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BEHIND THE CONCEPT OF F-CONTRACTION

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Abstract. F-contraction has been a widely investigated problem in the Fixed Point Theory during the last decade. There are different results regarding generalizations and modifications of F-contraction in various settings, along with the results concerning application of those concepts, mostly in the area of differential and difference equations, fractional calculus, etc. In this paper, it will be shown that there are abundant requests regarding the definition of F-contraction. In this way, the wider class of F contraction is formed and, for this new type of contraction, called simple F-contraction or sF-contraction, we prove the existence and uniqueness of the fixed point on a complete metric space. Similar results are derived for the modified concept of F-Suzuki contraction.
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1. Introduction

Wardowski in [21] presented a new type of contractive mapping named F-contraction and gave the proof of existence and uniqueness of a fixed point for the class of F-contractions on a complete metric space. Additionally, it was also emphasized that Banach contraction is a special type of a F-contraction for a specifically chosen function $F(x) = \ln x$, $x \in (0, \infty)$. In that way the results of [21] extend the famous Banach fixed point theorem. Even though this result is of a newer date, originated in 2012., the scientific community has shown a specific interest in this topic. Several authors modified and generalized the concept of F-contraction by changing the prerequisites for the mapping F, introducing different contractive conditions depending on several different distances, observing the new setting (b-matric space, cone metric space, partial metric space, fuzzy metric space, etc.), discussing the case of multivalued mappings or combining all of the above. Compellingly, most of the published articles regarding this topic were substantiated with applications of theoretical results as in the area od differential, difference equations, fractals theory, fractal calculus, functional equations, homotopy theory, etc. We address the reader to just a few chosen references on this topic ([1, 2, 3, 4, 7, 8, 6, 5, 9, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23]), among many others.

Definition 1.1. [21] Let $F : (0, \infty) \to \mathbb{R}$ be a function fulfilling the following conditions:

(F_1) F is strictly increasing, i.e., $0 < x < y \implies F(x) < F(y)$;
(F_2) For each sequence $(x_n) \subseteq (0, \infty)$,

$$\lim_{n\to\infty} x_n = 0 \Leftrightarrow \lim_{n\to\infty} F(x_n) = -\infty;$$

(F_3) There exists $k \in (0,1)$, such that $\lim_{x\to 0^+} x^k F(x) = 0$.

We denote by $\mathcal{F}$ the set of all functions $F : (0, \infty) \to \mathbb{R}$ satisfying $(F_1) - (F_3)$. Taking into the account a class $\mathcal{F}$, F-contraction is defined as follows:

Definition 1.2. Let (X, d) be a metric space and $T : X \to X$ a mapping. If there exist $F \in \mathcal{F}$ and $\tau > 0$ such that, for all $x, y \in X$,

$$d(Tx, Ty) > 0 \implies \tau + F(d(Tx, Ty)) \leq F(d(x, y)),$$

then a mapping T is called a F-contraction.

Theorem 1.3. *Let (X, d) be a complete metric space and $T : X \mapsto X$ a F-contraction. Then T has a unique fixed point $x^* \in X$ and, for every $x \in X$, a sequence $(T^n x)$ is convergent to x^*.*

As mentioned, this result was modified in several different manners. We will discuss only on the characterization of the family $\mathcal{F}$ and we will prove that it is possible to omit some conditions in the definition of class $\mathcal{F}$ without losing any of the already obtained results regarding existence and uniqueness of a fixed point.
We will also give a modification of F-Suzuki contraction and analysis of results concerning different types of F-contraction.
In [16], the authors gave existence and uniqueness theorem for the F-Suzuki type contraction but for a different class of functions $\mathcal{F}^*$.

Definition 1.4. Let $F : (0, \infty) \to \mathbb{R}$ be a function fulfilling the following conditions:

(F_1) F is strictly increasing, i.e., $0 < x < y \implies F(x) < F(y)$;
(F_2^*) $\inf_{x\in(0,\infty)} F(x) = -\infty$;
(F_3^*) F is a continuous function.

We denote by $\mathcal{F}^*$ the set of all functions $F : (0, \infty) \to \mathbb{R}$ satisfying $(F_1), (F_2^*)$ and (F_3^*). This notation differs form [16] since, there, the authors held on the same notation $\mathcal{F}$.

