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FIXED POINT RESULTS FOR MULTI-VALUED FENG-LIU CONTRACTIONS IN $b-$METRIC SPACES

CRISTIAN CHIFU* AND GABRIELA PETRUŞEL**

*Babeş-Bolyai University Cluj-Napoca, Department of Business,
Horea Street, No.7, 400134, Cluj-Napoca, Romania
E-mail: cristian.chifu@ubbcluj.ro

**Babeş-Bolyai University Cluj-Napoca, Department of Business,
Horea Street, No.7, 400134, Cluj-Napoca, Romania
E-mail: gabriela.petrusel@ubbcluj.ro

Abstract. The purpose of this paper is to give some fixed point results for multi-valued Feng-Liu contractions in $b-$metric spaces, as well in $b-$metric spaces endowed with a graph. We will provide existence and approximation theorems for generalized multi-valued Feng-Liu contraction. Data dependence, well-posedness and Ulam-Hyers stability for the fixed point inclusion are also studied. As an application to the coincidence problem for two multi-valued operators is given.
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1. Introduction and preliminaries

1.1. **Introduction.** The metric fixed point theory for multi-valued mappings practically started in 1969 when S.B. Nadler Jr. published a multi-valued variant of the well-known Banach-Caccioppoli contraction principle. The result was improved, one year later, by H. Covitz and S.B. Nadler Jr. in a paper published in Israel Journal of Mathematics. This result, known in the literature, as the Multi-valued Contraction Principles, states that any self multi-valued contraction on a complete metric space has at least one fixed point. A generalization of this result for the case of complete b-metric spaces was given by S. Czerwik in 1998.

A very interesting approach, in the theory of fixed points in some general structures, was given by J. Jachymski [9] and G. Gwóźdź-lukawska, J. Jachymski [7], by using the context of metric spaces endowed with a graph.

One of the most interesting extension of the Multi-valued Contraction Principle was given by Y. Feng and S. Liu in 2006, see [6]. Several other results of this type were considered in the recent literature, mainly for the case of complete metric spaces, see [13], [12], [16], [15], [14] and the references therein. For some results in complete b-metric spaces see [11] and [17].

In this paper, we will prove some fixed point results for multi-valued Feng-Liu contractions in b-metric spaces, as well in b-metric spaces endowed with a graph. We will provide existence and approximation theorems for generalized multi-valued Feng-Liu contraction, both the linear and the nonlinear form case. Data dependence, well-posedness and Ulam-Hyers stability for the fixed point inclusion are studied, too.

The structure of the paper is the following: In Section 2 we are providing our main result in b-metric spaces. Section 3 is dedicated to the study of some stability properties. Section 4 presents an application to the coincidence problem for two multi-valued operators. In Section 5 we present some results in the context of b-metric spaces endowed with a graph.

1.2. **Preliminaries.** We shall start by presenting some basic notions and fundamental results in the literature, see [2] and [21]. Let $\mathbb{R}$, $\mathbb{N}$ denote the set of real numbers and positive integers, respectively. Further, we set $\mathbb{R}_0^+ = [0, \infty)$ $\mathbb{R}_0^\infty = \mathbb{R}_0^+ \cup \{\infty\}$ and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

Definition 1.1. ([5]) Let X be a nonempty set and let $s \geq 1$ be a given real number. A functional $d : X \times X \to \mathbb{R}_0^+$ is said to be a b-metric if the following conditions are satisfied:

(1) $d(x, y) = 0$ if and only if $x = y$;
(2) $d(x, y) = d(y, x)$;
(3) $d(x, z) \leq s[d(x, y) + d(y, z)]$, for all $x, y, z \in X$.

In this case the pair (X, d, s) is called a b-metric space.

Remark 1.1. The class of b-metric spaces is larger than the class of metric spaces since a b-metric space is a metric space when s=1. For more details and examples of b-metric spaces, see e.g. [3, 1, 5, 4, 8].

Let us consider the following families of subsets of a b-metric space (X, d, s).

$$P(X) = \{Y \subset X | \, |Y \neq \emptyset\}, P_b(X) = \{Y \in P(X) | \ Y \text{ is bounded }\},$$

$$P_{cl}(X) = \{Y \in P(X) | \ Y \text{ is closed}\}, P_{cp}(X) = \{Y \in P(X) | \ Y \text{ is compact}\}.$$

Let us consider the following functionals defined on $P(X) \times P(X)$:

(1) **the gap functional**

$$D(A, B) = inf\{d(a, b) \mid a \in A, \ b \in B\};$$

(2) **the excess generalized functional**

$$\rho(A, B) = sup\{D(a, B) \mid a \in A\};$$

(3) **the Pompeiu-Hausdorff generalized functional:**

$$H(A, B) = max\{\rho(A, B), \rho(B, A)\};$$

In the sequel, the following results are useful for some of the proofs in the paper.

