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FIXED POINTS ON PARTIALLY ORDERED QUASI-METRIC SPACES

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Abstract. In this paper we prove new fixed point results in partially ordered bicomplete quasi-metric spaces. Our results extends/generalized celebrated results, on the one hand, by Nieto and Rodríguez-López for contraction mappings in partially ordered complete metric spaces and, on the other hand, by Schellekens for contraction mappings in bicomplete quasi-metric spaces. Moreover, it is also shown that neither our assumptions can be weakened nor our results can be deduced from the celebrated Kleene's fixed point theorem. Finally, an application of our results to the asymptotic analysis of recurrence equations is given.
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1. Introduction and preliminaries

Among celebrated fixed point results that allows us to prove the existence of solutions to functional equations, we can highlight two.

On the one hand, we have the Banach fixed point result which can be stated as follows ([11]):

Theorem 1.1. *Let* (X, d) *be a complete metric spaces. Let* $f : X \to X$ *be a mapping satisfying*

$$d(f(x), f(y)) \leq kd(x, y) \tag{1.1}$$

for all $x, y \in X$ *and some* $k \in [0, 1[$, *then there exists a unique* $x^* \in X$ *such that* $Fix(f) = \{x \in X : f(x) = x\} = x^*$.

We assume that the reader is familiar with basics of metric fixed point theory.

According to [12], the preceding result was obtained mainly in order to provide the existence of solution to differential equations by means of the so-called successive approximations scheme, i.e., the solution to the aforesaid kind of equations is obtained as the limit, with respect to the topology induced by the metric $\tau(d)$, of the sequence $(f^n(x_0))_{n\in\mathbb{N}}$ where x_0 is any point in X. Many applications of Theorem 1.1 to other type of functional equations have been given in the literature (see, for instance, [11, 12]).

On the other hand, we have the so-called Kleene's fixed point theorem. This result allows us to develop a fixed point induction principle in partially ordered sets. In order to state it, let us recall a few pertinent notions.

Following [9], a partially ordered set is a pair $(Y, \preceq)$ such that Y is a nonempty set and $\preceq$ is a reflexive, antisymmetric and transitive binary relation on Y.

If $(Y, \preceq)$ is a partially ordered set and $X \subseteq Y$, then an upper bound for X in $(Y, \preceq)$ is an element $y \in Y$ such that $x \preceq y$ for all $x \in X$. An element $z \in X$ is least in $(X, \preceq)$ provided that $z \preceq x$ for all $x \in X$. Thus, the least upper bound for X in $(Y, \preceq)$, if exists, is an element $z \in Y$ which is an upper bound for X and, in addition, it is least in the set $(UB(X), \preceq)$, where $UB(X) = \{u \in X : u \text{ is an upper bound for } X\}$.

According to [5], a partially ordered set $(X, \preceq)$ is called chain-complete provided that every increasing sequence has a least upper bound. Of course, a sequence $(x_n)_{n\in\mathbb{N}}$ is called increasing whenever $x_n \preceq x_{n+1}$ for all $n \in \mathbb{N}$, where $\mathbb{N}$ stands for the set of positive integer numbers. In addition, a mapping from a partially ordered set $(X, \preceq)$ into itself is called $\preceq$-continuous if the least upper bound of the sequence $(f(x_n))_{n\in\mathbb{N}}$ is $f(x)$ for every increasing sequence $(x_n)_{n\in\mathbb{N}}$ whose least upper bound exists and is x.

Now Kleene fixed point [5] can be restated as:

Theorem 1.2. *Let $(X, \preceq)$ be a chain-complete partially ordered set. Let f be a $\preceq$-continuous mapping from $(X, \preceq)$ into itself. If there exists $x_0 \in X$ such that $x_0 \preceq f(x_0)$, then f has a fixed point x^* which is least in $(Fix(f) \cap \uparrow_{\preceq} x_0, \preceq)$, where $\uparrow_{\preceq} x_0 = \{x \in X : x_0 \preceq x\}$.*

The preceding result was introduced mainly in order to develop a mathematical technique that provides the meaning of recursive specifications in denotational semantics for programming languages as the fixed point of a functional equation which is obtained as the least upper bound, with respect to the partial order $\preceq$ on X, of the iterative sequence $(f^n(x_0))_{n\in\mathbb{N}}$ where x_0 is any point in X with $x_0 \preceq f(x_0)$. We refer the reader to [21] for a detailed treatment of the topic.

Taking into account the importance of the exposed celebrated results, Nieto and Rodríguez-López posed the question whether it is possible to capture the spirit of both results in one fixed point result. They provided a positive answer to such a question.

Theorem 1.3 ([14]). *Let $(X, d, \preceq)$ be a partially ordered complete metric spaces. If $f : X \to X$ is a continuous and monotone function such that there exist $k \in [0, 1[$ and $x_0 \in X$ with $x_0 \preceq f(x_0)$ and*

$$d(f(x), f(y)) \leq kd(x, y) \tag{1.2}$$

for all $x, y \in X$ such that $y \preceq x$, then f has a fixed point.

Observe that a partially ordered complete metric space $(X, d, \preceq)$ is a complete metric space (X, d) endowed with a partial order $\preceq$ on X.

Although partially ordered metric spaces has been shown to be a useful tool in the aforesaid realms, the symmetry of metrics limits their utility in many other applied fields. Concretely, in [19] Schellekens asked the question about the possibility of developing a mathematical fixed point technique in order to analyze the asymptotic behavior of the complexity of algorithms which was different from Kleene's technique, i.e., in such a way that the technique was based on a quantitative approach able to provide information of the improvement in complexity when an algorithm is replaced by another one. The answer to the posed question is affirmative and the aforementioned technique is based on the notion of quasi-metric (see Section 5 for a fuller treatment of the topic).

A quasi-metric [13] (see also [10]), on a (nonempty) set X is a function $d : X \times X \to \mathbb{R}^+$ such that for all $x, y, z \in X$:

$$\begin{array}{ll} \text{(i)} & d(x, y) = d(y, x) = 0 \Leftrightarrow x = y, \\ \text{(ii)} & d(x, z) \leq d(x, y) + d(y, z). \end{array}$$

Notice that a metric d on X is exactly a quasi-metric satisfying $d(x, y) = d(y, x)$ for all $x, y \in X$.

Every quasi-metric d on a set X induces a T_0 topology $\tau(d)$ on X which has as a base the family of all open d-balls $\{B_d(x, r) : x \in X,\ r > 0\}$. Here $B_d(x, r) = \{y \in X : d(x, y) < r\}$ for all $x \in X$ and $r > 0$. According to [10, 13], a quasi-metric space (X, d) is said to be T_1 provided that $d(x, y) = 0 \Leftrightarrow x = y$.

A (T_1) quasi-metric space is a pair (X, d) such that X is a (nonempty) set and d is a (T_1) quasi-metric on X.

If d is a quasi-metric on a set X, then the functions d^{-1} and d^s defined on $X \times X$ by $d^{-1}(x, y) = d(y, x)$ and $d^s(x, y) = \max\{d(x, y), d(y, x)\}$ is a quasi-metric and a metric on X, respectively. It must be pointed out that the quasi-metric d^{-1} is known as the conjugated quasi-metric of d.

According to [10, 13], a quasi-metric space (X, d) is bicomplete provided that the metric space (X, d^s) is complete.

Every quasi-metric space (X, d) becomes a partially ordered set endowed with the so-called specialization partial order $\preceq_d$, which is defined by $x \preceq_d y \Leftrightarrow d(x, y) = 0$ (see [10, 13]). Notice that when the quasi-metric d is T_1 (or a metric), then the specialization partial order $\preceq_d$ is exactly the flat one, i.e., $x \preceq_d y \Leftrightarrow x = y$.

The Schellekens fixed point technique was based on the following quasi-metric extension of the Banach fixed point theorem that also tries to capture the essence of Kleene's fixed point theorem.

