non empty compact subsets of X, respectively.

Definition 1.1. [46] A point $u \in X$ is said to be fixed point of the multivalued mapping $\mathcal{J} : X \to P(X)$ if $u \in \mathcal{J}u$.

We will write $F_{\mathcal{J}}$ to denote the collection of all fixed points of the multivalued mapping $\mathcal{J}$. For the definition of a generalized Pompieu-Hausdorff metric, we refer the reader to see [46]. Let (X, ρ) be a metric space and $R, S \in CL(X)$. Then we write H_ρ to denote the a mapping $\mathcal{M}_\rho : CL(\aleph) \times CL(\aleph) \to \mathbb{R}_+ (= [0, \infty))$ is said to be generalized Pompieu-Hausdorff metric on $CL(X)$ induced by ρ and if

$$\mathcal{M}_\rho(R,S) = \begin{cases} \max\big\{\sup_{r\in R}\mathcal{N}_\rho(r,S),\ \sup_{s\in S}\mathcal{N}_\rho(s,R)\big\}, & \text{if maximum exists },\\ \infty, & \text{otherwise,}\end{cases}$$

where $\mathcal{N}_\rho(r,S)$ to denote $\inf\limits_{s\in S}\rho(r,s)$. Furthermore, we write $\Delta_\rho(R,S)$ to denote $\sup\big\{\rho(r,s) : r \in R, s \in S\big\}$ and we write $\Delta_\rho(R)$ to mean $\Delta_\rho(R,R)$. Next, we move to the definition of b-metric space.

Definition 1.2. [30] Let $\aleph \neq \emptyset$ and λ be a constant such that $\lambda \in [1, \infty)$. A mapping $\rho_\lambda : \aleph \times \aleph \to \mathbb{R}_+$ is said to be a b-metric with coefficient λ if $\forall\ \vartheta_1, \vartheta_2, \vartheta_3 \in \aleph$ the following assertions hold:

(1) $\rho_\lambda(\vartheta_1, \vartheta_2) = 0 \Leftrightarrow \vartheta_1 = \vartheta_2$;
(2) $\rho_\lambda(\vartheta_1, \vartheta_2) = \rho_\lambda(\vartheta_2, \vartheta_1)$;
(3) $\rho_\lambda(\vartheta_1, \vartheta_2) \leq \lambda\big[\rho_\lambda(\vartheta_1, \vartheta_3) + \rho_\lambda(\vartheta_3, \vartheta_2)\big]$.

Moreover, the pair $(\aleph, \rho_\lambda)$ is said to be a b-metric space.

To see the definition of a Cauchy sequence, convergent sequence, completeness in a b-metric space, we refer the reader to see [50]. Next, we state the following three important lemmas. In the proof of our main results, we will use Lemma-1.8, Lemma-1.10 of [29] and Lemma-3 of [50].

Lemma 1.1. [29] *Let $(\aleph, \rho_\lambda)$ be a b-metric space with coefficient $\lambda \in [1, \infty)$. Let $S \in CL(\aleph)$ and suppose that $\exists\ r \in \aleph$ such that $\mathcal{N}_{\rho_\lambda}(r, S) > 0$. Then $\exists$ a $s \in S$ such that $\rho_\lambda(r, s) < \kappa \mathcal{N}_{\rho_\lambda}(r, S)$, where $\kappa > 1$.*

Lemma 1.2. [29] *Let $(\aleph, \rho_\lambda)$ be a b-metric space with coefficient $\lambda \in [1, \infty)$. Let $S \in P(\aleph)$ and $\vartheta \in \aleph$. Then $\mathcal{N}_{\rho_\lambda}(\vartheta, S) = 0$ if and only if $\vartheta \in \overline{S}$.*

Lemma 1.3. [50] *Let $(\aleph, \rho_\lambda)$ be a b-metric space with coefficient $\lambda \in [1, \infty)$. Let $R, S, T \in CB(\aleph)$. Then for $\vartheta, \varsigma \in \aleph$, the following assertions hold:*

(1) $\mathcal{N}_{\rho_\lambda}(\vartheta, R) \leq \rho_\lambda(\vartheta, \varsigma),\ \forall\ \varsigma \in R$;
(2) $\mathcal{N}_{\rho_\lambda}(\vartheta, R) \leq \mathcal{M}_{\rho_\lambda}(S, R),\ \forall\ \vartheta \in S$;
(3) $\mathcal{M}_{\rho_\lambda}(R, R) = 0$;
(4) $\mathcal{M}_{\rho_\lambda}(S, R) = \mathcal{M}_{\rho_\lambda}(R, S)$;
(5) $\mathcal{M}_{\rho_\lambda}(S, R) \leq \lambda\big[\mathcal{M}_{\rho_\lambda}(S, T) + \mathcal{M}_{\rho_\lambda}(T, R)\big]$;
(6) $\mathcal{N}_{\rho_\lambda}(\vartheta, R) \leq \lambda\big[\rho_\lambda(\vartheta, \varsigma) + \mathcal{N}_{\rho_\lambda}(\varsigma, R)\big]$.

Now, we move to the definition of α-admissible multivalued mapping of type S.

Definition 1.3. [50] Let $X \neq \emptyset$ and $s \in [1, \infty)$. Suppose that $\mathcal{J} : X \to CL(X)$ and $\alpha : X \times X \to \mathbb{R}_+$ are two given mappings. Then $\mathcal{J}$ is called an multivalued α-admissible mapping of type S if for every $u \in X$ and $v \in \mathcal{J}u$ with $\alpha(u, v) \geq s$ implies $\alpha(v, w) \geq s$, for every $w \in \mathcal{J}v$.

Next, we state the definition of triangular α-admissible mapping.

Definition 1.4. [16] Let $\mathcal{J} : X \to X$ and $\alpha : X \times X \to \mathbb{R}_+$ be two given mappings. Then $\mathcal{J}$ is said to be triangular α-admissible mapping if it satisfies the following two assertions:

(1) $\mathcal{J}$ is α-admissible;
(2) $\alpha(u, v) \geq 1$ and $\alpha(v, w) \geq 1 \Rightarrow \alpha(u, w) \geq 1$.

Motivated by these work, here we introduce multivalued triangular α-admissible mappings of type Λ.

Definition 1.5. Let $X \neq \emptyset$ and λ be a real number such that $\lambda \geq 1$. Assume that $\mathcal{J} : X \to CL(X)$ and $\alpha : X \times X \to \mathbb{R}_+$ be two mappings. Then $\mathcal{J}$ is said to be multivalued triangular α-admissible mappings of type Λ if

(1) for $u_1 \in X$ and $u_2 \in \mathcal{J}u_1$ with $\alpha(u_1, u_2) \geq \lambda$, we have $\alpha(u_2, u_3) \geq \lambda$ for each $u_3 \in \mathcal{J}u_2$;
(2) for any $u_1, u_2, u_3 \in X$ with $\alpha(u_1, u_2) \geq \lambda$ and $\alpha(u_2, u_3) \geq \lambda$ implies $\alpha(u_1, u_3) \geq \lambda$.

Example 1.1. Let $X = \{1,2,3,4,5\}$ and $\lambda = 2$. Let us define two functions $\alpha : X \times X \to \mathbb{R}_+$ and $\mathcal{J} : X \to CL(X)$ by

$$\mathcal{J}(1) = \{2,3\}, \mathcal{J}(2) = \{1,3\}, \mathcal{J}(3) = \{1,2\}, \mathcal{J}(4) = \{5\}, \mathcal{J}(5) = \{4\},$$

and

$$\alpha(u,v) = \begin{cases} u+v+1, & \text{if } u,v \in \{1,2,3\}, \\ \frac{|u-v|}{5}, & \text{otherwise.} \end{cases}$$

It can be easily checked that $\mathcal{J}$ is a multivalued triangular α-admissible mapping of type 2.

In a similar way for $\lambda \geq 1$, we can define the notion of multivalued triangular μ-sub admissible mappings of type Λ^{-1} by changing " $\geq$ " sign into " $\leq$ " and "λ" into "λ^{-1}".
Let X_1, X_2 be two metric spaces. Then a multivalued mapping $\mathcal{J} : X_1 \to P(X_2)$ is said to be closed if the graph $\mathcal{G}r(\mathcal{J}) = \{(u,v) : u \in X_1, v \in \mathcal{J}u\}$ is a closed subset of $X_1 \times X_2$. It is known that $\mathcal{J}$ has only closed values provided $\mathcal{J}$ is a closed multi-valued mapping. Next, we move to the definition of $(\mathcal{F},\mathcal{H})$-upper class function.

Definition 1.6. [17] A function $\mathcal{H} : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ is said to be a sub-class function of type I if $\tilde{a} \geq 1 \Rightarrow \mathcal{H}(1,b) \leq \mathcal{H}(\tilde{a},b),\ \forall\, b \in \mathbb{R}_+$.

Definition 1.7. [17] Let $\mathcal{F},\mathcal{H} : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ be two given functions. Then the pair $(\mathcal{F},\mathcal{H})$ is said to be an upper class if $\mathcal{H}$ is a sub-class function of type I, along with the following two conditions:

(1) $0 \leq \tilde{c} \leq 1 \Rightarrow \mathcal{F}(\tilde{c},r) \leq \mathcal{F}(1,r)$;
(2) $\mathcal{H}(1,b) \leq \mathcal{F}(1,r) \Rightarrow b \leq r,\ \forall\, b,r\ \in \mathbb{R}_+$.

Now, we state the definition of C-class function.

Definition 1.8 ([17], [51])**.** A continuous function $\mathsf{F} : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ is said to be a C-class function if for $\tilde{a}, \tilde{b} \in \mathbb{R}_+$ the following two assertions hold:

(1) $\mathsf{F}(\tilde{a},\tilde{b}) \leq \tilde{a}$;
(2) $\mathsf{F}(\tilde{a},\tilde{b}) = \tilde{a}$ implies either $\tilde{a} = 0$ or $\tilde{b} = 0$.

Let $X \neq \emptyset$ and $\mathcal{J} : X \to P(X)$ be a multivalued mapping. From now, we will write $\mathcal{J}_\lambda(u_0,\alpha,\mu)$ to denote the collection of all sequence $\{u_r\}_{r=0}^{\infty}$ that satisfy $u_{r+1} \in \mathcal{J}u_r$ together with $\alpha(u_r,u_{r+1}) \geq \lambda$ and $\mu(u_r,u_{r+1}) \leq \lambda^{-1}$, $\forall\ r \in \{0\} \cup \mathbb{N}$, where $\lambda \geq 1$. Now, consider a sequence $\omega = \{u_r\}_{r=0}^{\infty}$ in $\mathcal{J}_\lambda(u_0,\alpha,\mu)$. We say $\Omega_{k,l}(\omega) = \{u_k, u_{k+1}, \cdots, u_l\}$ is a ω-orbit starting at k and end at l, and $\Omega_{k,\infty}(\omega) = \{u_k, u_{k+1}, \cdots\}$ is a ω-orbit starting at k and end at ∞. Now, we state an important definition on orbitally completeness.

