Fixed Point Theory, 25(2024), No. 2, 495-506
DOI: 10.24193/fpt-ro.2024.2.03
http://www.math.ubbcluj.ro/~nodeacj/sfptcj.html

ON ENRICHED CYCLIC ITERATED FUNCTION SYSTEMS

RAVINDRA K. BISHT

Department of Mathematics, National Defence Academy,
Khadakwasla-411023, Pune, India
E-mail: ravindra.bisht@yahoo.com

Abstract. In this paper, we consider a new class of generalized enriched cyclic contraction and establish a related fixed point theorem in the setting of a Banach space. As an application to the fractals, we develop a new iterated function system (IFS) consisting of enriched cyclic $(b_n, \varphi_n(t), \beta_n(t))$-contractions.
Key Words and Phrases: Cyclic contraction, fractal, iterated function system.
2020 Mathematics Subject Classification: 47H10, 54H25.

1. Introduction

Fixed point theory plays a vital role in the study of iterated function systems (IFS) and fractals. The IFS was introduced by Hutchinson [20] and later generalized by Barnsley [2] (see also [4]). Contraction mappings are commonly used to generate mainstream fractals. However, the Banach contraction principle ensures the existence of an attractor of a finite number of mappings in the setting of a complete metric space. In recent years, many authors ([11, 13, 26, 29, 38, 37, 41, 46, 50]) have proven results on finite iterated function systems consisting of various contractive mappings [39]. In [14] (see references therein), the authors pointed out that the actual study of finite IFS of various generalized contractive mappings is possible only when the mappings satisfy a commutativity assumption.

The last two decades have witnessed a great resurgence of interest in the study of IFS theory. Various authors have expanded the IFS theory's conceptual framework to include generalized contractions, countable IFSs, multifunction systems, and more general spaces. Hata [19] constructed IFS using ϕ functions. The concept of infinite IFS was introduced by Wicks [49] (see also [15]). In 2001, Secelean [42] studied countable iterated function systems on a compact metric space (see also [43]). The concept of constructing new IFSs from different $F-$contractions was first put forth by Secelean [44]. The topological IFS attractor, a generalization of the traditional IFS attractor was first introduced in [3]. A multivalued technique was used by Leśniak [24] to create an infinite iteration function system. Secelean [45] defined on the space $l^\infty(X)$ of all bounded sequences of elements from X (mainly with supremum metric) with values in X (see also [27]). Iterated Multifunction Systems (IMS), a generalization of IFS, from a standard point-to-point contraction mapping to a set-valued operator was studied in [23] (see also [22, 36, 46, 48] and references therein). Other remarkable works in this very area have been studied in Gwóźdź-Lukowska and Jachymski [18], Mauldin and Urbański [28], Klimek and Kosek [22] and Leśniak [24]. A systematic overview of the existing literature on IFSs and their applications can be found in [16] (see also Leśniak et al. [25]).

Mihail and Miculescu [30, 31] took into consideration mappings from a finite Cartesian product $X \times X \times ... \times X$ into X as an alternative to self-mappings of a metric space X. Dumitru [12] used a generalised IFS made up of Meir-Keeler type mappings to refine Miculescu and Mihail's work (see also Strobin and Swaczyna [47]). A new IFS associated with generalized $\theta-$contraction, consisting of a finite collection of $\theta^{k_n}-$contraction functions on a complete metric product space was studied in Pasupathi et al. [35].

The notion of enriched contraction mappings was coined in [6, 7, 8, 9]. In [1], the authors introduced the notion of generalized enriched cyclic contraction mappings or $(b, \theta, a)-$generalized enriched cyclic contractions and studied convergence results associated with these mappings. In addition to it, they investigated the iterated function system (IFS) composed of generalized enriched cyclic contraction mappings. In [33], the authors studied cyclic contraction and cyclic ϕ-contraction [21] to construct new IFSs, namely cyclic hyperbolic IFS and cyclic ϕ-contraction IFS, respectively (see also [34]).

Let T be a self-mapping on a nonempty convex subset C of a Banach space. By an averaged mapping, we mean one of the form $T_\lambda = (1-\lambda)I + \lambda T : C \to C$, where $0 < \lambda < 1$, I is the identity operator. Generally, T and T_λ, both have the same fixed point set ($Fix(T_\lambda) = Fix(T)$). For other properties of T_λ one may refer to [17]. A mapping T is said to be asymptotically regular [10] on C if for each $x \in C$, $T^{n+1}x - T^n x \to 0$ as $n \to \infty$.

Let $\{A_1, ..., A_p\}$ denote a finite family of nonempty closed subsets of a normed space $(X, \|\cdot\|)$, where $p \in \mathbf{N}$, the set of positive integers.

Definition 1.1. ([21, 40]) Let X be a nonempty set, $p \in \mathbf{N}$, and T be a self mapping on X. A finite collection $\{A_j \subseteq X : j = 1, 2, 3, ..., p\}$ is called its cyclic representation of X with respect to T if

(a) $X = \bigcup_{j=1}^{p} A_j$,

(b) $T(A_j) \subseteq A_{j+1}$ for $j \in \{1, 2, 3, ..., p\}$, where $A_{p+1} = A_1$.

