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BEST PROXIMITY POINT RESULTS FOR PROXIMAL CONTRACTION IN TOPOLOGICAL SPACES

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Abstract. Let X be an arbitrary topological space and $g : X \times X \to \mathbb{R}$ be a real valued continuous function defined on $X \times X$. In this article, we introduce two notions like topologically Berinde weak proximal contraction and topologically proximal weakly contractive mapping with respect to g. We explore sufficient conditions for the existence and uniqueness of best proximity points for these classes of mappings. Moreover, in the last part of the paper, we show that the best proximity point theorem for topologically proximal weakly contractive mapping can be deduced from some fixed point theorems in topological spaces.

Key Words and Phrases: Best proximity point, fixed point, topological space, Berinde weak proximal contraction mapping, approximatively compact.
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1. Introduction

Let (M, ρ) be a metric space and C be a non-empty subset of M. Let $f : C \to C$ be a mapping. The mapping f is said to be a contraction if there exists $\alpha \in (0, 1)$ such that $\rho(f(x), f(y)) \leq \alpha\rho(x, y)$ for all $x, y \in C$. In the year 1922, Banach proved that if M is complete then the contraction map f has a unique fixed point in C. It is one of the pioneer results in metric fixed point theory as it gives sufficient conditions which will ensure the existence of solutions of the equation $f(x) = x$ in C. It also has a lot of applications in the areas of differential equations, integral equations, non-linear matrix equations and many more. In case of a self mapping $f : C \to C$, it is obvious that $f(C) \cap C \neq \emptyset$. On the other hand, if $f : A \to B$ is a mapping such that A, B are non-empty subsets of the metric space $M, A \neq B$ and $f(A) \cap A = \emptyset$ then the mapping f has no fixed points. So, in case of a non-self map, one seek for an element in the domain space whose distance from its image is minimum i.e., in this case, one interesting problem is to minimize $\rho(x, f(x))$ such that $x \in A$. Since $\rho(x, f(x)) \geq D(A, B) = \inf\{\rho(x, y) : x \in A, y \in B\}$, so, one can search for an element $x \in A$ such that $\rho(x, f(x)) = D(A, B)$. Best proximity point problems deal with this situation. For a self mapping, best proximity points coincide with fixed points. In the year 2011, Basha [1] introduced the notion of proximal contraction which is a generalization of contraction in case of self mapping. For more details about the best proximity points, one can refer to [2, 5, 9] and the references therein. The main thing is that, all these results are formulated in the framework of metric spaces or Banach spaces where the standard metric or norm plays an important role.

Recently in the year 2020, it is exciting that Raj and Piramatchi [7] presented a way in which we can extend the best proximity point results from standard metric spaces to topological spaces. In this paper, instead of taking metric space or normed space, we take an arbitrary topological space X and a real valued continuous function g defined on $X \times X$. We present our work in two sections. In the first section we introduce the notion of topologically Berinde weak proximal contraction and discuss the existence of best proximity points for this class of mappings. By an example, we have shown the necessity of defining the concept of topologically Berinde weak proximal contraction with respect to g. We show that though a mapping $f : A \to B$, where $A, B(\neq \emptyset) \subset X$, is a topologically Berinde weak proximal contraction with respect to a continuous function g, may not be a topologically Berinde weak proximal contraction with respect to another continuous function h. We also show that, there exists a topological space X with a continuous real valued function g, two non-empty disjoint subsets A, B of X and a mapping $f : A \to B$ such that f is topologically Berinde weak proximal contraction with respect to g, but if the topological space is metrizable with respect to a metric d then f is not Berinde weak proximal contraction with respect to the metric d. Moreover, to build the best proximity point results for topologically Berinde weak proximal contraction, we have introduced the notion of approximatively g-compactness of a non-empty subset A in X with respect to another non-empty subset B and show by an example that the notion of approximatively g-compactness for topological spaces is more general than the notion of approximatively compactness for metric spaces [1].

In the second section, we introduce the notion of topologically weakly contractive mapping with respect to g and topologically proximal weakly contractive mapping with respect to g. We also set up two different concepts for the existence and uniqueness of best proximity point for topologically proximal weakly contractive mapping with respect to g. In the first concept, we prove the existence and uniqueness of best proximity point for this class of mappings using g-completeness property. In the second concept, we introduce a new notion of g-isometry which is more general than the notion of isometry in metric spaces.