Definition 1.9. Let (X,ρ_λ) be a b-metric space with coefficient λ. Suppose that $\alpha : X \times X \to \mathbb{R}_+$, $\mu : X \times X \to \mathbb{R}_+$ are α-admissible and μ-sub admissible mappings of type Λ, respectively. Also, consider a multivalued mapping $\mathcal{J}$ from X to $P(X)$. Then, the b-metric space (X,ρ_λ) is said to be $\mathcal{J}$-orbitally complete if for any Cauchy sequence in $\mathcal{J}_\lambda(u_0,\alpha,\mu)$ converges in X for any $u_0 \in X$.

Karapınar et al. [41] introduced the definition of interpolative Hardy-Rogers (HR) type contractions as following.

Definition 1.10. [41] Let (X, ρ) be a metric space. A mapping $\mathcal{J} : X \to X$ is said to be an interpolative Hardy-Rogers (HR) type contractions if $\exists$ a $\lambda \in [0, 1)$ and $\beta_1, \beta_2, \beta_3 \in (0, 1)$ with $\beta_1 + \beta_2 + \beta_3 < 1$, such that

$$\rho(\mathcal{J}u, \mathcal{J}v) \leq \lambda\big[\rho(u, v)\big]^{\beta_1} \cdot \big[\rho(u, \mathcal{J}u)\big]^{\beta_2} \cdot \big[\rho(v, \mathcal{J}v)\big]^{\beta_3} \cdot \Big[\frac{\rho(u, \mathcal{J}v) + \rho(v, \mathcal{J}u)}{2}\Big]^{1-\beta_1-\beta_2-\beta_3},$$

for all $u, v \in X$ with $u, v \in X \setminus F_{\mathcal{J}}$.

Consequently, by using the above definition, Karapınar et al. proved the following theorem in usual metric space.

Theorem 1.4. [41] *Let (X, ρ) be a complete metric space. Suppose that $\mathcal{J}$ is an interpolative Hardy-Rogers type contraction. Then, $\mathcal{J}$ has a fixed point in X.*

Motivated from the current ongoing research on multivalued mapping, α-admissible mappings, Maia type results, and results on b-metric spaces (mentioned in the beginning of the introduction), next we are going to extend and generalize Theorem 1.4 as well as P-contraction proposed by Popescu [49].

2. Main results

We now start this section by introducing one new definition.

Definition 2.1. Let (X, ρ_λ) be a b-metric space with coefficient λ. Let $\alpha, \mu : X \times X \to \mathbb{R}_+$ and $\mathcal{J} : X \to CB(X)$ be three given mappings. We say $\mathcal{J}$ is a multi-valued interpolative $(\alpha, \mu, \mathcal{H}, \mathcal{F})_{\theta_\gamma}$-contraction of HR-type w.r.t ρ_λ if $\exists$ a $\gamma \in \mathbb{R}_+ \setminus \{0\}$ and $\beta_1, \beta_2, \beta_3 \in (0, 1)$ with $\beta_1 + \beta_2 + \beta_3 < 1$ such that the following holds

$$\mathcal{H}\big(\alpha(u, v), \Delta_{\rho_\lambda}\big(\mathcal{J}u, \mathcal{J}v\big)\big) \leq \mathcal{F}\big(\mu(u, v), \theta_\gamma\big(I_{HR}(u, v)\big)\big), \ \forall\ u, v\ \in X \setminus F_{\mathcal{J}},$$

where the pair $(\mathcal{F}, \mathcal{H})$ is an upper class function, θ_γ is an increasing function from $\mathbb{R}_+$ to $\mathbb{R}_+$ satisfying (i) $\gamma\theta_\gamma(a) < a, , \forall a > 0$; (ii) $\lim\limits_{k\to\infty} \theta_\gamma^k(a) = 0,\ \forall\ a > 0$; (iii) $\lim\limits_{a\to\infty}(a - \gamma\theta_\gamma(a)) = +\infty$ and

$$I_{HR}(u, v) =$$

$$= \Big(\big[\rho_\lambda(u, v)\big]^{\beta_1} \cdot \big[\mathcal{N}_{\rho_\lambda}(u, \mathcal{J}u)\big]^{\beta_2} \cdot \big[\mathcal{N}_{\rho_\lambda}(v, \mathcal{J}v)\big]^{\beta_3} \cdot \Big[\frac{\mathcal{N}_{\rho_\lambda}(u, \mathcal{J}v) + \mathcal{N}_{\rho_\lambda}(v, \mathcal{J}u)}{2}\Big]^{1-\beta_1-\beta_2-\beta_3}\Big).$$

From onwards, we write $\mathcal{Q}(u, v)$ to mean

$$\max\big\{\rho_\lambda(u, v), \mathcal{N}_{\rho_\lambda}(u, \mathcal{J}u), \mathcal{N}_{\rho_\lambda}(v, \mathcal{J}v), \mathcal{N}_{\rho_\lambda}(u, \mathcal{J}v), \mathcal{N}_{\rho_\lambda}(v, \mathcal{J}u)\big\}.$$

Now, we are in a position to state and proof our first main result.

Theorem 2.1. *Let (X, ρ_λ) be a b-metric space with coefficient λ and (X, δ) be another b-metric space. Let $\mathcal{J} : X \to CB(X)$ be a multivalued mapping. Assume that the following assertions hold:*

(A_1) *$\mathcal{J}$ is a multivalued triangular α-admissible mapping of type λ and multivalued triangular μ-sub admissible mapping of type λ^{-1};*

(A_2) *there exists a $u_0 \in X$ such that $u_1 \in \mathcal{J}u_0$ such that $\alpha(u_0, u_1) \geq \lambda$ and $\mu(u_0, u_1) \leq \lambda^{-1}$;*

(A_3) *$\mathcal{J}$ is a multi-valued interpolative $(\alpha, \mu, \mathcal{H}, \mathcal{F})_{\theta_\gamma}$-contraction of HR-type w.r.t ρ_λ;*

(A_4) *$\delta(u, v) \leq \rho_\lambda(u, v), \ \forall \ u, v \in X$;*

(A_5) *(X, δ) is a $\mathcal{J}$ orbitally complete mapping;*

(A_6) *$\mathcal{J}$ is a closed multivalued mapping w.r.t δ.*

Then, $\mathcal{J}$ has a fixed point in X, i.e., $F_{\mathcal{J}} \neq \emptyset$.

Proof. We divide the proof into few steps.

Step I: By assumption (A_2), there exists a point $u_0 \in X$ and $u_1 \in \mathcal{J}u_0$ such that $\alpha(u_0, u_1) \geq \lambda$ and $\mu(u_0, u_1) \leq \lambda^{-1}$. If $u_0 = u_1$ or $u_1 \in \mathcal{J}u_1$, then we have nothing to show, since u_1 is a fixed point of $\mathcal{J}$. So, we assume $u_0 \neq u_1$ and $u_1 \notin \mathcal{J}u_1$. Now, by (A_1), $\mathcal{J}$ is a multivalued triangular α-admissible and μ- sub admissible mapping. Consequently, for $u_1 \in \mathcal{J}u_0$ with $\alpha(u_0, u_1) \geq \lambda$ and $\mu(u_0, u_1) \leq \lambda^{-1}$ implies $\alpha(u_1, u_2) \geq \lambda$ and $\mu(u_1, u_2) \leq \lambda^{-1}$ for $u_2 \in \mathcal{J}u_1$. Again, if $u_1 = u_2$ or $u_2 \in \mathcal{J}u_2$ then we are done. So, we assume $u_1 \neq u_2$ and $u_2 \notin \mathcal{J}u_2$. Continuing in this way, we can obtain a sequence $\omega = \{u_k\}_{k=0}^{\infty}$ such that $u_{k+1} \in \mathcal{J}u_k$, $u_k \neq u_{k+1}$ and $u_{k+1} \notin \mathcal{J}u_{k+1}$ with $\alpha(u_k, u_{k+1}) \geq \lambda, \mu(u_k, u_{k+1}) \leq \lambda^{-1}$. Therefore, we conclude $\omega = \{u_k\}_{k=0}^{\infty} \in \mathcal{J}_\lambda(u_0, \alpha, \mu)$.

Step II: Our next intention is to that for any sequence $\omega = \{u_k\}_{k=0}^{\infty} \in \mathcal{J}_\lambda(u_0, \alpha, \mu)$, $\exists$ a $t (\in \mathbb{N})$ satisfying $r < t \leq s$ such that $\rho_\lambda(u_r, u_t) = \Delta_\lambda(\Omega_{r,s}(\omega))$, where $r, s \in \{0\} \cup \mathbb{N}$. Let us consider a sequence $\omega = \{u_k\}_{k=0}^{\infty}$ in $\mathcal{J}_\lambda(u_0, \alpha, \mu)$ and two real numbers $r, s \in \{0\} \cup \mathbb{N}$ such that $r < s$. Then, we get $\alpha(u_k, u_{k+1}) \geq \lambda$, $\mu(u_k, u_{k+1}) \leq \lambda^{-1}$ and $u_{k+1} \in \mathcal{J}u_k$. Since $\mathcal{J}$ is a multivalued triangular α-admissible mapping of type λ and μ-sub admissible mapping of type λ^{-1}, consequently we obtain $\alpha(u_{k-1}, u_{l-1}) \geq \lambda$, $\mu(u_{k-1}, u_{l-1}) \leq \lambda^{-1}$ for $k, l \in \{0\} \cup \mathbb{N}$ with $k < l$. Since $\mathcal{J}$ is a multivalued interpolative $(\alpha, \mu, \mathcal{H}, \mathcal{F})_{\theta_\gamma}$-contraction of HR-type w.r.t ρ_λ, thus we have the following

$$
\begin{aligned}
&\mathcal{H}\big(1, \Delta_\lambda\big(\mathcal{J}u_{k-1}, \mathcal{J}u_{l-1}\big)\big) \\
&\leq \mathcal{H}\big(\alpha(u_{k-1}, u_{l-1}), \Delta_{\rho_\lambda}\big(\mathcal{J}u_{k-1}, \mathcal{J}u_{l-1}\big)\big) \\
&\leq \mathcal{F}\big(\mu(u_{k-1}, u_{l-1}), \theta_\gamma\big(I_{HR}\big(u_{k-1}, u_{l-1}\big)\big)\big) \\
&\leq \mathcal{F}\big(1, \theta_\gamma\big(I_{HR}\big(u_{k-1}, u_{l-1}\big)\big)\big).
\end{aligned}
$$

Consequently, from the definition of upper class function, we obtain

$$
\Delta_{\rho_\lambda}\big(\mathcal{J}u_{k-1}, \mathcal{J}u_{l-1}\big) \leq \theta_\gamma\big(I_{HR}\big(u_{k-1}, u_{l-1}\big)\big) \leq \theta_\gamma\big(\mathcal{Q}\big(u_{k-1}, u_{l-1}\big)\big). \tag{2.1}
$$

Now, from second property of θ_γ function, we can deduce that $\theta_\gamma(a) < a$ for $a > 0$. Hence, from (2.1), we obtain

$$
\Delta_{\rho_\lambda}\big(\mathcal{J}u_{k-1}, \mathcal{J}u_{l-1}\big) \leq \theta_\gamma\big(\mathcal{Q}\big(u_{k-1}, u_{l-1}\big)\big) < \mathcal{Q}\big(u_{k-1}, u_{l-1}\big). \tag{2.2}
$$

Since $u_k \in \mathcal{J}u_{k-1}$ and $u_l \in \mathcal{J}u_{l-1}$, consequently we obtain,

$$\rho_\lambda(u_k, u_l) \leq \Delta_{\rho_\lambda}\big(\mathcal{J}u_{k-1}, \mathcal{J}u_{l-1}\big), \tag{2.3}$$

and

$$\mathcal{Q}\big(u_{k-1}, u_{l-1}\big) \leq \Delta_{\rho_\lambda}(\Omega_{r,s}(\omega)), \text{ where } r < k < l \leq s. \tag{2.4}$$

Hence observing (2.1) – (2.4), we obtain the following

$$\rho_\lambda(u_k, u_l) < \Delta_{\rho_\lambda}(\Omega_{r,s}(\omega)),$$

which shows that $\exists$ some t, for which we have

$$\rho_\lambda(u_r, u_t) = \Delta_{\rho_\lambda}(\Omega_{r,s}(\omega)), \text{ where } t \in \mathbb{N} \text{ and } r < t \leq s.$$

Therefore, the claim of Step II has been established.