In 2022, the authors [1] introduced the notion of $(b, \theta, a)-$generalized enriched cyclic contraction and proved the following:

Theorem 1.2. (Theorem 2.0.4 [1]) *Let $(X, \|\cdot\|)$ be a Banach space and $T : \bigcup_{j=1}^{p} A_j \to \bigcup_{j=1}^{p} A_j$. Suppose T is a $(b, \theta, a)-$generalized enriched cyclic contraction, i.e.,*

(i) *$\{A_j : j = 1, 2, 3, ..., p\}$ is a cyclic representation of $\bigcup_{j=1}^{p} A_j$ with respect to T_λ,*

(ii) *there exist* $a, b \in [0, \infty)$ *and* $\theta \in [0, b+1)$ *such that for all* $x \in A_j, y \in A_{j+1}$ *for* $1 \leq j \leq p$,

$$\|b(x-y) + Tx - Ty\| \leq \theta\|x-y\| + a \cdot \{\|x - Tx\| + \|y - Ty\|\} \tag{1.1}$$

where $\lambda = \frac{1}{b+1}$, *provided that* $a \neq 1$ *and* $2a + \theta\lambda < 1$.

Then Fix(T)= x^* *for some* $x^* \in \bigcap_{j=1}^{p} A_j$. *The Krasnoselskij iteration given by* $x_{n+1} = T_{\frac{1}{b+1}} x_n$, $n = 0, 1, 2, \ldots$, *converges to the unique fixed point* x^* *of* T *where* x_0 *is in* $\bigcup_{j=1}^{p} A_j$.

In Theorem 1.2 (Theorem 2.0.4 of [1]) above, the authors also calculated prior and posterior estimates besides studying the rate of convergence of Krasnoselskij iteration. We observe that Theorem 1.2 (Theorem 2.0.4 of [1]) holds for $a \in [0, 1/2)$ instead of $a \in [0, \infty)$. In view of $0 \leq \theta < (b+1)$ and $\lambda(b+1) = 1$, we get $0 \leq \theta\lambda < 1$. But $2a + \theta\lambda < 1$ implies $a \in [0, 1/2)$. Therefore the conditions $a \in [0, \infty)$ and $a \neq 1$ assumed in the above theorem can be replaced with $a \in [0, 1/2)$.

Based on the above observation, we now present a generalized version of Theorem 1.2 by replacing Condition (1.1) with a more general condition. The following classes of functions were studied in [17] (see also [32]): Let $b \in [0, \infty)$ and let $\mathcal{U}$ denote the class of all mappings $\varphi : [0, \infty) \to [0, \infty)$ satisfying

(a) $\varphi(t) < t(b+1)$ for all $t > 0$,
(b) φ is upper semi-continuous, that is $t_n \to t \geq 0 \Rightarrow \limsup_{n\to\infty} \varphi(t_n) \leq \varphi(t)$.

Also, let $\mathcal{B}$ denote the class of functions $\beta : [0, \infty) \to [0, \infty)$ satisfying the following condition, $\limsup_{t\to 0} \beta(t) < \infty$.

2. Main results

We begin this section by the following definition:

Definition 2.1. A mapping $T : \bigcup_{j=1}^{p} A_j \to \bigcup_{j=1}^{p} A_j$ is called a $(b, \varphi(t), \beta(t))$-generalized enriched cyclic contraction if it satisfies the following conditions:

(i) $\{A_j : j = 1, 2, 3, ..., p\}$ is a cyclic representation of $\bigcup_{j=1}^{p} A_j$ with respect to T_λ,
(ii) *there exist* $0 \leq b < \infty$, $\varphi \in \mathcal{U}$, $\beta_1, \beta_2 \in \mathcal{B}$, $\ell_1, \ell_2 > 0$ such that for each $x \in A_j, y \in A_{j+1}$ for $1 \leq j \leq p$,

$$\|b(x-y) + Tx - Ty\| \leq \varphi(\|x-y\|) + \beta_1\left(\frac{1}{b+1}\|x - Tx\|\right) \cdot \|x - Tx\|^{\ell_1} + \beta_2\left(\frac{1}{b+1}\|y - Ty\|\right) \cdot \|y - Ty\|^{\ell_2}. \tag{2.1}$$

where $\lambda = \frac{1}{b+1}$.

Our first main result is the following:

Theorem 2.2. *Let $(X, \|\cdot\|)$ be a Banach space and $T : \bigcup_{j=1}^{p} A_j \to \bigcup_{j=1}^{p} A_j$. Suppose T is a $(b, \varphi(t), \beta(t))$-generalized enriched cyclic contraction and $T_{\frac{1}{b+1}}$ is an asymptotically regular mapping. If T is a continuous, then T has a unique fixed point $x^* \in \bigcap_{j=1}^{p} A_j$. The Krasnoselskij iteration given by $x_{n+1} = T_{\frac{1}{b+1}} x_n$, $n = 0, 1, 2, \ldots$, converges to the unique fixed point x^* of T where x_0 is in $\bigcup_{j=1}^{p} A_j$.*

