Fixed Point Theory, 25(2024), No. 2, 459-472
DOI: 10.24193/fpt-ro.2024.2.01
http://www.math.ubbcluj.ro/~nodeacj/sfptcj.html

A NEW METHOD FOR THE CONSTRUCTION OF FRACTALS VIA BEST PROXIMITY POINT THEORY

MUSTAFA ASLANTAS*, HAKAN SAHIN** AND ISHAK ALTUN***

*Department of Mathematics, Faculty of Science,
Çankırı Karatekin University, 18100 Çankırı, Turkey
E-mail: maslantas@karatekin.edu.tr

**Department of Mathematics, Faculty of Engineering and Natural Sciences,
Bursa Technical University, 16310 Bursa, Turkey
E-mail: hakan.sahin@btu.edu.tr

***Department of Mathematics, Faculty of Engineering and Natural Sciences,
Kırıkkale University, 71450 Yahsihan, Kırıkkale, Turkey
E-mail: ishakaltun@yahoo.com

Abstract. In this paper, taking into account the P-property in the best proximity point theory, we present a new and interesting construction method that is different from the method given in [3] for fractals. First, we introduce the concept of a generalized iterated function system (in short GIFS) constructed by a finite family of λ-contractions. Then, we present our main theorem in which sufficient conditions are determined to obtain a fractal which is also an attractor of the mentioned system. Finally, we support our results with some illustrative and attractive examples.
Key Words and Phrases: Fractals, best proximity point, P-property, iterated function systems.
2020 Mathematics Subject Classification: 28A80, 54H25, 47H10.

1. Introduction and preliminaries

Geometric modeling of many irregular patterns in nature is a difficult process in computer graphics. A significant class of these patterns emerges from physical phenomena such as plants, clouds, trees. Since these patterns possess infinitely non-smooth, highly structured geometries, standard geometry is useless to model these objects. The concept of the fractal, which has enormous potential to model these objects, was introduced by Mandelbrot [11]. One of the famous examples of fractals is the Cantor set. To construct a classical Cantor set, let's start with the segment $I_0 = [0, 1]$ and remove an open interval of length $\frac{1}{3}$ from center. Then, we have two closed intervals $\left[0, \frac{1}{3}\right]$ and $\left[\frac{2}{3}, 1\right]$. Let $I_1 = \left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right]$. Repeating this process nth times, we have the set I_n as the union of 2^n closed intervals whose the length of each 3^{-n}. Then, the Cantor set E is the intersection of I_n for all $n \in \mathbb{N}$, that is, $E = \bigcap_n I_n$.

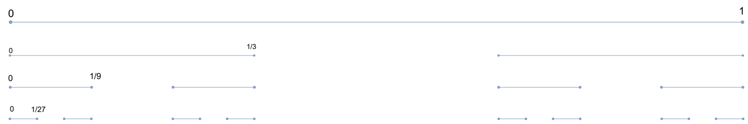
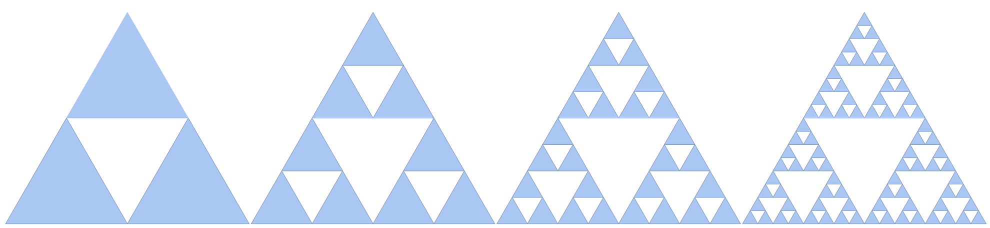


FIGURE 1. Towards to Cantor set

Other famous fractals examples are the Koch curve and Sierpinski triangle (see Figure 2 and Figure 3).

Stage 1 Stage 2 Stage 3 Stage 4

FIGURE 2. Towards to Koch curve



Stage 1 Stage 2 Stage 3 Stage 4

FIGURE 3. Towards to Sierpinski triangle

However, we know only a few methods to construct fractals. One of the most common methods is the iterated function system established by Hutchinson [10]. An iterated function system (in short IFS) consists of a metric space together with a finite contraction mapping set. If we start with any compact subset of the metric space and apply these mappings iteratively under certain conditions, we will come close to a fixed compact subset called an attractor of the iterated function system or a fractal. Further, IFS is an effective method for the construction of a wide variety of geometric objects. Hence, there are many studies on this topic in the literature [1, 9, 12, 17, 18]. In this context, Hutchinson [10] obtained a fundamental theorem for an attractor of an IFS based on the Banach fixed point result [5], and so the system became more popular. Now, we remind some basic concepts related to iterated function systems:

We denote the family of all nonempty compact subsets of a metric space (Λ, ρ) by $C(\Lambda)$. Then, the mapping $h : C(\Lambda) \times C(\Lambda) \to [0, \infty)$ defined by

$$h(P, Q) = \max\{\delta(P, Q), \delta(Q, P)\}$$

for all $P, Q \in C(\Lambda)$ is the Pompeiu-Hausdorff metric induced by ρ where

$$\begin{aligned}\delta(P,Q) &= \sup\{\rho(\varkappa, Q) : \varkappa \in P\} \\ &= \sup\{\inf\{\rho(\varkappa,\eta) : \eta \in Q\} : \varkappa \in P\}.\end{aligned}$$

It is well known that $(C(\Lambda), h)$ is complete whenever (Λ, ρ) is complete.

