Fixed Point Theory, 25(2024), No. 2, 531-544
DOI: 10.24193/fpt-ro.2024.2.06
http://www.math.ubbcluj.ro/~nodeacj/sfptcj.html

APPROXIMATE FIXED POINTS IN METRIC SPACES

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Abstract. The chances for a mapping taken at random from a given set of mappings to have approximate fixed points are studied in this paper. We start from the discrete case to range more abstract spaces as metric measure spaces. Initial insights for this work are elementary, and some of the observations may already be known. At the same time, they seem to point the way to deeper questions and raise the potential for future study.

Key Words and Phrases: Discrete probability, metric measure spaces, approximate fixed points, random mappings.
2020 Mathematics Subject Classification: 54H25, 47H10, 51K05, 60C05.

1. Introduction

This paper contains some observations about the connection between mappings taken at random and the probability of the existence of fixed points and approximate fixed points in certain metric spaces.

The concept of approximate fixed point is straightforward. Let f be a mapping of a metric space (X, d) into itself. Then, if $\varepsilon > 0$, a point $x \in X$ is said to be an *ε-approximate fixed point of f* if $d(x, f(x)) \leq \varepsilon$. The mapping f is said to have an *approximate fixed point* if f has an ε-approximate fixed point for every $\varepsilon > 0$. This of course is equivalent to saying:

$$\inf \{d(x, f(x)) : x \in X\} = 0.$$

A *fixed point* of f is a point $x \in X$ for which $f(x) = x$; thus x is an ε-approximate fixed point of f for every $\varepsilon > 0$. For throughout expositions on metric fixed point theory the interested reader may check any of the two monographs [2, 4] and references therein.

The concept of a *random mapping or mapping taken at random* in this context is more subtle and may be thought of in two ways. If S and X are sets, the usual and most obvious way to define a random mapping is to simply say that a mapping $f : S \to X$ is *random* if it is arbitrarily chosen from the family X^S of all mappings from

S into X. In studying the existence of approximate fixed points, another approach is also useful. Assume (X, d, μ) is a metric measure space, where μ is a probability measure on X (i.e., a finite positive measure for which $\mu(X) = 1$). In this context, one might say that $f : S \to X$ is random (or taken at random) if given any $x \in X$ and any measurable subset M of X, the only information we know about $f(x)$ is that the probability that $f(x)$ lies in M is simply $\mu(M)$.

We begin with a simple example involving only concepts of finite probability theory. Here a finite set is a collection of n events each having probability $1/n$ of occurring. No distance is needed but we can assume the discrete distance is considered. The symbol $\mathbb{N}$ denotes the collection of natural numbers.

Example 1.1 Let $n, k \in \mathbb{N}$ and

$$A = \{a_1, \cdots, a_n\} \text{ and } B = \{a_1, \cdots, a_n, \cdots, a_{n+k}\}.$$

Assume also that all points are distinct, and suppose $f : A \to B$. Then the probability that f has at least one fixed point is

$$1 - \left(\frac{n+k-1}{n+k}\right)^n.$$

Indeed, if a given point $a_i \in A$ is not fixed it must be mapped into any of $n+k-1$ points out of $n+k$ possibilities. Thus the probability that it is not fixed is

$$\left(\frac{n+k-1}{n+k}\right).$$

Therefore the probability that no point of A is fixed is:

$$\left(\frac{n+k-1}{n+k}\right)^n.$$

It follows that probability that at least one point is left fixed is

$$1 - \left(\frac{n+k-1}{n+k}\right)^n.$$

This simple example enlightens our first approach to approximate fixed points given in the next section.

2. Approximate fixed points in an interval

We now move to a different question, beginning with the simplest case. We assume the uniform probability over finite sets. Taking $A = B$ in Example 1.1 yields the following:

Theorem 2.1 *Suppose $f : A \to A$ is a mapping taken at random from A^A, where A is a set with n elements. Then the probability that f has at least one fixed point is*

$$1 - \left(\frac{n-1}{n}\right)^n.$$

This, additionally, leads to the following:

Theorem 2.2 *Suppose $f_i : A \to A$, $i = 1, \cdots, k$, are k mappings taken at random from A^A, where A is a set with n elements. Then the probability that at least one of the mappings has at least one fixed point is given by*

$$1 - \left(\frac{n-1}{n}\right)^{nk}.$$

These ideas can be applied to obtain approximate fixed point theorems in nondiscrete cases. We see it next. Take $I = [0, 1]$ and suppose f is any mapping of I into itself. Let $\varepsilon > 0$ and choose n to be the smallest $n \in \mathbb{N}$ such that $\frac{1}{n} \le 2\varepsilon$. Let $P_n = \{a_i\}$, where $a_i = i/n$, $i = 0, 1, \cdots, (n-1)$, be a partition of I into n equal subintervals each of length $\frac{1}{n}$, and define $h : [0, 1) \to [0, 1)$ by taking