Lemma 1.1. *Let (X, d, s) be a b-metric space with constant $s > 1$ and $B \in P_{cl}(X)$. Assume that there exists $x \in X$ such that $D(x, B) > 0$. Then there exists $y \in B$ such that*

$$d(x, y) \leq qD(x, B),$$

where $q > 1$.

Let (X, d, s) be a b-metric space and let $T : X \to P(X)$ be a multi-valued operator.

Definition 1.2. A point $x \in X$ is called fixed point for T if $x \in Tx$. The set $Fix\,(T) = \{x \in X : x \in Tx\}$ is called the fixed point set of T.

Definition 1.3. A function $f : X \to \mathbb{R}$ is called lower semi-continuous if for any $(x_n) \subset X$ and $x \in X$, the following implication holds

$$x_n \to x, n \to \infty \Longrightarrow f\,(x) \leq \underline{\lim}_{n\to\infty} f\,(x_n)\,.$$

Let (X, d, s) be a b-metric space and Δ be the diagonal of $X \times X$. Let G be a directed graph, such that the set $V(G)$ of its vertices coincides with X and $\Delta \subseteq E(G)$, where $E(G)$ is the set of the edges of the graph. Assume also that G has no parallel edges and, thus, one can identify G with the pair $(V(G), E(G))$. A b-metric space (X, d, s) endowed with a directed graph G having the above properties will be called a graph b-metric space and denoted by (X, d, s, G).

2. Fixed point results in b-metric spaces

Let us recall first the notion of multi-valued $\alpha-$contraction of *Feng-Liu* type in the context of a b-metric space. See also [6].

Let $T : X \to P(X)$ and $\beta \in (0, 1)$. Define

$$I^x_\beta = \{y \in Tx \;|\beta d\,(x, y) \leq D\,(x, Tx)\}\,.$$

Remark 2.1. Based on Lemma 1.1, $I^x_\beta \neq \emptyset$, for every $x \in X$.

Definition 2.1. Let (X, d, s) be a b-metric space and $T : X \to P(X)$ be a multi-valued operator. T is called multi-valued $\alpha-$contraction of *Feng-Liu* type if there exists $\alpha \in (0, \beta)$ such that, for each $x \in X$, there is $y \in I^x_\beta$ for which

$$D\,(y, Ty) \leq \alpha d\,(x, y)\,.$$

Theorem 2.1. *Let (X, d, s) be a complete b-metric space and let $T : X \to P_{cl}(X)$ be a multi-valued $\alpha-$contraction of Feng-Liu type. If T has a closed graph or $f : X \to \mathbb{R}^+_0, f\,(x) = D\,(x, Tx)$ is lower semi-continuous, then $Fix(T) \neq \emptyset$.*

Proof. Let $x_0 \in X$. If $D\,(x_0, Tx_0) = 0$, then $x_0 \in Tx_0$ and the proof is done. Suppose $D\,(x_0, Tx_0) > 0$.

Hence, there exists $x_1 \in I^{x_0}_\beta$, i.e. $x_1 \in Tx_0$ and

$$\beta d\,(x_0, x_1) \leq D\,(x_0, Tx_0)$$

such that

$$D\,(x_1, Tx_1) \leq \alpha d\,(x_0, x_1) \tag{2.1}$$

If $D\,(x_1, Tx_1) = 0$, then $x_1 \in Tx_1$ and the proof is done. Suppose $D\,(x_1, Tx_1) > 0$.