2. Main results

Before going further we first recall the following definitions from [7].

Definition 2.1. [7] Let A, B be non-empty subsets of a topological space X and $g : X \times X \to \mathbb{R}$ be a continuous function. Define

$$D_g(A, B) = \inf\{|g(x, y)| : x \in A,\ y \in B\}.$$

Definition 2.2. [7] Let A, B be non-empty subsets of a topological space X and $g : X \times X \to \mathbb{R}$ be a continuous function. A point $x \in A \cup B$ of the mapping $f : A \cup B \to A \cup B$ is called a best proximity point of f with respect to g if $|g(x, f(x))| = D_g(A, B)$.

For further developments, we take the definitions and concepts of g-convergence, g-Cauchy, g-completeness and g-closedness from [10].

We present an example to show that there exists a non-empty subset A of X such that A is not closed with respect to the usual topology, but is g-closed for some real valued continuous function g defined on $X \times X$.

Example 2.3. Consider $\mathbb{R}$ with the usual topology and let $g : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be defined by $g(x, y) = x - y + 1$. Let $A = (\frac{1}{2}, \infty)$. Then A is not closed with respect to the usual topology in $\mathbb{R}$. Let $\{x_n\}$ be a sequence in A which is g-convergent to $x \in \mathbb{R}$. Then

$$\begin{aligned} &|g(x_n, x)| \to 0 \text{ as } n \to \infty, \\ \Longrightarrow\ &|x_n - x + 1| \to 0 \text{ as } n \to \infty, \\ \Longrightarrow\ &x_n \to (x - 1) \text{ as } n \to \infty. \end{aligned}$$

But, since $\{x_n\}$ is a sequence in $(\frac{1}{2}, \infty)$ so, we have $x - 1 \geq \frac{1}{2}$. This shows that $x \geq \frac{3}{2}$ and A is g-closed.

2.1. Best proximity point theorem for topologically Berinde weak proximal contraction.

Now we would like to introduce the notion of topologically Berinde weak proximal contraction as follows:

Definition 2.4. Let X be a topological space and $g : X \times X \to \mathbb{R}$ be a continuous function. Let (A, B) be a pair of non-empty g-closed subsets of X. A non-self mapping $f : A \to B$ is said to be topologically Berinde weak proximal contraction with respect to g if there exists $\lambda \in [0, 1)$ and $\mu \in [0, \infty)$ such that for all $x, y, u, v \in A$ with $|g(u, f(x))| = D_g(A, B)$ and $|g(v, f(y))| = D_g(A, B)$ we have

$$\frac{1}{1 + \lambda + \mu}|g^*\left(x, f(x)\right)| \leq |g(x, y)| \Longrightarrow |g(u, v)| \leq \lambda|g(x, y)| + \mu|g^*\left(f(x), y\right)|,$$

where we define $\Big|g^*(x, y)\Big| = \Big|g(x, y)\Big| - D_g(A, B)\ \forall\ (x, y) \in A \times B$.

Note 2.5. If the topological space X is metrizable with respect to a metric d, then by taking $g = d$ in Definition 2.4, we get the notion of Berinde weak proximal contraction for standard metric spaces introduced by Gabeleh in [4].

Now we provide an example of a non-self mapping f defined on a non-empty subset of a topological space X such that f is a topologically Berinde weak proximal contraction with respect to a continuous function g but f is not topologically Berinde weak proximal contraction with respect to another continuous function h defined on $X \times X$.

Example 2.6. Let $X = \mathbb{R}^2$ with the usual topology and $g : X \times X \to \mathbb{R}$ be defined by $g((x, y), (u, v)) = y - v$. Then g is a continuous function. Suppose

$$A = \Big\{(0,0), (0,1), (0,2)\Big\} \text{ and } B = \{(1,-1), (1,3), (1,5)\}.$$