Proof. For $\lambda = \frac{1}{b+1} \in (0, 1)$. By (2.1), we get

$$\left\| \left(\frac{1-\lambda}{\lambda} \right)(x-y) + Tx - Ty \right\|$$

$$\leq \varphi(\|x-y\|) + \beta_1\left(\frac{1}{\lambda}\|x-Tx\|\right) \cdot \|x-Tx\|^{\ell_1} + \beta_2\left(\frac{1}{\lambda}\|y-Ty\|\right) \cdot \|y-Ty\|^{\ell_2},$$

which can be written in an equivalent form as

$$\begin{aligned} \|T_\lambda x - T_\lambda y\| \leq \lambda \cdot \varphi(\|x-y\|) + \beta_1(\|x-T_\lambda x\|) \cdot \|x-T_\lambda x\|^{\ell_1} \\ + \beta_2(\|y-T_\lambda y\|) \cdot \|y-T_\lambda y\|^{\ell_2}. \end{aligned} \tag{2.2}$$

Let $x_0 \in X$ and let $x_{n+1} = T_\lambda x_n$, $n = 0, 1, 2, \ldots$ If $x_{n+1} = x_n$ for some $n \in \mathbf{N} \cup \{0\}$, then $(1-\lambda)x_n + \lambda T x_n = x_n$, so $Tx_n = x_n$ is the fixed point of T. Suppose $x_{n+1} \neq x_n$ for all $n \geq 0$. Assume that $\{x_n\}$ is not a Cauchy sequence. Then there exists $\varepsilon > 0$ and $n_i, m_i \in \mathbf{N}$ such that $m_i > n_i \geq i$ and

$$\|x_{n_i} - x_{m_i}\| \geq \varepsilon \text{ for } i = 1, 2, \ldots$$

. We may choose m_i to be as small as possible, such that

$$\|x_{n_i} - x_{m_i - 1}\| < \varepsilon.$$

In view of triangle inequality, we have for each $i \in \mathbf{N}$ that

$$\varepsilon \leq \|x_{n_i} - x_{m_i}\| \leq \|x_{n_i} - x_{m_i-1}\| + \|x_{m_i-1} - x_{m_i}\| < \varepsilon + \|x_{m_i-1} - x_{m_i}\|. \tag{2.3}$$

It follows from asymptotic regularity that

$$\lim_{i\to\infty} \|x_{n_i} - x_{m_i}\| = \varepsilon.$$

From (2.2), we have for each i

$$\begin{aligned} \|x_{n_i} - x_{m_i}\| \leq & \|x_{n_i} - x_{n_i+1}\| + \|x_{m_i} - x_{m_i+1}\| + \|x_{n_i+1} - x_{m_i+1}\| \\ & \leq \|x_{n_i} - x_{n_i+1}\| + \|x_{m_i} - x_{m_i+1}\| + \lambda \cdot \varphi(\|x_{n_i} - x_{m_i}\|) \\ & + \beta_1(\|x_{n_i} - x_{n_i+1}\|) \cdot \|x_{n_i} - x_{n_i+1}\|)^{\ell_1} \\ & + \beta_2(\|x_{m_i} - x_{m_i+1}\|) \cdot \|x_{m_i} - x_{m_i+1}\|^{\ell_2}. \end{aligned}$$

It follows from $\mathcal{B}$ that there exist $i_1 \geq i_0 \in \mathbf{N}$ and $L > 0$ such that

$$\beta_1(\|x_{n_i} - x_{n_i+1}\|) \leq L, \quad \text{and} \quad \beta_2(\|x_{m_i} - x_{m_i+1}\|) \leq L$$

for all $i \geq i_1$. Thus, by above inequalities, we have for all $i \geq i_1$,

$$\begin{aligned}\|x_{n_i} - x_{m_i}\| \leq & \|x_{n_i} - x_{n_i+1}\| + \|x_{m_i} - x_{m_i+1}\| + \lambda \cdot \varphi(\|x_{n_i} - x_{m_i}\|) + \\ & + L\{\|x_{n_i} - x_{n_i+1}\|^{\ell_1} + \|x_{m_i} - x_{m_i+1}\|^{\ell_2}\}.\end{aligned}$$

Making $i \to \infty$, using asymptotic regularity and upper semi-continuity of φ, we get

$$0 < \varepsilon = \lim_{i\to\infty} \|x_{n_i} - x_{m_i}\| \leq \limsup_{i\to\infty} \lambda \cdot \varphi(\|x_{n_i} - x_{m_i}\|) \leq \lambda \cdot \varphi(\varepsilon) < \varepsilon,$$

which is a contradiction. Hence $\{x_n\}$ is a Cauchy sequence in $\bigcup_{j=1}^{p} A_j$. Following the similar arguments given in ([1], Theorem 2.0.4), we conclude that $x^* \in \bigcap_{j=1}^{p} A_j$. Using the continuity of T_λ, we immediately obtain $x^* = T_\lambda x^*$, so by the property, $Tx^* = x^*$. Uniqueness of x^* follows easily.