There exists $x_2 \in I_\beta^{x_1}$, i.e. $x_2 \in Tx_1$ and

$$\beta d(x_1, x_2) \leq D(x_1, Tx_1) \tag{2.2}$$

such that

$$D(x_2, Tx_2) \leq \alpha d(x_1, x_2) \tag{2.3}$$

From (2.1) and (2.2) we have:

$$d(x_1, x_2) \leq \frac{1}{\beta} D(x_1, Tx_1) \leq \frac{\alpha}{\beta} d(x_0, x_1). \tag{2.4}$$

There exists $x_3 \in I_\beta^{x_2}$, i.e. $x_3 \in Tx_2$ and

$$\beta d(x_2, x_3) \leq D(x_2, Tx_2) \tag{2.5}$$

such that

$$D(x_3, Tx_3) \leq \alpha d(x_2, x_3) \tag{2.6}$$

From (2.3), (2.4) and (2.5) we obtain

$$d(x_2, x_3) \leq \frac{1}{\beta} D(x_2, Tx_2) \leq \frac{\alpha}{\beta} d(x_1, x_2) \leq \left(\frac{\alpha}{\beta}\right)^2 d(x_0, x_1).$$

By induction, we obtain that there exists $x_{n+1} \in I_\beta^{x_n}$, i.e. $x_{n+1} \in Tx_n$, and $\beta d(x_n, x_{n+1}) \leq D(x_n, Tx_n)$, such that

$$D(x_{n+1}, Tx_{n+1}) \leq \alpha d(x_n, x_{n+1})$$

Hence

$$d(x_n, x_{n+1}) \leq \left(\frac{\alpha}{\beta}\right)^n d(x_0, x_1).$$

According to [10] since $\frac{\alpha}{\beta} < 1$, the sequence $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in the complete space (X, d, s) and hence, there exists $x^* \in X$ such that $x_n \to x^*$ as $n \to \infty$.

If T has a closed graph, then we get immediately that $x^* \in Tx^*$.

Suppose that $f : X \to \mathbb{R}_0^+, f(x) = D(x, Tx)$ is lower semi-continuous, $f(x_n) = D(x_n, Tx_n)$. Since

$$D(x_{n+1}, Tx_{n+1}) \leq \alpha d(x_n, x_{n+1}) \leq \frac{\alpha}{\beta} D(x_n, Tx_n) < D(x_n, Tx_n),$$

$(f(x_n))_{n\in\mathbb{N}}$ is a decreasing sequence and converges to 0. Hence

$$0 \leq f(x^*) \leq \underline{\lim}_{n\to\infty} f(x_n) = 0$$

$$f(x^*) = 0 \Longleftrightarrow D(x^*, Tx^*) = 0.$$

Hence, $x^* \in Tx^*$, and the proof is complete. □

We recall now the strong retraction displacement condition for the case of multi-valued operators.

Definition 2.2. Let (X, d, s) be a b-metric space and let $T : X \to P(X)$ be a multi-valued operator such that $Fix(T) \neq \emptyset$. T satisfies the strong retraction displacement condition if there exists $c > 0$ and a set retraction $r : X \to Fix(T)$ such that

$$d(x, r(x)) \leq cD(x, Tx), \text{for all } x \in X.$$

Theorem 2.2. *In the conditions of Theorem 2.1, if additionally* $1 \leq s < \frac{\beta}{\alpha}$, *then* T *satisfies the following strong retraction displacement condition*

$$d(x, r(x)) \leq \frac{s^2}{\beta - s\alpha} D(x, Tx), \text{for all } x \in X.$$

Proof. Let $x_0 \in X$. From Theorem 2.1, for every $x_0 \in X$ the sequence $x_{n+1} := T(x_n)$ is convergent and its limit, denoted by $x^*(x_0) \in X$, has the property that $x^*(x_0) \in Fix(T)$. Now,

$$\begin{aligned} d(x_n, x_{n+p}) &\leq sd(x_n, x_{n+1}) + sd(x_{n+1}, x_{n+p}) \\ &\leq sd(x_n, x_{n+1}) + s^2 d(x_{n+1}, x_{n+2}) + s^2 d(x_{n+2}, x_{n+p}) \\ &\leq sd(x_n, x_{n+1}) + s^2 d(x_{n+1}, x_{n+2}) + ... + s^p d(x_{n+p-1}, x_{n+p}) \\ &\leq \left(s\left(\frac{\alpha}{\beta}\right)^n + s^2\left(\frac{\alpha}{\beta}\right)^{n+1} + ... + s^p\left(\frac{\alpha}{\beta}\right)^{n+p-1} \right) d(x_0, x_1) \\ &= s\left(\frac{\alpha}{\beta}\right)^n \left(1 + \frac{s\alpha}{\beta} + ... + \left(\frac{s\alpha}{\beta}\right)^{p-1}\right) d(x_0, x_1) \\ &= s\left(\frac{\alpha}{\beta}\right)^n \frac{1 - \left(\frac{s\alpha}{\beta}\right)^p}{1 - \frac{s\alpha}{\beta}} d(x_0, x_1) \end{aligned}$$