Step III: In this step, we wish to show that any sequence in $\mathcal{J}_\lambda(u_0, \alpha, \mu)$ is bounded. Let us consider a sequence $\omega = \{u_k\}_{k=0}^{\infty} \in \mathcal{J}_\lambda(u_0, \alpha, \mu)$. In this step, first we make an observation that for the arbitrary sequence $\omega = \{u_k\}_{k=0}^{\infty}$, we get

$$\Omega_{0,1}(\omega) \subseteq \Omega_{0,2}(\omega) \subseteq \Omega_{0,3}(\omega) \subseteq \cdots,$$

which shows

$$\Delta_{\rho_\lambda}(\Omega_{0,1}(\omega)) \leq \Delta_{\rho_\lambda}(\Omega_{0,2}(\omega)) \leq \Delta_{\rho_\lambda}(\Omega_{0,3}(\omega)) \leq \cdots .$$

Hence, we obtain that $\big\{\Delta_{\rho_\lambda}(\Omega_{0,s}(\omega))\big\}_{s=1}^{\infty}$ is a non-decreasing sequence. To establish our claim, it is enough to prove that $\big\{\Delta_{\rho_\lambda}(\Omega_{0,s}(\omega))\big\}_{s=1}^{\infty}$ has an upper bound. From Step II, we can say that for any fixed $s \in \mathbb{N}$ $\exists$ a $t \in \mathbb{N}$ such that $\rho_\lambda(u_0, u_t) = \Delta_{\rho_\lambda}(\Omega_{0,s}(\omega))$ with $0 < t \leq s$. Now we consider two cases.

Case 1: If $t = 1$ implies $\rho_\lambda(u_0, u_1) = \Delta_{\rho_\lambda}(\Omega_{0,s}(\omega))$ holds $\forall\ s \in \{0\} \cup \mathbb{N}$, then we have nothing to show. Otherwise, we can arrive into case-2.

Case 2: Suppose that $t > 1$ corresponding to any fixed $s \in \{0\} \cup \mathbb{N}$. Then, we obtain

$$\rho_\lambda(u_0, u_t) = \Delta_{\rho_\lambda}(\Omega_{0,s}(\omega)).$$

Now, from the b-metric property, we know the following inequality,

$$\rho_\lambda(u_0, u_t) \leq \lambda \rho_\lambda(u_0, u_1) + \lambda \rho_\lambda(u_1, u_t).$$

Again, $u_1 \in \mathcal{J}u_0$ and $u_t \in \mathcal{J}u_{t-1}$. Consequently, we have

$$\rho_\lambda(u_0, u_t) \leq \lambda \rho_\lambda(u_0, u_1) + \lambda \Delta_{\rho_\lambda}(\mathcal{J}u_0, \mathcal{J}u_{t-1}). \tag{2.5}$$

Since $\omega = \{u_k\}_{k=0}^{\infty} \in \mathcal{J}_\lambda(u_0, \alpha, \mu)$, so by applying transitivity property of the mapping α and μ, we have $\alpha(u_0, u_r) \geq \lambda$ and $\mu(u_0, u_r) \leq \lambda^{-1}$, $\forall\ r \in \mathbb{N}$. Again, $\mathcal{J}$ is a multi-valued interpolative $(\alpha, \mu, \mathcal{H}, \mathcal{F})_{\theta_\gamma}$-contraction of HR-type w.r.t ρ_λ. Thus, we have

$$\begin{aligned}
&\mathcal{H}\big(1, \Delta_{\rho_\lambda}\big(\mathcal{J}u_0, \mathcal{J}u_{t-1}\big)\big) \\
&\leq \mathcal{H}\big(\alpha(u_0, u_r), \Delta_{\rho_\lambda}\big(\mathcal{J}u_0, \mathcal{J}u_{t-1}\big)\big) \\
&\leq \mathcal{F}\big(\mu(u_0, u_{t-1}), \theta_\gamma\big(I_{HR}\big(u_0, u_{t-1}\big)\big)\big) \\
&\leq \mathcal{F}\big(1, \theta_\gamma\big(I_{HR}\big(u_0, u_{t-1}\big)\big)\big).
\end{aligned}$$

Consequently, from the definition of upper class function, we have

$$\Delta_{\rho_\lambda}\big(\mathcal{J}u_0, \mathcal{J}u_{t-1}\big) \leq \theta_\gamma\big(I_{HR}\big(u_0, u_{t-1}\big)\big) < \theta_\gamma\big(\mathcal{Q}\big(u_0, u_{t-1}\big)\big).$$

Again, we have

$$\begin{aligned}&\mathcal{Q}(u_0, u_{t-1}) \leq \Delta_{\rho_\lambda}(\Omega_{0,s}(\omega))\\ &\Rightarrow \Delta_{\rho_\lambda}\big(\mathcal{J}u_0, \mathcal{J}u_{t-1}\big) \leq \theta_\gamma\big(\Delta_{\rho_\lambda}(\Omega_{0,s}(\omega))\big).\end{aligned}$$

Thus, from (2.5), we get

$$\begin{aligned}&\rho_\lambda(u_0, u_t) \leq \lambda\rho_\lambda(u_0, u_1) + \lambda\theta_\gamma\big(\Delta_{\rho_\lambda}(\Omega_{0,s}(\omega))\big)\\ &\Rightarrow \rho_\lambda(u_0, u_t) - \lambda\theta_\gamma\big(\Delta_{\rho_\lambda}(\Omega_{0,s}(\omega))\big) \leq \lambda\rho_\lambda(u_0, u_1)\end{aligned}$$

$$\Rightarrow \Delta_{\rho_\lambda}(\Omega_{0,s}(\omega)) - \lambda\theta_\gamma\big(\Delta_{\rho_\lambda}(\Omega_{0,s}(\omega))\big) \leq \lambda\rho_\lambda(u_0, u_1). \tag{2.6}$$

It is clear from (2.6) that $\Delta_{\rho_\lambda}(\Omega_{0,s}(\omega))-\lambda\theta_\gamma\big(\Delta_{\rho_\lambda}(\Omega_{0,s}(\omega))\big)$ is bounded by $\lambda\rho_\lambda(u_0, u_1)$ for any $s \in \mathbb{N}$. Next, we suppose that $\big\{\Delta_{\rho_\lambda}(\Omega_{0,s}(\omega))\big\}_{s=1}^{\infty}$ is not bounded, i.e., $\lim\limits_{s\to\infty} \Delta_{\rho_\lambda}(\Omega_{0,s}(\omega)) = \infty$. Then, from third property of θ_γ function, we have

$$\lim_{s\to\infty} \Delta_{\rho_\lambda}(\Omega_{0,s}(\omega)) - \lambda\theta_\gamma\big(\Delta_{\rho_\lambda}(\Omega_{0,s}(\omega))\big) = +\infty,$$

a contradiction. Thus, we deduce that $\big\{\Delta_{\rho_\lambda}(\Omega_{0,s}(\omega))\big\}_{s=1}^{\infty}$ has an upper bound, i.e., bounded.

Step IV: From step-I, we have obtained a sequence $\omega = \big\{u_k\big\}_{k=0}^{\infty} \in \mathcal{J}_\lambda(u_0, \alpha, \mu)$. Choose two natural numbers r and s with $r < s$, and observe that $\alpha(u_r, u_s) \geq \lambda$ and $\mu(u_r, u_s) \leq \lambda^{-1}$, since $\mathcal{J}$ is a multivalued triangular α-admissible and μ-sub admissible mappings of type λ, and λ^{-1} respectively. Again, from step-II, $\exists$ a $t(\in \mathbb{N})$ with $r < t \leq s$ such that

$$\rho_\lambda(u_r, u_t) = \Delta_{\rho_\lambda}(\Omega_{r,s}(\omega)). \tag{2.7}$$

Now, since $\mathcal{J}$ is a multi-valued interpolative $(\alpha, \mu, \mathcal{H}, \mathcal{F})_{\theta_\gamma}$-contraction of HR-type w.r.t ρ_λ. Thus, we have

$$\begin{aligned}&\mathcal{H}\big(1, \Delta_{\rho_\lambda}\big(\mathcal{J}u_r, \mathcal{J}u_s\big)\big)\\ &\leq \mathcal{H}\big(\alpha(u_r, u_s), \Delta_{\rho_\lambda}\big(\mathcal{J}u_r, \mathcal{J}u_s\big)\big)\\ &\leq \mathcal{F}\big(\mu(u_r, u_s), \theta_\gamma\big(I_{HR}(u_r, u_s)\big)\big)\\ &\leq \mathcal{F}\big(1, \theta_\gamma\big(I_{HR}(u_r, u_s)\big)\big).\end{aligned}$$

Hence, by using the definition of upper class function, we get

$$\Delta_{\rho_\lambda}\big(\mathcal{J}u_r, \mathcal{J}u_s\big) \leq \theta_\gamma\big(I_{HR}(u_r, u_s)\big). \tag{2.8}$$

Again, from (2.7) and (2.8), we can write the following

$$\rho_\lambda(u_{r+1}, u_{s+1}) \leq \theta_\gamma\big(\mathcal{Q}(u_r, u_s)\big) \leq \theta_\gamma\big(\Delta_{\rho_\lambda}\big(\Omega_{r,s}(\omega)\big)\big) = \theta_\gamma\big(\rho_\lambda\big(u_r, u_t\big)\big). \tag{2.9}$$

Since, $t-1$ is a natural number satisfying, $r-1 < t-1 \leq s-1$, consequently by applying the transitivity property of the mapping α and μ, we can obtain that $\alpha(u_{r-1}, u_{r-1}) \geq \lambda$, and $\mu(u_{r-1}, u_{r-1}) \leq \lambda^{-1}$. Again, $\mathcal{J}$ is a multi-valued interpolative