$$h(x) = a_i \Leftrightarrow x \in [a_i, a_{i+1}), \; i = 0, \cdots, n-1.$$

Now let $A = \{a_1, \cdots, a_n\}$ and define $\tilde{f} : A \to A$ by taking $\tilde{f}(a_i) = h \circ f(a_i)$, for $i = 1, \cdots, n$. Since $\tilde{f}$ is a mapping of P_n into P_n (which can be considered to have been taken at random), by the above theorem, the probability that $\tilde{f}$ has at least one fixed point is given by

$$1 - \left(\frac{n-1}{n}\right)^{n}.$$

However, $\tilde{f}(a_i) = a_i$ implies that $|a_i - f(a_i)| \le \varepsilon$. Thus a fixed point of $\tilde{f}$ yields an ε-approximate fixed point of the original mapping f.

This implies that, for $\varepsilon > 0$, given a mapping f of I into I there is always a finite subset of I such that f restricted to this set has probability at least $1 - \left(\frac{n-1}{n}\right)^n$ to have an ε-approximate fixed point (actually, the same finite subset works for any such f in this case). Then we have the following definition.

Definition 2.3 Let X be a metric space. Let F be a nonempty finite subset of X. We denote by $\alpha_\varepsilon(F)$ the supremum of the numbers α such that, given a mapping f of X into X at random, it is the case that f restricted to F has at least probability α to have an ε-approximate fixed point. Then we say that the ε-approximate fixed point probability constant of X, denoted as $\alpha_\varepsilon(X)$, is the supremum of all $\alpha_\varepsilon(F)$ when F is a nonempty finite subset of X.

After the previous reasoning for I and the above definition, we have the following result.

Theorem 2.4 *If I is the real unit interval and $\varepsilon > 0$, then*

$$\alpha_\varepsilon(I) \ge 1 - \left(\frac{n-1}{n}\right)^{n}.$$

We can still push farther this scenery, since $\lim_{n\to\infty} \left(\frac{n-1}{n}\right)^n = e^{-1}$ in a decreasing way, we have that $\alpha_\varepsilon(I) \ge 1 - e^{-1}$ for any $\varepsilon > 0$. This brings the next definition up.

Definition 2.5 Let X be a metric space. Then the approximate fixed point probability constant of X, denoted as $\alpha(X)$, is the infimum of all $\alpha_\varepsilon(X)$ when $\varepsilon > 0$.

Therefore, from above we deduce the next theorem.

Theorem 2.6 *Let I be the unit real interval with its usual metric, then*

$$\alpha(I) \geq 1 - e^{-1}.$$

Remark 2.7 The heuristic idea behind this result is that given $f\colon [0,1] \to [0,1]$ at random, the probability that it has an approximate fixed points is at least $1 - e^{-1}$ which is larger than 0.63.

Notice that it takes no changes in the proof of Theorem 2.6 to deduce the next corollary (see also Theorem 2.11 below).

Corollary 2.8 *Let I be any bounded nontrivial real interval with its usual metric, then*

$$\alpha(I) \geq 1 - e^{-1}.$$

Suppose now that $[a,b]$ is a subinterval of $[0,1]$ and suppose that f is any mapping of $[0,1]$ into $[0,1]$. We can define in a similar way as Definition 2.5 the probability that f has an approximate fixed point in $[a,b]$ by calculating the above probability constants for mappings restricted to $[a,b]$, that is, looking for the ε-approximate fixed point in $[a,b]$. The corresponding approximate fixed point probability constant will be denoted as $\alpha([a,b],I)$ in this case. We wonder about a lower estimate of this constant next.

Given $\varepsilon > 0$ we can follow a reasoning as the one given in the proof of Theorem 2.6 to fix n and a partition $A = \{a_1, \cdots, a_n\}$ of $[0,1]$ as above. Let $l = b - a$, then there are roughly (asymptotically as $n \to \infty$) $n \cdot l$ points of A in $[a,b]$. By considering Example 1.1, we can deduce that the probability that $\tilde{f}$ has an ε-approximate fixed point is at least

$$1 - \left(\frac{n-1}{n}\right)^{ln}.$$

As consequence, we can deduce the next theorem.

Theorem 2.9 *Let I be the unit closed real interval and a and b real numbers such that $0 \leq a < b \leq 1$. Take $l = b - a$, then*

$$\alpha([a,b],I) \geq 1 - e^{-l}.$$

Remark 2.10 Notice that the lower bound given in the above theorem is larger to $l(1 - e^{-1})$, which is the one we find if we think of the problem in pure probabilistic terms: probability that the ε-approximate fixed point falls in $[a,b]$ given that we know the probability that there is an ε-approximate fixed point in $[0,1]$. Of course, the nature of the problem we are dealing with is far more complicated than that.