Because $s < \frac{\beta}{\alpha}$, we get that $\frac{\alpha s}{\beta} < 1$. Now, since

$$d(x_n, x^*(x_0)) \leq s\left[d(x_n, x_{n+p}) + d(x_{n+p}, x^*(x_0))\right] \leq$$

$$s^2\left(\frac{\alpha}{\beta}\right)^n \cdot \frac{\beta d(x_0, x_1)}{\beta - s\alpha} + sd(x_{n+p}, x^*(x_0)).$$

Taking the limit when $p \to \infty$, we obtain

$$d(x_n, x^*(x_0)) \leq s^2\left(\frac{\alpha}{\beta}\right)^n \cdot \frac{\beta d(x_0, x_1)}{\beta - s\alpha}.$$

If we consider $n = 0$, then we have

$$d(x_0, x^*(x_0)) \leq \frac{s^2\beta}{\beta - s\alpha} d(x_0, x_1) \leq \frac{s^2}{\beta - s\alpha} D(x_0, Tx_0). \tag{2.7}$$

Let us define $r : X \to Fix(T), r(x) = x^*(x)$. Hence, from (2.7) we have

$$d(x, r(x)) \leq \frac{s^2}{\beta - s\alpha} D(x, Tx), \text{for all } x \in X. \tag{2.8}$$

Hence T satisfies the strong retraction displacement condition. □

3. Stability properties: Well-posedness, Ulam-Hyers stability and data dependence for the fixed point inclusion

For the notion and some results related to fixed point theory (for both single-valued and multi-valued operators) see [18], [19], [16], [15], [21] and [20].

Let (X, d, s) be a b-metric space and let $T : X \to P(X)$ be a multi-valued operator. Consider the fixed point inclusion

$$x \in Tx \tag{3.1}$$

Definition 3.1. Suppose that $Fix(T) \neq \emptyset$ and let $r : X \to Fix(T)$ be a set retraction. Then, the fixed point inclusion (3.1) is called well-posed with respect to r if for each $x^* \in Fix(T)$ and for every sequence $(y_n)_{n\in\mathbb{N}} \subset r^{-1}(x^*)$ such that $D(y_n, Ty_n) \to 0$, as $n \to \infty$, we have that $y_n \to x^*$, as $n \to \infty$.

Definition 3.2. The fixed point inclusion (3.1) is called Ulam-Hyers stable if there exists $c > 0$, such that for every $\varepsilon > 0$ and for each $\varepsilon-$fixed point $y^* \in X$ (i.e. $D(y^*, Ty^*) \leq \varepsilon$), there exists $x^* \in Fix(T)$ such that $d(y^*, x^*) \leq c\varepsilon$.

Definition 3.3. Let $F : X \to P(X)$ be a multi-valued operator with the following properties:

(i) $Fix(F) \neq \emptyset$

(ii) there exists $\eta > 0$ such that $H(Tx, Fx) \leq \eta$, for all $x \in X$.

Then, the fixed point inclusion (3.1) has the data dependence property if for each $y^* \in Fix(F)$, there exists $x^* \in Fix(T)$ such that $d(y^*, x^*) \leq c\eta$ for some $c > 0$.

Theorem 3.1. *Let* (X, d, s) *be a complete b-metric space and let* $T : X \to P_{cl}(X)$ *be a multi-valued operator. In the conditions of Theorem 2.1 the fixed point inclusion* (3.1) *is well-posed with respect to the set retraction generated by the relation (2.8) has the Ulam-Hyers property and satisfies the data dependence property.*

Proof. The conditions of Theorem 2.1 assure the fact that $Fix(T) \neq \emptyset$ and T satisfies the strong retraction displacement condition, i.e. $d(x, r(x)) \leq \frac{s^2}{\beta - s\alpha} D(x, Tx)$, for all $x \in X$.