$(\alpha, \mu, \mathcal{H}, \mathcal{F})_{\theta_\gamma}$-contraction of HR-type, so we have

$$\begin{aligned}
&\mathcal{H}\big(1, \Delta_{\rho_\lambda}\big(\mathcal{J}u_{r-1}, \mathcal{J}u_{t-1}\big)\big) \\
&\le \mathcal{H}\big(\alpha(u_{r-1}, u_{t-1}), \Delta_{\rho_\lambda}\big(\mathcal{J}u_{r-1}, \mathcal{J}u_{t-1}\big)\big) \\
&\le \mathcal{F}\big(\mu(u_{r-1}, u_{t-1}), \theta_\gamma\big(I_{HR}\big(u_{r-1}, u_{t-1}\big)\big)\big) \\
&\le \mathcal{F}\big(1, \theta_\gamma\big(I_{HR}\big(u_{r-1}, u_{t-1}\big)\big)\big).
\end{aligned}$$

Thus, by applying the definition of upper class function, we get

$$\Delta_{\rho_\lambda}\big(\mathcal{J}u_{r-1}, \mathcal{J}u_{t-1}\big) \le \theta_\gamma\big(I_{HR}\big(u_{r-1}, u_{t-1}\big)\big). \tag{2.10}$$

From (2.10), one can obtain the following,

$$\rho_\lambda(u_r, u_t) \le \theta_\gamma\big(\mathcal{Q}(u_{r-1}, u_{t-1})\big) \le \theta_\gamma\big(\Delta_{\rho_\lambda}\big(\Omega_{r-1,t-1}(\omega)\big)\big) \le \theta_\gamma\big(\Delta_{\rho_\lambda}\big(\Omega_{r-1,s}(\omega)\big)\big).$$

Thus, from (2.9) and (2.10), we have

$$\rho_\lambda(u_{r+1}, u_{s+1}) \le \theta_\gamma(\rho_\lambda(u_r, u_t)) \le \theta_\gamma^2(\Delta_{\rho_\lambda}(\Omega_{r-1,s}(\omega))).$$

Continuing in this way, one can get the following

$$\begin{aligned}
&\rho_\lambda(u_{r+1}, u_{s+1}) \\
&\le \theta_\gamma^2(\Delta_{\rho_\lambda}(\Omega_{r-1,s}(\omega))) \\
&\le \theta_\gamma^3(\Delta_{\rho_\lambda}(\Omega_{r-2,s}(\omega))) \le \theta_\gamma^4(\Delta_{\rho_\lambda}(\Omega_{r-3,s}(\omega))) \le \cdots \le \theta_\gamma^{r+1}(\Delta_{\rho_\lambda}(\Omega_{0,s}(\omega))).
\end{aligned} \tag{2.11}$$

But, from step-III, we have already concluded that $\big\{\Delta_{\rho_\lambda}(\Omega_{0,s}(\omega))\big\}_{s=1}^{\infty}$ has an upper bound. Let us consider β as such upper bound. Hence, from (2.11), we have

$$\rho_\lambda(u_{r+1}, u_{s+1}) \le \theta_\gamma^{r+1}(\beta). \tag{2.12}$$

Now, in (2.11), if we make $r \to \infty$, then by the second property of the θ_γ function, we have $\lim\limits_{r\to\infty} \rho_\lambda(u_{r+1}, u_{s+1}) = 0$, i.e., $\omega = \big\{u_k\big\}_{k=0}^{\infty}$ is a Cauchy sequence in (X, ρ_λ). Now, we move to the last step of our proof.

Step V: Now, from (A_4), we have $\lim\limits_{r\to\infty} \delta(u_{r+1}, u_{s+1}) = 0$. Therefore, the sequence $\omega = \big\{u_k\big\}_{k=0}^{\infty}$ is a Cauchy sequence in (X, δ) also. Again, by our assumption (A_5), (X, δ) is a $\mathcal{J}$-orbitally complete b-metric space. Consequently, $\omega = \big\{u_k\big\}_{k=0}^{\infty}$ is a convergent sequence in (X, δ), i.e., $\exists$ an element $u^*(\in X)$ such that $\delta(u_k, u^*) \to 0$ as $k \to \infty$. From (A_6), $\mathcal{J}$ is a closed multivalued mapping w.r.t δ. Hence, we obtain, $u^* \in \mathcal{J}u^*$. Thus $\mathcal{J}$ has a fixed point in X, i.e., $F_{\mathcal{J}} \ne \emptyset$. $\square$

Next, we do an observation which shows that by choosing some particular values as well as mappings in Theorem 2.1, one can easily derive the contraction given in Definition 1.10 as well as Theorem 1.4.

An observation:

In the statement of Theorem 2.1, if we consider $\mathcal{J}$ as a single valued mapping from X to X instead of a multivalued mapping, $\mathcal{H}(\tilde{a}, \tilde{b}) = \tilde{b}$, $\mathcal{F}(s,t) = t$, $\alpha(u,v) = \mu(u,v) = 1$, $\forall\ u, v \in X$, $\theta_\gamma(t) = kt$, where $k \in [0,1)$ with $\gamma = 1$, and lastly instead of two b-metric spaces only a single metric space with $\rho_\lambda = \delta$, then Theorem 2.1 reduces to Theorem 1.4.

Now, we give an example to support Theorem 2.1.

Remark 2.1. We can derive many results as a corollary on multi-valued interpolative HR-contraction for two different b-metrics by assigning different types of values as well as functions in the statement of Theorem 2.1. We are skipping those results due to the length of the paper.

Example 2.1. Let $X = B_1 \cup B_2$, where $B_1 = \left\{\frac{1}{16}, \frac{1}{8}, \frac{1}{4}, \frac{1}{3}, \frac{1}{2}, 1\right\}$ and $B_2 = \left\{2\right\}$. Let $\delta, \rho_\lambda : X \times X \to \mathbb{R}_+$ be two mappings defined by

$$\delta(u,v) =\mid u-v \mid^2, \ \forall\ u,v \in X;$$

$$\rho_\lambda(u,v) = \begin{cases} 0, & \text{if u = v,} \\ \mid u-v \mid^2 +1, & \text{if } u \neq v. \end{cases}$$

Clearly, δ and ρ_λ are two b-metrics with coefficient 2. Next, we define a multivalued mapping $\mathcal{J} : X \to CB(X)$ as

$$\mathcal{J}(u) = \begin{cases} \left\{\frac{1}{16}\right\}, & \text{if } u \in B_1, \\ \left\{\frac{1}{3}, 1\right\}, & \text{if } u \in B_2. \end{cases}$$

We now consider two mappings $\alpha, \mu : X \times X \to \mathbb{R}_+$ as

$$\alpha(u,v) = \begin{cases} 4, & \text{if } u,v \in B_1 \text{ with } u \leq v, \\ \frac{1}{5}, & \text{if } u = \frac{1}{3}, v = 2, \\ 0, & \text{otherwise.} \end{cases}$$

$$\mu(u,v) = \begin{cases} \frac{3}{10}, & \text{if } u,v \in B_1 \text{ with } u \leq v, \\ 1.4, & \text{if } u = \frac{1}{3}, v = 2, \\ 0, & \text{otherwise.} \end{cases}$$

Let $\theta_\gamma : \mathbb{R}_+ \to \mathbb{R}_+$ be a mapping defined by

$$\theta_\gamma(t) = \frac{2}{5}t \text{ with } \gamma = 2.$$

We, also consider two mappings $\mathcal{H}, \mathcal{F} : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ which are defined by

$$\mathcal{H}(u,v) = \big(u+l\big)^v, \mathcal{F}(s,t) = \big(1+l\big)^{st}, \text{ with } l = 1.5.$$

Let us consider three real numbers $\beta_1, \beta_2, \beta_3$ as $\beta_1 = \frac{1}{3}, \beta_2 = \frac{1}{5}, \beta_3 = \frac{1}{9}$. Then, we have $\beta_1 + \beta_2 + \beta_3 \approx 0.644$ and $1 - (\beta_1 + \beta_2 + \beta_3) \approx 0.355$.
Now, we have to consider the following three cases.
Case 1: Let $u, v \in B_1$ with $u \leq v$. We have

$$\mathcal{H}\big(\alpha(u,v), \Delta_{\rho_\lambda}\big(\mathcal{J}u, \mathcal{J}v\big)\big) = (4+1.5)^{\Delta_{\rho_\lambda}(\mathcal{J}u,\mathcal{J}v)}, \text{ and}$$
$$\mathcal{F}(\mu(u,v), \theta_\gamma(I_{HR}(u,v))) = (1+1.5)^{\mu(u,v)\cdot\theta_\gamma(I_{HR}(u,v))}.$$

Now, observe that

$$\Delta_{\rho_\lambda}(\mathcal{J}u, \mathcal{J}v) = 0$$

and
$I_{HR}(u,v) =$

$$= \big[\rho_\lambda(u,v)\big]^{\beta_1} \cdot \big[\mathcal{N}_{\rho_\lambda}(u,\mathcal{J}u)\big]^{\beta_2} \cdot \big[\mathcal{N}_{\rho_\lambda}(v,\mathcal{J}v)\big]^{\beta_3} \cdot \Big[\frac{\mathcal{N}_{\rho_\lambda}(u,\mathcal{J}v)+\mathcal{N}_{\rho_\lambda}(v,\mathcal{J}u)}{2}\Big]^{1-\beta_1-\beta_2-\beta_3} > 1$$

implies $\theta_\gamma(I_{HR}(u,v)) > 0$, and consequently $\mu(u,v) \cdot \theta_\gamma(I_{HR}(u,v)) > 0$.
Hence, we get

$$\mathcal{H}\big(\alpha(u,v), \Delta_{\rho_\lambda}\big(\mathcal{J}u,\mathcal{J}v\big)\big) < \mathcal{F}(\mu(u,v), \theta_\gamma(I_{HR}(u,v))).$$

Case 2: If $u \in B_1 \backslash \big\{\frac{1}{3}\big\}$ and $v = 2$, then

$$\mathcal{H}\big(\alpha(u,v), \Delta_{\rho_\lambda}\big(\mathcal{J}u,\mathcal{J}v\big)\big) = (0+1.5)^{\Delta_{\rho_\lambda}(\mathcal{J}u,\mathcal{J}v)} = 1, \text{ and}$$
$$\mathcal{F}(\mu(u,v), \theta_\gamma(I_{HR}(u,v))) = (1+1.5)^{\mu(u,v)\cdot\theta_\gamma(I_{HR}(u,v))} = 1.$$

Therefore, we have

$$\mathcal{H}\big(\alpha(u,v), \Delta_{\rho_\lambda}\big(\mathcal{J}u,\mathcal{J}v\big)\big) = \mathcal{F}(\mu(u,v), \theta_\gamma(I_{HR}(u,v))).$$

Case 3: If $u = \frac{1}{3}$ and $v = 2$, then

$$\mathcal{H}\big(\alpha(u,v), \Delta_{\rho_\lambda}\big(\mathcal{J}u,\mathcal{J}v\big)\big)$$
$$= (\frac{1}{5} + 1.5)^{\Delta_{\rho_\lambda}(\mathcal{J}u,\mathcal{J}v)} = (1.7)^{\Delta_{\rho_\lambda}(\{\frac{1}{16}\},\{\frac{1}{3},1\})} \approx (1.7)^{1.87890625} \approx 2.710141$$

and
$I_{HR}(u,v)$

$$= \big[\rho_\lambda(\frac{1}{3},2)\big]^{\frac{1}{3}} \cdot \big[\mathcal{N}_{\rho_\lambda}(\frac{1}{3},\frac{1}{16})\big]^{\frac{1}{5}} \cdot \big[\mathcal{N}_{\rho_\lambda}(2,\{\frac{1}{3},1\})\big]^{\frac{1}{9}} \cdot \Big[\frac{\mathcal{N}_{\rho_\lambda}(\frac{1}{3},\{\frac{1}{3},1\}) + \mathcal{N}_{\rho_\lambda}(2,\frac{1}{16})}{2}\Big]^{0.355}$$
$$\approx (1.55744) \cdot (1.01425) \cdot (1.080060) \cdot (1.359838)$$
$$\approx 2.320018.$$

Now, $\theta_\gamma(I_{HR}(u,v)) \approx 0.928007$, $\mu(u,v)\theta_\gamma(I_{HR}(u,v)) \approx 1.2992098$, and

$$\begin{aligned} &\mathcal{F}(\mu(u,v), \theta_\gamma(I_{HR}(u,v))) \\ &= (1+1.5)^{\mu(u,v)\theta_\gamma(I_{HR}(u,v))} \\ &\approx (2.5)^{1.2992098} \\ &\approx 3.288573. \end{aligned}$$

Hence, we obtain

$$\mathcal{H}\big(\alpha(u,v), \Delta_{\rho_\lambda}\big(\mathcal{J}u,\mathcal{J}v\big)\big) < \mathcal{F}(\mu(u,v), \theta_\gamma(I_{HR}(u,v))).$$

One can see that $\delta(u,v) \le \rho_\lambda(u,v),\ \forall\ u,v \in X$. In fact, it can be easily checked that all the conditions of Theorem 2.1 are satisfied. Hence, the mapping $\mathcal{J}$ has a fixed point. Here, $\frac{1}{16}$ is the fixed point of the mapping $\mathcal{J}$.