Theorem 1.4 ([19]). *Let (X, d) be a bicomplete complete quasi-metric space and let f be a mapping from X into itself such that there exists $k \in [0, 1[$ with*

$$d(f(x), f(y)) \leq k d(x, y) \tag{1.3}$$

for all $x, y \in X$*. Then there exists a unique* $x^* \in X$ *such that* $Fix(f) = x^*$*. Moreover, if there exists* $x_0 \in X$ *such that* $x_0 \preceq_d f(x_0)$ *then* x^* *is the least upper bound of* $(f^n(x_0))_{n\in\mathbb{N}}$ *in* $(X, \preceq_d)$ *and, thus,* $x^* \in \uparrow_{\preceq_d} x_0$*.*

Since Schellekens proved the preceding result, the fixed point theory on bicomplete quasi-metric spaces has aroused interest. The reader can find recent works in this direction in [1, 2, 4, 6, 15, 16, 18, 17]. Motivated, on the one hand, by the applicability of bicomplete quasi-metric spaces to Computer Science and, on the other hand, by the fact that Theorem 1.4 only provides a partial extension of Theorem 1.3 to the context of quasi-metric spaces, our main objective in this paper is to analyze the possibility of extending the aforesaid result in such a way that Theorems 1.4 and 1.3 can be retrieved as particular cases and, in addition, the spirit of Banach's and Kleene's fixed point results are preserved. Hence the remainder of the paper is organized as follows:

In Section 2 we present new general extensions of the aforementioned results and we show that the the assumptions in the statements of our results cannot be weakened. Moreover, we reveal that our results cannot be deduced from the celebrated Kleene's fixed point theorem. In addition, conditions that guarantee the uniqueness of fixed point are provided. Furthermore, many examples are given in order to illustrate our results. Section 3, deals with order boundedness of quasi metric space endowed with the specialization partial order. In Section 4, we study the relationship of newly obtained results with Kleene's fixed point theorem. Section 5 is devoted to give an application. Concretely, we introduce a new fixed point technique for getting upper and lower asymptotic bounds for the solution to a kind of recurrence equations which appears in a natural way in asymptotic analysis of algorithms.

2. The extension results

In this section our aim is to extend Theorem 1.3 to the context of partially ordered quasi-metric spaces and provide a general version of Theorem 1.4. To this end, from now on, given a quasi-metric space (X, d), we will say that a mapping $f : X \to X$ is continuous if it is continuous from $(X, \tau(d))$ into itself. Moreover, we will say that f is conjugate continuous if it is continuous from $(X, \tau(d^{-1}))$ into itself. Furthermore, we will say that f is s-continuous when it is continuous from $(X, \tau(d^s))$ into itself.

A natural way of extending Theorem 1.3 to the asymmetric context consists of replacing in its statement partially ordered complete metric spaces by partially ordered bicomplete quasi-metric spaces. Thus one can conjecture that the next result would be desirable.

"Let $(X, d, \preceq)$ be a partially ordered bicomplete quasi-metric space. If $f : X \to X$ is a continuous and monotone function such that there exist $k \in [0, 1[$ and $x_0 \in X$ with $x_0 \preceq f(x_0)$ and

$$d(f(x), f(y)) \leq kd(x, y) \tag{2.1}$$

for all $x, y \in X$ such that $y \preceq x$, then f has a fixed point."

However, the next example shows that such a result does not hold.

Example 2.1. Consider the set $\mathbb{R}$ endowed with the usual order $\preceq$ and with the upper-quasi-metric $d_u : \mathbb{R} \times \mathbb{R} \to \mathbb{R}^+$ given by $d_u(x, y) = \max\{y - x, 0\}$ for all $x, y \in \mathbb{R}$. It is clear that $(\mathbb{R}, \preceq, d_u)$ is a partially ordered bicomplete quasi-metric space. Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x + 1$ for all $x \in \mathbb{R}$. Then it is easily seen that f is a continuous monotone function satisfying condition (2.1) (above) for all $x \geq y$ and, in addition, that $0 \preceq f(0)$. Function f has no fixed point.

Notice that if a mapping f holds that

$$d^s(f(x), f(y)) \leq kd(x, y)$$

for all $y \preceq x$, then it holds the contractive condition (2.1).

In the light of the above remark another natural way to extend Theorem 1.3 would be as follows:

"Let $(X, d, \preceq)$ be a partially ordered bicomplete quasi-metric space. If $f : X \to X$ is a continuous and monotone function such that there exist $k \in [0, 1[$ and $x_0 \in X$ with $x_0 \preceq f(x_0)$ and

$$d^s(f(x), f(y)) \leq kd(x, y)$$

for all $y \preceq x$, then f has a fixed point."

Nevertheless, the next example shows that such a result does not hold in our context.

Example 2.2. Consider the quasi-metric space $([1, \infty[, d_l)$, where

$$d_l(x, y) = \max\{x - y, 0\} \text{ for all } x, y \in [1, \infty[.$$

It is clear that the quasi-metric space $([1, \infty[, d_l)$ is bicomplete. Next consider, the partial order $\preceq_*$ defined on $[1, \infty[$ as follows:

$x \preceq_* y \Leftrightarrow$ there exists $n \in \mathbb{N}$ such that $x, y \in [n, n + 1[$ and $x \preceq y$.

Define the mapping $f : [1, \infty[\to [1, \infty[$ by $f(x) = \frac{n+1}{2} + \frac{x}{2}$ for all $x \in [n, n + 1[$. It is not hard to see that f is monotone with respect to $\preceq_*$ and that $1 \preceq_* f(1) = \frac{3}{2}$. Moreover, a straightforward computation gives that f is continuous from $([1, \infty[, d_l)$ into itself. Furthermore

$$\frac{x - y}{2} = d_l^s(f(x), f(y)) \leq \frac{1}{2} d_l(x, y)$$

for all $x, y \in [1, \infty[$ such that $y \preceq_* x$. Of course, f has no fixed point.

Inspired by the fact that the quasi-metric spaces provided by Examples 2.1 and 2.2 are only T_0 we will propose an extension of Theorem 1.3 for T_1 quasi-metric spaces. To this end, the following result will play a crucial role.

Theorem 2.3. *Let (X, d) be a bicomplete T_1 quasi-metric space. Let $f : X \to X$ be a function such that there exist $k \in [0, 1[$ and $x_0 \in X$ such that*

$$d^s(f^n(x_0), f^{n+1}(x_0)) \leq kd(f^n(x_0), f^{n-1}(x_0)) \tag{2.2}$$

for all $n \in \mathbb{N}$. If f is continuous, then f has a fixed point.

Proof. By condition (2.2) we immediately obtain that

$$d^s(f^{n+1}(x_0), f^n(x_0)) \leq k^n d(f(x_0), x_0)$$

for all $n \in \mathbb{N}$. It follows that $(f^n(x_0))_{n\in\mathbb{N}}$ is a Cauchy sequence in (X, d^s). Indeed, let $m, n \in \mathbb{N}$. Of course we can assume that $m \geq n$. Then

$$\begin{aligned} d^s(f^m(x_0), f^n(x_0)) &\leq d(f^m(x_0), f^{m-1}(x_0)) + \cdots + d(f^{n+1}(x_0), f^n(x_0)) \\ &\leq \left(k^{m-1} + k^{m-2} \cdots + k^n\right) d(f(x_0), x_0) \\ &= \frac{k^n - k^m}{1-k} d(f(x_0), x_0) \\ &\leq \frac{k^n}{1-k} d(f(x_0), x_0). \end{aligned}$$

Since (X, d) is a bicomplete quasi-metric space there exists $y \in X$ such that $(f^n(x_0))_{n\in\mathbb{N}}$ is convergent to y in (X, d^s).

Next we prove that y is a fixed point of f. To this end, let $\epsilon > 0$. Then, by the continuity of f, there is $n_1 \in \mathbb{N}$ such that $d(f(y), f(f^n(x_0))) < \frac{\epsilon}{2}$ for all $n \geq n_1$. The fact that y is the limit of the sequence in (X, d^s) implies that there exists $n_2 \in \mathbb{N}$ such that $d^s(f^{n+1}(x_0), y) < \frac{\epsilon}{2}$ for all $n \geq n_2$. Hence we have that

$$\begin{aligned} d(f(y), y) &\leq d(f(y), f(f^n(x_0))) + d(f^{n+1}(x_0), y) \\ &\leq d(f(y), f(f^n(x_0))) + d^s(f^{n+1}(x_0), y) \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

for all $n \geq \max\{n_1, n_2\}$. So $d(f(y), y) = 0$. Since (X, d) is a T_1 quasi-metric space we conclude that $y = f(y)$. □

In the light of the preceding result we obtain the promised extension of Theorem 1.3.

Corollary 2.4. *Let $(X, d, \preceq)$ be a partially ordered bicomplete T_1 quasi-metric space. Let $f : X \to X$ be a monotone function such that there exist $k \in [0, 1[$ and $x_0 \in X$ with $x_0 \preceq f(x_0)$ and*

$$d^s(f(x), f(y)) \leq kd(x, y) \tag{2.3}$$

for all $x, y \in X$ such that $y \preceq x$. If f is continuous, then f has a fixed point.