(i) **Well-posedness**

Since $Fix(T) \neq \emptyset$, there exists $x^* \in Fix(T), i.e., x^* \in Tx^*$. Let us consider the sequence $(y_n)_{n\in\mathbb{N}} \subset r^{-1}(x^*)$ such that $D(y_n, Ty_n) \to 0$, as $n \to \infty$. Now, for each $n \in \mathbb{N}$, $r(y_n) = x^*$. By the strong retraction displacement condition we have

$$d(y_n, r(y_n)) \leq \frac{s^2}{\beta - s\alpha} D(y_n, Ty_n),$$

and hence

$$d(y_n, x^*) \leq \frac{s^2}{\beta - s\alpha} D(y_n, Ty_n).$$

Thus, $d(y_n, x^*) \to 0$ as $n \to \infty$. The proof is complete.

(ii) **Ulam-Hyers stability**

Let $\varepsilon > 0$ and $y^* \in X$ such that $D(y^*, Ty^*) \leq \varepsilon$. By the strong retraction displacement condition we have

$$d(y^*, r(y^*)) \leq \frac{s^2}{\beta - s\alpha} D(y^*, Ty^*) \leq \frac{s^2 \varepsilon}{\beta - s\alpha}.$$

Now, there exists $x^* \in Fix(T)$ such that $r(y^*) = x^*$. Hence

$$d(y^*, x^*) \leq \frac{s^2 \varepsilon}{\beta - s\alpha},$$

and the proof is complete.

(iii) **Data dependence**

Let $F : X \to P(X)$ be a multi-valued operator with the following properties:

(i) $Fix(F) \neq \emptyset$;

(ii) there exists $\eta > 0$ such that $H(Tx, Fx) \leq \eta$, for all $x \in X$.

Since $Fix(F) \neq \emptyset$, there exists $y^* \in Fix(F)$. Let us denote $x^* = r(y^*)$. By the strong retraction displacement condition we have

$$d(y^*, x^*) = d(y^*, r(y^*)) \leq \frac{s^2}{\beta - s\alpha} D(y^*, Ty^*) \leq \frac{s^2}{\beta - s\alpha} H(Fy^*, Ty^*) \leq \frac{s^2}{\beta - s\alpha}\eta,$$

and, thus, the proof is complete. □

4. An application to the coincidence problem for two multi-valued operators

Let $(X, d_1, s), (Y, d_2, s)$ be two b-metric spaces and $S, T : X \to P(Y)$ be two multi-valued operators. We consider the coincidence problem as follows: find a solution of the coincidence problem for S and T, i.e., a pair $(x^*, y^*) \in X \times Y$, such that

$$y^* \in S(x^*) \cap T(x^*). \tag{4.1}$$

We denote:

$$CP(S, T) := \{(x, y) \in X \times Y | y \in T(x) \cap S(x)\}.$$

Suppose S is onto, i.e., for every $y \in Y$ there exists $x \in X$ such that $y \in S(x)$. Let $F : X \times Y \to P(X \times Y)$ be defined by the formula $F(x, y) = S^{-1}(y) \times T(x)$, where $S^{-1} : Y \to P(X)$ is given by $S^{-1}(y) := \{x \in X : y \in S(x)\}$. Then, it is easy to see that $CP(S, T) = Fix(F)$. Indeed, if $z = (x, y) \in Fix(F)$, then $(x, y) \in S^{-1}(y) \times T(x)$, or equivalently $y \in S(x)$ and $y \in T(x)$. Thus, $z = (x, y) \in CP(S, T)$.

Let $\beta \in (0, 1)$. We define

$$I_\beta^{(x,y)} = \left\{u \in S^{-1}(y) \mid \beta d_1(x, u) \leq D_{d_1}\left(x, S^{-1}(y)\right)\right\},$$

$$J_\beta^{(x,y)} = \left\{v \in T(x) \mid \beta d_2(y, v) \leq D_{d_2}(y, T(x))\right\}.$$

and

$$K_\beta^{(x,y)} := I_\beta^{(x,y)} \times J_\beta^{(x,y)}.$$

Obvious, $K_\beta^z \neq \emptyset$, for every $z = (x, y) \in X \times Y$.

Then, we have the following existence result for the coincidence problem.

Theorem 4.1. *Let* (X, d_1, s) *and* (Y, d_2, s) *be two complete b-metric spaces. Let* $T, S : X \to P(Y)$ *be two multi-valued operators, such that* S *is onto. We suppose:*

(i) T *has closed graph and, for each* $(x, y) \in X \times Y$ *and each* $u \in X$*, there exist* $k_2 > 0$ *and* $v \in J_\beta^{(x,y)}$ *such that*
$$D_{d_2}(v, T(u)) \leq k_2 d_2(y, v);$$

(ii) S^{-1} *has closed graph and, for each* $(x, y) \in X \times Y$ *and each* $v \in Y$*, there exists* $k_1 > 0$ *and* $u \in I_\beta^{(x,y)}$ *such that*
$$D_{d_1}(u, S^{-1}(v)) \leq k_1 d(x, u);$$

(iii) $k := \max\{k_1, k_2\} < \beta$.