Then $D_g(A, B) = 1$. Let $f : A \to B$ be a mapping defined by

$$f((0,0)) = f((0,1)) = (1,3) \text{ and } f((0,2)) = (1,5).$$

Now $|g(u, f(x))| = 1 \Longrightarrow u = (0,2)$ and $x \in \{(0,0), (0,1)\}$. There are four cases:
Case 1. $u = v = (0,2)$, $x = (0,0)$, $y = (0,1)$. We have

$$\frac{1}{1+0.5+1}|g^*(x, f(x))| = \frac{2}{2.5} \le |g(x,y)|$$

$$\Longrightarrow |g(u,v)| = 0 \le \frac{3}{2} = \frac{1}{2}|g(x,y)| + |g^*(f(x), y)|.$$

Case 2. $u = v = (0,2)$, $x = (0,1)$, $y = (0,0)$. We have

$$\frac{1}{1+0.5+1}|g^*(x, f(x))| = \frac{1}{2.5} \le |g(x,y)|$$

$$\Longrightarrow |g(u,v)| = 0 \le \frac{5}{2} = \frac{1}{2}|g(x,y)| + |g^*(f(x), y)|.$$

Case 3. $u = v = (0,2)$, $x = (0,0) = y$. We have

$$\frac{1}{1+0.5+1}|g^*(x, f(x))| = \frac{2}{2.5} > 0 = |g(x,y)|.$$

Case 4. $u = v = (0,2)$, $x = (0,1) = y$. We have

$$\frac{1}{1+0.5+1}|g^*(x, f(x))| = \frac{1}{2.5} > 0 = |g(x,y)|.$$

Therefore, f is a topologically Berinde weak proximal contraction with respect to g with $\lambda = \frac{1}{2}$ and $\mu = 1$.
Now let $h : X \times X \to \mathbb{R}$ defined by $h((x,y),(u,v)) = \min\{y, v\}$. Then $D_h(A, B) = 1$. Again $|h(u, f(x))| = 1 \Longrightarrow u = (0,1)$ and $x \in \{(0,0), (0,1), (0,2)\}$. Now by taking $u = v = (0,1)$, $x = (0,1)$ and $y = (0,0)$, then $|h(u, f(x))| = 1 = |h(v, f(y))|$. But we have

$$\frac{1}{1+\lambda+\mu}|h^*(x, f(x))| = 0 = |h(x,y)|$$

and

$$|h(u,v)| = 1 > \lambda|h(x,y)| + \mu|h^*(f(x), y)|$$

for any $\lambda \in [0,1)$ and $\mu \in [0,\infty)$. Therefore, f is not a topologically Berinde weak proximal contraction with respect to h.

From Note 2.5, it is clear that the notion of topologically Berinde weak proximal contraction with respect to a real valued continuous function is an extension of Berinde weak proximal contraction from metric space to topological space. The next example also clarifies this fact.

Example 2.7. Let $X = \mathbb{R}$ with the usual topology and $g : X \times X \to \mathbb{R}$ be defined by $g(x, y) = x + y$. Then g is a continuous function. Suppose $A = \{0, 3, 5, 7\}$ and $B = \{1, 2, 9\}$. Then $D_g(A, B) = 1$. Let $f : A \to B$ be a mapping defined by $f(0) = 2$, $f(3) = 1$ $f(5) = 9 = f(7)$. Now $|g(u, f(x))| = 1 \Longrightarrow u \in \{0\}$ and $x \in \{3\}$.

$u = 0, v = 0$, $x = 3$, $y = 3$ and $|g(0, f(3))| = D_g(A, B)$ and $|g(0, f(3))| = D_g(A, B)$ We have

$$\frac{1}{1 + 0.5 + 1}|g^*(3, f(3)| = \frac{3}{2.5} \leq 6 = |g(3, 3)|$$
$$\Longrightarrow |g(u, v)| = 0 \leq 3 + 3 = \frac{1}{2}|g(3, 3)| + |g^*(f(3), 3)|.$$

Therefore, f is a topologically Berinde weak proximal contraction w.r.t g with $\lambda = \frac{1}{2}$ and $\mu = 1$.

Now, we take the usual metric d on $\mathbb{R}$. So $D(A, B) = 1$. Now $d\,(u, f(x)) = 1 \Longrightarrow u \in \{0, 3\}$ and $x \in \{0, 3\}$. Consider the case when $u = 3, v = 0$, $x = 0$, $y = 3$ and $d\,(3, f(0)) = D(A, B)$ and $d\,(0, f(3)) = D(A, B)$. We have

$$\frac{1}{1 + \lambda + \mu}d^*\,(3, f(3)) = \frac{1}{1 + \lambda + \mu} \leq 3 = d(0, 3)$$

But,

$$|g(u, v)| = 3 \not\leq 3\lambda = \lambda d(0, 3) + \mu d^*\,(f(0), 3)$$

for any $\lambda \in [0, 1)$ and $\mu \in [0, \infty)$. Therefore f is not a topologically Berinde weak proximal contraction w.r.t d.