Definition 1.5. Let (X, d) be a metric space and $T : X \to X$ a mapping. If there exist $F \in \mathcal{F}^*$ and $\tau > 0$ such that, for all $x, y \in X$,

$$d(Tx, Ty) > 0 \implies \tau + F(d(Tx, Ty)) \leq F(d(x, y)),$$

then a mapping T is called a F^*-contraction.

It is important to compare those two definitions of F-contractions, their differences and similarities. In [19], the author presents an equivalent of (F_2) and offers a new type of prerequisite, already noted as (F_2^*), based on the following lemma:

Lemma 1.6. [19] *If $F:(0,\infty)\mapsto\mathbb{R}$ is an increasing mapping and (t_n) a sequence of positive numbers, then*

(a) If $\lim_{n\to\infty}F(t_n)=-\infty$, then $\lim_{n\to\infty}t_n=0$;
(b) If $\inf F=-\infty$ and $\lim_{n\to\infty}t_n=0$, then $\lim_{n\to\infty}F(t_n)=-\infty$.

Therefore, the condition (F_2^*) is only a more simply formulated equivalent of (F_2) (note that $\inf F=\inf_{x\in(0,\infty)}F(x)$). Obviously, if (F_2) holds, we have (F_2^*) and the opposite follows from the Lemma 1.6 part (b). Hence, replacing (F_2) by the equivalent condition (F_2^*) will not in any case change the definition of the class $\mathcal{F}$ or F-contraction. On the other side, the class $\mathcal{F}^*$ is different than $\mathcal{F}$ since (F_3) and (F_3^*) are not related. Same as in the case of F-contraction defined by Wardowski, this modification of F-contraction by Piri and Kumam also has an unique fixed point on a complete metric space.

Theorem 1.7. [16] *Let (X,d) be a complete metric space and $T:X\mapsto X$ a F^*-contraction. Then T has a unique fixed point $x^*\in X$ and, for every $x\in X$, a sequence (T^nx) is convergent to x^*.*

In the previously mentioned paper, the authors introduced the notion on F-Suzuki type contraction and presented appropriate fixed point result.

Definition 1.8. Let (X,d) be a metric space. A mapping $T:X\mapsto X$ is said to be an F^*-Suzuki contraction if there exist $\tau>0$ and $F\in\mathcal{F}^*$ such that for all $x,y\in X$ with $Tx\neq Ty$

$$\frac{1}{2}d(x,Tx)<d(x,y)\implies\tau+F(d(Tx,Ty))\le F(d(x,y)),\ x,y\in X. \tag{1.1}$$

Theorem 1.9. [16] *Let (X,d) be a complete metric space and $T:X\mapsto X$ a F^*-Suzuki contraction. Then T has a unique fixed point $x^*\in X$ and, for every $x\in X$, a sequence (T^nx) is convergent to x^*.*

Up to authors knowledge, there is no corresponding fixed point result for the F-Suzuki contraction taking $F\in\mathcal{F}$. We will obtain this result as a corollary in the next section.

Since the discussion concerning almost F-contraction, generalized F-contraction, Fisher or Ćirić type F-contraction and similar modifications of F-contractions would go in a similar way as for F-contraction and F-Suzuki contraction, we will not discuss on those topics.

2. Main results

Definition 2.1. A set $\mathcal{F}^\natural$ is the set of all strictly increasing functions $F:(0,\infty)\to\mathbb{R}$.

Therefore, $\mathcal{F}^\natural$ the set of all functions $F:(0,\infty)\to\mathbb{R}$ satisfying (F_1) and $\mathcal{F},\mathcal{F}^*\subseteq\mathcal{F}^\natural$. In accordance with newly defined class of functions, we define a simple F-contraction or sF-contraction.

Definition 2.2. Let (X,d) be a metric space and $T: X \to X$ a mapping. If there exist $F \in \mathcal{F}^{\natural}$ and $\tau > 0$ such that, for all $x, y \in X$,

$$d(Tx,Ty) > 0 \implies \tau + F(d(Tx,Ty)) \leq F(d(x,y)), \tag{2.1}$$

then a mapping T is called a simple F-contraction (sF-contraction).

The main result is dedicated to the existence and uniqueness of a fixed point for a simple F-contraction.