Then there exists at least one solution of coincidence problem (4.1).

Proof. Let $F : X \times Y \to P(X \times Y)$ be defined by $F(x, y) = S^{-1}(y) \times T(x)$. Then, by the above considerations we have that $CP(S, T) = Fix(F)$. Thus, it is enough to prove that $Fix(T) \neq \emptyset$. We consider on $Z := X \times Y$ the scalar metric $\tilde{d}(z, w) := d_1(x, u) + d_2(y, v)$, for $z = (x, y), w = (u, v) \in Z$. Denote by $\tilde{D}$ the gap functional generated by $\tilde{d}$, i.e., $\tilde{D}((x, y), A \times B) = D_{d_1}(x, A) + D_{d_2}(y, B)$.

By our hypotheses, we get that for every $z = (x, y) \in Z := X \times Y$ there exists $w = (u, v) \in K_\beta^z$ such that
$$\tilde{D}(w, F(w)) \leq k\tilde{d}(z, w).$$
Hence, F is a multi-valued k-contraction of Feng-Liu type on Z. By Theorem 2.1 we get that $Fix(F) \neq \emptyset$, which gives the desired conclusion of this theorem. □

5. Fixed point results in b-metric spaces endowed with a graph

Let (X, d, s, G) be a graph b-metric space, $T : X \to P(X)$ be a multi-valued operator and $\beta \in (0, 1)$. Define the following sets:
$$X_T = \{x \in X \mid \text{for all } y \in Tx \text{ we have } (x, y) \in E(G)\}$$
$$J_\beta^x = \{y \in Tx \mid (x, y) \in E(G) \text{ and } \beta d(x, y) \leq D(x, Tx)\}$$

Lemma 5.1. $X_T \neq \emptyset \Rightarrow J_\beta^x \neq \emptyset$*, for each* $x \in X_T$.

Proof. Let $x_0 \in X_T$. Then, for all $y \in Tx_0$ we have that $(x_0, y) \in E(G)$. Using Lemma 1.1 with $q = \frac{1}{\beta}$ we obtain there exists $y_0 \in Tx_0$ such that $\beta d(x_0, y_0) \leq D(x_0, Tx_0)$. Thus, $y_0 \in J_\beta^{x_0}$. □

Definition 5.1. Let (X, d, s, G) be a graph b-metric space. We say that (X, d, s, G) has the property (A) if, for any sequence $(x_n)_{n\in\mathbb{N}} \subset X$, with $x_n \to x$, as $n \to \infty$ and $(x_n, x_{n+1}) \in E(G)$, for $n \in \mathbb{N}$, we have $(x_n, x) \in E(G)$.

Definition 5.2. Let (X, d, s, G) be a graph b-metric space and $T : X \to P(X)$ be a multi-valued operator. T is called multi-valued $\alpha - G-$contraction of *Feng-Liu* type if there exists $\alpha \in (0, \beta)$ such that for each $x \in X$, there is $y \in J_\beta^x$ for which
$$D(y, Ty) \leq \alpha d(x, y).$$

Theorem 5.1. *Let (X,d,s,G) be a complete graph b-metric space such that (X,d,s,G) has the property (A) and let $T : X \to P_{cl}(X)$ be a multi-valued $\alpha-G-$contraction of Feng-Liu type. Suppose that $X_T \neq \emptyset$. If T has a closed graph or the map $f : X \to \mathbb{R}_0^+, f(x) = D(x,Tx)$ is lower semi-continuous, then $Fix(T) \neq \emptyset$.*