Definition 2.8. [4] Let A, B be two non-empty subsets of a metric space (X, d). Then A is said to be approximatively compact with respect to B if for every sequence $\{x_n\} \subset A$ satisfying $d(y, x_n) \to d(y, A)$ as $n \to \infty$, for some $y \in B$, has a convergent subsequence where $d(y, A) = \inf\{d(y, x) : x \in A\}$.

Now we introduce the notion of approximatively g-compact set in a topological space X with respect to another non-empty set as follows:

Definition 2.9. Let X be a topological space and $g : X \times X \to \mathbb{R}$ be a continuous function. Let (A, B) be a pair of non-empty subsets of X. Then A is said to be approximatively g-compact with respect to B if for every sequence $\{x_n\}_{n\in\mathbb{N}}$ of A satisfying the condition that $|g(x_n, y)| \to D_g(A, y)$ as $n \to \infty$ for some $y \in B$, has a g-convergent subsequence $\{x_{n_k}\}$ which is g-convergent to a point in A where define $D_g(A, y) = \inf\{|g(t, y)| : t \in A\}$.

Note 2.10. If the topological space X is metrizable with respect to a metric d, then by taking $g = d$ in Definition 2.9, we get the notion of approximatively compactness for standard metric spaces introduced by Basha in [1].

From Definition 2.9 and Note 2.10, it is clear that the notion of approximatively g-compactness with respect to g is more general than the notion of approximatively compactness for metric spaces. To validate this statement, we provide an example to show that there exist two non-empty subsets A, B of a topological space X such that A is not approximatively compact with respect to B but A is approximatively g-compact with respect to B for some real-valued continuous function g defined on $X \times X$.

Example 2.11. Let $X = \mathbb{R}$ with the usual topology, $A = \{\frac{1}{n} : n \in \mathbb{N}\}$ and $B = \{0, \frac{1}{2}\}$. Let $g : X \times X \to \mathbb{R}$ be defined by $g((x, y)) = x - y + \frac{1}{2}$. Then g is a continuous function. Now, $D_g(A, 0) = \frac{1}{2}$ and $D_g(A, \frac{1}{2}) = 0$. Here, the set A is not approximatively compact with respect to B, because, for the sequence $\{\frac{1}{n}\}$, there exists $0 \in B$ such that $d(0, x_n) \to D(0, A) = 0$ as $n \to \infty$ but $\{\frac{1}{n}\}$ has no convergent subsequence. On the other hand, only sequence $\{x_n\}$ which satisfy the condition that $\Big|g(x_n, y)\Big| \to D_g(A, y)$ for some $y \in B$, has infinite range. Now let $\{x_n\}$ be a sequence in A with infinite range. Then there exists $\frac{1}{2} \in B$ such that $\Big|g(x_n, \frac{1}{2})\Big| \to D_g(A, \frac{1}{2}) = 0$ as $n \to \infty$. Let, on the contrary, for any subsequence (x_{n_k}) of $\{x_n\}$ and for any $x \in A$, x_{n_k} is not g-convergent to x. So

$$\Big|g(x_{n_k}, x)\Big| \nrightarrow 0 \text{ as } k \to \infty$$

$$\implies |x_{n_k} - x + \frac{1}{2}| \nrightarrow 0 \text{ as } k \to \infty.$$

But this is a contradiction since the above convergence is true for $x = \frac{1}{2}$. So $\{x_n\}$ has a g-convergent subsequence which is g-convergent to an element of A. So, A is approximatively g-compact with respect to B.