Theorem 2.3. *Let (X,d) be a complete metric space and $T: X \mapsto X$ a simple F-contraction. Then T has a unique fixed point $x^* \in X$ and, for every $x \in X$, a sequence $(T^n x)$ converges to x^*.*

Proof. Let $F \in \mathcal{F}^{\natural}$ and $\tau > 0$ such that (2.1) is fulfilled. We will separate the further discussion depending on the value of $\inf F$.
(i) Assume that (F_2) holds, thus $\inf F = -\infty$. As F is strictly increasing, there exists the inverse function $F^{-1} : (-\infty, \sup F) \setminus A \mapsto (0,\infty)$ which is also increasing. The lower boundary of the domain of F^{-1} (or range of F) is determined based on the infimum assumption and the common fact that the increasing function defined on $(0,\infty)$ (or any set unbounded from the above as a domain) does not have a maximum, meaning that its supremum must be out of its range and A is (countable) set of discontinuities of function F. We will use a transformed condition (2.1)

$$F\left(d(Tx,Ty)\right) \leq F\left(d(x,y)\right) - \tau,\ x, y \in X. \tag{2.2}$$

Choose $x_0 \in X$ arbitrary and form a sequence $(x_n) \subseteq X$ such that for any natural n $x_n = Tx_{n-1} = T^n x_0$. If $x_n = x_{n+1}$, then x_n is a fixed point of T and existence part is done. Otherwise, by using the principle of mathematical induction, we have

$$F\left(d(x_n, x_{n+1})\right) \leq F\left(d(x_0,x_1)\right) - n\tau. \tag{2.3}$$

Indeed, (2.3) holds for $n = 1$ thanks to (2.2).
Assume that (2.3) holds for some $n > 1$, and consider $F\left(d(x_{n+1}, x_{n+2})\right)$.

$$\begin{aligned} F\left(d(x_{n+1}, x_{n+2})\right) &= F\left(d(Tx_n, Tx_{n+1})\right) \\ &\leq F\left(d(x_n, x_{n+1})\right) - \tau \\ &\leq \left(F\left(d(x_0,x_1)\right) - n\tau\right) - \tau \\ &= F\left(d(x_0,x_1)\right) - (n+1)\tau. \end{aligned}$$

Therefore, by the principle of mathematical induction, the inequality (2.3) holds for any natural number n.
As $\lim_{n\to\infty} F\left(d(x_0,x_1)\right) - n\tau = -\infty$ holds knowing that F is strictly increasing function and that $\inf F = -\infty$, it follows $\lim_{n\to\infty} d(x_n, x_{n+1}) = 0$. Assume that it is not a Cauchy sequence and that there exists $\varepsilon > 0$ and strictly increasing sequences $(n_k), (m_k) \subseteq \mathbb{N}$ such that $n_k < m_k$, $k \in \mathbb{N}$,

$$d(x_{n_k}, x_{m_k}) \geq \varepsilon \,\text{and}\, d(x_{n_k}, x_{m_k - 1}) < \varepsilon,$$

where n_k is minimal such that those subsequences exist. Meaning,

$$n_k = \min\{n \geq k \mid d(x_n, x_m) > \varepsilon \wedge m > n\},$$

and

$$m_k = \min\{m > n_k \mid d(x_{n_k}, x_m) > \varepsilon\}.$$

Since A, the set of discontinuities of a mapping F, is countable, we may claim that $\varepsilon \in (0,\infty) \setminus A$. In order to justify this assertation, we may observe arbitrary close $\varepsilon' < \varepsilon$ such that $\varepsilon' \in (0,\infty) \setminus A$ due to the fact that discontinuities of F are jump discontinuities and $A = iso(A)$ is a set of isolated points with respect to the usual (metric) topology on the real line and its relative topology of positives. Easily we can obtain

$$\lim_{k\to\infty} d(x_{n_k}, x_{m_k}) = \lim_{k\to\infty} d(x_{n_k-1}, x_{m_k-1}) = \lim_{k\to\infty} d(x_{n_k+1}, x_{m_k+1}) = \varepsilon.$$