From now on, we will focus on the case where the initial and final sets of mappings under consideration coincide. There are only two crucial assumptions in the proofs of the above results. The first is that f is a mapping chosen at random from the set of all mappings of $[0,1]$ into $[0,1]$. The second is that $[0,1]$ is viewed as a probability measure space using Lebesgue measure, and so the probability that a randomly chosen point in $[0,1]$ lies in a given measurable subset S of $[0,1]$ is simply the Lebesgue measure of S. In view of this, it is clear that the result should have wider meaning. We begin by replacing the unit interval $[0,1]$ with a path in $\mathbb{R}^n$. A *path* in $\mathbb{R}^n$ is the image of a continuous mapping $\gamma : [a, b] \to X$ where $[a, b]$, $a \leq b$, is a real line interval. The length $\ell(\gamma)$ is the infimum of the lengths

$$\sum_{i=1}^{\mu} d(\gamma(x_i), \gamma(x_j)).$$

When this infimum is finite the path is said to be *rectifiable*. An argument essentially identical to the above yields the following result.

Theorem 2.11 *Let P be a rectifiable path in $\mathbb{R}^n$ with $\ell(P) = \lambda$, and let $f : P \to P$ be an arbitrary mapping. Consider the Hausdorff measure of dimension 1 on P, then*

$$\alpha(P) \geq 1 - e^{-1}.$$

Proof. Replace ε with ε/λ and follow the proof of Theorem 2.6 along with the new measure.

We now turn to more intricate sets. In the following statement we assume that a set P in a metric space can be decomposed into a finite number of rectifiable paths P_i, $i = 1, \cdots, N$, having respective lengths $\ell(P_i)$, where $\sum_{i-1}^{N} \ell(P_i) < \infty$. We also assume that each two paths intersect in at least 1 and in at most 2 points. It is now possible to think of P as the union of a finite number of connected metric spaces with the distance between each two points x and y which lie in a common component taken to be the length of the shortest path joining x and y.

Theorem 2.12 *Let (X, d) be a metric space and let $P = \{P_i\}_{i=1}^{N}$, $N \geq 1$, be a (possibly disconnected) subset of X consisting of a collection of N rectifiable paths, each having finite length. Suppose each two of paths P_i and P_j of P either have empty intersection, or they intersect as follows. If P_i and P_j lie in a common component of P and $i \neq j$, then $P_i \cup P_j$ either forms a simple closed curve having two common endpoints, or a single path with one endpoint of P_i coinciding with one endpoint of P_j. Then*

$$\alpha(P) \geq 1 - e^{-1}.$$

Proof. Let $p_j = \ell(P_j) / \sum_{i=1}^{N} \ell(P_i)$, $j = 1, \cdots, N$. Now normalize the total length of the structure P so that its total length is 1; thus if $p_i = \ell(P_i)$ then $\sum_{i=1}^{N} p_i = 1$. Then given $x \in P_i$, the probability that $f(x) \in P_i$ is precisely p_i. By assumption, the probability that a given $x \in P_i$ is an ε-approximate fixed point for f is now equal to the probability that x is an ε-approximate fixed point for f given that $f(x)$ lies in P_i times the probability that $f(x)$ lies in P_i. This probability is at least $\left(1 - e^{-1}\right) p_i$. Since the sets P_i are disjoint (except possibly for common endpoints) the total probability that an arbitrary point $x \in P$ is an ε-approximate fixed point for f is at least

$$\sum_{i=1}^{N} \left(1 - e^{-1}\right) p_i = 1 - e^{-1}.$$

(This is precisely the line of reasoning in the proof of Theorem 2.6 except that a finite number of points are ignored. Given that the underlying set is uncountable, this has no impact since the probability that any given point $x \in P$ is an ε-approximate fixed point for f is zero.)

3. Metric measure spaces

Up to this point we have been mainly dealing with finite sets under the uniformly distributed probability and direct consequences out of them. On the other hand, certain probability measures have been shown to be related to the idea of how to choose a mapping at random in a natural way. A primary example, as we have seen, may be given by the Lebesgue measure on real line intervals (or 1-Hausdorff measure on a rectifiable path). In this section we take up the more general case by considering arbitrary maps defined on a metric measure space consisting of a triplet (X, d, μ), where X is a set, d is a metric on X and μ is a finite positive measure on X. We assume further that μ is a probability measure and so $\mu(X) = 1$. For a metric measure space, we assume that balls are always measurable sets. The question considered here is again to obtain a lower bound for $\alpha(X)$.

We set next the precise idea of mapping taken at random in this context.

Definition 3.1 Let (X, d, μ) be a metric measure space and suppose μ is a probability measure. A mapping f of an arbitrary set into X is said to be a random mapping or a mapping taken at random if given $x \in S$ and a measurable subset H of X, the information we know about f is that the probability that $f(x) \in H$ is $\mu(H)$.