Proof. It suffices to observe that the sequence $(f^n(x_0))_{n\in\mathbb{N}}$ is increasing in $(X, \preceq)$ and, in addition, it satisfies that

$$d^s(f^n(x_0), f^{n+1}(x_0)) \leq kd(f^n(x_0), f^{n-1}(x_0)) \tag{2.4}$$

for all $n \in \mathbb{N}$. Theorem 2.3 provides the conclusion. □

Observe that when the quasi-metric space in statement of Corollary 2.4 is exactly a metric space then Theorem 1.3 is obtained as a consequence and, in addition, it provides an extension of Theorem 1.4. It must be stressed that Theorem 2.3 and Corollary 2.4 give less information about the fixed point than Theorem 1.4. This is a consequence of the fact that a general partial order $\preceq$ is considered in the statement of the aforesaid results instead of the particular specialization partial order $\preceq_d$.

Example 4.3 shows that in the most general context the fixed point fails to belong to $Fix(f) \cap \uparrow_{\preceq} x_0$ and it is not least in $(Fix(f) \cap \uparrow_{\preceq} x_0, \preceq)$ in general, respectively (see Section 4). Moreover, Example 2.12 shows that, contrary to Theorem 1.4, the uniqueness is not guaranteed by our new theorem (see Section 2).

Notice that if a mapping f holds that

$$d^s(f(x), f(y)) \leq kd(x, y)$$

for all $y \preceq x$, then it holds also the inequality

$$d^s(f(x), f(y)) \leq kd^s(x, y) \tag{2.5}$$

for all $y \preceq x$. Hence it seems natural to wonder whether Theorem 1.4 could be retrieved as a particular case of a metric version of Theorem 2.3 or Corollary 2.4 in the spirit of Theorem 1.3. However, the answer to the posed question is negative because there exist continuous functions that are not s-continuous functions such as the next example shows.

Example 2.5. Let $X = [0, \infty)$ endowed with the usual partial order $\preceq$ and let d be a quasi-metric on X which is defined by the following

$$d(x, y) = \begin{cases} y - x & \text{if } x \preceq y \\ x & \text{otherwise} \end{cases}$$

for all $x, y \in X$. Define the mapping $f : X \to X$ by $f(x) = x+1$. The sequence $(\frac{1}{n})_{n\in\mathbb{N}}$ converges to 0 according to $\tau(d^s)$, but the sequence $(f(\dfrac{1}{n}))_{n\in\mathbb{N}}$ does not converge to 1 according to $\tau(d^{-1})$. Indeed, $d(1 + \dfrac{1}{n}, 1) = 1 + \dfrac{1}{n}$ and the limit of $(1 + \dfrac{1}{n})_{n\in\mathbb{N}}$ with respect to $\tau(d^{-1})$ is 1. So f is not continuous according to $\tau(d^s)$.

Notice that Example 2.2 shows that the T_1 condition of the quasi-metric space cannot be relaxed in order to guarantee the existence of fixed point.

In the following example we show that we cannot delete the bicompleteness in Theorem 2.3.

Example 2.6. Consider the T_1 quasi-metric space $(]0, 1], d_u)$, where we have denoted by d_u the restriction of the upper-quasi-metric introduced in Example 2.1 to the set $]0, 1]$. It is not hard to check that it is not bicomplete. Consider, in addition, the specilization partial order $\preceq_{d_u}$ on $[0, 1]$. Define the mapping $f :]0, 1] \to]0, 1]$ by $f(x) = \frac{x}{2}$ for all $x \in]0, 1]$. Clearly, the sequence $(f^n(1))_n \in \mathbb{N}$ satisfies that

$$d_u^s(f^n(1), f^{n+1}(1)) \leq \frac{1}{2} d_u(f^n(1), f^{n-1}(1))$$

for all $n \in \mathbb{N}$. Of course, f is continuous. However, f has no fixed point.

In the next example we show that the contractive condition (2.2) cannot be relaxed in the statement of Theorem 2.3.

Example 2.7. Consider the T_1 bicomplete quasi-metric space $([0,1], d_S)$, where d_S is defined on $[0,1]$ by

$$d_S(x,y) = \begin{cases} y - x & \text{if } y \succeq x \\ 1 & \text{if } y \prec x \end{cases}$$

Moreover, consider the partial order $\preceq_=$ given on $[0,1]$ by $x \preceq_= y \Leftrightarrow x = y$.

Define the mapping $f : [0,1] \to [0,1]$ by $f(x) = \dfrac{x+1}{2}$

It is routine to check that f is continuous. Furthermore, there do not exist $x \in [0,1]$ and $k \in [0,1[$ such that the sequence $(f^n(x))_n \in \mathbb{N}$ satisfies that

$$d_S^s(f^n(x), f^{n+1}(x)) \leq \frac{1}{2} d_S(f^n(x), f^{n-1}(x))$$

for all $n \in \mathbb{N}$. Clearly, f has no fixed point.

The continuity of the mapping in the statement of Theorem 2.3 can be replaced by the conjugate continuity as the following result proves.

Theorem 2.8. *Let (X,d) bicomplete T_1 quasi-metric space. Let $f : X \to X$ be a function and there exist $k \in [0,1[$, $x_0 \in X$ such that the sequence $(f^n(x_0))_{n\in\mathbb{N}}$ satisfies that*

$$d^s(f^n(x_0), f^{n+1}(x_0)) \leq kd(f^n(x_0), f^{n-1}(x_0)) \tag{2.6}$$

for all $n \in \mathbb{N}$. If f is conjugate continuous, then f has a fixed point.

Proof. The same arguments provided in the proof of Theorem 2.3 remain valid in order to prove that the sequence $(f^n(x_0))_{n\in\mathbb{N}}$ is Cauchy in (X, d^s) and, thus it converges to $y \in X$ with respect to $\tau(d^s)$.

It remains to prove that y is a fixed point of f. To this end, let $\epsilon > 0$. Then, by the continuity of f, there is $n_1 \in \mathbb{N}$ such that $d(f(f^n(x_0)), f(y)) < \frac{\epsilon}{2}$ for all $n \geq n_1$. The fact that y is the limit of the sequence in (X, d^s) implies that there exists $n_2 \in \mathbb{N}$ such that $d^s(f^{n+1}(x_0), y) < \frac{\epsilon}{2}$ for all $n \geq n_2$. Hence we have that

$$\begin{aligned} d(y, f(y)) &\leq d(y, (f^{n+1}(x_0))) + d(f(f^n(x_0)), f(y)) \\ &\leq d^s(y, f^{n+1}(x_0)) + d(f(f^n(x_0)), f(y)) \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

for all $n \geq \max\{n_1, n_2\}$. So $d(y, f(y)) = 0$. Since (X,d) is a T_1 quasi-metric space we conclude that $y = f(y)$. □

The following example shows that one of the both continuities is necessary in Theorems 2.3 and 2.8.

Example 2.9. Consider the quasi-metric space $([1,\infty[, d_2)$, where

$$d_2(x,y) = \begin{cases} y - x & \text{if } y \succeq x \\ 2(x-y) & \text{if } y \prec x \end{cases}$$

It is clear that

$$|y - x| \leq d_2(x,y) \leq 2|y - x|$$

for all $x, y \in [1, \infty[$. So the quasi-metric space $([1, \infty[, d_2)$ is bicomplete. Next consider, the partial order $\preceq_*$ on $[1, \infty[$ also introduced in Example 2.2. Let f be the mapping introduced also in Example 2.2. Then f is monotone with respect to $\preceq_*$ and, besides, $1 \preceq_* f(1) = \frac{3}{2}$. Moreover, the sequence $(f^n(1))_{n\in\mathbb{N}}$ satisfies that

$$d_2^s(f^n(1), f^{n+1}(1)) \leq \frac{1}{2} d_2(f^n(1), f^{n-1}(1),$$

since

$$d_2^s(f(x), f(y)) \leq \frac{1}{2} d_2(x, y)$$

for all $x, y \in [1, \infty[$ such that $y \preceq_* x$. Furthermore, f is neither continuous nor conjugate continuous. Indeed, let $(x_n)_{n\in\mathbb{N}}$ be the sequence in $[1, \infty[$ given by $x_n = 2 - \frac{1}{n}$ for all $n \in \mathbb{N}$. A straightforward computation shows that $(x_n)_{n\in\mathbb{N}}$ converges to 2 with respect to $\tau(d_2^s)$ and, thus, with respect to $\tau(d_2)$ and $\tau(d_2^{-1})$. Nonetheless, $f(x_n)_{n\in\mathbb{N}}$ does not converge to $f(2)$ neither with respect to $\tau(d_2)$ nor with respect to $\tau(d_2^{-1})$. Indeed,

$$d_2(f(2), f(x_n)) = 1 + \frac{1}{n}$$

and

$$d_2(f(x_n), f(2))) = \frac{1}{2} + \frac{1}{2n}$$

for all $n \in \mathbb{N}$. It is easy to check that f has no fixed point.