Definition 1.1 Let (Λ, ρ) be a metric space and $f_i : \Lambda \to \Lambda$ be mappings for all $i = 1, 2, \cdots, N$. The system $\{\Lambda; f_i, i = 1, 2, \cdots, N\}$ is said to be an iterated function system if the mappings f_i are λ_i-contractions, that is, there exists λ_i in $[0, 1)$ such that for all $\varkappa, \eta \in \Lambda$

$$\rho(f_i\varkappa, f_i\eta) \le \lambda_i \rho(\varkappa, \eta)$$

for all $i = 1, 2, \cdots, N$.

Now, we present Hutchinson's famous result related to constructing a fractal via an IFS.

Theorem 1.2 *Let $\{\Lambda; f_i, i = 1, 2, \cdots, N\}$ be an IFS on a complete metric space (Λ, ρ). Then, the mapping $T : C(\Lambda) \to C(\Lambda)$ defined by*

$$TE = \bigcup_{i=1}^{N} f_i(E)$$

for all $E \in C(\Lambda)$ is a λ-contraction mapping on complete metric space $C(\Lambda)$ where $\lambda = \max\{\lambda_i : i = 1, 2, \cdots, N\}$. Further, for arbitrary set $Q \in C(\Lambda)$, it satisfies

$$\lim_{n\to\infty} T^n Q = P,$$

where P is the attractor of the IFS.

On the other hand, Basha and Veeramani [8] introduced a nice concept of best proximity point which extends the notion of fixed point. Let (Λ, ρ) be a metric space, $\emptyset \ne P, Q \subseteq \Lambda$ and $T : P \to Q$ be a mapping. If the intersection of P and Q is empty, then the mapping T cannot have a fixed point. Hence, it is reasonable to investigate the existence of a point $\varkappa \in P$ such that $\rho(\varkappa, T\varkappa) = \rho(P, Q)$ which is called a best proximity point of the mapping T. A best proximity point of the mapping T is both an optimal solution for the minimization problem $\min_{\varkappa\in P} \rho(\varkappa, T\varkappa)$ and a fixed point of it in case of $P = Q = \Lambda$. Due to these facts, this topic has been studied by many authors [2, 4, 7, 13, 15, 16]. We will use the following subsets in the rest of paper.

$$P_0 = \{\varkappa \in P : \rho(\varkappa, \eta) = \rho(P, Q) \text{ for some } \eta \in Q\}$$

and

$$Q_0 = \{\eta \in Q : \rho(\varkappa, \eta) = \rho(P, Q) \text{ for some } \varkappa \in P\},$$

where $\rho(P, Q) = \inf\{\rho(\varkappa, \eta) : \varkappa \in P \text{ and } \eta \in Q\}$.

The following lemma is important for our main result.

Lemma 1.3 [6] *Let P, Q, E and D be arbitrary compact subsets of a metric space (Λ, ρ). Then, we have*

$$h(P \cup E, Q \cup D) \le \max\{h(P, Q), h(E, D)\}.$$

Considering the concept of proximal λ-contraction, the first remarkable result showing the relation between the best proximity point and fractal has been obtained in [3]. Here, we will establish this relationship in a different approach, thanks to the λ-contractions and the following property.

Definition 1.4 [14] Let (Λ, ρ) be a metric space and $\emptyset \neq P, Q \subseteq \Lambda$. Then, the pair (P, Q) is said to have the P_ρ-Property if

$$\left.\begin{array}{l}\rho(u_1, v_1) = \rho(P, Q) \\ \rho(u_2, v_2) = \rho(P, Q)\end{array}\right\} \Longrightarrow \rho(u_1, u_2) = \rho(v_1, v_2)$$

for all $u_1, u_2 \in P$ and $v_1, v_2 \in Q$.

Our main result in this paper is based on the following best proximity point theorem given in [14].

Theorem 1.5 [14] *Let (Λ, ρ) be a complete metric space, $\emptyset \neq P, Q \subseteq \Lambda$ with $P_0 \neq \emptyset$. Assume that the pair (P, Q) has the P_ρ-Property and P, Q are closed. If $f : P \to Q$ is λ-contraction mapping satisfying $f(P_0) \subseteq Q_0$, then f has a unique best proximity point.*

Remark 1.6 From the proof of Theorem 1.5, it can be seen that the sequence $\{\varkappa_n\}$ constructed by

$$\rho(\varkappa_n, f(\varkappa_{n-1})) = \rho(P, Q)$$

for all $n \in \mathbb{N}$ with the initial point $\varkappa_0 \in P_0$ converges to best proximity point of f.