First of all,

$$\begin{aligned} \varepsilon &\leq d(x_{n_k}, x_{m_k}) \\ &\leq d(x_{n_k}, x_{n_k-1}) + d(x_{n_k-1}, x_{m_k-1}) + d(x_{m_k-1}, x_{m_k}), \end{aligned}$$

and when we let $k \to \infty$, we get $\lim_{k\to\infty} d(x_{n_k}, x_{m_k}) = \varepsilon$. Further on,

$$\begin{aligned} d(x_{n_k-1}, x_{m_k-1}) &\geq d(x_{n_k}, x_{m_k}) - d(x_{n_k-1}, x_{n_k}) - d(x_{m_k}, x_{m_k-1}) \\ d(x_{n_k-1}, x_{m_k-1}) &\leq d(x_{n_k-1}, x_{n_k}) + d(x_{n_k}, x_{m_k}) + d(x_{m_k}, x_{m_k-1}), \end{aligned}$$

so $\lim_{k\to\infty} d(x_{n_k-1}, x_{m_k-1}) = \varepsilon$. As $k \to \infty$, we get

$$\begin{aligned} F(\varepsilon) &\leq F\left(d(x_{n_k}, x_{m_k})\right) \\ &\leq F\left(d(x_{n_{k-1}}, x_{m_{k-1}})\right) - \tau \\ &\leq F(\varepsilon) - \tau, \end{aligned}$$

which is not possible.
Accordingly, for $\inf F = -\infty$, $(x_n) \subseteq X$ is a Cauchy sequence in a complete metric space and for this reason convergent. Denote with $x^* \in X$ the limit of the sequence (x_n). In addition,

$$\begin{aligned} F\left(d(x_{n+1}, Tx^*)\right) &= F\left(d(Tx_n, Tx^*)\right) \\ &\leq F\left(d(x_n, x^*)\right) - \tau. \end{aligned}$$

Recall that $\lim_{n\to\infty} d(x_n, x^*) = 0$, then $\lim_{n\to\infty} F(d(x_n, x^*)) - \tau = -\infty$ and $\lim_{n\to\infty} d(x_{n+1}, Tx^*) = 0$. As x^* is already the limit of the sequence (x_n), we get $Tx^* = x^*$.

(ii) If $\inf F \in \mathbb{R}$, then let $b = \inf\{F(d(x,y)) \mid x, y \in X, x \neq y\}$ and note that this set is nonempty whenever $|X| > 1$, so $b \in \mathbb{R}$. If $|X| = 1$, then T has a unique fixed point since it is a self-mapping (indeed identic). Observe sequences $(x_n), (y_n) \subseteq X$ such that $F(d(x_n, y_n)) < b + \frac{1}{n}$, $n \in \mathbb{N}$. We will differ two cases, first of them when there exists subsequence such that $Tx_{n_k} \neq Ty_{n_k}$, $k \in \mathbb{N}$. In that case,

$$\begin{aligned} b \leq F(d(Tx_{n_k}, Ty_{n_k}) &\leq F(d(x_{n_k}, y_{n_k})) - \tau \\ &< b + \frac{1}{n_k} - \tau. \end{aligned}$$

Hence, as $k \to \infty$, we get $b \leq b - \tau$, so this case is impossible.
If $\tau \geq 1$, then $Tx_n = Ty_n$, $n \in \mathbb{N}$, since otherwise

$$\begin{aligned} b \leq F(d(Tx_n, Ty_n) &\leq F(d(x_n, y_n)) - \tau \\ &< b + \frac{1}{n} - \tau \\ &< b, \end{aligned}$$

which yields to the contradiction. If this is not the case, then there exists some $n_0 \in \mathbb{N}$ such that

$$\frac{1}{n_0} < \tau < \frac{1}{n_0 - 1},$$

and

$$d(x_n, y_n) < b + \frac{1}{n} \implies T^n x_n = T^n y_n,\ n \geq n_0.$$

For arbitrary $x \in X$ such that there exists some $y \in X \setminus \{x\}$ such that $d(x, y) < b + \frac{1}{n_0}$, define an open ball with the center x and radius t where $F(t) < b + \frac{1}{n_0}$:

$$U = K(x, t) = \{y \in X \mid d(x, y) < t\}.$$

The mapping F is locally constant, i.e., is constant on U for each specifically chosen x.
Let $z \in X$ such that $Ty = z$ for any $y \in U$. If there exists some $n \in \mathbb{N}$ such that $T^n z = T^{n+1} z$, then $T^n z$ is a fixed point of T. Alternatively, we can apply (2.1) for any two successors in the iterative sequence $(T^n z)$ and similarly as in (2.3), we have