As a consequence of Theorem 2.8 we obtain;

Corollary 2.10. *Let $(X, d, \preceq)$ be a partially ordered bicomplete T_1 quasi-metric space. Let $f : X \to X$ be a monotone mapping such that there exist $k \in [0, 1[$ and $x_0 \in X$ with $x_0 \preceq f(x_0)$ and*

$$d^s(f(x), f(y)) \leq kd(x, y)$$

for all $x, y \in X$ such that $y \preceq x$. If f is conjugate continuous, then f has a fixed point.

It is worth to mention that versions of our results, Theorems 2.3 and 2.8, can be obtained simply adapting appropriately the contractive condition (2.2) by this one

$$d^s(f^n(x), f^{n+1}(x_0)) \leq kd(f^{n-1}(x_0), f^n(x_0))$$

for all $n \in \mathbb{N}$ or interchanging the condition "$x_0 \preceq f(x_0)$" by "$f(x_0) \preceq x_0$"in the statements of Corollaries 2.4 and 2.10.

In the metric case the contractive condition

$$d(f(x), f(y)) \leq kd(x, y)$$

for all $y \preceq x$, is equivalent to

$$d(f(x), f(y)) \leq kd(x, y)$$

for all $x, y \in X$ such that either $x \preceq y$ or $y \preceq x$. Of course the equivalence between the both contractive conditions is due to the symmetry of the distance function. The preceding fact inspires the natural question whether in the quasi-metric case the inequality

$$d^s(f(x), f(y)) \leq kd(x, y)$$

for all $y \preceq x$, is equivalent to the following one

$$d^s(f(x), f(y)) \leq kd(x, y)$$

for all $x, y \in X$ such that either $x \preceq y$ or $y \preceq x$. Clearly the second contractive condition implies the first one. However, the next example shows that they are not equivalent.

Example 2.11. Consider the partially ordered quasi-metric space $(\mathbb{R}^+, d_2, \preceq)$, where d_2 is defined as the quasi-metric introduced in Example 2.9. Define the mapping $f : \mathbb{R}^+ \to \mathbb{R}^+$ by $f(x) = \frac{x}{2}$ for all $x \in \mathbb{R}^+$. Then it is clear that

$$d_2^s(f(x), f(y)) \leq \frac{1}{2} d_2(x, y)$$

for all $y \preceq x$. But does not exist $k \in [0, 1[$ such that the mapping f holds for all $x \preceq y$ the inequality below

$$d_2^s(f(x), f(y)) \leq kd_2(x, y).$$

Indeed, if $x \preceq y$, then

$$y - x = d_2^s(f(x), f(y))) \leq kd_2(x, y) = k(y - x).$$

So $1 \preceq k$.

Uniqueness

In spite of the fact that Theorem 1.4 yields uniqueness of the fixed point, our next example shows that Theorem 2.3 and Corollary 2.4 do not assure such a uniqueness as it happens with Theorems 1.2 and 1.3.

Example 2.12. Consider the T_1 bicomplete quasi-metric space $(d_S, [0, 1])$ introduced in Example 2.7. It is clear that the mapping $f : [0, 1] \to [0, 1]$ defined by $f(x) = x$ for all $x \in [0, 1]$ is monotone with respect to $\preceq_=$ and, in addition, $x \preceq_= f(x)$ for all $x \in [0, 1]$. Thus $(f^n(x))_{n \in \mathbb{N}}$ is increasing in $([0, 1], \preceq_=)$ for all $x \in [0, 1]$. Besides, f is continuous with respect to $\tau(d_S)$. Furthermore,

$$0 = d_S^s(f^n(x_0), f^{n+1}(x_0)) \leq kd_S(f^n(x_0), f^{n-1}(x_0)) = 0$$

for all $n \in [0, 1[$ and for all $x_0 \in [0, 1]$ and, in addition,

$$0 = d_S^s(f(x), f(y)) \leq kd_S(x, y) = 0$$

for all $x, y \in [0, 1]$ such that $y \preceq_= x$. Finally, it is obvious that $[0, 1]$ matches up with the set of fixed points of f.

In the light of the preceding example we provide sufficient conditions that guarantee the uniqueness of fixed point. To this end, let us recall that a partially ordered set $(X, \preceq)$ is called upward (downward) directed set provided that for each $x, y \in X$ there exists $z \in X$ such that $x \preceq z$ ($z \preceq x$) and $y \preceq z$ ($z \preceq y$)(see, for instance, [8]).

Proposition 2.13. *Let $(X, d, \preceq)$ be a partially ordered quasi-metric space such that partially ordered set $(X, \preceq)$ is directed (upward or downward). Let $f : X \to X$ be a monotone mapping such that there exist $k \in [0, 1[$ and*

$$d^s(f(x), f(y)) \leq kd(x, y)$$

for all $x, y \in X$ such that $y \preceq x$. Assume that f has a fixed point $x^* \in X$. Then the following assertions hold:

1) *f has a unique fixed point.*
2) *If x^* is the limit of the sequence $(f^n(x_0))_{n\in\mathbb{N}}$ with respect to $\tau(d^s)$, then x^* is the limit of $(f^n(x))_{n\in\mathbb{N}}$ with respect to $\tau(d^s)$ for all $x \in X$.*

Proof. 1). We show the uniqueness. Assume with this aim that the partially ordered set $(X, \leq)$ is upward directed. Suppose that $y^* \in Fix(f)$ with $x^* \neq y^*$. Since $(X, \leq)$ is upward directed there exists $z \in X$ such that $x^* \leq z$ and $y^* \leq z$. Since f is monotone we have that $f^n(x^*) \leq f^n(z)$ and $f^n(y^*) \leq f^n(z)$. It follows, by the contractive condition, that

$$\begin{aligned} d^s(x^*, y^*) = d^s(f^n(x^*), f^n(y^*)) &\leq d^s(f^n(z), f^n(x^*)) + d^s(f^n(z), f^n(y^*)) \\ &\leq k^n\left[d(z, y^*) + d(z, x^*)\right] \end{aligned}$$

for all $n \in \mathbb{N}$. Whence we have that $d^s(x^*, y^*) = 0$. Thus we conclude that $x^* = y^*$.

2). We prove that $(f^n(x))_{n\in\mathbb{N}}$ converges to x^* with respect to $\tau(d^s)$ for all $x \in X$. To this end, assume that $(X, \leq)$ is upward directed and let $x \in X$. Then there exists $z \in X$ such that $x^* \leq z$ and $x \leq z$. Then $f^n(x^*) \leq f^n(z)$ and $f^n(x) \leq f^n(z)$ for all $n \in \mathbb{N}$. Hence we get that

$$d^s(f^n(x), x^*) \leq d^s(f^n(z), f^n(x)) + d^s(f^n(z), f^n(x^*)) \leq k^n\left[d(z, x^*) + d(z, x)\right].$$

Whence we deduce that $(f^n(x))_{n\in\mathbb{N}}$ converges to x^* with respect to $\tau(d^s)$.

Similar arguments can be applied to get the thesis of the result whenever the partially ordered set $(X, \leq)$ is downward directed. □

It must be stressed that Proposition 2.13 provides the information about the fixed point, uniqueness and "global attraction", that is also provided by the classical Banach fixed point theorem (Theorem 1.3) and Schellekens fixed point theorem (Theorem 2.3). Moreover, Proposition 2.13 is related to Theorem 3.1 provided in [3] for contractions in the context of relational metric spaces.

Example 2.12 shows that the directedness of the partially ordered set cannot be relaxed in order to guarantee the uniqueness of fixed point.

In the following example we show that the contractive condition in the statement of Proposition 2.13 cannot be exchanged by the contractive condition in the statement of Theorem 2.3 in order to obtain the uniqueness of the fixed point.

Example 2.14. Let $(X, d_S, \preceq)$ be the partially ordered bicomplete quasi-metric space such that $X = \{0, 1\}$, where d_S is the restriction of the quasi-metric introduced in Example 2.7 to X and $\preceq$ is the usual partial order $\preceq$ on X. It is clear that $(X, d_S, \preceq)$ is directed (upward and downward). Define the mapping $f : X \to X$ by $f(0) = 0$ and $f(1) = 1$. Of course f is monotone and it has 1 a fixed point. Moreover, it is not hard to check that

$$d_S^s(f^n(x_0), f^{n+1}(x_0)) \leq k d_S(f^n(x_0), f^{n-1}(x_0))$$

for all $x_0 \in X$. Of course, X matches up with the set of fixed points of f.