Next, we proceed to our second main result of this paper, which is based on "P-contraction" involving Maia type result. To discuss our second main result, we need the definition of altering distance function, and for this we refer the reader to see [32]. To establish our second main result, first we introduce the following definition.

Definition 2.2. Let (X, ρ_λ) be a b-metric space with coefficient λ. Let $\alpha : X \times X \to \mathbb{R}_+$ and $\mathcal{J} : X \to CB(X)$ be two given mappings. We say $\mathcal{J}$ is a multi-valued P-contraction via C-class function w.r.t ρ_λ if

$$\xi(\lambda^3 \Delta_{\rho_\lambda}(\mathcal{J}u, \mathcal{J}v)) \leq \mathsf{F}(\xi(\mathsf{P}(u, v)), \eta(\mathsf{P}(u, v))),$$

holds for all $u, v \in X$ with $\alpha(u, v) \geq \lambda$, where ξ, η are altering distance functions, F is a C-class function and

$$\mathsf{P}(u, v) = \rho_\lambda(u, v) + \mid \mathcal{N}_{\rho_\lambda}(u, \mathcal{J}u) - \mathcal{N}_{\rho_\lambda}(v, \mathcal{J}v) \mid .$$

Now, we are in a position to state and proof our second main result.

Theorem 2.2. *Let (X, ρ_λ) be a b-metric space with coefficient $\lambda \in (1, \infty)$ and (X, δ) be another b-metric space. Suppose $\beta \in (0, 1)$ be any fixed real number such that $\beta^{-1} < \lambda^3$. Let $\mathcal{J} : X \to CB(X)$ be a multivalued mapping. Assume that the following assertions hold:*

(D_1) *$\mathcal{J}$ is a multivalued triangular $\alpha -$ admissible mapping of type λ;*

(D_2) *$\exists$ a point $u_0 \in X$ and $u_1 \in \mathcal{J}u_0$ such that $\alpha(u_0, u_1) \geq \lambda$;*

(D_3) *$\mathcal{J}$ is a multivalued $\mathsf{P} -$ contraction via C-class function w.r.t ρ_λ;*

(D_4) *$\delta(u, v) \leq \rho_\lambda(u, v),\ \forall\ u, v \in X$;*

(D_5) *(X, δ) is a $\mathcal{J}$ orbitally complete mapping;*

(D_6) *$\mathcal{J}$ is a closed multivalued mapping w.r.t δ.*

Then, $\mathcal{J}$ has a fixed point in X, i.e., $F_{\mathcal{J}} \neq \emptyset$.

Proof. By condition (D_2), $\exists$ a point $u_0 \in X$ and $u_1 \in \mathcal{J}u_0$ such that $\alpha(u_0, u_1) \geq \lambda$. Again, $\mathcal{J}$ is a multivalued P-contraction via C-class function w.r.t ρ_λ, consequently we have

$$\xi(\lambda^3 \Delta_{\rho_\lambda}(\mathcal{J}u_0, \mathcal{J}u_1)) \leq \mathsf{F}(\xi(\mathsf{P}(u_0, u_1)), \eta(\mathsf{P}(u_0, u_1))),$$

where

$$\mathsf{P}(u_0, u_1) = \rho_\lambda(u_0, u_1) + \mid \mathcal{N}_{\rho_\lambda}(u_0, \mathcal{J}u_0) - \mathcal{N}_{\rho_\lambda}(u_1, \mathcal{J}u_1) \mid .$$

Clearly, $u_1 \in \mathcal{J}u_1$, then we have nothing to show. Hence, we assume $u_1 \notin \mathcal{J}u_1$. Again,

$$\lambda^3 \mathcal{N}_{\rho_\lambda}(u_1, \mathcal{J}u_1) \leq \lambda^3 \Delta_{\rho_\lambda}(\mathcal{J}u_0, \mathcal{J}u_1),$$

and hence $\beta^{-1}\mathcal{N}_{\rho_\lambda}(u_1, \mathcal{J}u_1) \leq \lambda^3 \mathcal{N}_{\rho_\lambda}(u_1, \mathcal{J}u_1)$. By using Lemma-1.8 of [29], we can find a $u_2 \in \mathcal{J}u_1$ such that

$$\rho_\lambda(u_1, u_2) < \beta^{-1}\mathcal{N}_{\rho_\lambda}(u_1, \mathcal{J}u_1).$$

From (D_1), $\mathcal{J}$ is a multivalued triangular α-admissible mapping of type λ which implies $\alpha(u_1, u_2) \geq \lambda$. Next, we assume that $u_2 \notin \mathcal{J}u_2$ otherwise there is nothing to show. Since $\alpha(u_1, u_2) \geq \lambda$, consequently by (D_3), we get

$$\xi(\lambda^3 \Delta_{\rho_\lambda}(\mathcal{J}u_1, \mathcal{J}u_2)) \leq \mathsf{F}(\xi(\mathsf{P}(u_1, u_2)), \eta(\mathsf{P}(u_1, u_2))),$$

where

$$\mathsf{P}(u_1, u_2) = \rho_\lambda(u_1, u_2) + \mid \mathcal{N}_{\rho_\lambda}(u_1, \mathcal{J}u_1) - \mathcal{N}_{\rho_\lambda}(u_2, \mathcal{J}u_2) \mid .$$

By similar argument, we can find a $u_3 \in \mathcal{J}u_2$ such that

$$\rho_\lambda(u_2, u_3) < \beta^{-1}\mathcal{N}_{\rho_\lambda}(u_2, \mathcal{J}u_2) \text{ with } \alpha(u_2, u_3) \geq \lambda.$$

Continuing in this way, we can find a sequence $\{u_r\}_{r=0}^{\infty}$ such that $u_r \notin \mathcal{J}u_r$, $u_{r+1} \in \mathcal{J}u_r$ and

$$\rho_\lambda(u_r, u_{r+1}) < \beta^{-1}\mathcal{N}_{\rho_\lambda}(u_r, \mathcal{J}u_r) \text{ with } \alpha(u_r, u_{r+1}) \geq \lambda. \tag{2.13}$$

Next, we claim that $\mathcal{N}_{\rho_\lambda}(u_{r+1}, \mathcal{J}u_{r+1}) \leq \mathcal{N}_{\rho_\lambda}(u_r, \mathcal{J}u_r)$ for all $r \in \{0\} \cup \mathbb{N}$. Suppose on the contrary, i.e., $\exists$ a $k \in \{0\} \cup \mathbb{N}$ such that $\mathcal{N}_{\rho_\lambda}(u_{k+1}, \mathcal{J}u_{k+1}) > \mathcal{N}_{\rho_\lambda}(u_k, \mathcal{J}u_k)$. Thus, we have the following

$$\begin{aligned}
&\xi(\lambda^3\mathcal{N}_{\rho_\lambda}(u_{k+1}, \mathcal{J}u_{k+1})) \\
&\leq \xi(\lambda^3\Delta_{\rho_\lambda}(\mathcal{J}u_k, \mathcal{J}u_{k+1})) \leq \mathsf{F}(\xi(\mathsf{P}(u_k, u_{k+1})), \eta(\mathsf{P}(u_k, u_{k+1}))) \leq \xi(\mathsf{P}(u_k, u_{k+1})).
\end{aligned}$$

Since ξ is a non-decreasing function, i.e., we have

$$\lambda^3\mathcal{N}_{\rho_\lambda}(u_{k+1}, \mathcal{J}u_{k+1}) \leq \mathsf{P}(u_k, u_{k+1}),$$

implies

$$\begin{aligned}
\lambda^3\mathcal{N}_{\rho_\lambda}(u_{k+1}, \mathcal{J}u_{k+1}) &\leq \rho_\lambda(u_k, u_{k+1}) + \mid \mathcal{N}_{\rho_\lambda}(u_k, \mathcal{J}u_k) - \mathcal{N}_{\rho_\lambda}(u_{k+1}, \mathcal{J}u_{k+1}) \mid \\
&= \rho_\lambda(u_k, u_{k+1}) + \mathcal{N}_{\rho_\lambda}(u_{k+1}, \mathcal{J}u_{k+1}) - \mathcal{N}_{\rho_\lambda}(u_k, \mathcal{J}u_k).
\end{aligned}$$

Also, we can write

$$\begin{aligned}
(\lambda^3 - 1)\mathcal{N}_{\rho_\lambda}(u_{k+1}, \mathcal{J}u_{k+1}) &\leq \rho_\lambda(u_k, u_{k+1}) - \mathcal{N}_{\rho_\lambda}(u_k, \mathcal{J}u_k) \\
&< \beta^{-1}\mathcal{N}_{\rho_\lambda}(u_k, \mathcal{J}u_k) - \mathcal{N}_{\rho_\lambda}(u_k, \mathcal{J}u_k), \text{ by (2.13)} \\
&= (\beta^{-1} - 1)\mathcal{N}_{\rho_\lambda}(u_k, \mathcal{J}u_k) \\
&< (\beta^{-1} - 1)\mathcal{N}_{\rho_\lambda}(u_{k+1}, \mathcal{J}u_{k+1}).
\end{aligned}$$