2. Main results

We begin this section by revising some notions and definitions related to best proximity point theory. Let (Λ, ρ) be a metric space and $\emptyset \neq P, Q \subseteq \Lambda$. Throughout this paper, we will use the following subfamilies of $C(P)$ and $C(Q)$, respectively

$$(C(P))_0 = \{E \in C(P) : h(E, D) = H\left(C(P), C(Q)\right) \text{ for some } D \in C(Q)\}$$

and

$$(C(Q))_0 = \{D \in C(Q) : h(E, D) = H\left(C(P), C(Q)\right) \text{ for some } E \in C(P)\}$$

where

$$H\left(C(P), C(Q)\right) = \inf\left\{h(E, D) : E \in C(P) \text{ and } D \in C(Q)\right\}.$$

Definition 2.1 Let (Λ, ρ) be a metric space and $\emptyset \neq P, Q \subseteq \Lambda$. Then, the pair $(C(P), C(Q))$ is said to have the P_h-Property if

$$\left.\begin{array}{l}h(U_1, V_1) = H(C(P), C(Q)) \\ h(U_2, V_2) = H(C(P), C(Q))\end{array}\right\} \Longrightarrow h(U_1, U_2) = h(V_1, V_2)$$

for all $U_1, U_2 \in C(P)$ and $V_1, V_2 \in C(Q)$.

The following lemmas are crucial for our main result.

Lemma 2.2 [3] *Let P, Q be nonempty subsets of a metric space (Λ, ρ) and $P_0 \neq \emptyset$. Then, we get*

$$H\left(C(P), C(Q)\right) = \rho(P, Q).$$

Lemma 2.3 [3] *Let (Λ, ρ) be a metric space and $\emptyset \neq P, Q \subseteq \Lambda$. If $P_0 \neq \emptyset$, then we have $(C(P))_0 \neq \emptyset$.*

Lemma 2.4 [3] *Let (Λ, ρ) be a metric space and $\emptyset \neq P, Q \subseteq \Lambda$ with $P_0 \neq \emptyset$. Then, we have $(C(P))_0 \subseteq C(P_0)$.*

The reverse of the inclusion given in Lemma 2.4 is not true in general as it can be seen in Example 1 in [3]. However, we have the following lemma with the help of P_ρ-property.

Lemma 2.5 *Let (Λ, ρ) be a metric space and $\emptyset \neq P, Q \subseteq \Lambda$ with $P_0 \neq \emptyset$. If the pair (P, Q) has the P_ρ-Property, then we have $(C(P))_0 = C(P_0)$.*

Proof. From Lemma 2.4, we have $(C(P))_0 \subseteq C(P_0)$. Now, let $E \in C(P_0)$ be an arbitrary set. Then, since E is a subset of P_0, there exists $\eta_\varkappa \in Q_0$ such that

$$\rho(\varkappa, \eta_\varkappa) = \rho(P, Q)$$

for each $\varkappa \in E$. Now, we show that $\eta_\varkappa \in Q_0$ is unique point satisfying $\rho(\varkappa, \eta_\varkappa) = \rho(P, Q)$ for each $\varkappa \in E$. Suppose that there exist $\eta_\varkappa$ and $\eta'_\varkappa \in Q_0$ with $\eta_\varkappa \neq \eta'_\varkappa$ for some $\varkappa \in E$ such that

$$\begin{aligned} \rho(\varkappa, \eta_\varkappa) &= \rho(P, Q) \\ \rho(\varkappa, \eta'_\varkappa) &= \rho(P, Q). \end{aligned}$$

Since the pair (P, Q) has the P_ρ-Property, we have

$$0 = \rho(\varkappa, \varkappa) = \rho(\eta_\varkappa, \eta'_\varkappa).$$

This contradicts our assumption. Now, define the following subset of Q_0:

$$W = \{\eta_\varkappa : \rho(\varkappa, \eta_\varkappa) = \rho(P, Q) \text{ for each } \varkappa \in E\}.$$

We claim that W is a compact subset of Q satisfying $h(E, W) = \rho(P, Q)$.

To show the compactness of W, let us consider an arbitrary sequence $\{\eta_n\}$ in W. Then, from the definition of W, there exists $\{\varkappa_n\}$ in E such that

$$\rho(\varkappa_n, \eta_n) = \rho(P, Q) \tag{2.1}$$

for all $n \geq 1$. Also, since E is a compact set, there exists a subsequence $\{\varkappa_{n_k}\}$ of $\{\varkappa_n\}$ such that

$$\varkappa_{n_k} \to \varkappa^* \text{ as } k \to \infty \tag{2.2}$$

for some $\varkappa^* \in E$. Then, from the definition of W, there exists η^* in W such that

$$\rho(\varkappa^*, \eta^*) = \rho(P, Q). \tag{2.3}$$

Now, since the pair (P, Q) has the P_ρ-Property, from (2.1) and (2.3) we have

$$\rho(\varkappa_{n_k}, \varkappa^*) = \rho(\eta_{n_k}, \eta^*) \tag{2.4}$$

for all $k \geq 1$. Taking limit $k \to \infty$ in equation (2.4), we get

$$\eta_{\varkappa_{n_k}} \to \eta^* \text{ as } k \to \infty.$$

Hence, $\{\eta_n\}$ has a convergent subsequence in W, and so W is compact.