Next results can be obtained from Proposition 2.13.

Corollary 2.15. *Let $(X, d, \preceq)$ be a partially ordered bicomplete T_1 quasi-metric space such that partially ordered set $(X, \preceq)$ is directed upward. Let $f : X \to X$ be a monotone function such that there exist $k \in [0, 1[$ and $x_0 \in X$ with $x_0 \preceq f(x_0)$ and*

$$d^s(f(x), f(y)) \leq kd(x, y)$$

for all $x, y \in X$ such that $y \preceq x$. If f is either continuous or conjugate continuous, then f has a unique fixed point x^* and, in addition, the sequence $(f^n(x))_{n\in\mathbb{N}}$ converges to x^* with respect to $\tau(d^s)$ for all $x \in X$.

Proof. The existence of fixed point x^* is provided by Corollary 2.4 and Corollary 2.10. Moreover, the same results give that such a fixed point is provided as the limit of the sequence $(f^n(x_0))_{n\in\mathbb{N}}$. The uniqueness and the convergence condition follow from Proposition 2.13. □

It must be stressed that, in a similar way, a version of the preceding result can be obtained when we have $f(x_0) \preceq x_0$ and the partially ordered quasi-metric space $(X, d, \preceq)$ is downward directed.

3. Order boundness

As pointed out in Section 1, on account of [13], every quasi-metric space (X, d) becomes a partially ordered set endowed with the so-called specialization partial order $\preceq_d$, which is defined by $x \preceq_d y \Leftrightarrow d(x, y) = 0$. Those quasi-metric spaces endowed with the specialization partial order enjoy the following outstanding property, whose proof can be found in [20, Proposition 2.1].

Proposition 3.1 ([20])**.** *Let (X, d) be a quasi-metric space and let $(x_n)_{n\in\mathbb{N}}$ be an increasing sequence in $(X, \preceq_d)$. If $(x_n)_{n\in\mathbb{N}}$ converges to $x \in X$ with respect to $\tau(d^{-1})$, then x is an upper bound of $(x_n)_{n\in\mathbb{N}}$. Moreover, if x is an upper bound of $(x_n)_{n\in\mathbb{N}}$ and $(x_n)_{n\in\mathbb{N}}$ converges to x with respect to $\tau(d)$, then x is the least upper bound of $(x_n)_{n\in\mathbb{N}}$.*

The preceding result inspires the following notion that we have named $\preceq$-bounded. Thus we will say that a sequence $(x_n)_{n\in\mathbb{N}}$ in a partially ordered quasi-metric space $(X, d, \preceq)$ is upward $\preceq$-bounded whenever it is increasing in $(X, \preceq)$ and the fact that it converges to $x \in X$ with respect to $\tau(d^{-1})$ implies that x is an upper bound of $(x_n)_{n\in\mathbb{N}}$ in $(X, \preceq)$. Based on the new notion we give new results which allow us to relax the T_1 condition and the continuity of the mapping in the statements of Theorems 2.3 and 2.8.

Theorem 3.2. *Let $(X, d, \preceq)$ be a partially ordered bicomplete quasi-metric space. Let $f : X \to X$ be a function such that there exist $k \in [0, 1[$ and $x_0 \in X$ such that $(f^n(x_0))_{n\in\mathbb{N}}$ is an upward $\preceq$-bounded sequence in $(X, \preceq)$ and*

$$d^s(f(z), f^{n+1}(x_0)) \leq kd(z, f^n(x_0)) \tag{3.1}$$

for all $z \in X$ with $f^n(x_0) \preceq z$ for any $n \in \mathbb{N}$. Then f has a fixed point y which is an upper bound of $(f^n(x_0))_{n\in\mathbb{N}}$ in $(X, \preceq)$ and, thus, $y \in \uparrow_{\preceq} f(x_0)$.

Proof. Since $(f^n(x_0))_{n\in\mathbb{N}}$ is an upward $\preceq$-bounded sequence we have that it is increasing and, thus, similar reasoning to that used in the proof of Theorem 2.3 allows us to show that the sequence $(f^n(x_0))_{n\in\mathbb{N}}$ is a Cauchy sequence in (X, d^s). Since (X, d) is a bicomplete quasi-metric space there exists $y \in X$ such that $(f^n(x_0))_{n\in\mathbb{N}}$ is convergent to y in (X, d^s). So it is convergent to y with respect to $\tau(d^{-1})$. The fact that the sequence $(f^n(x_0))_{k\in\mathbb{N}}$ is upward $\preceq$-bounded yields that y is an upper bound of $(f^n(x_0))_{n\in\mathbb{N}}$ in $(X, \preceq)$ and, thus, $f^n(x_0) \preceq y$ for all $n \in \mathbb{N}$, i.e., $y \in\uparrow_{\preceq} f(x_0)$. Moreover, by contractive condition (3.1), we get that $d(f(y), f(f^n(x_0))) \leq d(y, f^n(x_0))$ for all $n \in \mathbb{N}$.

Next we prove that $d^s(f(y), y) = 0$ and, thus, that $f(y) = y$. To this end, consider $\epsilon > 0$. Then there exists $n_0 \in \mathbb{N}$ such that $d^s(y, f^{n_k}(x_0)) < \frac{\epsilon}{2}$ for all $n_k \geq n_0$. Moreover we have that $d^s(f(y), f(f^n(x_0))) \leq kd(y, f^n(x_0))$ for all $n \geq n_0$, since y is an upper bound of $(f^n(x_0))_{n\in\mathbb{N}}$. Whence we deduce, by contractive condition (3.1), that

$$\begin{aligned} d^s(f(y), y) &\leq d^s(f(y), f(f^n(x_0))) + d^s(f^{n+1}(x_0), y) \\ &\leq kd(y, f^n(x_0)) + d^s(f^{n+1}(x_0), y) \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \end{aligned}$$

for all $n \geq n_0$. Hence $d^s(f(y), y) = 0$ and $f(y) = y$. □

Notice that Example 2.9 shows that in the statement of Theorem 3.2 we cannot omit the order-boundness for increasing sequences.

When the partial order in the statement of Theorem 3.2 is exactly the specialization order we retrieve as a particular case the results below. Observe that such results extend Theorem 1.4 and refine Theorem 3.2.

Corollary 3.3. *Let (X, d) be a bicomplete quasi-metric space. Let $f : X \to X$ be a function such that there exist $k \in [0, 1[$ and $x_0 \in X$ such that $(f^n(x_0))_{n\in\mathbb{N}}$ is increasing in $(X, \preceq_d)$ and*

$$d^s(f(y), f^{n+1}(x_0)) \leq kd(y, f^n(x_0)) \tag{3.2}$$

for all $z \in X$ with $f^n(x_0) \preceq_d z$ for any $n \in \mathbb{N}$. Then f has a fixed point y which is the least upper bound of $(f^n(x_0))_{n\in\mathbb{N}}$ in $(X, \preceq_d)$ and, thus, $y \in\uparrow_{\preceq_d} f(x_0)$.

Proof. The existence of $y \in X$ which is a fixed point of f and, in addition, y is an upper bound of $(f^n(x_0))_{n\in\mathbb{N}}$ in $(X, \preceq_d)$ is guaranteed by Theorem 3.2 and Proposition 3.1. Notice that the proof of Theorem 3.2 gives that $(f^n(x_0))_{n\in\mathbb{N}}$ is convergent to y in (X, d^s). Hence $(f^n(x_0))_{n\in\mathbb{N}}$ is convergent to y with respect to $\tau(d)$. Next we prove that y is the least upper bound of $(f^n(x_0))_{n\in\mathbb{N}}$. Indeed, assume that there exists $z \in X$ which is an upper bound of $(f^n(x_0))_{n\in\mathbb{N}}$ and $z \preceq_d y$. Then $d(f^n(x_0), z)$ for all $n \in \mathbb{N}$ and $d(z, y) = 0$. Moreover, for all $n \in \mathbb{N}$, we have that

$$d(y, z) \leq d(y, f^n(x_0)) + d(f^n(x_0), z) = d(y, f^n(x_0)).$$

It follows that $d(y, z) = 0$, since $(f^n(x_0))_{n\in\mathbb{N}}$ is convergent to y with respect to $\tau(d)$. We conclude that y is the least upper bound of $(f^n(x_0))_{n\in\mathbb{N}}$ in $(X, \preceq_d)$. □

In the light of the preceding corollary we obtain the following results whose proof we omit.