3. An application to IFS

As an application of the results presented in Section 2, we now study an iterated function system (IFS) composed of $(b, \varphi(t), \beta(t))$-generalized enriched cyclic contraction mappings. Let (X, d) be a complete metric space. Let $P(X)$ and $C(X)$ be the collection of all non-empty subsets and compact subsets of X, respectively. For $A, B \in C(X)$, let us define

$$D(A, B) = \sup_{x\in A} \inf_{y\in B} d(x, y).$$

Define the functional $H : C(X) \times C(X) \to [0, \infty)$ by

$$H(A, B) = \max\{D(A, B), D(B, A)\}.$$

The mapping H is called Pompeiu-Hausdorff metric on $C(X)$ [5] induced by d. The metric space $(C(X), H)$ is complete (compact) provided that (X, d) is complete (compact). Suppose that $T : X \to X$ is a continuous mapping. As the image of $A \in C(X)$ under T is compact, there is a natural way to define the induced mapping $T^\star : C(X) \to C(X)$ by $T^\star(A) := T(A)$, for all $A \in C(X)$, where $T(A)$ denotes the image of A under T. A generalized IFS consists of a Banach space $(X, \|\cdot\|)$ and a finite family $\{A_1, ..., A_p\}$ of nonempty closed subsets of X together with a finite set of continuous $(b_n, \varphi_n(t), \beta_n(t))$-generalized enriched cyclic contractions $T^{(n)} : \bigcup_{j=1}^{p} A_j \to X$ for $n \in \{1, 2, .., N\}, N \in \mathbf{N}$. The set function $F : \bigcup_{j=1}^{p} C(A_j) \to \bigcup_{j=1}^{p} C(A_j)$ defined by

$$F(B) = \bigcup_{j=1}^{N} T_{\lambda_n}^{(n)\star}(B)$$

is called an associated Hutchinson operator. The fixed point of F is called an attractor of a generalized IFS [1, 33].

Lemma 3.1. [2] *If $\{C_i\}_{i\in\Lambda}, \{D_i\}_{i\in\Lambda}$, are two finite collections of sets in $(C(X), H)$, then*

$$H(\bigcup_{i\in\Lambda} C_i, \bigcup_{i\in\Lambda} D_i) \leq \max_{i\in\Lambda} H(C_i, D_i),$$

where $\Lambda = \{1, 2, 3, ..., N\}$.

Lemma 3.2. [33] *Let (X, d) be a complete metric space. If A is a closed subset of X, then $C(A)$ is also closed subset of the complete metric space $(C(X), H)$.*

Theorem 3.3. *Let $T : \bigcup_{j=1}^{p} A_j \to \bigcup_{j=1}^{p} A_j$ be a $(b, \varphi(t), 0)$-generalized enriched cyclic contraction, i.e.,*

$$\|b(x-y) + Tx - Ty\| \leq \varphi(\|x-y\|).$$

Then the induced map $T^\star : \bigcup_{j=1}^{p} C(A_j) \to \bigcup_{j=1}^{p} C(A_j)$ satisfies the following conditions:

1. *for each $j \in \{1, 2, ..., p\}, A \in C(A_j)$ and $B \in C(A_{j+1})$, we have*

$$H(b(A) + T^\star(A), b(B) + T^\star(B)) \leq \varphi(H(A, B)).$$

2. *$\{C(A_j) : j = 1, 2, 3, ..., p\}$ is a cyclic representation of $\bigcup_{j=1}^{p} C(A_j)$ with respect to $T^\star_\lambda$ provided that T_λ is continuous, where $\lambda = \frac{1}{b+1}$.*

Proof. For $\lambda = \frac{1}{b+1}$, $(b, \varphi(t), 0)$-generalized enriched cyclic contraction can be written as

$$\|T_\lambda x - T_\lambda y\| \leq \lambda\varphi(\|x-y\|).$$

Let $A \in C(A_j)$, for some $j \in \{1, 2, ..., p\}$. Since $\{A_j : j = 1, 2, 3, ..., p\}$ is a cyclic representation of $\bigcup_{j=1}^{p} A_j$ with respect to T_λ, therefore $T_\lambda(A) \subseteq A_{j+1}$. By continuity of T_λ, $T_\lambda(A)$ is a compact set and hence $T_\lambda(A) \in C(A_{j+1})$. For all $j \in \{1, 2, ..., p\}$, this infers $T_\lambda(C(A_j)) \subseteq C(A_{j+1})$.

Take $A \in C(A_j)$ and $B \in C(A_{j+1})$ for some $j \in \{1, 2, ..., p\}$. We first claim

$$D(b(A) + T(A), b(B) + T(B)) \leq \varphi(D(A, B)). \tag{3.1}$$

In view of $\lambda = \frac{1}{b+1}(b > 0)$, we have

$$D\left(\left(\frac{1}{\lambda} - 1\right)(A) + T(A), \left(\frac{1}{\lambda} - 1\right)(B) + T(B)\right) \leq \varphi(D(A, B)), \tag{3.2}$$

and hence

$$D((1-\lambda)(A) + T(A), (1-\lambda)(B) + T(B)) \leq \lambda\varphi(D(A, B)).$$

This implies that

$$D(T_\lambda(A), T_\lambda(B)) \leq \lambda\varphi(D(A, B)).$$

Also, using assumptions of φ, we get

$$D(T_\lambda(A), T_\lambda(B)) = \sup_{x\in A} \inf_{y\in B} |T_\lambda x - T_\lambda y| \leq \sup_{x\in A} \inf_{y\in B} \varphi(|T_\lambda x - T_\lambda y|) \leq \varphi(D(A, B))).$$