Proof. For $x_0 \in X_T$ there exists $x_1 \in J_\beta^{x_0}$ such that $x_1 \in Tx_0$, $(x_0,x_1) \in E(G)$ and

$$\beta d(x_0,x_1) \leq D(x_0,Tx_0) \tag{5.1}$$

and

$$D(x_1,Tx_1) \leq \alpha d(x_0,x_1). \tag{5.2}$$

For $x_1 \in X$ there exists $x_2 \in J_\beta^{x_0}$ (i.e., $x_2 \in Tx_1$ and $(x_1,x_2) \in E(G)$,

$$\beta d(x_1,x_2) \leq D(x_1,Tx_1)) \tag{5.3}$$

such that

$$D(x_2,Tx_2) \leq \alpha d(x_1,x_2). \tag{5.4}$$

By (5.1)-(5.4) we get that

$$d(x_1,x_2) \leq \frac{\alpha}{\beta} d(x_0,x_1) \leq \frac{\alpha}{\beta^2} D(x_0,Tx_0). \tag{5.5}$$

By this procedure we obtain inductively a sequence $(x_n)_{n\in\mathbb{N}}$ with the property that $x_{n+1} \in J_\beta^{x_n}$ (i.e., $x_{n+1} \in Tx_n, (x_n,x_{n+1}) \in E(G)$ and $\beta d(x_n,x_{n+1}) \leq D(x_n,Tx_n)$, and satisfying the relation

$$D(x_{n+1},Tx_{n+1}) \leq \alpha d(x_n,x_{n+1}) \tag{5.6}$$

As a consequence, we get that

$$d(x_n,x_{n+1}) \leq \left(\frac{\alpha}{\beta}\right)^n d(x_0,x_1) \leq \left(\frac{\alpha}{\beta}\right)^n \frac{1}{\beta} D(x_0,Tx_0). \tag{5.7}$$

According to [10], since $\frac{\alpha}{\beta} < 1$, the sequence $(x_n)_{n\in\mathbb{N}}$ is Cauchy in the complete b-metric space (X,d,s) and, hence, there exists $x^* \in X$ such that $x_n \to x^*$ as $n \to \infty$. Using property (A) we obtain $(x_n,x^*) \in E(G)$.

If T has a closed graph, then, since $x_{n+1} \in Tx_n$ we get that $x^* \in Tx^*$.

On the other hand, if $f : X \to \mathbb{R}_0^+, f(x) = D(x,Tx)$ is lower semi-continuous, then since

$$D(x_{n+1},Tx_{n+1}) \leq \alpha d(x_n,x_{n+1}) \leq \frac{\alpha}{\beta} D(x_n,Tx_n) < D(x_n,Tx_n),$$

the sequence $(f(x_n))_{n\in\mathbb{N}}, f(x_n) = D(x_n,Tx_n)$ is decreasing and converges to 0. Hence

$$0 \leq f(x^*) \leq \underline{\lim}_{n\to\infty} f(x_n) = 0.$$

Thus, we get that $f(x^*) = 0$ and so $x^* \in Tx^*$. This completes the proof. □

In the case of a b-metric space endowed with a graph, the strong retraction displacement condition takes the following form.

Definition 5.3. Let (X, d, s, G) be a graph b-metric space and $T : X \to P(X)$ be a multi-valued operator such that $X_T \neq \emptyset$ and $Fix(T) \neq \emptyset$. Then T satisfies the strong G-retraction displacement condition if there exists $c > 0$ and a set retraction $r : X_T \to Fix\,(T)$ such that

$$d\,(x, r\,(x)) \leq cD\,(x, Tx)\,, \text{ for all } x \in X_T.$$

Following a similar approach to that from Theorem 2.2, by the relation (5.7) we get the following result.

Theorem 5.2. *In the conditions of Theorem 5.1, if* $1 \leq s < \frac{\beta}{\alpha}$, *then* T *satisfies the following strong* G*-retraction displacement condition:*

$$d\,(x, r\,(x)) \leq \frac{s^2}{\beta - s\alpha} D\,(x, Tx)\,, \text{ for all } x \in X_T.$$

Let us consider now some stability properties (well-posedness, Ulam-Hyers stability and data dependence) for the case of multi-valued Feng-Liu operators in b-metric spaces endowed with a graph.

Definition 5.4. Let (X, d, s, G) be a graph b-metric space and $T : X \to P(X)$ be a multi-valued operator, such that $X_T \neq \emptyset$ and $Fix(T) \neq \emptyset$. Then, the fixed point inclusion $x \in T(x)$ is called Ulam-Hyers G-stable if there exists $c > 0$ and a set retraction $r : X_T \to Fix(T)$, such that for every $\varepsilon > 0$ and every $z \in X_T$ with the property $D(z, Tz) \leq \varepsilon$ there exists $x^* \in Fix(T)$ with the property $d(x^*, z) \leq c\varepsilon$.