Corollary 3.4. *Let* (X, d) *be a bicomplete quasi-metric space. Let* $f : X \to X$ *be a monotone function such that there exist* $k \in [0, 1[$ *and* $x_0 \in X$ *such that* $(f^n(x_0))_{n\in\mathbb{N}}$ *is increasing in* $(X, \preceq_d)$ *and*

$$d(f(y), f^{n+1}(x_0)) \leq kd(y, f^n(x_0)) \tag{3.3}$$

for all $z \in X$ *with* $f^n(x_0) \preceq_d z$ *for any* $n \in \mathbb{N}$. *Then* f *has a fixed point* y *which is the least upper bound of* $(f^n(x_0))_{n\in\mathbb{N}}$ *in* $(X, \preceq_d)$ *and, thus,* $y \in \uparrow_{\preceq_d} f(x_0)$.

Corollary 3.5. *Let* (X, d) *be a bicomplete quasi-metric space. Let* $f : X \to X$ *be a monotone function with respect to* $\preceq_d$ *such that there exist* $k \in [0, 1[$ *and* $x_0 \in X$ *with* $x_0 \preceq_d f(x_0)$ *and*

$$d(f(x), f(y)) \leq kd(x, y) \tag{3.4}$$

for all $x, y \in X$ *such that* $y \preceq_d x$. *Then* f *has a fixed point* y *which is the least upper bound of* $(f^n(x_0))_{n\in\mathbb{N}}$ *in* $(X, \preceq_d)$ *and, thus,* $y \in \uparrow_{\preceq_d} x_0$.

In the following, a partially ordered quasi-metric space $(X, d, \preceq)$ will be called upward $\preceq$-bounded provided that every increasing sequence is upward $\preceq$-bounded.

Of course, by Proposition 3.1, every bicomplete quasi-metric space (X, d) is an instance of upward $\preceq_d$-bounded quasi-metric space. However, every partially ordered bicomplete quasi-metric space $(X, d, \preceq)$ is not always upward $\preceq$-bounded such as the next example shows.

Example 3.6. Consider the bicomplete quasi-metric space $(\mathbb{R}, d_l)$, where d_l is the quasi-metric defined by $d_l(x, y) = \max\{x - y, 0\}$ for all $x, y \in \mathbb{R}$. Endow $\mathbb{R}$ with the partial order $\preceq$ defined by $x \preceq y \Leftrightarrow y \leq x$. Clearly $d_l^{-1} = d_u$. Take the sequence $(x_n)_n \in \mathbb{N}$ with $x_n = -n$ for all $n \in \mathbb{N}$. Then it is clear that such a sequence is increasing in $(\mathbb{R}, \preceq)$ and convergent to 0 with respect to $\tau(d_u)$. However, 0 is not an upper bound of $(x_n)_n \in \mathbb{N}$ in $(\mathbb{R}, \preceq)$.

The next results are obtained as a particular case of Theorem 3.2 when the partially ordered quasi-metric space $(X, d, \preceq)$ is upward $\preceq$-bounded.

Corollary 3.7. *Let* $(X, d, \preceq)$ *be an upward* $\preceq$*-bounded bicomplete quasi-metric space. Let* $f : X \to X$ *be a function such that there exist* $k \in [0, 1[$ *and* $x_0 \in X$ *such that* $(f^n(x_0))_{n\in\mathbb{N}}$ *is increasing and*

$$d^s(f(y), f^{n+1}(x_0)) \leq kd(y, f^n(x_0)) \tag{3.5}$$

for all $z \in X$ *with* $f^n(x_0) \preceq z$ *for any* $n \in \mathbb{N}$.*Then* f *has a fixed point* y *which is an upper bound of* $(f^n(x_0))_{n\in\mathbb{N}}$ *in* $(X, \preceq)$ *and, thus,* $y \in \uparrow_{\preceq} f(x_0)$.

Corollary 3.8. *Let* $(X, d, \preceq)$ *be an upward* $\preceq$*-bounded bicomplete quasi-metric space. Let* $f : X \to X$ *be a monotone function such that there exist* $k \in [0, 1[$ *and* $x_0 \in X$ *such that* $x_0 \preceq f(x_0)$ *and*

$$d^s(f(y), f^{n+1}(x_0)) \leq kd(y, f^n(x_0)) \tag{3.6}$$

for all $z \in X$ *with* $f^n(x_0) \preceq z$ *for any* $n \in \mathbb{N}$.*Then* f *has a fixed point* y *which is an upper bound of* $(f^n(x_0))_{n\in\mathbb{N}}$ *in* $(X, \preceq)$ *and, thus,* $y \in \uparrow_{\preceq} x_0$.

It is worth to mention that versions of the exposed results adapting appropriately the notion of upward $\preceq$-boundedness to the case of decreasing sequences (downward $\preceq$-boundness), interchanging the contractive condition (3.1) for this one

$$d(f(y), f^{n+1}(x_0)) \leq kd(f^n(x_0), y) \tag{3.7}$$

for all $y \in X$ with $y \preceq_d f^n(x_0)$ and for any $n \in \mathbb{N}$, and considering the condition "$f(x_0) \preceq x_0$".

4. Kleene's fixed point theorem: The relationship

It seems natural to wonder if our main results (Theorems 2.3, 2.8 and 3.2) can be derived from the celebrated fixed point theorem of Kleene (Theorem 1.2).

However, this question has a negative answer, even when the specialization order is under consideration, such as Example 4.1 and 4.2 show.

Example 4.1. Let $([0, 1[, d_S)$ be the bicomplete quasi-metric space where the quasi-metric d_S has been introduced in Example 2.7. Consider the usual partial order $\preceq$ on $[0, 1[$. Moreover, consider the sequence $(x_n)_{n\in\mathbb{N}}$ in $[0, 1[$ given by $x_n = 1 - \frac{1}{2n}$ for all $n \in \mathbb{N}$. It is clear that sequence $(x_n)_{n\in\mathbb{N}}$ is increasing in $([0, 1[, \preceq)$. However, it is obvious that it has not least upper bound in $([0, 1[, \preceq)$. Therefore $([0, 1[, \preceq)$ is not chain-complete.

Example 4.2. Consider the bicomplete quasi-metric space $([0, \infty[, d_l)$, where the quasi-metric d_l is defined on $[0, \infty[$ by $d_l(x, y) = \max\{x - y, 0\}$. Then it is clear that $x \preceq_{d_l} y \Leftrightarrow x \preceq y$, where $\preceq$ denotes the usual partial order on $[0, \infty[$. Consider the sequence $(x_n)_{n\in\mathbb{N}}$ in $[0, \infty[$ given by $x_n = n$ for all $n \in \mathbb{N}$. Obviously $(x_n)_{n\in\mathbb{N}}$ is increasing in $([0, \infty[, \preceq_{d_l})$ but it has not least upper bound. Consequently, $([0, \infty[, \preceq_{d_l}$ $)$ is not chain-complete.

A natural question, that Theorems 2.3, 2.8 and 3.2 do not answer, is whether the fixed point guaranteed by the aforementioned results is least in $(Fix(f) \cap \uparrow_{\preceq} x_0, \preceq)$. In fact we show that the aforementioned results do not assure that the fixed point belongs to $\uparrow_{\preceq} x_0$ even if the partial order $\preceq$ is exactly $\preceq_d$.

Example 4.3. Let $(X, d_S, \preceq_=)$ be the partially ordered T_1 bicomplete quasi-metric space such that $X = \{0, 1\}$, d_S is the quasi-metric introduced in Example 2.14 and $\preceq_=$ the partial order given by $x \preceq_= y \Leftrightarrow x = y$. Define the mapping $f : X \to X$ by $f(1) = 1$ and $f(0) = 1$. Of course, f is continuous and conjugate continuous. Moreover, $(f^n(0))_{n\in\mathbb{N}}$ is increasing in $(X, \preceq_=)$ and

$$d_S^s(f^n(0), f^{n+1}(0)) \leq kd_S(f^n(0), f^{n-1}(0))$$

holds for all $n \in \mathbb{N}$ with $n \in \mathbb{N}$. Clearly 1 is the fixed point provided by Theorems 2.3 and 2.8. Observe, in addition, that $(X, d_S, \preceq_=)$ is upward $\preceq_=$-bounded and that

$$d^s(f(1), f^{n+1}(0)) \leq kd(1, f^n(0))$$

for all $n \in \mathbb{N}$. Theorem 3.2 gives 1 as fixed point of f. However $1 \notin \uparrow_{\preceq_=} 0$.

In the light of the preceding example we provide conditions that warranty that the fixed point is least in $(Fix(f) \cap \uparrow_{\preceq} x_0, \preceq)$ when the specialization partial order is considered.