Similarly,

$$D(T_\lambda(B), T_\lambda(A)) \leq \lambda\varphi(D(B, A)).$$

Also,

$$D(b(A)+T(A),b(B)+T(B))\le \varphi(D(A,B)).$$

Note that

$$H(b(A)+T(A),b(B)+T(B))$$
$$=\max\{D(b(A)+T(A),b(B)+T(B)),D(b(B)+T(B),b(A)+T(A))\}.$$

By definition of $T^\star$ and (3.1), we have

$$H(b(A)+T^\star(A),b(B)+T^\star(B))\le \varphi(H(A,B)).$$

Theorem 3.4. *Let $T^{(n)}:\bigcup_{j=1}^{p}A_j\to\bigcup_{j=1}^{p}A_j$, where $n\in\{1,2,..,N\}$. Assume that*

(1) *each $T^{(n)}$ is $(b_n,\varphi_n(t),0)$-generalized enriched cyclic contraction mapping,*

(2) *$\{A_j : j=1,2,3,...,p\}$ is a cyclic representation of $\bigcup_{j=1}^{p}A_j$ with respect to $T^{(n)}\lambda_n$, for all $n=1,2,..,N$, where $N\in\mathbf{N}$.*

Then the map $F:\bigcup_{j=1}^{p}C(A_j)\to\bigcup_{j=1}^{p}C(A_j)$ defined by

$$F(A)=\bigcup_{j=1}^{N}T_{\lambda_n}^{(n)\star}(A)$$

satisfies the following conditions:

(i) $\{C(A_j) : j=1,2,3,...,p\}$ is a cyclic representation of $\bigcup_{j=1}^{p}C(A_j)$ with respect to F,

(ii) for all $1\le j\le p$, $A\in C(A_j)$ and $B\in C(A_{j+1})$, we have

$$H(F(A),F(B))\le\varphi(H(A,B)),$$

where $\varphi(t)=\max_{1\le n\le N}\lambda_n\varphi_n(t)$ and $\lambda_n=\frac{1}{b_n+1}$ provided that $T_{\lambda_n}^{(n)}$ are continuous mappings for all $n\in\{1,2,..,N\}$.

Proof. Let $C\in C(A_j)$ for some j. By Theorem 3.3, $\{C(A_j) : j=1,2,3,...,p\}$ is a cyclic representation of $\bigcup_{j=1}^{p}C(A_j)$ with respect to $T_{\lambda_n}^{(n)\star}$ for all n.Therefore, $F(C)\in C(A_{j+1})$. This implies $F(C(A_j))\subseteq C(A_{j+1})$ for each j. Using Theorem 3.3, we have

$$H(b_n(A)+T^{(n)\star}(A),b_n(B)+T^{(n)\star}(B))\le\varphi_n(H(A,B)),$$

for each $j\in\{1,2,...,p\},n\in\{1,2,..,N\},A\in C(A_j)$ and $B\in C(A_{j+1})$. For $\lambda_n=\frac{1}{b_n+1}$ and $n\in\{1,2,..,N\}$, the above inequality reduces to

$$H(T_{\lambda_n}^{(n)\star}(A),T_{\lambda_n}^{(n)\star}(B))\le\lambda_n\varphi(H(A,B)).$$

Let $A\in C(A_j)$ and $B\in C(A_{j+1})$ for some j. In view of Lemma 3.1, we have

$$H(F(A),F(B))=H\left(\bigcup_{n=1}^{N}\{T_{\lambda_n}^{(n)\star}(A)\},\bigcup_{n=1}^{N}\{T_{\lambda_n}^{(n)\star}(B)\}\right)\le\varphi(H(A,B)),$$

where $\varphi(t)=\max_{1\le n\le N}\lambda_n\varphi_n(t)$.

Now we prove the existence of the attractor for an IFS with $(b_n, \varphi_n(t), 0)$-generalized enriched cyclic contraction mappings.

Corollary 3.5. *Let (X, d) be a complete metric space and $T^{(n)} : \bigcup_{j=1}^{p} A_j \to \bigcup_{j=1}^{p} A_j$, $n \in \{1, 2, .., N\}$. If $T^{(n)}$ satisfies (1) and (2) of Theorem 3.4, then the mapping F defined in Theorem 3.4 has a unique attractor $\mathbf{A}$. Moreover, $\mathbf{A} = \lim_{m\to\infty} F^m(B)$, for any $B \in \bigcup_{j=1}^{p} C(A_j)$, provided that $T^{(n)}_{\lambda_n}$ are continuous mappings for all $n \in \{1, 2, .., N\}$.*

Proof. Since (X, d) is a complete metric space, $(C(X), H)$ is a complete metric space. Utilizing Lemma 3.2, we get $C(A_j)$ is nonempty closed subset of $C(X)$ for every j. By Theorem 3.4 and metric version of Theorem 2.2 (take $b = 0$ and $\beta_1(t) = \beta_2(t) = 0$), F has a unique attractor $\mathbf{A}$ and $\mathbf{A} = \lim_{m\to\infty} F^m(B)$, for any $B \in \bigcup_{j=1}^{p} C(A_j)$.