$$b \leq F(d(T^n z, T^{n+1} z)) \leq F d(z, Tz) - n\tau,\ n \in \mathbb{N}.$$

By letting $n \to \infty$, it follows that this is impossible since $-n\tau \to -\infty$ as $n \to \infty$.
From (i) and (ii), follows the existence of a fixed point for the mapping T.
Assume that there is some $y \in X$ a fixed point of the mapping T different than x^*. In that case, because $d(x^*, y) > 0$,

$$F(d(x^*, y)) = F\left(d(Tx^*, Ty)\right) \leq F\left(d(x^*, y)\right) - \tau < F\left(d(x^*, y)\right),$$

which is impossible, so there is no fixed point of T except x^*. □

Remark 2.4. Observe that the previous result is obtainable if we assume that F is increasing, meaning $0 < x \leq y \implies F(x) \leq F(y)$, instead of being strictly increasing as in (F_1).

In [16], the authors discussed on the relation between F-contraction and F^* contraction, and gave adequate examples. Par example, $F(t) = \frac{1}{t^n}$, $t \in (0, \infty)$, is continuous but it does not fulfill (F_3) for $n \geq 1$. On the other hand, $F(t) = -\left(\frac{1}{t+[t]}\right)^m$, $t \in (0, \infty)$, assuming that $m \in (0, \frac{1}{a})$ for some $a > 1$, is not continuous, but (F_3) holds for any $k \in (\frac{1}{a}, 1)$. And of course, there are functions like $F(t) = \ln(t)$, $t \in (0, \infty)$, being both in $\mathcal{F}$ and $\mathcal{F}^*$.
When we talk about the relation between simple F-contraction on one side and F-contraction, respectively F^*-contraction, on the other, evident conclusion is presented

in next two lemmas. proofs are omitted since they are direct consequence of definitions of F-contraction, F^*-contraction and simple F-contraction.

Lemma 2.5. *Any F-contraction is a sF-contraction.*

Lemma 2.6. *Any F^*-contraction is a sF-contraction.*

Example 2.7. By introducing the simple F-contraction, we have constructed a superset of $\mathcal{F}$ and $\mathcal{F}^*$. There are also functions in $\mathcal{F}^\natural \setminus (\mathcal{F} \cup \mathcal{F}^*)$, like

$$F(t) = \begin{cases} -\frac{1}{t}, & \text{if } t \in (0,1), \\ -\frac{1}{2t}, & \text{if } t \in [1,\infty). \end{cases}.$$

For any $k \in (0,1)$,

$$t^k F(t) = \begin{cases} -\frac{1}{t^{1-k}}, & \text{if } t \in (0,1), \\ -\frac{1}{2t^{1-k}}, & \text{if } t \in [1,\infty). \end{cases},$$

and $\lim_{t\to 0+} t^k F(t) = -\infty$. It is also discontinuous, so $F \notin \mathcal{F} \cup \mathcal{F}^*$, but it is strictly increasing, hence $F \in \mathcal{F}^\natural$. Also, $\inf F = -\infty$.

Example 2.8. We will give another example, when $\inf F \neq -\infty$. Let

$$F(t) = \begin{cases} -3 + t, & \text{if } t \in (0,1), \\ -\frac{1}{t}, & \text{if } t \in [1,\infty). \end{cases}.$$

It has a discontinuity at $t = 1$ and

$$t^k F(t) = \begin{cases} -3t^k + t^{k+1}, & \text{if } t \in (0,1), \\ -\frac{1}{t^{1-k}}, & \text{if } t \in [1,\infty). \end{cases},$$

Observe that if we choose for F to be bounded below, then (F_3) holds for any $k \in (0,1)$. But this function still is not in $\mathcal{F}$ since it does not fulfill (F_2). However, F is strictly increasing function, so an element of $\mathcal{F}^\natural$.

Remark 2.9. Mapping $T : X \mapsto X$ on a metric space (X,d) is called contractive mapping if

$$d(Tx,Ty) < d(x,y).$$

If T is a sF-contraction, then

$$F(d(Tx,Ty)) \le F(d(x,y)) - \tau < F(d(x,y))$$

along with the fact that F is strictly increasing leads to the conclusion that T is a contractive mapping.
Same remark holds for F-contraction and F^*-contraction having in mind previous corollaries.