Now, for the second part of the claim, we have to show that $h(E, W) = \rho(P, Q)$. Considering the set W, we have

$$\begin{aligned} \rho(P,Q) &\leq \rho(\varkappa, W) \\ &= \inf\{\rho(\varkappa, \zeta) : \zeta \in W\} \\ &\leq \rho(\varkappa, \eta_\varkappa) \\ &= \rho(P,Q) \end{aligned}$$

for all $\varkappa \in E$, and hence

$$\begin{aligned} \delta(E,W) &= \sup\{\rho(\varkappa, W) : \varkappa \in E\} \\ &= \rho(P,Q). \end{aligned}$$

On the other hand, let $\zeta \in W$ be an arbitary point. Then, from the construction of W, we say that there exists $\varkappa_\zeta \in E$ such that

$$\rho(\varkappa_\zeta, \zeta) = \rho(P, Q).$$

Therefore, we have

$$\begin{aligned} \rho(P,Q) &\leq \rho(\zeta, E) \\ &= \inf\{\rho(\zeta, u) : u \in E\} \\ &\leq \rho(\zeta, \varkappa_\zeta) \\ &= \rho(P,Q) \end{aligned}$$

for all $\zeta \in W$, and hence

$$\begin{aligned} \delta(W,E) &= \sup\{\rho(\zeta, E) : \zeta \in W\} \\ &= \rho(P,Q). \end{aligned}$$

Thus, we have

$$\begin{aligned} h(E,W) &= \max\{\delta(E,W), \delta(W,E)\} \\ &= \rho(P,Q). \end{aligned}$$

Therefore, we get $E \in (C(P))_0$, and so $(C(P))_0 = C(P_0)$.

Lemma 2.6 *Let (Λ, ρ) be a metric space and $\emptyset \neq P, Q \subseteq \Lambda$ with $P_0 \neq \emptyset$. If the pair (P, Q) has the P_ρ-Property, then the pair $(C(P), C(Q))$ has the P_h-Property.*

Proof. Let $U_1, U_2 \in C(P)$ and $V_1, V_2 \in C(Q)$ satisfying

$$\begin{aligned} h(U_1, V_1) &= H(C(P), C(Q)) \\ h(U_2, V_2) &= H(C(P), C(Q)). \end{aligned}$$

Then, from Lemma 2.2 we have

$$\begin{aligned} h(U_1, V_1) &= \rho(P,Q) \\ h(U_2, V_2) &= \rho(P,Q). \end{aligned}$$

Therefore, we get

$$\delta(U_1, V_1) = \rho(P, Q), \tag{2.5}$$

$$\delta(V_1, U_1) = \rho(P, Q), \tag{2.6}$$

and

$$\delta(U_2, V_2) = \rho(P, Q), \tag{2.7}$$

$$\delta(V_2, U_2) = \rho(P, Q). \tag{2.8}$$

Hence, since V_1 is compact, from (2.5) for all $u_1 \in U_1$ there exists $v_{u_1} \in V_1$ such that

$$\rho(u_1, v_{u_1}) = \rho(P, Q).$$

Similarly, from (2.8), for all $v_2 \in V_2$, there exists $u_{v_2} \in U_2$ such that

$$\rho(v_2, u_{v_2}) = \rho(P, Q).$$

Since the pair (P, Q) has P_ρ-Property, we have

$$\rho(u_1, u_{v_2}) = \rho(v_{u_1}, v_2).$$

Hence, we get

$$\begin{aligned} \rho(u_1, U_2) &\leq \rho(u_1, u_{v_2}) \\ &= \rho(v_{u_1}, v_2). \end{aligned}$$

Then, we have

$$\begin{aligned} \rho(u_1, U_2) &\leq \inf \{\rho(v_{u_1}, v_2) : v_2 \in V_2\} \\ &= \rho(v_{u_1}, V_2) \\ &\leq \delta(V_1, V_2), \end{aligned}$$

and so from last inequality we have

$$\begin{aligned} \delta(U_1, U_2) &= \sup \{\rho(u_1, U_2) : u_1 \in U_1\} \\ &\leq \delta(V_1, V_2). \end{aligned} \tag{2.9}$$

On the other hand, similarly, from (2.6) and (2.7), we get

$$\delta(V_1, V_2) \leq \delta(U_1, U_2). \tag{2.10}$$

Therefore, from (2.9) and (2.10) we have

$$\delta(V_1, V_2) = \delta(U_1, U_2).$$

Similarly, we can obtain

$$\delta(V_2, V_1) = \delta(U_2, U_1),$$

and so we have

$$\begin{aligned} h(U_1, U_2) &= \max\{\delta(U_1, U_2), \delta(U_2, U_1)\} \\ &= \max\{\delta(V_1, V_2), \delta(V_2, V_1)\} \\ &= h(V_1, V_2). \end{aligned}$$

This shows that the pair $(C(P), C(Q))$ has the P_h-Property.

Lemma 2.7 *Let (Λ, ρ) be a metric space and $\emptyset \neq P, Q \subseteq \Lambda$ with $P_0 \neq \emptyset$. Assume that the pair (P, Q) has the P_ρ-Property and $f_i : P \to Q$ are continuous mappings satisfying*

$f_i(P_0) \subseteq Q_0$ for all $i = 1, 2, \cdots, N$. Then, for the mapping $T : C(P) \to C(Q)$ defined as

$$TU = \bigcup_{i=1}^{N} f_i(U) \tag{2.11}$$

we have $T(C(P_0)) \subseteq C(Q_0)$.