4. Examples

Example 4.1. Let $A_1 = [0, 2], A_2 = [1, 3]$ and $0 < \lambda < 1$. Define $T^{(1)}, T^{(2)} : A_1 \cup A_2 \to \mathbb{R}$ by

$$T^{(1)}(x) := \begin{cases} \frac{17-(1+8\lambda)x}{8(1-\lambda)}, & 0 \le x \le 2, \\ \frac{15-8\lambda x}{8(1-\lambda)}, & 2 \le x \le \frac{11}{4}, \\ \frac{37-8(1+\lambda)x}{8(1-\lambda)}, & \frac{11}{4} \le x \le 3, \end{cases}$$

and

$$T^{(2)}(x) := \begin{cases} \frac{10-(1+8\lambda)x}{8(1-\lambda)}, & \text{if } 0 \le x \le 2, \\ \frac{1-\lambda x}{1-\lambda}, & \text{if } 2 \le x \le \frac{11}{4}, \\ \frac{15-4(1+\lambda)x}{4(1-\lambda)}, & \text{if } \frac{11}{4} \le x \le 3. \end{cases}$$

For $\lambda = \frac{1}{2}$, we get

$$T^{(1)}(x) := \begin{cases} \frac{17-5x}{4}, & \text{if } 0 \le x \le 2, \\ \frac{15-4x}{4}, & \text{if } 2 \le x \le \frac{11}{4}, \\ \frac{37-12x}{4}, & \text{if } \frac{11}{4} \le x \le 3, \end{cases}$$

and

$$T^{(2)}(x) := \begin{cases} \frac{10-5x}{4}, & \text{if } 0 \le x \le 2, \\ 2-x, & \text{if } 2 \le x \le \frac{11}{4}, \\ \frac{15-6x}{2}, & \text{if } \frac{11}{4} \le x \le 3. \end{cases}$$

If one considers $b = 1$, then $\lambda = \frac{1}{2}$, gives that

$$T^{(1)}_{\frac{1}{2}}(x) := \begin{cases} \frac{17-x}{4} & \text{if } 0 \le x \le 2, \\ \frac{15}{8} & \text{if } 2 \le x \le \frac{11}{4}, \\ \frac{37-8x}{8} & \text{if } \frac{11}{4} \le x \le 3, \end{cases}$$

and

$$T^{(2)}_{\frac{1}{2}}(x) := \begin{cases} \frac{10-x}{8} & \text{if } 0 \le x \le 2, \\ 1 & \text{if } 2 \le x \le \frac{11}{4}, \\ \frac{15-4x}{4} & \text{if } \frac{11}{4} \le x \le 3. \end{cases}$$

Following Pashupathi [33] (see also [1]), it can easily be shown that

(i) $\{A_1, A_2\}$ is a cyclic representation of $A_1 \cup A_2$ with respect to $T^{(n)}_{\lambda_n}$, for $n = 1, 2$.

(ii) $T^{(1)}$ and $T^{(2)}$ are $(1, t/3, 0)$-generalized enriched contraction mappings for $\lambda = 1/2$.

Therefore, the mapping F defined in Corollary 3.5 has a unique fixed point $\mathbf{A}$. The attractor $\mathbf{A}$ of the cyclic $(b_n, \varphi_n(t), 0)$-generalized enriched IFS $I_C = \{(A_1 \cup A_2) : T^{(1)}, T^{(2)}\}$ is similar to a Cantor set for $[1, 2]$ with 8 sub-intervals, and retaining first and last sub-intervals at each stage (see Figure 1).

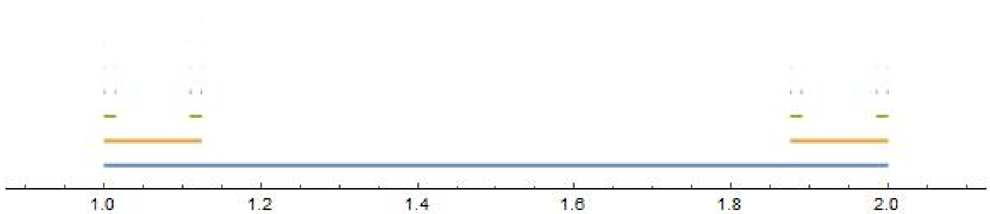

Figure 1. Cantor set for $[1, 2]$ with 8 subintervals

Example 4.2. Let $X = \mathbb{R}^2$, $A_1 = [-1, 1] \times \mathbb{R}$, $A_2 = [-1/2, 1] \times \mathbb{R}$ and $0 < \lambda < 1$. Define

$$T^{(1)}(x, y) = \left(\frac{(0.5 - \lambda)x}{1 - \lambda}, \frac{x + (0.6 - \lambda)y}{1 - \lambda} \right)$$

and

$$T^{(2)}(x, y) = \left(\frac{(0.5 - \lambda)x + 0.5}{1 - \lambda}, \frac{-x - (0.5 + \lambda)y + 1}{1 - \lambda} \right).$$

For $\lambda = \frac{1}{2}$, we get

$$T^{(1)}(x, y) = \Big(0, 2x + 0.2y) \Big)$$

and

$$T^{(2)}(x, y) = \Big(1, -3x - 2y + 2 \Big).$$

If we take $b = 1$, then $\lambda = \frac{1}{2}$, gives

$$T^{(1)}_{\frac{1}{2}}(x, y) = (0.5x, x + 0.6y) \quad \text{and} \quad T^{(2)}_{\frac{1}{2}}(x, y) = (0.5x + 0.5, -x - 0.5y + 1).$$

Following Pashupathi [33] (see also [1]), it can easily be shown that

(i) $\{A_1, A_2\}$ is a cyclic representation of $A_1 \cup A_2$ with respect to $T^{(n)}_{\lambda_n}$, for $n = 1, 2$.