Thus, we have

$$\mathcal{N}_{\rho_\lambda}(u_{k+1}, \mathcal{J}u_{k+1}) < \frac{(\beta^{-1} - 1)}{(\lambda^3 - 1)}\mathcal{N}_{\rho_\lambda}(u_{k+1}, \mathcal{J}u_{k+1}),$$

which is a contradiction. Consequently, our claim is justified, i.e., $\mathcal{N}_{\rho_\lambda}(u_{r+1}, \mathcal{J}u_{r+1}) \leq \mathcal{N}_{\rho_\lambda}(u_r, \mathcal{J}u_r)$ for all $r \in \{0\} \cup \mathbb{N}$. Therefore $\{\mathcal{N}_{\rho_\lambda}(u_r, \mathcal{J}u_r)\}_{r=0}^{\infty}$ is a decreasing sequence bounded below by 0, i.e., $\exists$ a $\tilde{b} \in [0, \infty)$ such that $\lim\limits_{r\to\infty} \mathcal{N}_{\rho_\lambda}(u_r, \mathcal{J}u_r) = \tilde{b}$. Since $\alpha(u_r, u_{r+1}) \geq \lambda$ for all $r \in \{0\} \cup \mathbb{N}$, thus we have

$$\begin{aligned}
&\xi(\rho_\lambda(u_{r+1}, u_{r+2})) \\
&\leq \xi(\beta^{-1}\mathcal{N}_{\rho_\lambda}(u_{r+1}, \mathcal{J}u_{r+1})) \\
&\leq \xi(\lambda^3\Delta_{\rho_\lambda}(\mathcal{J}u_r, \mathcal{J}u_{r+1})) \\
&\leq \mathsf{F}(\xi(\mathsf{P}(u_r, u_{r+1})), \eta(\mathsf{P}(u_r, u_{r+1}))) \\
&\leq \xi(\mathsf{P}(u_r, u_{r+1})),
\end{aligned} \tag{2.14}$$

where $\mathsf{P}(u_r, u_{r+1}) = \rho_\lambda(u_r, u_{r+1}) + \mid \mathcal{N}_{\rho_\lambda}(u_r, \mathcal{J}u_r) - \mathcal{N}_{\rho_\lambda}(u_{r+1}, \mathcal{J}u_{r+1}) \mid$. Now, considering limit as $r \to \infty$ in (2.14), we get

$$\begin{aligned}
\xi(\lim_{r\to\infty} \rho_\lambda(u_r, u_{r+1})) &\leq \mathsf{F}(\xi(\lim_{r\to\infty} \rho_\lambda(u_r, u_{r+1})), \eta(\lim_{r\to\infty} \rho_\lambda(u_r, u_{r+1}))) \\
&\leq \xi(\lim_{r\to\infty} \rho_\lambda(u_r, u_{r+1})) \\
\Rightarrow \mathsf{F}(\xi(\lim_{r\to\infty} \rho_\lambda(u_r, u_{r+1})), \eta(\lim_{r\to\infty} \rho_\lambda(u_r, u_{r+1}))) &= \xi(\lim_{r\to\infty} \rho_\lambda(u_r, u_{r+1})),
\end{aligned}$$

which shows that either $\xi(\lim_{r\to\infty} \rho_\lambda(u_r, u_{r+1})) = 0$ or $\eta(\lim_{r\to\infty} \rho_\lambda(u_r, u_{r+1})) = 0$ implies

$$\lim_{r\to\infty} \rho_\lambda(u_r, u_{r+1}) = 0.$$

Observe that, since $\mathcal{J}$ is a multivalued triangular α-admissible mapping with $\alpha(u_r, u_{r+1}) \geq \lambda,\ \forall\ r \in \{0\} \cup \mathbb{N}$ implies $\alpha(u_r, u_s) \geq \lambda$ for all $r, s \in \{0\} \cup \mathbb{N}$ with $r < s$. Now our motive is to show that $\{u_r\}_{r=0}^{\infty}$ is a Cauchy sequence. Suppose on the contrary, Then there exists a $\tau(> 0)$ such that $\rho_\lambda(u_{r(l)}, u_{s(l)}) \geq \tau$ and $\rho_\lambda(u_{r(l)}, u_{s(l)-1}) < \tau$ for $s(l) > r(l) > l$. Consequently, we have the following

$$\begin{aligned}
\tau &\leq \rho_\lambda(u_{r(l)}, u_{s(l)}) \\
&\leq \lambda[\rho_\lambda(u_{r(l)}, u_{s(l)-1}) + \rho_\lambda(u_{s(l)-1}, u_{s(l)})] \\
&\leq \lambda[\tau + \rho_\lambda(u_{s(l)-1}, u_{s(l)})].
\end{aligned} \tag{2.15}$$

By considering lim sup in (2.15), we get

$$\tau \leq \overline{\lim_{l\to\infty}} \rho_\lambda(u_{r(l)}, u_{s(l)}) < \lambda\tau. \tag{2.16}$$

Again,

$$\rho_\lambda(u_{r(l)}, u_{s(l)}) \leq \lambda[\rho_\lambda(u_{r(l)}, u_{r(l)+1}) + \rho_\lambda(u_{r(l)+1}, u_{s(l)})], \tag{2.17}$$

and

$$\rho_\lambda(u_{r(l)+1}, u_{s(l)}) \leq \lambda[\rho_\lambda(u_{r(l)+1}, u_{r(l)}) + \rho_\lambda(u_{r(l)}, u_{s(l)})]. \tag{2.18}$$

Now considering lim sup in (2.17), (2.18), and using (2.16), we have

$$\frac{\tau}{\lambda} \leq \overline{\lim_{l\to\infty}} \rho_\lambda(u_{r(l)+1}, u_{s(l)}) < \lambda^2\tau. \tag{2.19}$$

In a similar way, we get

$$\frac{\tau}{\lambda} \leq \overline{\lim_{l\to\infty}} \rho_\lambda(u_{r(l)}, u_{s(l)+1}) < \lambda^2\tau. \tag{2.20}$$

Again, by using (2.19) and (2.20), one can show that

$$\frac{\tau}{\lambda^2} \leq \overline{\lim_{l\to\infty}} \rho_\lambda(u_{r(l)+1}, u_{s(l)+1}) < \lambda^3\tau. \tag{2.21}$$

Since, $\mathcal{J}$ is a multi valued P-contraction via C-class function w.r.t ρ_λ with $\alpha(u_r, u_s) \geq \lambda,\ \forall\ r, s \in \{0\} \cup \mathbb{N}$. Consequently, we have

$$\begin{aligned}
&\xi(\rho_\lambda(u_{r(l)+1}, u_{s(l)+1})) \\
&\leq \xi(\lambda^3 \Delta_{\rho_\lambda}(\mathcal{J}u_{r(l)}, \mathcal{J}u_{s(l)})) \\
&\leq \mathsf{F}(\xi(\mathsf{P}(u_{r(l)}, u_{s(l)})), \eta(\mathsf{P}(u_{r(l)}, u_{s(l)}))) \\
&\leq \xi(\mathsf{P}(u_{r(l)}, u_{s(l)})),
\end{aligned} \tag{2.22}$$

where $\mathsf{P}(u_{r(l)}, u_{s(l)}) = \rho_\lambda(u_{r(l)}, u_{s(l)}) + \mid \mathcal{N}_{\rho_\lambda}(u_{r(l)}, \mathcal{J}u_{r(l)}) - \mathcal{N}_{\rho_\lambda}(u_{s(l)}, \mathcal{J}u_{s(l)}) \mid$. Now, observe that

$$0 \leq \mid \mathcal{N}_{\rho_\lambda}(u_{r(l)}, \mathcal{J}u_{r(l)}) - \mathcal{N}_{\rho_\lambda}(u_{s(l)}, \mathcal{J}u_{s(l)}) \mid \leq \mathcal{N}_{\rho_\lambda}(u_{r(l)}, \mathcal{J}u_{r(l)}) \leq \rho_\lambda(u_{r(l)}, u_{r(l)+1}).$$

Making $l \to \infty$ in the above inequality, we get

$$\lim_{l\to\infty} \mid \mathcal{N}_{\rho_\lambda}(u_{r(l)}, \mathcal{J}u_{r(l)}) - \mathcal{N}_{\rho_\lambda}(u_{s(l)}, \mathcal{J}u_{s(l)}) \mid = 0.$$

Now considering lim sup in (2.22), we get

$$\begin{aligned} &\xi(\lambda\tau) \leq \mathsf{F}(\xi(\lambda\tau), \eta(\lambda\tau)) \leq \xi(\lambda\tau) \\ &\Rightarrow \mathsf{F}(\xi(\lambda\tau), \eta(\lambda\tau)) = \xi(\lambda\tau) \\ &\Rightarrow \text{either } \xi(\lambda\tau) = 0 \text{ or } \eta(\lambda\tau) = 0. \end{aligned}$$

Since $\lambda \in (1, \infty)$ so it implies $\tau = 0$, a contradiction to the fact that $\tau > 0$. Hence $\{u_r\}_{r=0}^{\infty}$ is a Cauchy sequence in (X, ρ_λ), i.e., for $r < s$ $\rho_\lambda(u_r, u_s) \to 0$ as $r \to \infty$. Now, from (D_4), we have $\lim\limits_{r\to\infty} \delta(u_{r+1}, u_{s+1}) = 0$. Therefore, the sequence $\{u_r\}_{r=0}^{\infty}$ is a Cauchy sequence in (X, δ) also. Again, by our assumption (D_5), (X, δ) is a $\mathcal{J}$-orbitally complete b-metric space. Consequently, $\{u_r\}_{r=0}^{\infty}$ is a convergent sequence in (X, δ), i.e., $\exists$ an element $u^*(\in X)$ such that $\delta(u_r, u^*) \to 0$ as $r \to \infty$. From (D_6), $\mathcal{J}$ is a closed multivalued mapping w.r.t δ. Hence, we obtain, $u^* \in \mathcal{J}u^*$. Thus $\mathcal{J}$ has a fixed point in X, i.e., $F_{\mathcal{J}} \neq \emptyset$. ☐

Next, we wish to show the uniqueness of fixed point.

Property (U): For any $u, v \in F_{\mathcal{J}}$, we have $\alpha(u, v) \geq \lambda$.

Theorem 2.3. *Assume that all the hypotheses Theorem 2.2 are satisfied together with $\alpha(u, v) \geq \lambda\ \forall\ u, v \in F_{\mathcal{J}}$, then there is a only one point in $F_{\mathcal{J}}$, i.e., fixed point of $\mathcal{J}$ is unique.*

Proof. Let u, v be any two elements in $F_{\mathcal{J}}$ with $\alpha(u, v) \geq \lambda$. Then, since $\mathcal{J}$ is a multi-valued P-contraction via C-class function w.r.t ρ_λ, consequent we have

$$\xi(\lambda^3 \Delta_{\rho_\lambda}(\mathcal{J}u, \mathcal{J}v)) \leq \mathsf{F}(\xi(\mathsf{P}(u, v)), \eta(\mathsf{P}(u, v))), \tag{2.23}$$

where

$$\mathsf{P}(u, v) = \rho_\lambda(u, v) + \mid \mathcal{N}_{\rho_\lambda}(u, \mathcal{J}u) - \mathcal{N}_{\rho_\lambda}(v, \mathcal{J}v) \mid .$$

Since $u \in \mathcal{J}u$ and $v \in \mathcal{J}v$, so it gives $\mathcal{N}_{\rho_\lambda}(u, \mathcal{J}u) = \mathcal{N}_{\rho_\lambda}(v, \mathcal{J}v) = 0$. Thus, from (2.23), we get

$$\xi(\rho_\lambda(u, v)) \leq \xi(\lambda^3 \Delta_{\rho_\lambda}(\mathcal{J}u, \mathcal{J}v)) \leq \mathsf{F}(\xi(\rho_\lambda(u, v)), \eta(\rho_\lambda(u, v))) \leq \xi(\rho_\lambda(u, v)),$$

implies $\mathsf{F}(\xi(\rho_\lambda(u, v)), \eta(\rho_\lambda(u, v))) = \xi(\rho_\lambda(u, v))$. Now, by second property of C-class function, we have either $\xi(\rho_\lambda(u, v)) = 0$ or $\eta(\rho_\lambda(u, v)) = 0$. Thus, we obtain $\rho_\lambda(u, v) = 0 \Rightarrow u = v$. Hence, $F_{\mathcal{J}}$ is singleton. ☐

In the next section, we discuss an application of our proposed first new fixed point result.