Proof. Let $E \in C(P_0)$ be an arbitrary set. Then E is a compact subset of P_0. Since $f_i(P_0) \subseteq Q_0$ for all $i = 1, 2, \cdots, N$, we have

$$f_i(E) \subseteq f_i(P_0) \subseteq Q_0.$$

Also, since f_i are continuous mappings and E is compact, we have $f_i(E)$ are compact for all $i = 1, 2, \cdots, N$. Therefore

$$TE = \bigcup_{i=1}^{N} f_i(E)$$

is a compact subset of Q_0, and so $TE \in C(Q_0)$. Hence, the proof is done.

Remark 2.8 Under the assumptions of Lemma 2.7, we also have

$$T((C(P))_0) \subseteq (C(Q))_0$$

by using Lemma 2.5.

Definition 2.9 Let (Λ, ρ) be a metric space, $\emptyset \neq P, Q \subseteq \Lambda$ and $f_i : P \to Q$ be mappings for all $i = 1, 2, \cdots, N$. The system $\{P, Q; f_i, i = 1, 2, \cdots, N\}$ is said to be a generalized iterated function system (in short GIFS) if for all $i = 1, 2, \cdots, N$, $f_i : P \to Q$ are λ_i-contraction mappings.

Theorem 2.10 *Let (Λ, ρ) be a complete metric space and $\{P, Q; f_i : i = 1, 2, \cdots, N\}$ be a GIFS, where P and Q are closed subsets of Λ. Assume that $P_0 \neq \emptyset$ and $f_i(P_0) \subseteq Q_0$ for all $i = 1, 2, \cdots, N$. If the pair (P, Q) has the P_ρ-property, then the mapping $T : C(P) \to C(Q)$ given by (2.11) has a best proximity point E in $C(P)$. Moreover, the sequence $\{E_n\}$ constructed by*

$$h(E_n, TE_{n-1}) = H(C(P), C(Q))$$

for all $n \in \mathbb{N}$ with the initial point $E_0 \in C(P_0)$ converges to E with respect to h.

Remark 2.11 The subset E in Theorem 2.10 is called best attractor of the GIFS.

Proof of Theorem 2.10. Since P and Q are closed subsets of Λ, $C(P)$ and $C(Q)$ are closed subsets of the complete metric space $(C(\Lambda), h)$. From Lemma 2.3, Lemma 2.5 and Remark 2.8 we have $(C(P))_0 \neq \emptyset$ and $T((C(P))_0) \subseteq (C(Q))_0$. Further, from Lemma 2.6 the pair $(C(P), C(Q))$ has the P_h-Property. Now, we want to show that T is λ-contraction, that is,

$$h(TU_1, TU_2) \leq \lambda h(U_1, U_2)$$

for all $U_1, U_2 \in C(P)$ holds where $\lambda = \max\{\lambda_1, \lambda_2, \cdots, \lambda_N\}$. Let $U_1, U_2 \in C(P)$ be arbitrary sets. Since f_i are λ_i-contraction mappings for all $i = 1, 2, ..., N$, we have

$$\begin{aligned} \delta(f_i(U_1), f_i(U_2)) &= \max\{\min\{\rho(f_i\varkappa, f_i\eta) : \eta \in U_2\} : \varkappa \in U_1\} \\ &\leq \max\{\min\{\lambda_i\rho(\varkappa, \eta) : \eta \in U_2\} : \varkappa \in U_1\} \\ &= \lambda_i \max\{\min\{\rho(\varkappa, \eta) : \eta \in U_2\} : \varkappa \in U_1\} \\ &= \lambda_i\delta(U_1, U_2) \\ &\leq \lambda\delta(U_1, U_2) \end{aligned} \tag{2.12}$$

for all $i = 1, 2, ..., N$. Similarly, we get

$$\delta(f_i(U_2), f_i(U_1)) \leq \lambda\delta(U_2, U_1) \tag{2.13}$$

for all $i = 1, 2, ..., N$. From (2.12) and (2.13) we have

$$\begin{aligned} h\left(f_i(U_1), f_i(U_2)\right) &= \max\{\delta(f_i(U_1), f_i(U_2)), \delta(f_i(U_2), f_i(U_1))\} \\ &\leq \max\{\lambda\delta(U_1, U_2), \lambda\delta(U_2, U_1)\} \\ &= \lambda \max\{\delta(U_1, U_2), \delta(U_2, U_1)\} \\ &= \lambda h(U_1, U_2) \end{aligned}$$

for all $i = 1, 2, ..., N$. Hence, from Lemma 1.3 we get

$$\begin{aligned} h(TU_1, TU_2) &= h\left(\bigcup_{i=1}^{N} f_i(U_1), \bigcup_{i=1}^{N} f_i(U_2)\right) \\ &\leq \max\left\{h(f_i(U_1), f_i(U_2)) : i \in \{1, 2, \cdots, N\}\right\} \\ &\leq \lambda h(U_1, U_2). \end{aligned}$$

Therefore, T is λ-contraction mapping. Hence, all assumptions of Theorem 1.5 are satisfied, and so T has a unique best proximity point in $C(P)$. Therefore, the GIFS $\{P, Q; f_i : i = 1, 2, \cdots, N\}$ has a unique best attractor E in $C(P)$. Further, from Remark 1.6 the sequence $\{E_n\}$ constructed by

$$h(E_n, TE_{n-1}) = H(C(P), C(Q))$$

for all $n \in \mathbb{N}$ with the initial point $E_0 \in C(P_0)$ converges to E with respect to h.