From this definition we can set that, given $(c_n)_{n=1}^N \subset X$, the probability that at least one of these points is an ε-approximate fixed point of an arbitrary map $f : X \to X$ is given by

$$1 - \prod_{n=1}^{N} (1 - \mu(B(c_n; \varepsilon))).$$

Now, for each finite set $(c_n)_{n=1}^N$ and $\varepsilon > 0$, consider the set

$$C\left((c_n)_{n=1}^N, \varepsilon\right) = \{f \in X^X : \exists\, n_0 \in \{1, \cdots, N\} \text{ with } d(c_{n_0}, f(c_{n_0})) \leq \varepsilon\}.$$

Therefore, we can say in an heuristic way that the probability that f has a ε-approximate fixed point is at least as large as the probability that $f \in C\left((c_n)_{n=1}^N, \varepsilon\right)$. (This is analogous to the reasoning applied in Theorem 2.6.)

We now turn to an extension of Theorem 2.6. We say that a metric measure space (X, d, μ) (as before, assuming $\mu(X) = 1$) has the *boundary condition* if $\mu(\partial B(x; r)) = 0$ for any closed ball $B(x; r)$. This assures the following continuity property:

Lemma 3.2 *Let (X, d, μ) be a metric measure space with the boundary condition. Then $\alpha(r) = \mu(B(x; r) \cap C)$ is continuous for any $x \in X$ and $C \subseteq X$. Consequently $\alpha(r) = \mu((B(x; r) \setminus B(x; r_0)) \cap C)$ is also continuous.*

Now we extend Theorem 2.6 to metric measure spaces.

Theorem 3.3 *Suppose (X, d, μ) is a compact metric measure space with the boundary condition. Let $\delta > 0$. Then for every $\varepsilon > 0$ there exists a finite collection of points $(c_n)_{n=1}^N$ in X such that the probability that $f \in C\left((c_n)_{n=1}^N, \varepsilon\right)$, for random $f : X \to$*

X, is at least equal to $1-e^{-1}+\delta$. In particular,

$$\alpha(X) \geq 1-e^{-1}.$$

Proof. Let $\varepsilon>0$ and let $\{B(x_n;\varepsilon/2)\}_{n=1}^{N}$ be an $\varepsilon/2$ covering of X. Now let

$$\begin{aligned} B_1 &= B(x_1;\varepsilon/2), \\ B_2 &= B(x_2;\varepsilon/2)\setminus B_1, \\ &\vdots \\ B_i &= B(x_i;\varepsilon/2)\setminus \cup_{j=1}^{i-1} B_j, \\ &\vdots \\ B_N &= B(x_N;\varepsilon/2)\setminus \cup_{j=1}^{N-1} B_j. \end{aligned}$$

We assume without loss of generality that $\mu(B_i)>0$ for each i and choose $M>0$ so that

$$M^{-1} < \min\{\mu(B_i) : i=1,\cdots,N\}.$$

Now, from Lemma 3.2, for each $i=1$ take $c_1^1=x_1$ we can choose r_1^1 so that

$$\mu\left(B\left(c_1^1;r_1^1\right)\right)=\frac{1}{M}.$$

Let

$$C_1^1 = B\left(x_1;r_1^1\right).$$

If

$$\mu\left(B_1\setminus C_1^1\right)\geq \frac{1}{M}$$

then choose r_1^2 so that

$$\mu\left(B\left(x_1;r_1^2\right)\setminus C_1^1\right)=\frac{1}{M},$$

i.e., so that

$$\mu\left(B\left(x_1;r_1^2\right)\setminus B\left(x_1;r_1^1\right)\right)=\frac{1}{M}.$$

Continue in this way until we arrive at step $J(1)$ at which

$$\mu\left(B_1\setminus \cup_{j=1}^{J(1)} C_1^j\right) < \frac{1}{M}.$$

Now set $A_1 = B_1\setminus \cup_{j=1}^{J(1)-1} C_1^j$ and choose $a_1\in A_1$. (Simply disregard A_1 if $A_1=\emptyset$.)

Now proceed in the same fashion with x_2 and B_2, with the difference that each of the sets C_2^j, $j=1,\cdots,J(2)$, are intersected with B_2. Proceed in this way until reaching x_N and corresponding sets $\left\{C_N^j\right\}_{j=1}^{J(N)}$. In this way the original set is partitioned into $\sum_{i=1}^{N} J(i)$ disjoint sets, each having diameter less than ε and measure equal to $\frac{1}{M}$ along with N sets each having measure strictly less than $\frac{1}{M}$. The process results in the following:

- A collection of disjoint sets $\left(C_i^j\right)$ with $1 \leq i \leq N$ and $1 \leq j \leq J(i)$ such that $\mu\left(C_i^j\right) = 1/M$ for each i, j.
- A collection of points $c_i^j \in C_i^j$ for each i, j with $C_i^j \subseteq B\left(c_i^j; \varepsilon\right)$. (In particular the sets C_i^j each have diameter $\leq \varepsilon$.)
- N sets A_i, $1 \leq i \leq N$, with $\mu(A_i) < 1/M$ for each i, and points a_i such that $A_i \subseteq B(a_i; \varepsilon)$.
- The family

$$\left\{C_1^1, C_1^2, \cdots, C_1^{J(1)}, C_2^1, \cdots, C_2^{J(2)}, C_N^1, \cdots, C_N^{J(N)}, A_1, \cdots, A_N\right\}$$

form a disjoint covering of the set X.