In the upcoming theorem, we present a best proximity point result for topologically Berinde weak proximal contractions and for the purpose of the Theorem, we recall the following definitions from [10] as follows:

$$A_g = \{x \in A : |g(x, y)| = D_g(A, B) \text{ for some } y \in B\}.$$

$$B_g = \{y \in B : |g(x, y)| = D_g(A, B) \text{ for some } x \in A\}.$$

Theorem 2.12. *Let X be a g-complete topological space where $g : X \times X \to \mathbb{R}$ is a continuous function such that $g(x, y) = 0 \implies x = y, |g(x, y)| = |g(y, x)|$ and $|g(x, z)| \leq |g(x, y)| + |g(y, z)|$ for all $x, y, z \in X$. Let (A, B) be a pair of non-empty g-closed subsets of X such that A_g is non-empty. Let $f : A \to B$ be a topologically Berinde weak proximal contraction w.r.t g with $\lambda \in [0, 1)$ and $\mu \in [0, \infty)$ such that $f(A_g) \subseteq B_g$. If B is approximatively g-compact w.r.t A, then*

1. *there exists a best proximity point $p^* \in A_g$ of f and for any element $p_0 \in A_g$, the sequence $\{p_n\}$ satisfying $|g(p_{n+1}, f(p_n))| = D_g(A, B)$ converges to p^*;*
2. *moreover, if $(1 - \lambda - \mu) > 0$ then the best proximity point p^* is unique.*

Proof. Let $p_0 \in A_g$. Since $f(A_g) \subseteq B_g$, we have $f(p_0) \in B_g$. So, there exists $p_1 \in A_g$ such that $|g(p_1, f(p_0))| = D_g(A, B)$. Similarly as $f(p_1) \in B_g$, so there exists $p_2 \in A_g$ such that $|g(p_2, f(p_1))| = D_g(A, B)$. Continuing this process, we get a sequence $\{p_n\} \subseteq A_g$ such that

$$\Big|g(p_{n+1}, f(p_n))\Big| = D_g(A, B) \ \forall \ n \geq 0.$$

We claim that the sequence $\{p_n\}$ is a g-Cauchy sequence. Now

$$|g(p_0, f(p_0))| \leq |g(p_0, p_1)| + |g(p_1, f(p_0))| = |g(p_0, p_1)| + D_g(A, B).$$

We have $|g(p_1, f(p_0))| = D_g(A, B) = |g(p_2, f(p_1))|$ and

$$\frac{1}{1 + \lambda + \mu}|g^*(p_0, f(p_0))| \leq |g^*(p_0, f(p_0))| \leq |g(p_0, p_1)|.$$

As f is a topologically Berinde weak proximal contraction w.r.t g, so conclude that

$$|g(p_1, p_2)| \leq \lambda|g(p_0, p_1)| + \mu|g^*(p_1, f(p_0))|.$$

In a similar manner

$$|g(p_1, f(p_1))| \leq |g(p_1, p_2)| + |g(p_2, f(p_1))| = |g(p_1, p_2)| + D_g(A, B).$$

$|g(p_2, f(p_1))| = D_g(A, B) = |g(p_3, f(p_2))|$ and $\frac{1}{1+\lambda+\mu}|g^*(p_1, f(p_1))| \leq |g(p_1, p_2)|$. Therefore,

$$|g(p_2, p_3)| \leq \lambda|g(p_1, p_2)| + \mu|g^*(p_2, f(p_1))| \leq \lambda|g(p_1, p_2)| \leq \lambda^2|g(p_0, p_1)|.$$

So, by induction, we get $|g(p_n, p_{n+1})| \leq \lambda^n|g(p_0, p_1)|$. Suppose $m > 0$. Now,

$$\begin{aligned} |g(p_n, p_{n+m})| &\leq |g(p_n, p_{n+1})| + |g(p_{n+1}, p_{n+2})| + \cdots + |g(p_{n+m-1}, p_{n+m})| \\ \implies |g(p_n, p_{n+m})| &\leq \left(\lambda^n + \lambda^{n+1} + \cdots + \lambda^{n+m-1}\right)|g(p_0, p_1)| \\ \implies |g(p_n, p_{n+m})| &\leq \frac{\lambda^n(1 - \lambda^m)}{(1 - \lambda)}|g(p_0, p_1)| \to 0 \text{ as } n, m \to \infty. \end{aligned}$$

This shows that the sequence $\{p_n\}$ is a g-Cauchy sequence. Since X is g-complete and A is g-closed, so the sequence $\{p_n\}$ is g-convergent to a point $p^* \in A$. Besides, we have

$$D_g(p^*, B) \leq \lim_{n\to\infty}[|g(p^*, p_{n+1})| + |g(p_{n+1}, f(p_n))|] = D_g(p^*, B)$$