Now, we present some examples to illustrate and support our main result.

Example 2.12 Let $\Lambda = \mathbb{R}^2$ be endowed with the Euclidean metric ρ. Consider the following closed subsets of Λ,

$$P = \{(\varkappa, \eta) : \ \eta = \varkappa \text{ and } 0 \leq \varkappa \leq 1\}$$

and

$$Q = \left\{(\varkappa, \eta) : \eta = \varkappa - \sqrt{2} \text{ and } 0 \leq \varkappa \leq 2\right\}.$$

Then, we have $\rho(P, Q) = 1$, $P_0 = P$ and

$$Q_0 = \left\{(\varkappa, \eta) : \eta = \varkappa - \sqrt{2} \text{ and } \frac{\sqrt{2}}{2} \leq \varkappa \leq \frac{2 + \sqrt{2}}{2}\right\}.$$

Also, the pair (P,Q) has the P_ρ-property. To see this, let $u_1,u_2\in P$ and $v_1,v_2\in Q$ be arbitrary points satisfying

$$\begin{aligned}\rho(u_1,v_1)=\rho(P,Q)=1\\ \rho(u_2,v_2)=\rho(P,Q)=1.\end{aligned}\tag{2.14}$$

Then, from (2.14) we have $u_1=(\varkappa_1,\varkappa_1)$ and $u_2=(\varkappa_2,\varkappa_2)$, $v_1=\left(\frac{2\varkappa_1+\sqrt{2}}{2},\frac{2\varkappa_1-\sqrt{2}}{2}\right)$ and $v_2=\left(\frac{2\varkappa_2+\sqrt{2}}{2},\frac{2\varkappa_2-\sqrt{2}}{2}\right)$. In this case, we have

$$\rho(u_1,u_2)=\rho(v_1,v_2).$$

Now, define the mappings $f_1,f_2:P\to Q$ as follows:

$$f_1(\varkappa,\eta)=\left(\frac{\varkappa}{3}+\frac{\sqrt{2}}{2},\frac{\eta}{3}-\frac{\sqrt{2}}{2}\right)$$

and

$$f_2(\varkappa,\eta)=\left(\frac{\varkappa}{3}+\frac{4+3\sqrt{2}}{6},\frac{\eta}{3}+\frac{4-3\sqrt{2}}{6}\right).$$

Then, it is clear that $f_i(P_0)\subseteq Q_0$ for $i=1,2$. Further, these mappings are $\frac{1}{3}$-contractions. Hence, the system $\{P,Q;f_1,f_2\}$ is a GIFS. Then, all assumptions of Theorem 2.10 hold, and so $T:C(P)\to C(Q)$ defined as

$$TE=\bigcup_{i=1}^{2}f_i(E)$$

has a unique best proximity point in $C(P)$. Therefore, the GIFS $\{P,Q;f_1,f_2\}$ has a unique best attractor and the sequence $\{E_n\}$ constructed by

$$h(E_n,TE_{n-1})=H(C(P),C(Q))$$

for all $n\in\mathbb{N}$ with the initial set $E_0\in C(P_0)$ converges to this best attractor with respect to h.

Now we want to construct a few steps of the mentioned sequence $\{E_n\}$ with the initial set $E_0=P\in C(P_0)$. In this case, we have

$$E_1=\left\{(\varkappa,\eta):\eta=\varkappa\text{ and }\varkappa\in\left[0,\frac{1}{3}\right]\bigcup\left[\frac{2}{3},1\right]\right\},$$

$$E_2=\left\{(\varkappa,\eta):\eta=\varkappa\text{ and }\varkappa\in\left[0,\frac{1}{3^2}\right]\bigcup\left[\frac{2}{3^2},\frac{1}{3}\right]\bigcup\left[\frac{6}{3^2},\frac{7}{3^2}\right]\bigcup\left[\frac{8}{3^2},1\right]\right\},$$

$$E_3=\left\{(\varkappa,\eta):\eta=\varkappa\text{ and }\varkappa\in\left[0,\frac{1}{3^3}\right]\bigcup\left[\frac{2}{3^3},\frac{1}{3^2}\right]\bigcup\cdots\bigcup\left[\frac{26}{3^3},1\right]\right\},$$

$$\vdots$$

$$E_n=\left\{(\varkappa,\eta):\eta=\varkappa\text{ and }\varkappa\in\left[0,\frac{1}{3^n}\right]\bigcup\left[\frac{2}{3^n},\frac{1}{3^{n-1}}\right]\bigcup\cdots\bigcup\left[\frac{(3^n-1)}{3^n},1\right]\right\}.$$

Deduce that $\lim_{n\to\infty}E_n$ which is the best attractor of GIFS $\{P,Q;f_1,f_2\}$ is the Cantor set. Further, Figure 4 shows a few steps of the sequence $\{E_n\}$:

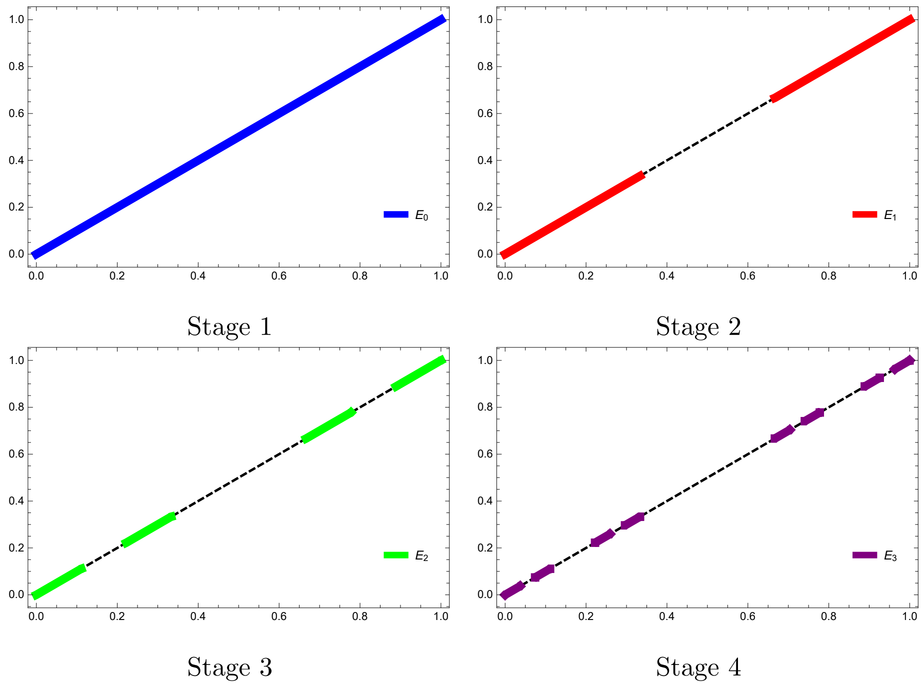


FIGURE 4. A few steps of the sequence $\{E_n\}$

Example 2.13 Let $\Lambda = \mathbb{R}^3$ be endowed with the taxicab metric ρ. Consider the following closed subsets of Λ

$$P = \{(\varkappa, \eta, 0) : \ \varkappa, \eta \in [0, 1]\}$$

and

$$Q = \{(\varkappa, \eta, 1) : \ \varkappa, \eta \in [0, 1]\} .$$

Then, we have $\rho(P, Q) = 1$, $P_0 = P$ and $Q_0 = Q$. Also, it is easy to see that the pair (P, Q) has the P_ρ-property. Define the mappings $f_1, f_2 : P \to Q$ as follows:

$$f_1((\varkappa, \eta, 0)) = \left(\frac{\varkappa}{2}, \frac{\eta}{2}, 1\right)$$

and

$$f_2((\varkappa, \eta, 0)) = \left(\frac{\varkappa}{2} + \frac{1}{2}, \frac{\eta}{2} + \frac{1}{2}, 1\right).$$

Then, it is clear that $f_i(P_0) \subseteq Q_0$ for $i = 1, 2$. Further, these mappings are $\frac{1}{2}$-contractions. Hence, the system $\{P, Q; f_1, f_2\}$ is a GIFS, and so all hypotheses of Theorem 2.10 are satisfied. Therefore, $T : C(P) \to C(Q)$ defined as

$$TE = \bigcup_{i=1}^{2} f_i(E)$$

has a unique best proximity point in $C(P)$. So, the GIFS $\{P, Q; f_1, f_2\}$ has a unique best attractor and the sequence $\{E_n\}$ constructed by

$$h(E_n, TE_{n-1}) = H(C(P), C(Q))$$

for all $n \in \mathbb{N}$ with the initial set $E_0 \in C(P_0)$ converges to this best attractor with respect to h.

Now we want to construct a few steps of the mentioned sequence $\{E_n\}$ with the initial set $E_0 = P \in C(P_0)$. In this case, we have

$$\begin{aligned} E_1 &= \left\{\left(\frac{\varkappa}{2}, \frac{\eta}{2}, 0\right) : \varkappa, \eta \in [0,1]\right\} \bigcup \left\{\left(\frac{\varkappa+1}{2}, \frac{\eta+1}{2}, 0\right) : \varkappa, \eta \in [0,1]\right\} \\ &= \bigcup_{i=0}^{1} \left\{\left(\frac{\varkappa+i}{2}, \frac{\eta+i}{2}, 0\right) : \varkappa, \eta \in [0,1]\right\}, \end{aligned}$$

$$E_2 = \bigcup_{i=0}^{2^2-1} \left\{\left(\frac{\varkappa+i}{2^2}, \frac{\eta+i}{2^2}, 0\right) : \varkappa, \eta \in [0,1]\right\}$$

$$E_3 = \bigcup_{i=0}^{2^3-1} \left\{\left(\frac{\varkappa+i}{2^3}, \frac{\eta+i}{2^3}, 0\right) : \varkappa, \eta \in [0,1]\right\}$$

$$E_4 = \bigcup_{i=0}^{2^4-1} \left\{\left(\frac{\varkappa+i}{2^4}, \frac{\eta+i}{2^4}, 0\right) : \varkappa, \eta \in [0,1]\right\}$$

$$\vdots$$

$$E_n = \bigcup_{i=0}^{2^n-1} \left\{\left(\frac{\varkappa+i}{2^n}, \frac{\eta+i}{2^n}, 0\right) : \varkappa, \eta \in [0,1]\right\}.$$

Deduce that

$$\lim_{n\to\infty} E_n = \{(\zeta, \zeta, 0) : \zeta \in [0,1]\}$$

is the best attractor of GIFS $\{P, Q; f_1, f_2\}$. Further, Figure 5 shows a few steps of the sequence $\{E_n\}$.