Proposition 4.4. *Let (X, d) be a bicomplete quasi-metric space. Let $f : X \to X$ be a monotone function such that there exist $k \in [0, 1[$ and $x_0 \in X$ such that the sequence $(f^n(x_0))_{n \in \mathbb{N}}$ is increasing in $(X, \preceq_d)$ and*

$$d(f(y), f^{n+1}(x_0)) \leq kd(y, f^n(x_0)) \tag{4.1}$$

for all $y \in X$ with $f^n(x_0) \preceq_d y$ for any $n \in \mathbb{N}$. Then f has a fixed point which is least in $(Fix(f) \cap \uparrow_{\preceq_d} f(x_0), \preceq_d)$. Moreover, if $x_0 \preceq f(x_0)$ then the fixed point is least in $(Fix(f) \cap \uparrow_{\preceq_d} x_0, \preceq_d)$.

Proof. Corollary 3.4 gives that f has a fixed point y which is the least upper bound of $(f^n(x_0))_{n \in \mathbb{N}}$ in $(X, \leq_d)$ and, thus, that $y \in \uparrow_{\leq_d} f(x_0)$. Moreover, the fixed point y is the limit of the sequence $(f^n(x_0))_{n \in \mathbb{N}}$ with respect to $\tau(d^s)$. Since $(f^n(x_0))_{n \in \mathbb{N}}$ is an increasing sequence and it is convergent with respect to $\tau(d^{-1})$ we have that y is an upper bound of $(f^n(x_0))_{n \in \mathbb{N}}$. Thus $y \in (Fix(f) \cap \uparrow_{\leq_d} f(x_0))$. Now we prove that y is least in $(Fix(f) \cap \uparrow_{\leq_d} f(x_0), \leq_d)$. Suppose that z is another fixed point such that $f(x_0) \leq_d z$. Since f is monotone and $z \in Fix(f)$ we obtain that $f^n(x_0) \leq_d z$ for all $n \in \mathbb{N}$ and, thus, that $d(f^n(x_0), z) = 0$ for all $n \in \mathbb{N}$. Now let $\epsilon > 0$. Since $(f^n(x_0))_{n \in \mathbb{N}}$ converges to y with respect to $\tau(d^s)$ there is $n_0 \in \mathbb{N}$ such that $d(y, f^n(x_0)) < \epsilon$ for all $n \geq n_0$. Hence we have that

$$d(y, z) \leq d(y, f^n(x_0)) + d(f^n(x_0), z) < \epsilon$$

for all $n \geq n_0$. So $d(y, z) = 0$. Therefore $y \leq_d z$ and y is least in $(Fix(f) \cap \uparrow_{\leq_d} f(x_0), \leq_d)$. Notice that the fact that $y \in \uparrow_{\leq_d} x_0$ when $y \in \uparrow_{\leq_d} f(x_0)$ and $x_0 \leq_d f(x_0)$ provides that a simple adaptation of the proof allows us to show that y is least in $(Fix(f) \cap \uparrow_{\leq_d} x_0, \leq_d)$. □

As pointed out earlier, an appropriate version of the previous result can be stated when downward $\preceq_d$-boundedness and contractive condition (3.7) are under consideration.

5. An application to asymptotic analysis of recurrence equations

According to [7], recurrence equations appear in a natural way in complexity analysis of algorithms. Let us recall that such an analysis is based on determining the quantity of resources needed by the algorithm in order to solve the problem for which it has been designed. Typical resources, playing a central role in complexity analysis, are running time of computing and the required memory space. Usually in order to represent mathematically the resources consumed by an algorithm A a function $f_A : \mathbb{N} \to (0, \infty]$ is associated to A in such a way that $f(n)$ matches up with the quantity of resource (the complexity) taken by the algorithm to solve the problem when the input data is of size n.

In general to get an exact expression of the function f_A is unnecessary and it is enough to provide asymptotic upper and lower bounds for f_A. With this aim, the following asymptotic formalism is required.

Given $g : \mathbb{N} \to (0, \infty]$, the statement $f_A \in \mathcal{O}(g)$, means that there exist $n_0 \in \mathbb{N}$ and $c \in \mathbb{R}^+$ such that $f_A(n) \preceq cg(n)$ for all $n \in \mathbb{N}$ with $n_0 \preceq n$. Hence g allows to give an asymptotic upper bound of f_A. Similarly, $f_A \in \Omega(g)$ means that there exists $n_0 \in \mathbb{N}$ and $c \in \mathbb{R}^+$ such that $cg(n) \preceq f(n)$ for all $n_0 \preceq n$ and, hence, that the function g gives an asymptotic lower bound of f. If $f_A \in \mathcal{O}(g) \cap \Omega(g)$, then $f_A \in \Theta(g)$ and g yields a tight asymptotic bound of f_A. In all cases, note that through g we provide an "approximate" information about f_A.

In many cases, the mapping f_A that gives the amount of resource taken by the algorithm A to solve the problem can be obtained as the solution to a recurrence equation. In 1995, Schellekens [19] developed a a fixed point technique in order to contribute to the topological foundation of asymptotic complexity analysis of algorithms based on quasi-metric spaces and the so-called complexity space. Let us recall, that the complexity spaces is exactly the quasi-metric space $(\mathcal{C}, d_{\mathcal{C}})$, where

$$\mathcal{C} = \{f : \mathbb{N} \to \mathbb{R}^+ : \sum_{n=1}^{\infty} 2^{-n} f(n) < \infty\} \tag{5.1}$$

and $d_{\mathcal{C}}$ is defined by

$$d_{\mathcal{C}}(f, g) = \sum_{n=1}^{\infty} 2^{-n} \max\{\frac{1}{g(n)} - \frac{1}{f(n)}, 0\} \tag{5.2}$$

On account of [19], the most important clases, computability point of view, of algorithms satisfy the "convergence condition" $\sum_{n=1}^{\infty} 2^{-n} f(n) < \infty$.

Clearly the set $\mathcal{C}$ becomes a partially ordered set when we endow it with the partial order $\preceq_{\mathcal{C}}$ given by $f \preceq_{\mathcal{C}} g \iff f(n) \preceq g(n)$ for all $n \in \mathbb{N}$. Consider any functions $f, g \in \mathcal{C}$. Then, it is clear that there exist $f(n) \vee g(n)$ for all $n \in \mathbb{N}$ and, thus, we have that $f \preceq_{\mathcal{C}} f \vee g$ and $g \preceq_{\mathcal{C}} f \vee g$. Therefore, the partially ordered set $(\mathcal{C}, \preceq_{\mathcal{C}})$ is directed upward.

Observe that, $d_{\mathcal{C}}(f, g) = 0 \iff f(n) \preceq g(n)$ for all $n \in \mathbb{N}$ and thus $\preceq_{d_{\mathcal{C}}} = \preceq_{\mathcal{C}}$. Hence the value $d_{\mathcal{C}}(f, g) = 0$ can be interpreted as there is an improvement in complexity when the algorithm whose complexity is represented by the function g is replaced by the algorithm whose complexity is represented by the function f.

Shcellekens [19], showed that Theorem 1.4 can be applied to analyze the asymptotic behavior of those algorithms whose resources are represented by a function which satisfies a recurrence equation as follows:

$$T(n) = \begin{cases} c & \text{if } n = 1 \\ aT(n-1) + h(n) & \text{if } n \geq 2 \end{cases}, \tag{5.3}$$

where $c > 0$, $a > 1$ and $h \in \mathcal{C}$ such that $h(n) < \infty$ for all $n \in \mathbb{N}$.

Notice that the original class of algorithms analyzed by Schellekens is the Divide and Conquer one which can satisfies a recurrence equation that can be retrieved as a particular case from the preceding one.

In order to show the applicability of Theorem 1.4 Schellekens proved (see also [18, 17]) that $(\mathcal{C}_c, d_{\mathcal{C}})$ is a bicomplete quasi-metric space, where

$$\mathcal{C}_c = \{f \in \mathcal{C} : f(1) = c\},$$

and introduced the monotone function $\Psi : \mathcal{C}_c \to \mathcal{C}_c$ defined by

$$\Psi_T(f)(n) = \begin{cases} c & \text{if } n = 1 \\ af(n-1) + h(n) & \text{if } n \geq 2 \end{cases} \tag{5.4}$$

for all $f \in \mathcal{C}_c$. It is clear that $f_T \in \mathcal{C}_c$ is a solution to (5.3) if and only if it is a fixed point of Ψ_T. Moreover, he showed that Ψ_T satisfies

$$d_{\mathcal{C}}(\Psi_T(f), \Psi_T(g)) \leq \frac{1}{2a} d_{\mathcal{C}}(f, g) \tag{5.5}$$

for all $f, g \in \mathcal{C}_c$. Finally, he showed that $f_T \in \mathcal{O}(g)$ and $f_T \in \Omega(f)$ making use of the conditions $\Psi_T(g) \leq_{\mathcal{C}} g$ and $f \leq_{\mathcal{C}} \Psi_T(f)$, respectively.