(ii) $T^{(1)}$ and $T^{(2)}$ are $(1, 2t, 0)$-generalized enriched contraction mappings for $\lambda = 1/2$.

Therefore, the mapping F defined in Corollary 3.5 has a unique fixed point $\mathbf{A}$. The attractor $\mathbf{A}$ of the cyclic $(b_n, \varphi_n(t), 0)$-generalized enriched IFS $\{(X, A_1, A_2); T^{(1)}, T^{(2)}\}$ is the graph of the fractal function similar to [33].

5. Conclusion

In this paper, we have generalized the concept of a generalized enriched cyclic contraction mappings or $(b, \theta, a)-$generalized enriched cyclic contraction introduced in [1]. Our results improve and extend various results in the literature including those in ([1, 6, 7, 8, 9, 17, 33]). In 2017, Van Dung and Petruşel [14] proposed a revision to the theorems on iterated function systems that consist of Chatterjea, Kannan, and Reich contractions. The authors [14] added commutativity assumptions on the functions, which led to a more comprehensive framework for the analysis of such systems. It would be interesting to study IFS of $(b, \varphi(t), \beta(t))$-generalized enriched cyclic contraction mappings. We pose this as an open problem.

Acknowledgements. The author would like to thank the referee for his/her valuable suggestions for the improvement of the paper.

References

[1] M. Abbas, M.R. Anjum, H. Iqbal, *Generalized enriched cyclic contractions with application to generalized iterated function system*, Chaos, Solitons & Fractals, **154**(2022), 111591.

[2] M.F. Barnsley, *Fractals Everywhere*, Academic Press, Boston, 1988.

[3] T. Banakh, M. Nowak, *A 1−dimensional, Peano continuum which is not an IFS attractor*, Proc. Amer. Math. Soc., **141**(2013), no. 3, 931-935.

[4] M.F. Barnsley, S. Demko, *Iterated function systems and the global construction of fractals*, Proc. Roy. Soc. London Ser A, **399**(1985), 243-275.

[5] V. Berinde, M. Pacurar, *The role of the Pompeiu-Hausdorff metric in fixed point theory*, Creat. Math. Inform., **22**(2013), 35-42.

[6] V. Berinde, M. Pacurar, *Approximating fixed points of enriched contractions in Banach spaces*, J. Fixed Point Theory Appl., **22**(2020), no. 2, 1-10.

[7] V. Berinde, M. Pacurar, *Kannan's fixed point approximation for solving split feasibility and variational inequality problems*, J. Comput. Appl. Math., (2020), 377-427.

[8] V. Berinde, M. Pacurar, *Approximating fixed points of enriched Chatterjea contractions by Krasnoselskij iterative algorithm in Banach spaces*, J. Fixed Point Theory Appl., **23**(2021), 23-66.

[9] V. Berinde, M. Pacurar, *Fixed point theorems for enriched Ćirić-Reich-Rus contractions in Banach spaces and convex metric spaces*, Carpathian J. Math., **37**(2021), 173-184.

[10] F.E. Browder, W.V. Peryshyn, *The solution by iteration of nonlinear functional equations in Banach spaces*, Bull. Amer. Math. Soc., **72**(1966), 571-575.

[11] C. Chifu, A. Petruşel, *Multivalued fractals and generalized multivalued contractions*, Chaos, Solitons & Fractals, **70**(36)(2008), 203-210.

[12] D. Dumitru, *Generalized iterated function systems containing Meir-Keeler functions*, An. Univ. Bucureşti, Math., **58**(2009), 3-15.

[13] N.V. Dung, *Answers to questions on Ćirić type theorems*, Fractals, **25**(2017), no. 1, 1-9.

[14] N.V. Dung, A. Petruşel, *On iterated function systems consisting of Kannan maps, Reich maps, Chatterjea type maps, and related results*, J. Fixed Point Theory Appl., **19**(2017), 2271-2285.