We now give a lower bound for the number $\sum_{i=1}^N J(i)$:

$$\begin{aligned} 1 &= \mu(X) = \mu\left(\cup_{j=1}^{J(i)}\left(\cup_{i=1}^N C_i^j\right)\bigcup\left(\cup_{i=1}^N A_i\right)\right) \\ &= \sum_{1\leq i\leq N; 1\leq j\leq J(i)} \mu\left(C_i^j\right) + \sum_{1=1}^N \mu(A_i) \\ &= \sum_{i=1}^N \frac{J(i)}{M} + \sum_{1=1}^N \mu(A_i) \\ &\leq \left(\sum_{1\leq j\leq N} J(i)\right)\frac{1}{M} + \frac{N}{M}. \end{aligned}$$

This implies

$$\sum_{i=1}^N J(i) \geq M - N. \tag{3.1}$$

Let f be a mapping taken at random of X into X. For each $1 \leq i \leq N$ and $1 \leq j \leq J(i)$, the probability that $f(c_i^j) \notin C_i^j$ equals $1 - 1/M$, and for each i the probability that $f(a_i) \notin A_i$ lies in the interval $[0, 1/M)$. Therefore, we can set the probability that none of the points (c_i^j) and (a_i) are not in the corresponding sets C_i^J and A_i as

$$\begin{aligned} p &= \prod_{1\leq i\leq N;:1\leq j\leq J(i)} P(f(c_i^j) \notin C_i^j) \times \prod_{i=1}^N P(a_i \notin A_i) \\ &= \left(1 - \frac{1}{M}\right)^{\sum_{i=1}^N J(i)} \times \prod_{i=1}^N P(a_i \notin A_i) \end{aligned}$$

From (3.1) and the fact that probabilities are no larger than 1,

$$p \leq \left(1 - \frac{1}{M}\right)^{M-N} \leq \left(1 - \frac{1}{M}\right)^{M - \frac{N}{M}},$$

where N is fixed and M can move to infinite. This gives a sequence converging to e^{-1} as M goes to infinity.

Therefore, given $\delta > 0$ it is possible to fix M large enough so that $p \in (1 - e^{-1} - \delta, 1 - e^{-1} + \delta)$. This yields a finite collection of points (c_n) (these are points (c_i^j) and (a_i) in the proof) such that

$$P(((c_n), \varepsilon)) \geq 1 - e^{-1} + \delta.$$

Remark 3.4 An upper bound can be found for $\sum_{i=1}^{N} J(i)$ as follows, but it does not seem to be of much help since it leads to an upper bound for the probability we are looking for.

$$1 = \mu(X) \geq \mu(\cup_{1 \leq i \leq N;\, 1 \leq j \leq J(i)} C_i^j) = \frac{1}{M} \sum_{i=1}^{N} J(i).$$

Therefore

$$\sum_{i=1}^{N} J(i) \leq M. \tag{3.2}$$

Then

$$\begin{aligned} p &\geq \left(1 - \frac{1}{M}\right)^{\sum_{i=1}^{N} J(i)} \times \left(1 - \frac{1}{M}\right)^{N} \\ &\geq \left(1 - \frac{1}{M}\right)^{M} \times \left(1 - \frac{1}{M}\right)^{N}. \end{aligned}$$

This gives an increasing sequence that converges to e^{-1} as M goes to infinity.

Remark 3.5 Notice that in all cases under consideration, except for Theorem 2.9, we have obtained a same lower bound for $\alpha(X)$, which in fact is larger than 0.63. This is not surprising if we realize that,results from Section 2 are particular cases of Theorem 3.3. It is plausible that higher bounds could be found for these spaces.

4. Comments on invariant sets and graphs

We show in this section some applications of the preceding. We begin with the following simple example. Suppose $\Sigma = \{S_1, S_2\}$ where S_1 and S_2 are nonempty bounded closed convex subsets of $\mathbb{R}^2$, and suppose $f : S_1 \cup S_2 \to S_1 \cup S_2$ is continuous. Suppose also that:

$$\text{Given } i \in \{1,2\} \text{ there exists } j \in \{1,2\} \text{ such that } f(S_i) \subseteq S_j. \tag{4.1}$$

What additional conditions assure that f has a fixed point? The condition $S_1 \cap S_2 \neq \emptyset$ provides an obvious answer. This is because $S_1 \cap S_2 \neq \emptyset$, in conjunction with condition (4.1) and continuity of f, implies that either $f : S_i \to S_i$ for some $i \in \{1,2\}$, or $f : S_1 \cap S_2 \to S_1 \cap S_2$. On the other hand, if $S_1 \cap S_2 = \emptyset$ then the probability that a mapping taken at random of S_i into S_j has a fixed point is 3/4.