3. Application

3.1. **Generalized data dependence problem.** Let $\mathcal{J}_1, \mathcal{J}_2 : X \to CB(X)$ be two given multivalued mappings. Suppose that for $\sigma > 0$, $\mathsf{H}_{\rho_\lambda}(\mathcal{J}_1 u, \mathcal{J}_2 u) \leq \sigma$ holds $\forall\, u \in X$. Also, suppose that the fixed point set of the mappings $\mathcal{J}_1, \mathcal{J}_2$ are non-empty, i.e., $F_{\mathcal{J}_1}, F_{\mathcal{J}_2} \neq \emptyset$. Then to measure the distance between the sets $F_{\mathcal{J}_1}$ and $F_{\mathcal{J}_2}$ is known as data dependence problem. There are different types of data dependence results available in the literature. Below, we prove the following theorem to discuss the generalized data dependence problem for two b-metric spaces.

Now, before going to our next theorem, first we slightly modify the contraction given in Definition 2.1 in the following way.

Modified contraction:

$$\mathcal{H}\big(\alpha(u,v), \lambda^{\varrho}\Delta_{\rho_\lambda}\big(\mathcal{J}u, \mathcal{J}v\big)\big) \leq \mathcal{F}\big(\mu(u,v), \theta_\gamma\big(I^{\lambda}_{HR}(u,v)\big)\big),\ \forall\, u,v\ \in X \setminus F_{\mathcal{J}},$$

where $\lambda, \varrho \in (1,\infty)$, the pair $(\mathcal{F},\mathcal{H})$ is an upper class function, θ_γ as defined in Definition 2.1 and

$$I^{\lambda}_{HR}(u,v) =$$

$$= \big[\rho_\lambda(u,v)\big]^{\beta_1} \cdot \big[\mathcal{N}_{\rho_\lambda}(u,\mathcal{J}u)\big]^{\beta_2} \cdot \big[\mathcal{N}_{\rho_\lambda}(v,\mathcal{J}v)\big]^{\beta_3} \cdot \Big[\frac{\mathcal{N}_{\rho_\lambda}(u,\mathcal{J}v) + \mathcal{N}_{\rho_\lambda}(v,\mathcal{J}u)}{2\lambda}\Big]^{1-\beta_1-\beta_2-\beta_3}.$$

Theorem 3.1. *Let (X,ρ_λ) and (X,δ_τ) be two b-metric spaces with coefficient λ, τ respectively. Let $\mathcal{J}_1, \mathcal{J}_2 : X \to CB(X)$ be two set-valued mappings such that for $\sigma > 0$, $\mathsf{H}_{\rho_\lambda}(\mathcal{J}_1 u, \mathcal{J}_2 u) \leq \sigma$ holds $\forall\, u \in X$. Assume that the following assertions hold:*
(E_1) $\mathcal{J}_2$ is a multivalued triangular α-admissible mapping of type λ and multivalued triangular μ-sub admissible mapping of type λ^{-1};
(E_2) assume that $F_{\mathcal{J}_1} \neq \emptyset$ together with $\alpha(u,v) \geq \lambda$ and $\mu(u,v) \leq \lambda^{-1},\ \forall\ u \in F_{\mathcal{J}_1}$ and $v \in \mathcal{J}_2 u$;
(E_3) $\mathcal{J}_2$ is a modified multi-valued interpolative $(\alpha,\mu,\mathcal{H},\mathcal{F})_{\theta_\gamma}$-contraction of HR-type w.r.t ρ_λ;
(E_4) $\delta_\tau(u,v) \leq \rho_\lambda(u,v),\ \forall\ u,v \in X$;
(E_5) (X,δ_τ) is a $\mathcal{J}$ orbitally complete mapping;
(E_6) $\mathcal{J}$ is a closed multivalued mapping w.r.t. δ_τ.

Then, $F_{\mathcal{J}_2} \neq \emptyset$. Moreover, $\sup_{z \in F_{\mathcal{J}_1}} \mathcal{N}_{\delta_\tau}(z, F_{\mathcal{J}_2}) \leq \frac{\tau\lambda^{\varrho+1}\sigma}{1-\lambda\beta}$, where $\beta = \frac{1}{\lambda^{\varrho}}$.

Proof. Since, $F_{\mathcal{J}_1} \neq \emptyset$, so let us consider a point $z_0 \in F_{\mathcal{J}_1}$. Now $\alpha(z_0,u) \geq \lambda$, $\mu(z_0,u) \leq \lambda^{-1}$, $\forall\ u \in \mathcal{J}_2 z_0$, since $\mathcal{J}z_0 \neq \emptyset$. Also, we have $\Delta_{\rho_\lambda}\big(\mathcal{J}u,\mathcal{J}v\big) \leq \lambda^{\varrho}\Delta_{\rho_\lambda}\big(\mathcal{J}u,\mathcal{J}v\big)$ and $\theta_\gamma(I^{\lambda}_{HR}(u,v)) \leq \theta_\gamma(I_{HR}(u,v))$. Now, one can observe that $\mathcal{J}_2$ enjoys all the assertions of Theorem 2.1. Consequently, $\mathcal{J}_2$ has a fixed point, i.e., $F_{\mathcal{J}_2} \neq \emptyset$. Since $\mathcal{J}_1 z_0, \mathcal{J}_2 z_0 \in CB(X)$ and $\lambda^{\varrho} > 1$, there exists a $z_1 \in \mathcal{J}_2 z_0$ such that

$$\rho_\lambda(z_0,z_1) \leq \lambda^{\varrho}\mathcal{N}_{\rho_\lambda}(z_0,\mathcal{J}_2 z_0) \leq \lambda^{\varrho}\mathsf{H}_{\rho_\lambda}(\mathcal{J}_1 z_0, \mathcal{J}_2 z_0). \tag{3.1}$$

Moreover, we have $\alpha(z_0,z_1) \geq \lambda, \mu(z_0,z_1) \leq \lambda^{-1}$. Since, $\mathcal{J}_2$ satisfies (E_3), consequently we get

$$\begin{aligned}&\mathcal{H}\big(1,\lambda^{\varrho}\Delta_{\rho_\lambda}\big(\mathcal{J}_2z_0,\mathcal{J}_2z_1\big)\big)\\&\leq \mathcal{H}\big(\alpha(z_0,z_1),\lambda^{\varrho}\Delta_{\rho_\lambda}\big(\mathcal{J}_2z_0,\mathcal{J}_2z_1\big)\big)\\&\leq \mathcal{F}\big(\mu(z_0,z_1),\theta_\gamma\big(I^{\lambda}_{HR}(z_0,z_1)\big)\big)\\&\leq \mathcal{F}\big(1,\theta_\gamma\big(I^{\lambda}_{HR}(z_0,z_1)\big)\big),\end{aligned}$$

which implies

$$\lambda^{\varrho}\Delta_{\rho_\lambda}\big(\mathcal{J}_2z_0,\mathcal{J}_2z_1\big) \leq \theta_\gamma\big(I^{\lambda}_{HR}(z_0,z_1)\big).$$

Furthermore, from the above inequality, we can also write the following

$$\lambda^{\varrho}\mathcal{N}_{\rho_\lambda}(z_1,\mathcal{J}_2z_1) \leq \theta_\gamma\big(\max\{\rho_\lambda(z_0,z_1),\mathcal{N}_{\rho_\lambda}(z_1,\mathcal{J}_2z_1)\}\big).$$

Clearly, if $\max\{\rho_\lambda(z_0,z_1),\mathcal{N}_{\rho_\lambda}(z_1,\mathcal{J}_2z_1)\} = \mathcal{N}_{\rho_\lambda}(z_1,\mathcal{J}_2z_1)$, then we arrive at a contradiction. Again, $\exists$ a $z_2 \in \mathcal{J}_2z_1$ such that

$$\rho_\lambda(z_1,z_2) \leq \frac{1}{\lambda^{\varrho}}\theta_\gamma(\rho_\lambda(z_0,z_1)) \leq \frac{1}{\lambda^{\varrho}}\rho_\lambda(z_0,z_1)$$

with $\alpha(z_1,z_2) \geq \lambda,\ \mu(z_1,z_2) \leq \lambda^{-1}$, since $\mathcal{J}_2$ is a triangular α-admissible and μ-subadmissible mapping of type λ and λ^{-1}, respectively. Continuing in this way, one can obtain

$$\rho_\lambda(z_{r+1},z_{r+2}) \leq \frac{1}{\lambda^{\varrho}}\rho_\lambda(z_r,z_{r+1}) \leq \cdots \leq \beta^{r+1}\rho_\lambda(z_0,z_1),\ \forall\ r \in \{0\}\cup\mathbb{N}, \tag{3.2}$$

where $\beta = \frac{1}{\lambda^{\varrho}}$. Clearly from (3.2), one can easily show that $\{z_r\}_{r=0}^{\infty}$ is a Cauchy sequence in (X,ρ_λ), i.e., $\rho_\lambda(z_r,z_s) \to 0$ as $r,s \to \infty$. But according to the given condition $\delta_\tau(u,v) \leq \rho_\lambda(u,v)\ \forall\ u,v \in X$. Hence, $\rho_\lambda(z_r,z_s) \to 0 \Rightarrow \delta_\tau(z_r,z_s) \to 0$ as $r,s \to \infty$. But (X,δ_τ) is a $\mathcal{J}_2$-orbitally complete. Hence $\exists$ a $z^* \in X$ such that $\delta_\tau(z_r,z^*) \to 0$ as $r \to \infty$. Since $\mathcal{J}_2$ is closed w.r.t δ_τ, i.e., $z^* \in \mathcal{J}_2z^*$. Also, from inequality (3.1), we get

$$\rho_\lambda(z_0,z_1) \leq \lambda^{\varrho}\mathsf{H}_{\rho_\lambda}(\mathcal{J}_1z_0,\mathcal{J}_2z_0) \leq \lambda^{\varrho}\sigma.$$

Again for $r,s \in \mathbb{N}$ with $r < s$, we have the following

$$\begin{aligned}\delta_\tau(z_r,z_{r+s}) &\leq \rho_\lambda(z_r,z_{r+s})\\&\leq \lambda\rho_\lambda(z_r,z_{r+1}) + \lambda^2\rho_\lambda(z_{r+1},z_{r+2}) + \cdots + \lambda^{s-1}\rho_\lambda(z_{r+s-1},z_{r+s})\\&\leq \lambda\beta^r\rho_\lambda(z_0,z_1) + \lambda^2\beta^{r+1}\rho_\lambda(z_0,z_1) + \cdots + \lambda^{s-1}\beta^{r+s-1}\rho_\lambda(z_0,z_1)\\&\leq \lambda\beta^r[1+\lambda\beta+(\lambda\beta)^2+\cdots+(\lambda\beta)^{s-1}]\rho_\lambda(z_0,z_1)\\&= \lambda\beta^r\frac{1-(\lambda\beta)^s}{1-\lambda\beta}\rho_\lambda(z_0,z_1).\end{aligned} \tag{3.3}$$