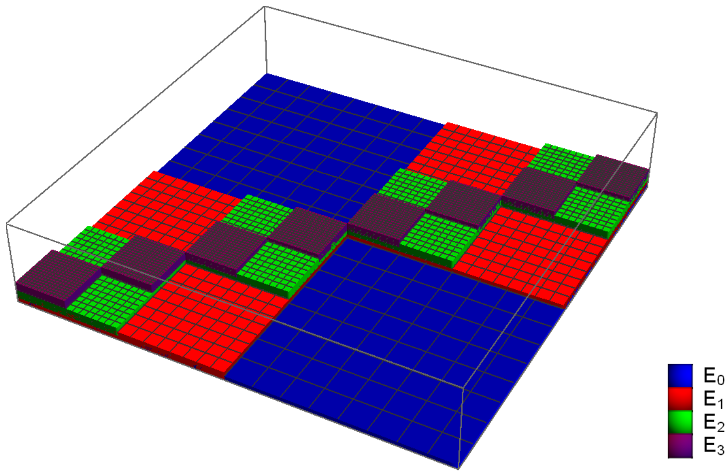


FIGURE 5. A few steps of the sequence $\{E_n\}$

Taking $P = Q = \Lambda$ in Theorem 2.10, we can deduce Theorem 1.2 which is the well known result of constructing a fractal via IFS.

Acknowledgement. The authors thank to the referee for his pertinent comments and suggestions which help us to improve the manuscript.

REFERENCES

[1] M. Abbas, T. Nazir, *Attractor of the generalized contractive iterated function system*, Mathematical Analysis and Applications: Selected Topics, John Wiley and Sons, Ltd, 2018, 401-428.
[2] I. Altun, M. Aslantas, H. Sahin, *Best proximity point results for p-proximal contractions*, Acta Math. Hungar., **162**(2020), 393-402.
[3] I. Altun, H. Sahin, M. Aslantas, *A new approach to fractals via best proximity point*, Chaos, Solitons Fractals, **146**(2021), 110850.
[4] M. Aslantas, H. Sahin, I. Altun, *Best proximity point theorems for cyclic p-contractions with some consequences and applications*, Nonlinear Anal. Model. Control, **26**(2021), 113-129.
[5] S. Banach, *Sur les opérations dans les ensembles abstraits et leur applications aux équations intégrales*, Fund. Math., **3**(1922), 133-181.
[6] M.F. Barnsley, *Fractals Everywhere*, Academic Press, New York, 1988.
[7] S.S. Basha, *Extensions of Banach's contraction principle*, Numer. Funct. Anal. Optim., **31**(2010), 569-576.
[8] S.S. Basha, P. Veeramani, *Best approximations and best proximity pairs*, Acta Sci. Math. (Szeged), **63** (1977), 289-300.
[9] C. Chifu, A. Petruşel, *Multivalued fractals and generalized multivalued contractions*, Chaos, Solitons Fractals, bf 6(2008), 203-210.
[10] J.E. Hutchinson, *Fractals and self similarity*, Indiana Univ Math. J., **30**(1981), 713-747.
[11] B. Mandelbrot, *Fractals, Form, Chance, and Dimension*, W.H. Freeman and Company, San Francisco, 1977.
[12] T. Nazir, S. Silvestrov, M. Abbas, *Fractals of generalized F-Hutchinson operator*, Waves, Wavelets and Fractals, **2**(2016), 29-40.

[13] V. Parvaneh, M.R. Haddadi, H. Aydi, *On best proximity point results for some type of mappings*, J. Function Spaces, **2020**(2020), Art. ID 6298138.
[14] V.S. Raj, *Best proximity point theorems for non-self mappings*, Fixed Point Theory, **14**(2013), 447-454.
[15] S. Reich, *Approximate selections, best approximations, fixed points, and invariant sets*, J. Math. Anal. Appl., **62**(1978), 104-113.
[16] H. Sahin, M. Aslantas, I. Altun, *Feng-Liu type approach to best proximity point results for multivalued mappings*, J. Fixed Point Theory Appl., **22**(2020), 11.
[17] N.A. Secelean, S. Mathew, D. Wardowski, *New fixed point results in quasi-metric spaces and applications in fractals theory*, Adv. Difference Equ., **2019**(2019), 1-23.
[18] N. Van Dung, A. Petruşel, *On iterated function systems consisting of Kannan maps, Reich maps, Chatterjea type maps, and related results*, J. Fixed Point Theory Appl., **19**(2017), 2271-2285.

Received: July 2, 2022; Accepted: March 23, 2023.