The same principle can be applied in the case of F-Suzuki contraction. We will define F-Suzuki contraction and simple F-Suzuki contraction.

Definition 2.10. Let (X,d) be a metric space. A mapping $T : X \mapsto X$ is said to be a simple F-Suzuki contraction (sF-Suzuki contraction) if there exists $\tau > 0$ and $F \in \mathcal{F}^\natural$ such that for all $x, y \in X$ with $Tx \neq Ty$

$$\frac{1}{2}d(x,Tx) < d(x,y) \implies \tau + F(d(Tx,Ty)) \leq F(d(x,y)),\ x,y \in X. \tag{2.4}$$

Obviously, any F-Suzuki contraction, and also F^*-Suzuki contraction is a simple F-Suzuki contraction.

Lemma 2.11. *F-Suzuki contraction is a simple F-Suzuki contraction.*

Lemma 2.12. *F^*-Suzuki contraction is a simple F-Suzuki contraction.*

The question that naturally raises is can we obtain analogous existence and uniqueness result for F-Suzuki contraction and simple F-Suzuki contraction as was already done for F^*-Suzuki contraction.

Theorem 2.13. *Let (X,d) be a complete metric space and $T : X \mapsto X$ a simple F-Suzuki contraction. Then T has a unique fixed point $x^* \in X$ and, for every $x \in X$, a sequence $(T^n x)$ is convergent to x^*.*

Proof. Choose arbitrary $x_0 \in X$ and observe the iterative sequence $x_n = Tx_{n-1}$, $n \in \mathbb{N}$. If $x_n = Tx_n$, then the existence part is done, otherwise, for any $n \in \mathbb{N}$,

$$\frac{1}{2}d(x_n,Tx_n) < d(x_n,x_{n+1}),$$

so

$$F\left(d(x_{n+1},x_{n+2})\right) \leq F\left(d(x_n,x_{n+1})\right) - \tau, \tag{2.5}$$

As in the proof of Theorem 2.3, by the principle of mathematical induction, we can prove that

$$F\left(d(x_n,x_{n+1})\right) \leq F\left(d(x_0,x_1)\right) - n\tau.$$

Further discussion will depend on the existence of a lower boundary of the range of F.

(i) If we suppose that (F_2) holds, then

$$\lim_{n\to\infty} d(x_n,x_{n+1}) = 0.$$

In order to prove that $(x_n) \subseteq X$ is a Cauchy sequence, thus convergent, assume contrary that there exists $\varepsilon > 0$ and strictly increasing sequences $(n_k), (m_k) \subseteq \mathbb{N}$ such that $n_k < m_k$, $k \in \mathbb{N}$,

$$d(x_{n_k},x_{m_k}) \geq \varepsilon \,\text{and}\, d(x_{n_k},x_{m_k-1}) < \varepsilon,$$

defined same as in the proof of Theorem 2.3. Let k_0 such that $d(x_{n_k},x_{n_k+1}) < \varepsilon$ for $k \geq k_0$, then, starting from k_0, we have

$$d(x_{n_k},x_{m_k}) \geq \varepsilon > \frac{1}{2}d(x_{n_k},x_{n_k+1}),$$

and therefore,

$$\begin{aligned} F(\varepsilon) &= \lim_{k\to\infty} F\left(d(x_{n_k+1}, x_{m_k+1})\right) \\ &\leq \lim_{k\to\infty} F\left(d(x_{n_k}, x_{m_k})\right) - \tau \\ &= F(\varepsilon) - \tau. \end{aligned}$$

Recall that $\lim_{k\to\infty} d(x_{n_k}, x_{m_k}) = \varepsilon$ and also

$$\begin{aligned} d(x_{n_k+1}, x_{m_k+1}) &\leq d(x_{n_k}, x_{m_k}) - d(x_{n_k+1}, x_{n_k}) - d(x_{m_k}, x_{m_k+1}) \\ d(x_{n_k+1}, x_{m_k+1}) &\leq d(x_{n_k+1}, x_{n_k}) + d(x_{n_k}, x_{m_k}) + d(x_{m_k}, x_{m_k+1}), \end{aligned}$$

so $\lim_{k\to\infty} d(x_{n_k+1}, x_{m_k+1}) = \varepsilon$. Hence we get the contradiction. Consequently, $(x_n) \subseteq X$ is a Cauchy sequence in a complete metric space, thus convergent. If $\lim_{n\to\infty} x_n = x^* \in X$, then

$$\begin{aligned} F\left(d(x_{n+1}, Tx^*)\right) &= F\left(d(Tx_n, Tx^*)\right) \\ &\leq F\left(d(x_n, x^*)\right). \end{aligned}$$