The situation is slightly more complicated if Σ consists of three elements. There are now two sufficient conditions (up to relabeling) which assure the existence of a fixed point for f :

(I) $S_1 \cap S_2 \cap S_3 \neq \emptyset$; and

(II) $S_1 \cap S_2 \neq \emptyset$ and $S_2 \cap S_3 \neq \emptyset$, and $S_3 \cap S_1 = \emptyset$.

A third possibility exists. When this occurs f may or may not have a fixed point.

(III) $S_1 \cap S_2 \neq \emptyset$, $S_2 \cap S_3 \neq \emptyset$, $S_3 \cap S_1 \neq \emptyset$, and $S_1 \cap S_2 \cap S_3 = \emptyset$.

When (I) occurs, f must have a fixed because $f : S_1 \cap S_2 \cap S_3 \to S_1 \cap S_2 \cap S_3$. Otherwise there are only two ways f can fail to have a fixed point. These are:

$$f : S_1 \to S_2,\ f : S_2 \to S_3,\ f : S_3 \to S_1;$$
$$f : S_1 \to S_3,\ f : S_3 \to S_2,\ f : S_2 \to S_1.$$

Assumption (II) in conjunction with condition (4.1) rules out each of these possibilities while either of these may occur under assumption (III). (One might also notice that there are 27 ways f can satisfy (4.1). The other 25 possibilities always assure that f has a fixed point. This means that if the mapping f takes elements of Σ randomly into subsets of them, then the probability that f will have a fixed point is strictly greater that 25/27.)

Our objective in this section is to extend the above very simple observations to infinite families $\{S_i\}$ of sets. To this end, let $\mathfrak{S}$ be a set and $\Sigma = \{S_\alpha, \alpha \in \mathfrak{I}\}$ a (possibly infinite) family of *nonempty* subsets of $\mathfrak{S}$. Suppose f is a mapping defined on $\cup\Sigma$ $(= \cup_i S_i)$ and taking values in $\mathfrak{S}$.

Definition 4.1 We say that the pair (Σ, f) is a fixed point structure if the following two conditions hold.

CI: The family Σ is closed under nonempty intersections.

FP: If $f : S_\alpha \to S_\alpha$ for $\alpha \in \mathfrak{I}$, then $f(x) = x$ for some $x \in S_\alpha$.

For intuitive purposes, one might think of Σ the very well-known cases given by: (i) a collection of nonempty compact convex subsets of a Banach space X and f a continuous mapping; or (ii) a collection bounded closed convex subsets of a uniformly convex Banach space X and f a nonexpansive mapping. Of course there are many other possibilities (check [2, 4]). This brings us to the following problem.

Problem 4.2 A Problem on Invariant Sets. Given (Σ, f) and some subfamily Σ' of Σ such that $f : \cup\Sigma' \to \cup\Sigma'$, find conditions on Σ' which assure that f has a fixed point. (Equivalently, find conditions which assure that $f : S_\alpha \to S_\alpha$ for some $\alpha \in \mathfrak{I}$.)

In the problem discussed at the outset, Σ is the family of nonempty bounded closed convex subsets of $\mathfrak{S} := \mathbb{R}^2$, $\Sigma' = \{S_1, S_2\}$, and $f : S_1 \cup S_2 \to S_1 \cup S_2$ is a continuous mapping. The desired condition is (4.1). To extend this one needs to adjust the above assumptions in such a way that a given mapping $f : \cup\Sigma \to \cup\Sigma$ will always map some member $S \in \Sigma$ into itself. One approach to this is to apply a well-known result from graph theory.

Recall that a *graph* G is an ordered pair (V, E) *where* V is a set and $E \subseteq V \times V$ is a binary relation on V. Elements of E are called *edges*. We assume the graph is *reflexive* and *undirected*, in the sense that $(a, a) \in E$ and for each $(a, b) \in V$,

$$(a, b) \in E \Rightarrow (a, b) = (b, a) \in E.$$

Also, given a graph $G = (V, E)$, a *path* in G is a sequence $a_0, a_1, \cdots, a_n, \cdots$ with $(a_i, a_{i+1}) \in E$ for each $i = 1, 2, \cdots$. A path $(a_0, \cdots, a_n)$ in G is said to have *length* n if $a_i \neq a_{i+1}$ for $i = 0, \cdots, n-1$. A *cycle* is a finite path $(a_0, \cdots, a_n)$ with $a_0 = a_n$. G is *connected* if there is a finite path joining any two of its vertices. Finally, a mapping $f : V \to V$ is said to be *edge-preserving* if $(a,b) \in E \Rightarrow (f(a), f(b)) \in E$.

We now turn to an application of the following classical result of graph theory.