[15] H. Fernau, *Infinite iterated function systems*, Math. Nachr., **170**(1994), 79-91.
[16] R. Ghosh, J. Marecek, *Iterated Function Systems: A Comprehensive Survey*, arXiv:2211.14661 (26 Nov 2022).
[17] J. Górnicki, R.K. Bisht, *Around averaged mappings*, J. Fixed Point Theory Appl., **23**(2021), no. 3,
[18] G. Gwóźdź-Łukowska, J. Jachymski, *The Hutchinson-Barnsley theory for infinite iterated function systems*, Bull. Aust. Math. Soc., **72**(2005), no. 3, 441-454.
[19] M. Hata, *On some properties of set-dynamical systems*, Proc. Jpn Acad. Ser. A, **61**(1985), 99-102.
[20] J. Hutchinson, *Fractals and self-similarity*, Indiana Univ. Math. J., **30**(1981), 713-747.
[21] W.A. Kirk, P.S. Srinivasan, P. Veeramani, *Fixed points for mappings satisfying cyclical contractive conditions*, Fixed Point Theory, **4**(2003), 79-89.
[22] M. Klimek, M. Kosek, *Generalized iterated function systems, multifunctions and Cantor sets*, Ann. Polon. Math., **96**(2009), no. 1, 25-41.
[23] H. Kunze, D. La Torre, E.R. Vrscay, *Contractive multifunctions, fixed point inclusions and iterated multifunction systems*, J. Math. Anal. Appl., **330**(2007), 159-173.
[24] K. Leśniak, *Infinite iterated function systems: a multivalued approach*, Bull. Pol. Acad. Sci. Math., **52**(2004), no. 1, 1-8.
[25] K. Leśniak, N. Snigireva, F. Strobin, *Weakly contractive iterated function systems and beyond: A manual*, J. Difference Equ. Appl., **26**:8, 1114-1173, DOI: 10.1080/10236198.2020.1760258
[26] E. Llorens-Fuster, A. Petruşel, J.C. Yao, *Iterated function systems and well-posedness*, Chaos, Solitons & Fractals, **41**(2009), 1561-1568.
[27] L. Maślanka, *On a typical compact set as the attractor of generalized iterated function systems of infinite order*, J. Math. Anal. Appl., **484**(2020), 123740.
[28] R.D. Mauldin, M. Urbański, *Dimensions and measure in infinite iterated function systems*, Proc. Lond. Math. Soc., **S3-73**(1996), no. 1, 105-154.
[29] R. Miculescu, A. Mihail, *Reich-type iterated function systems*, J. Fixed Point Theory Appl., **18**(2016), 285-296.
[30] A. Mihail, R. Miculescu, *Applications of fixed point theorems in the theory of generalized IFS*, Fixed Point Theory Appl. , **2008**(2008), 312876.
[31] A. Mihail, R. Miculescu, *Generalized IFSs on noncompact spaces*, Fixed Point Theory Appl., **20**(2010).
[32] L.V. Nguyen, *On fixed points of asymptotically regular mappings*, Rend. Circ. Mat. Palermo, II. Ser, **70**(2021), 709-719.
[33] P. Pashupathi, A.K.B. Chand, M.A. Navascués, *Cyclic iterated function systems*, J. Fixed Point Theory Appl., **22**(2020), 58.
[34] R. Pasupathi, A.K.B. Chand, M.A. Navascués, *Cyclic Meir-Keeler contraction and its fractals*, Numer. Funct. Anal. Optim., **42**(2021), 1-20.
[35] R. Pasupathi, M.A. Navascués, A.K.B. Chand, *Iterated functions systems composed of generalized $\theta-$contractions*, Fractal Fract., **5**(2021), 69.
[36] A. Petruşel, I.A. Rus, M.A. Şerban, *Fixed points, fixed sets and iterated multifunction systems for non-self multivalued operators*, Set-Valued Var. Anal., **23**(2015), 223-237.
[37] A. Petruşel, A. Soos, *Self-similar sets and fractals generated by Ćirić type operators*, J. Nonlinear Sci. Appl., **8**(2015), 1048-1058.
[38] B.V. Prithvi, S.K. Katiyar, *Interpolative operators: Fractal to multivalued fractal*, Chaos Solitons & Fractals, **164**(2022), 112449.
[39] B.E. Rhoades, *A comparison of various definitions of contractive mappings*, Trans. Am. Math. Soc., **226**(1977), 257-290.
[40] I.A. Rus, *Cyclic representations and fixed points*, Ann. T. Popoviciu Seminar Funct. Eq. Approx. Convexity., **3**(2005), 171-178.
[41] D.R. Sahu, A. Chakraborty, R.P. Dubey, *K-iterated function system*, Fractals, **18**(2010), 139-144.
[42] N.A. Secelean, *Countable iterated function systems*, Far East J. Dyn. Syst., **3**(2)(2001), 149-167.

[43] N.A. Secelean, *The existence of the attractor of countable iterated function systems*, Mediterr. J. Math., **9**(2012), 61-79.
[44] N.A. Secelean, *Iterated function systems consisting of $F-$contractions*, Fixed Point Theory Appl., **277**(2013), 13.
[45] N.A. Secelean, *Generalized iterated function systems on the space*, J. Math. Anal. Appl., **410**(2014), no. 2, 847-858.
[46] S.L. Singh, B. Prasad, A. Kumar, *Fractals via iterated functions and multifunctions*, Chaos, Solitons & Fractals, **39**(2009), 1224-1231.
[47] F. Strobin, J. Swaczyna, *On a certain generalization of the iterated function system*, Bull. Aust. Math. Soc., **87**(2013), 37-54.
[48] D. La Torre, F. Mendivil, *Iterated function systems on multifunctions and inverse problems*, J. Math. Anal. Appl., **340**(2008), 1649-1479.
[49] K.R. Wicks, *Fractals and Hyperspaces*, Lecture Notes in Mathematics, Vol. 1492, Berlin, Springer-Verlag, 1991.
[50] S. Xu, S. Cheng, Z. Zhou, *Reich's iterated function systems and well-posedness via fixed point theory*, Fixed Point Theory and Applications, **2015**(2015), 1-11.

Received: October 16, 2022; Accepted: April 8, 2023.