A few more clases of recurrence equations have been analyzed in [18, 17].

In the light of the exposed facts, our aim in the remainder of the section is, on the one hand, to show that the above specific technique can be formalized by means of our developed theory and, on the other hand, to extent such a technique in order to discuss the asymptotic behaviour of solutions to recurrence equations not considered before in [19, 18, 17]. To this end, we introduce the next result.

Proposition 5.1. *Let (X, d) be a bicomplete quasi-metric space such that $(X, \preceq_d)$ is upward directed. Let $f : X \to X$ be a monotone function with respect to $\preceq_d$ such that there exist $k \in [0, 1[$ with*

$$d(f(x), f(y)) \leq kd(x, y) \tag{5.6}$$

for all $x, y \in X$ such that $y \preceq_d x$. If there exists $x_0, y_0 \in X$ such that $x_0 \preceq_d f(x_0)$ and $f(y_0) \preceq_d y_0$, then f has a unique fixed point y with $y \in \uparrow_{\preceq_d} x_0 \cap \downarrow_{\preceq_d} y_0$.

Proof. Corollary 3.5 gives that f has a fixed point y such that $y \in \uparrow_{\leq_d} x_0$. Besides the same corollary yields that $(f^n(x_0))_{n\in\mathbb{N}}$ converges to y with respect to $\tau(d^s)$. It follows, by Proposition 2.13, that y is the unique fixed point of f and, in addition that y is the limit of $(f^n(x))_{n\in\mathbb{N}}$ with respect to $\tau(d^s)$ for all $x \in X$. It follows that, given $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $d(y, f^n(y_0)) \leq d^s(y, f^n(y_0)) < \epsilon$ for all $n \geq n_0$. Next we show that $y \in \downarrow_{\leq_d} y_0$. Indeed, the monotony of f provides that $f^n(y_0) \leq_d y_0$ and, hence, that $d(f^n(y_0), y_0) = 0$ for all $n \in \mathbb{N}$. Therefore

$$d(y, y_0) \leq d(y, f^n(y_0)) + d(f^n(y_0), y_0) < \epsilon$$

for all $n \geq n_0$. Whence we conclude that $y \in \downarrow_{\leq_d} y_0$. So $y \in \uparrow_{\leq_d} x_0 \cap \downarrow_{\leq_d} y_0$. □

Observe that the function Ψ_T satisfies contractive condition (5.5) and, thus, it fulfills the contractive condition in the statement of the preceding proposition, i.e.,

$$d_{\mathcal{C}}(\Psi_T(f), \Psi_T(g)) \leq \frac{1}{2a} d_{\mathcal{C}}(f, g)$$

for all $g \preceq_{\mathcal{C}} f$. So Schellekens technique can be retrieved as a particular case of the general technique introduced in Proposition 5.1. However, it must be pointed out that the aforementioned proposition only requires to check the contractive condition for those functions order related and not for all functions in $\mathcal{C}_c$, which is an improvement.

Next we consider the more general recurrence equation given by

$$T(n) = \begin{cases} c & \text{if } n = 1 \\ a(n)T(b(n)) + h(n) & \text{if } n \geq 2 \end{cases} \tag{5.7}$$

where $c \in \mathbb{R}^+$, $h \in \mathcal{C}$, $a : \mathbb{N} \to]0, \infty[$ with $a(n) > 1$ for all $n \in \mathbb{N}$ and $b : \mathbb{N} \to \mathbb{N}$ with $b(n) < n$ and $b(n) \preceq b(m)$ for all $n, m \in \mathbb{N}$ with $n \preceq m$.

Define the functional $\Psi : \mathcal{C}_c \to \mathcal{C}_c$ by

$$\Psi_T(f)(n) = \begin{cases} c & \text{if } n = 1 \\ a(n)f(b(n)) + h(n) & \text{if } n \geq 2 \end{cases} \tag{5.8}$$

for all $f \in \mathcal{C}_c$. It is clear that $f_T \in \mathcal{C}_c$ is a solution to (5.8) if and only if it is a fixed point of Ψ_T.

Obviously Ψ_T is monotone with respect to $\preceq_{d_\mathcal{C}}$ and

$$\begin{aligned} d_\mathcal{C}(\Psi_T(f), \Psi_T(g)) &= \sum_{n=1}^{\infty} 2^{-n} \max\left(\frac{1}{\Psi_T(g)(n)} - \frac{1}{\Psi_T(f)(n)}, 0 \right) \\ &= \sum_{n=1}^{\infty} 2^{-n} \left(\frac{1}{\Psi_T(g)(n)} - \frac{1}{\Psi_T(f)(n)} \right) \\ &= \sum_{n=1}^{\infty} 2^{-n} \left(\frac{a(n)f(b(n)) - a(n)g(b(n))}{(a(n)g(b(n)) + h(n))(a(n)f(b(n)) + h(n))} \right) \\ &\preceq \sum_{n=1}^{\infty} 2^{-n} \left(\frac{a(n)f(b(n)) - a(n)g(b(n))}{a^2(n)g(b(n))f(b(n))} \right) \\ &= \sum_{n=1}^{\infty} 2^{-n} \frac{1}{a(n)} \left(\frac{1}{g(b(n))} - \frac{1}{f(b(n))} \right) \\ &\preceq \frac{1}{2} \sum_{n=1}^{\infty} 2^{-n} \left(\frac{1}{g(b(n))} - \frac{1}{f(b(n))} \right) \\ &\preceq \frac{1}{2} d_\mathcal{C}(f, g) \end{aligned}$$

for all $f, g \in \mathcal{C}$ such that $g \preceq_{d_\mathcal{C}} f$. The existence and uniquness of the fixed point $f_T \in \mathcal{C}$ of Ψ_T follow from that $a(n) > 1$ for all $n \in \mathbb{N}$ and Corrolary 14 and Corrolary 40.

The next result, which can be derived immediately from Proposition 5.1, provides the technique which is able to yield the asymptotic bounds of a solution to recurrence equation (5.7).

Theorem 5.2. *If there exists $f, g \in \mathcal{C}_c$ such that $f \preceq_{d_\mathcal{C}} \Psi_T(f)$ and $\Psi_T(g) \preceq_{d_\mathcal{C}} g$, then the unique solution f_T to recurrence equation (5.7) satisfies $f_T \in \Omega(f) \cap \mathcal{O}(g)$.*

Finally, we illustrate the method given by the previous theorem. To this end, let consider the following recurrence equation

$$T(n) = \begin{cases} c & \text{if } n = 1 \\ an^2 + nT(n-1) & \text{if } n \geq 2 \end{cases}. \tag{5.9}$$

This recurrence equation can be recovered from (5.7) taking $a(n) = n$, $b(n) = n-1$ for all $n \in \mathbb{N}$ with $n \geq 2$ and $h(n) = an^2$ for all $n \in \mathbb{N}$.

Now let us consider the function $g_r \in \mathcal{C}_c$ given by

$$g_r(n) = \begin{cases} c & \text{if } n = 1 \\ rn! & \text{if } n \geq 2 \end{cases} \tag{5.10}$$

and take the functional Ψ_T given by

$$\Psi_T(f)(n) = \begin{cases} c & \text{if } n = 1 \\ an^2 + nf(n-1) & \text{if } n \geq 2 \end{cases} \tag{5.11}$$

for all $f \in \mathcal{C}_c$. Then, it is not hard to check that $g_r \preceq_{d_\mathcal{C}} \Psi_T(g_r)$ if and only if $r \preceq 2a + c$. Moreover, consider the function $g'_r \in \mathcal{C}_c$ given by

$$g'_r(n) = \begin{cases} c & \text{if } n = 1 \\ rn! - \dfrac{nr}{2} & \text{if } n \geq 2 \end{cases}. \tag{5.12}$$

Then, it is not hard to check that $\Psi_T(g'_r) \preceq_{d_\mathcal{C}} g'_r$ if and only if $r \geq \max\{4a + 2c, 6a\}$,

Therefore, by Theorem 5.2, we deduce that recurrence equation (5.7) has a unique solution $f_T \in \mathcal{C}_c$ such that $f_T \in \Omega(g_{2a+c}) \cap \mathcal{O}(g'_{\max\{4a+2c,6a\}})$. Consequently, $f_T \in \Theta(n!)$, which agrees with what is stipulated in the literature ([7]).

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