Theorem 4.3 [5] *Let $G = (V, E)$ be a reflexive graph which is connected, contains no cycles, and contains no infinite paths. Then every edge-preserving mapping f of V into itself leaves some edge invariant.*

See Espínola and Kirk [1] for a simple 'metric' proof of this result; also see Kirk [3] for further discussion. Notice in particular that the conclusion that f leaves some edge of G invariant means that either $f(a) = a$ for some vertex $a \in V$ or that $(f(a), f(b)) = (a, b)$ for some edge $(a, b) \in E$ with $a \neq b$. Also if f does not leave some vertex fixed then it is easy to see that the invariant edge (a, b) is unique, with $f(a) = b$ and $f(b) = a$.

Now assume Σ is a family of subsets of a given set $\mathfrak{S}$.

Definition 4.4 A chain joining A, B in Σ is a family $\{A_1, A_2, \cdots, A_n\}$ of distinct elements of Σ such that $A_1 = A$, $A_n = B$, and $A_i \cap A_{i+1} \neq \emptyset$ if $i = 1, \cdots, n-1$. Such a chain is said to have length n. An infinite chain in Σ is a family $\{A_1, A_2, \cdots\}$ of distinct elements of Σ for which $A_i \cap A_{i+1} \neq \emptyset$, $i = 1, 2, \cdots$. A cycle is a chain $\{A_1, A_2, \cdots, A_n\}$ of length at least three for which $A_1 = A_n$. A subfamily Σ' of Σ is said to be irreducible if $A \subseteq B \Rightarrow A = B$ whenever $A, B \in \Sigma'$. (In particular, if $A, B \in \Sigma'$ with $A \neq B$, and if $A \cap B \neq \emptyset$, then $A \cap B \notin \Sigma'$.) The subfamily Σ' is said to be connected if each two elements $A, B \in \Sigma'$ are joined by a chain in Σ'.

The following is now a consequence of Theorem 4.3.

Theorem 4.5 *Let $\mathfrak{S}$ be a set and let (Σ, f) be a fixed point structure in $\mathfrak{S}$. Let Σ' be a connected irreducible subfamily of Σ and suppose Σ' contains no cycles or infinite chains. Suppose also that if $A \in \Sigma'$ then there exists $B \in \Sigma'$ such that $f(A) \subseteq B$. Then f has a fixed point.*

Proof. Define a graph $G = (V, E)$ as follows. Take V to be the collection of elements of Σ' and E to be the pairs $(A, B) \in \Sigma' \times \Sigma'$ such that

$$(A, B) \in E \Leftrightarrow A \cap B \neq \emptyset.$$

Then G is reflexive. By assumption, for each $A \in \Sigma'$ there exists at least one $B \in \Sigma'$ such that $f(A) \subseteq B$. Introduce a new mapping $\tilde{f} : V \to V$ by setting $\tilde{f}(A) = B$ where B is any (fixed) member of Σ' containing $f(A)$. Now suppose $(A_1, A_2) \in E$. Then, by assumption, $A_1 \cap A_2 \neq \emptyset$, and there exist B_1 and B_2 in Σ' such that $f(A_i) \subseteq B_i$, $i = 1, 2$. Since $f(A_1 \cap A_2) \subseteq B_1 \cap B_2$ it follows that $B_1 \cap B_2 \neq \emptyset$ and hence $(B_1, B_2) = \left(\tilde{f}(A_1), \tilde{f}(A_2)\right) \in E$. Therefore $\tilde{f}$ is edge preserving. Also by assumption G contains no cycles or infinite paths. Thus, by Theorem 4.3, $\tilde{f}$ leaves some edge (A, B) of G invariant. This means that $(A, B) = \left(\tilde{f}(A), \tilde{f}(B)\right)$. There are two cases. Either $\tilde{f}(A) = A$ or $\tilde{f}(B) = B$, i.e., $f(A) \subseteq A$ or $f(B) \subseteq B$, in which case clearly f has a fixed point. Otherwise it must be the case that $f(A) \subseteq B$ and $f(B) \subseteq A$ and thus $f(A \cap B) \subseteq A \cap B$. But since $A \cap B \in \Sigma$ it follows in this case that $f(x) = x$ for some $x \in A \cap B$.

The assumption that Σ' contains neither cycles nor infinite paths is essential for the conclusion in Theorem 4.5. For example, suppose there exist distinct sets $A_1, A_2, A_3 \in$

Σ' such that $A_1 \cap A_2 \neq \emptyset$, $A_2 \cap A_3 \neq \emptyset$ and $A_3 \cap A_1 \neq \emptyset$. Then, taking $A_4 = A_1$, one obtains a fixed point free map by simply taking $f(a)$ to be a point in $A_{i+1} \backslash A_i$ for each $a \in A_i$, $i = 1, 2, 3$. For an infinite chain containing no cycles simply take A_i, $i = 1, 2, \cdots$, to all be distinct with $A_i \cap A_{i+1} \neq \emptyset$ and define f the same way.