Now, considering lim sup as $s \to \infty$ in (3.3), we have

$$\begin{aligned}&\frac{1}{\tau}\delta_\tau(z_r,z^*) \leq \frac{\lambda\beta^r}{1-\lambda\beta}\rho_\lambda(z_0,z_1)\\&\Rightarrow \delta_\tau(z_r,z^*) \leq \frac{\tau\lambda\beta^r}{1-\lambda\beta}\rho_\lambda(z_0,z_1)\end{aligned}$$

Now putting $r = 0$ in the above inequality, we have

$$\delta_\tau(z_0, z^*) \leq \frac{\tau\lambda}{1-\lambda\beta}\rho_\lambda(z_0, z_1) \leq \frac{\tau\lambda\lambda^\varrho}{1-\lambda\beta}\mathsf{H}_{\rho_\lambda}(\mathcal{J}_1 z_0, \mathcal{J}_2 z_0) \leq \frac{\tau\lambda^{\varrho+1}\sigma}{1-\lambda\beta}$$
$$\Rightarrow \mathcal{N}_{\delta_\tau}(z_0, F_{\mathcal{J}_2}) \leq \frac{\tau\lambda^{\varrho+1}\sigma}{1-\lambda\beta}.$$

Since $z_0 \in F_{\mathcal{J}_1}$ and it is arbitrary, consequently we have

$$\sup_{z\in F_{\mathcal{J}_1}} \mathcal{N}_{\delta_\tau}(z, F_{\mathcal{J}_2}) \leq \frac{\tau\lambda^{\varrho+1}\sigma}{1-\lambda\beta}. \qquad \square$$

3.2. **Generalized stability problem.** Let (X, ρ_λ) and (X, δ_τ) be two b-metric spaces with coefficient λ, τ respectively. Let us consider a sequence of multivalued mappings $\{\mathcal{J}_r : X \to CB(X)\}_{r=1}^{\infty}$ and another multivalued mapping $\{\mathcal{J} : X \to CB(X)\}$ such that $\mathcal{J}_r$ tends to $\mathcal{J}$ uniformly as $r \to \infty$, i.e., $\lim_{r\to\infty} \mathcal{J}_r = \mathcal{J}$ w.r.t ρ_λ. Suppose that $\{F_{\mathcal{J}_r}\}$, $\{F_{\mathcal{J}}\}$ be the collection of all fixed point sets of the mappings $\mathcal{J}_r$ and $\mathcal{J}$, respectively. We say that the fixed point set of $\mathcal{J}_r$, i.e., $F_{\mathcal{J}_r}$ are stable if $\mathsf{H}_{\delta_\tau}(F_{\mathcal{J}_r}, F_{\mathcal{J}}) \to 0$ as $r \to \infty$. To discuss our generalized stability problem, we need the following lemma.

Lemma 3.1. *Let (X, ρ_λ) be a continuous b-metric space. Let $\{\mathcal{J}_r : X \to CB(X)\}$ be a sequence of multivalued mappings which converges to a mapping $\{\mathcal{J} : X \to CB(X)\}$ uniformly w.r.t ρ_λ. If $\mathcal{J}_r$ satisfies the modified multi-valued interpolative $(\alpha, \mu, \mathcal{H}, \mathcal{F})_{\theta_\gamma}$-contraction of HR-type w.r.t ρ_λ for every $r \in \mathbb{N}$ then $\mathcal{J}$ also satisfies the same.*

Proof. For $u, v \in X \setminus F_{\mathcal{J}_r}$, we have the following

$$\mathcal{H}\big(\alpha(u,v), \lambda^\varrho \Delta_{\rho_\lambda}\big(\mathcal{J}_r u, \mathcal{J}_r v\big)\big) \leq \mathcal{F}\big(\mu(u,v), \theta_\gamma\big(I^\lambda_{HR}(u,v)\big)\big),$$

where $\lambda, \varrho \in (1, \infty)$, the pair $(\mathcal{F}, \mathcal{H})$ is an upper class function, θ_γ as defined in Definition 2.1 and

$$I^\lambda_{HR}(u,v) =$$
$$= \big[\rho_\lambda(u,v)\big]^{\beta_1} \cdot \big[\mathcal{N}_{\rho_\lambda}(u, \mathcal{J}_r u)\big]^{\beta_2} \cdot \big[\mathcal{N}_{\rho_\lambda}(v, \mathcal{J}_r v)\big]^{\beta_3} \cdot \Big[\frac{\mathcal{N}_{\rho_\lambda}(u, \mathcal{J}_r v) + \mathcal{N}_{\rho_\lambda}(v, \mathcal{J}_r u)}{2\lambda}\Big]^{1-\beta_1-\beta_2-\beta_3}.$$

Now, taking limit as $r \to \infty$ and keeping in mind that that $\mathcal{J}_r$ converges $\mathcal{J}$ uniformly w.r.t ρ_λ and ρ_λ is a continuous b-metric space, consequently we have the following

$$\mathcal{H}\big(\alpha(u,v), \lambda^\varrho \Delta_{\rho_\lambda}\big(\mathcal{J} u, \mathcal{J} v\big)\big) \leq \mathcal{F}\big(\mu(u,v), \theta_\gamma\big(I^\lambda_{HR}(u,v)\big)\big), \ \forall\ u, v\ \in X \setminus F_{\mathcal{J}},$$

where $\lambda, \varrho \in (1, \infty)$, the pair $(\mathcal{F}, \mathcal{H})$ is an upper class function, θ_γ as defined in Definition 2.1 and

$$I^\lambda_{HR}(u,v) =$$
$$= \big[\rho_\lambda(u,v)\big]^{\beta_1} \cdot \big[\mathcal{N}_{\rho_\lambda}(u, \mathcal{J} u)\big]^{\beta_2} \cdot \big[\mathcal{N}_{\rho_\lambda}(v, \mathcal{J} v)\big]^{\beta_3} \cdot \Big[\frac{\mathcal{N}_{\rho_\lambda}(u, \mathcal{J} v) + \mathcal{N}_{\rho_\lambda}(v, \mathcal{J} u)}{2\lambda}\Big]^{1-\beta_1-\beta_2-\beta_3}.$$

Hence $\mathcal{J}$ is a modified multi-valued interpolative $(\alpha, \mu, \mathcal{H}, \mathcal{F})_{\theta_\gamma}$-contraction of HR-type w.r.t ρ_λ. $\square$

Theorem 3.2. *Let* (X, ρ_λ) *and* (X, δ_τ) *be two b-metric spaces with coefficient* λ, τ *respectively. Let* $\{\mathcal{J}_r : X \to CB(X)\}$ *be a sequence of multivalued mappings which converges to a mapping* $\{\mathcal{J} : X \to CB(X)\}$ *uniformly w.r.t* ρ_λ. *Assume that every* $\{\mathcal{J}_r\}$ $(r \in \mathbb{N})$ *enjoys conditions* $(A_1) - (A_6)$ *in Theorem 2.1 and* $\mathcal{J}$ *enjoys every assertions from* (A_1) *to* (A_6) *except* (A_2). *Then* $F_{\mathcal{J}_r} \neq \emptyset$, $\forall\, r \in \mathbb{N}$. *Moreover, suppose that* $\alpha(u, v) \geq \lambda, \mu(u, v) \leq \lambda^{-1}$ *for every* $u \in F_{\mathcal{J}_r}$ $(r \in \mathbb{N})$ *and* $v \in \mathcal{J}u$ *or* $u \in F_{\mathcal{J}}$ *and* $v \in F_{\mathcal{J}_r}u$, $(r \in \mathbb{N})$. *Then fixed point sets of the sequence of multivalued mappings* $\{\mathcal{J}_r\}_{r=1}^{\infty}$ *are stable w.r.t* δ_τ.

Proof. Clearly $F_{\mathcal{J}_r} \neq \emptyset$, $\forall\, r \in \mathbb{N}$. By Lemma 3.1 and Theorem 3.1, we obtain $F_{\mathcal{J}} \neq \emptyset$. Let $\sigma_r = \sup_{u \in X} \mathsf{H}_{\rho_\lambda}(\mathcal{J}_r u, \mathcal{J}u)$, $\forall\, r \in \mathbb{N}$. By our assumption, $\mathcal{J}_r \to \mathcal{J}$ uniformly, i.e., we have

$$\lim_{r\to\infty} \sigma_r = \lim_{r\to\infty} \sup_{u \in X} \mathsf{H}_{\rho_\lambda}(\mathcal{J}_r u, \mathcal{J}u) = 0. \tag{3.4}$$

Now by Theorem 3.1, we have

$$\sup_{z \in F_{\mathcal{J}}} \mathcal{N}_{\delta_\tau}(z, F_{\mathcal{J}_r}) \leq \frac{\tau \lambda^{\varrho+1} \sigma_r}{1 - \lambda\beta}$$

and

$$\sup_{z \in F_{\mathcal{J}_r}} \mathcal{N}_{\delta_\tau}(z, F_{\mathcal{J}}) \leq \frac{\tau \lambda^{\varrho+1} \sigma_r}{1 - \lambda\beta},$$

which implies

$$\mathsf{H}_{\delta_\tau}(F_{\mathcal{J}_r}, F_{\mathcal{J}}) \leq \frac{\tau \lambda^{\varrho+1} \sigma_r}{1 - \lambda\beta}.$$

Clearly by using (3.4), from the above inequality, one can obtain

$$\lim_{r\to\infty} \mathsf{H}_{\delta_\tau}(F_{\mathcal{J}_r}, F_{\mathcal{J}}) = 0.$$

Therefore fixed point sets of the sequence of multivalued mappings $\{\mathcal{J}_r\}_{r=1}^{\infty}$ are stable w.r.t δ_τ. □

4. Conclusion

In this work, we have successfully extended the main result of Karapinar et al. [41], by involving the notion of multivalued triangular admissible/sub-admissible mappings of type Λ/Λ^{-1} along with upper class functions and Maia type result. We have studied the famous "P-contraction" by using C-class function, admissible mappings and Maia type result. We have given an example to support our main results. Lastly, we have discussed generalized data dependence and stability problem as an application.

5. An open problem

We have discussed our new results in the setting of two b-metric spaces. Our open problem is that can one extend our results into a more general setting, by using the same type of abstract spaces, for example in extended b-metric spaces (see [36]).

Acknowledgement. The authors are very grateful to Professor Dr. Adrian Petruşel (the editor in chief), Professor Dr. Erdal Karapınar and the anonymous reviewer for their valuable comments and several useful suggestions which improved the presentation of the paper. The first author (SKG) would like to thank his mother Mrs. Reba Ghosh for her continuous encouragement during the preparation of the manuscript.

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Received: August 19, 2023; Accepted: December 7, 2023.