Therefore, the condition (F_2) implies $\lim_{n\to\infty} d(x_{n+1}, Tx^*) = 0$ and jointly with $\lim_{n\to\infty} x_n = x^*$, we obtain $Tx^* = x^*$.
(ii) Elsewise, assume $\inf F \in \mathbb{R}$. If there exist some $x \in X$ and $n \in \mathbb{N}$ such that $T^n x = T^{n+1}x$, then $T^n x$ is a fixed point of the mapping T. Assume that such do not exist and denote with $b = \inf\{F(d(T^n x, T^{n+1}x)) \mid x \in X\} \in \mathbb{R}$ (the set is, due to the assumption, well-defined and non-empty). Let $n_0 \in \mathbb{N}$ such that $\tau > \frac{1}{n_0}$ and observe the sequence (x_m) such that

$$F(d(T^{n_m}x_m, T^{n_m+1}x_m)) < b + \frac{1}{m} \leq b + \frac{1}{n_0},\ m \geq n_0.$$

Moreover,

$$F(d(T^n x_m, T^{n+1}x_m)) < b + \frac{1}{n_0},\ n \geq n_m,\ m \geq n_0.$$

On the other hand, $\frac{1}{2}d(T^n x_m, T^{n+1}x_m) < d(T^n x_m, T^{n+1}x_m)$ implied by the assumption $T^n x_m \neq T^{n+1}x_m$, $n \geq n_m$, $m \geq n_0$, leads to

$$\begin{aligned} b &\leq F(d(T^{n_m+k}x_m, T^{n_m+k+1}x_m)) \\ &\leq F(d(T^{n_m}x_m, T^{n_m+1}x_m)) - k\tau \\ &< b + \frac{1}{n_0} - k\tau. \end{aligned}$$

As $k \to \infty$, the above inequalities lead to the contradiction.

Hence, in both (i) and (ii), we may conclude that there exists a fixed point of the mapping T.
Assume that there is some $y \in X$ a fixed point of the mapping T different than x^*. In that case, because $0 = \frac{1}{2}d(x^*, Tx^*) < d(x^*, y)$,

$$\tau + F\left(d(Tx^*, Ty)\right) \leq F(d(x^*, y)),$$

yields to $d(x^*, y) < d(x^*, y)$ which is not possible, so x^* is the unique fixed point of T. □

Corollary 2.14. *Let (X, d) be a complete metric space and $T : X \mapsto X$ a F-Suzuki contraction. Then T has a unique fixed point $x^* \in X$ and, for every $x \in X$, a sequence $(T^n x)$ is convergent to x^*.*

Proof. Recall that any F-Suzuki contraction is a simple F-Suzuki contraction. This result follows directly from Theorem 2.13. □

Similarly, we can obtain the main result of [16].

Corollary 2.15. *Let (X, d) be a complete metric space and $T : X \mapsto X$ a F^*-Suzuki contraction. Then T has a unique fixed point $x^* \in X$ and, for every $x \in X$, a sequence $(T^n x)$ is convergent to x^*.*

Proof. Any F^*-Suzuki contraction is a sF-Suzuki contraction. □

Previous results show that, in general, (F_2) and (F_3), analogously (F_2^*) and (F_3^*), can be omitted and we can still obtain existence and uniqueness of a fixed point for both F-contraction and F-Suzuki contraction. The comment regarding F being increasing instead of strictly increasing must be also taken into the consideration. In this way, the class of $\mathcal{F}$ is much wider without superfluous constraints and F-contraction still has a unique fixed point on a complete metric space and the sequence of successive approximations converges to the fixed point of F-contraction for arbitrarily chosen starting point. As mentioned at the beginning, there are numerous results concerning different concepts of F-contraction. Even though it was not possible to gather them all in this article, the same idea and adapted proof techniques are applicable in the most of them.

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