Some intriguing questions seem to arise from the above discussion. The fact that Σ contains no infinite chains does not rule out the possibility that Σ itself is infinite. However, suppose the fixed point structure Σ is finite in Theorem 4.5 and suppose the restriction that Σ' contains no cycles is removed. The question now becomes: What is the probability that f has a fixed point? (This, of course, is equivalent to asking what is the probability that f has a fixed edge in Theorem 4.5 under these weakened assumptions.) In particular, if Σ' contains n elements then the answer exceeds the probability that a mapping taken at random of an n element set into itself leaves some point invariant, and of course this probability, as it was explained earlier in this paper, is

$$1 - \left(\frac{n-1}{n} \right)^n .$$

As the proof of Theorem 4.5 suggests, this problem can be recast as a problem in graph theory; specifically, given a reflexive connected graph that has no infinite paths, what is the *probability* that an edge preserving map leave some edge invariant? This raises a host of related questions, specifically regarding the probability that a mapping of a finite graph into itself leaves some edge invariant, when it does have cycles. The complexity of the problems of this type becomes apparent at the earliest stages.

Remark 4.6 Consider a reflexive connected graph consisting of three distinct vertices $\{a_0, a_1, a_2\}$. What is the probability that a random mapping of these vertices into themselves leaves some edge invariant?

Notice that this is just a rephrasing of the problem discussed at the outset. There are 27 possible mappings. Of these 1 leaves each vertex fixed, 6 leave exactly two vertices fixed, 12 leave exactly one vertex fixed, and 8 mappings leave no vertex fixed. One sees immediately that the answer to the question is at least 19/27. We now must count the ways the mapping can leave an edge invariant but leave no vertex fixed. This can happen in 6 ways, e.g.,

$$\begin{aligned} a_0 &\longleftrightarrow a_1;\ a_2 \to a_0 \\ a_0 &\longleftrightarrow a_1;\ a_2 \to a_1. \end{aligned}$$

There are only two mappings that fail to leave a vertex fixed or an edge invariant. They are:

$$\begin{aligned} a_0 &\to a_1,\ a_1 \to a_2,\ a_2 \to a_0 \\ a_0 &\to a_2,\ a_2 \to a_1,\ a_1 \to a_0. \end{aligned}$$

Therefore the answer is 25/27.

Remark 4.7 This situation gets more complicated quickly. Consider a reflexive connected graph consisting of four distinct vertices $\{a_0, a_1, a_2, a_3\}$. What is the probability that a random mapping of these four vertices into themselves leaves some edge invariant?

There are exactly 12 different connected graphs having these vertices. Of these, two consist of one cycle each having respective edges:

$$\{(a_0, a_1), (a_1, a_2), (a_{2,}a_3), (a_3, a_0)\},$$
$$\{(a_0, a_2), (a_2, a_3), (a_3, a_1), (a_1, a_0)\}.$$

Four have exactly one cycle each consisting of the respective three edges:

$$\{(a_1, a_2), (a_2, a_3), (a_3, a_1)\};$$
$$\{(a_0, a_1), (a_1, a_2), (a_2, a_0)\};$$
$$\{(a_0, a_2), (a_2, a_3), (a_3, a_0)\};$$
$$\{(a_0, a_1), (a_{1,}a_3), (a_3, a_0)\}.$$

The remaining six have no cycles. These have respective edges

$$\{(a_0, a_1), (a_1, a_2), (a_2, a_3)\};$$
$$\{(a_0, a_1), (a_{0,}a_3), (a_2, a_3)\};$$
$$\{(a_0, a_1), (a_{0,}a_2), (a_2, a_3)\};$$
$$\{(a_0, a_3), (a_{0,}a_1), (a_1, a_2)\};$$
$$\{(a_0, a_2), (a_{0,}a_3), (a_1, a_2), (a_1, a_3)\};$$
$$\{(a_0, a_1), (a_{0,}a_2), (a_1, a_3), (a_2, a_3)\}.$$

This immediately implies that the probability is at least $6/12 = 1/2$ that a mapping taken at random of a connected graph into itself consisting of four distinct vertices will leave some edge invariant. However this fails to take account of the probability that such a mapping will have a fixed edge even if the graph has cycles.

Remark 4.8 The preceding examples suggest that there should be a general formula, especially one that applies to larger graphs, which provides an estimate that a random self-mapping the vertices of a connected graph leaves some edge invariant. This observation might deserve further study.

Acknowledgment. This work was initiated while the first author was visiting the Department of Mathematics at the University of Iowa in fall 2019. He wishes to thank this institution for the kind invitation and for providing nice working conditions during that stay. He also wishes to thank prof. Kirk for his endless generosity as well as for so many years of friendship and love. This work is dedicated to his memory.

The research of the first author was partially supported by the Ministerio de Ciencia, Innovación y Univesidades del Gobierno de España through the grant PGC2018-098474-B-C21, Plan Estatal 2017-2020 Generación Conocimiento - Proyectos I+D+i and Junta de Andalucía Proyectos I+D+i FEDER Andalucía 2014-2020 US-1380969 and P20-00637.

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Received: October 2, 2023; Accepted: January 10, 2024.