Definition 5.5. Let (X, d, s, G) be a graph b-metric space and $T : X \to P(X)$ be a multi-valued operator such that $X_T \neq \emptyset$ and $Fix(T) \neq \emptyset$. Then, the fixed point inclusion $x \in T(x)$ is called G-well-posed in the sense of Reich and Zaslavski if there exists a set retraction $r : X_T \to Fix(T)$ such that, for each $x^* \in Fix(T)$ and for any sequence $(y_n)_{n\in\mathbb{N}} \subset r^{-1}(x^*) \cap X_T$, with $D\,(y_n, Ty_n) \to 0$, we have that $y_n \to x^*$ as $n \to \infty$.

Definition 5.6. Let (X, d, s, G) be a graph b-metric space and $T : X \to P(X)$ be a multi-valued operator such that $X_T \neq \emptyset$ and $Fix(T) \neq \emptyset$. Then, the fixed point inclusion $x \in T(x)$ is called G-Ostrowski stable if there exists a set retraction $r : X_T \to Fix(T)$ such that, for each $x^* \in Fix(T)$ and for any sequence $(z_n)_{n\in\mathbb{N}} \subset r^{-1}(x^*) \cap X_T$, with $D\,(z_{n+1}, Tz_n) \to 0$, we have that $z_n \to x^*$ as $n \to \infty$.

Theorem 5.3. *Let* (X, d, s, G) *be a complete graph* b*-metric space and* $T : X \to P(X)$ *be a multi-valued operator. In the conditions of Theorem 5.2 the fixed point inclusion* (3.1) *has the following properties:*

(i) *is* G*-well-posed in the sense of Reich and Zaslavski;*
(ii) *is Ulam-Hyers* G*-stable;*
(iii) *satisfies the data dependence property.*

Proof. The conditions of Theorem 5.1 assure the fact that $X_T \neq \emptyset$ and $Fix(T) \neq \emptyset$. By Theorem 5.2 we obtain that there exists a set retraction $r : X_T \to Fix(T)$ such that T satisfies the strong G-retraction displacement condition:

$$d\,(x, r\,(x)) \leq \frac{s^2}{\beta - s\alpha} D\,(x, Tx)\,, \text{ for all } x \in X_T.$$

(i) **Well-posedness**

Let $x^* \in Fix(T)$. Let us consider the sequence $(y_n)_{n\in\mathbb{N}} \subset r^{-1}(x^*) \cap X_T$ such that $D(y_n, Ty_n) \to 0$, as $n \to \infty$. Now, for each $n \in \mathbb{N}$, $r(y_n) = x^*$. By the strong G-retraction displacement condition we have

$$d(y_n, x^*) = d(y_n, r(y_n)) \leq \frac{s^2}{\beta - s\alpha} D(y_n, Ty_n).$$

Thus, letting $n \to \infty$ in the above relation, we get $d(y_n, x^*) \to 0$.

(ii) **Ulam-Hyers stability**

Let $\varepsilon > 0$ and $y^* \in X_T$ such that $D(y^*, Ty^*) \leq \varepsilon$. Now, there exists $x^* \in Fix(T)$ such that $r(y^*) = x^*$. Then, by the strong retraction displacement condition we have

$$d(y^*, r(y^*)) \leq \frac{s^2}{\beta - s\alpha} D(y^*, Ty^*) \leq \frac{s^2\varepsilon}{\beta - s\alpha}.$$

Hence

$$d(y^*, x^*) \leq \frac{s^2\varepsilon}{\beta - s\alpha}.$$

Data dependence

Let $F : X \to P(X)$ be a multi-valued operator with the following properties:

(i) $Fix(F) \cap X_T \neq \emptyset$;

(ii) there exists $\eta > 0$ such that $H(Tx, Fx) \leq \eta$, for all $x \in X$.

Since $Fix(F) \cap X_T \neq \emptyset$, there exists $y^* \in Fix(F) \cap X_T$. Let us denote $x^* = r(y^*)$. We show that the assertion given in Definition 3.3 hold. Indeed, by the strong retraction displacement condition we have

$$d(y^*, x^*) = d(y^*, r(y^*)) \leq \frac{s^2}{\beta - s\alpha} D(y^*, Ty^*) \leq \frac{s^2}{\beta - s\alpha} H(Fy^*, Ty^*) \leq \frac{s^2}{\beta - s\alpha}\eta.$$

Thus, the proof is complete. □

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