So,

$$\lim_{n\to\infty}|g(p^*, f(p_n))| = D_g(p^*, B).$$

Since B is approximatively g-compact with respect to A, it follows that the sequence $\{f(p_n)\}$ has a subsequence $\{f(p_{n_k})\}$ converges to some element $q^* \in B$. Then

$$|g(p^*, q^*)| \leq \lim_{n\to\infty}[|g(p^*, p_{n_{k+1}})| + |g(p_{n_{k+1}}, f(p_{n_k}))| + |g(f(p_{n_k}), q^*)|] = D_g(A, B).$$

Therefore, $|g(p^*, q^*)| = D_g(A, B) \Rightarrow p^* \in A_g$. Since $f(A_g) \subseteq B_g$, there exists an element $r^* \in A_g$ such that $|g(r^*, f(p^*))| = D_g(A, B)$. We assert that

$$|g^*(p^*, f(p))| \leq \lambda|g(p^*, p)| + \mu|g^*(q^*, p)| \ \forall \ p \in A_g$$

with $p \neq p^*$. Let $p \in A_g$ be fixed such that $p \neq p^*$, by the fact that $f(A_g) \subseteq B_g$ there exists an element $s \in A_g$ such that

$$|g(s, f(p))| = D_g(A, B).$$

Since $p_n \to p^*$ there exists $N \in \mathbb{N}$ such that $|g(p_n, p^*)| \leq \frac{1}{3}|g(p, p^*)|\ \forall\ n \geq N$. Hence

$$\begin{aligned}
\frac{1}{1+\lambda+\mu}|g^*(p_n, f(p_n))| &\leq |g(p_n, p_{n+1})| + |g^*(p_{n+1}, f(p_n))| \\
&\leq |g(p_n, p^*)| + |g(p^*, p_{n+1})| \\
&\leq \frac{2}{3}|g(p, p^*)| \\
&= |g(p, p^*)| - \frac{1}{3}|g(p, p^*)| \\
&\leq |g(p, p^*)| - |g(p_n, p^*)| \\
&\leq |g(p, p_n)|.
\end{aligned}$$

Now we have, $|g(p_{n+1}, f(p_n))| = D_g(A, B) = |g(s, f(p))|$ and

$$\frac{1}{1+\lambda+\mu}|g^*(p_n, f(p_n))| \leq |g(p_n, p)|.$$

Since f is a topologically Berinde weak proximal contraction mapping w.r.t g, we deduce that

$$|g(p_{n+1}, s)| \leq \lambda|g(p_n, p)| + \mu|g^*(p, f(p_n))|\ \forall\ n \geq N.$$

Therefore

$$\begin{aligned}
|g(p^*, f(p))| &= \lim_{k\to\infty} |g(p_{n_{k+1}}, f(p))| \\
&\leq \lim_{k\to\infty} [|g(p_{n_{k+1}}, s)| + |g(s, f(p))|] \\
&\leq \lim_{k\to\infty} [\lambda|g(p_{n_k}, p)| + \mu|g^*(p, f(p_{n_k}))| + D_g(A, B)] \\
&= \lambda|g(p^*, p)| + \mu|g^*(p, q^*)| + D_g(A, B)
\end{aligned}$$

$$\implies |g^*(p^*, f(p))| \leq \lambda|g(p^*, p)| + \mu|g^*(p, q^*)|.$$

We obtain

$$\begin{aligned}
|g^*(p_{n_k}, f(p_{n_k}))| &\leq |g(p_{n_k}, p^*)| + |g^*(p^*, f(p_{n_k}))| \\
&\leq |g(p_{n_k}, p^*)| + \lambda|g(p^*, p_{n_k})| + \mu|g^*(p_{n_k}, q^*)| \\
&\leq (1+\lambda)|g(p_{n_k}, p^*)| + \mu[|g(p_{n_k}, p^*)| + |g^*(p^*, q^*)|] \\
&\leq (1+\lambda+\mu)|g(p_{n_k}, p^*)|
\end{aligned}$$

$$|g(p_{n_{k+1}}, f(p_{n_k}))| = D_g(A, B) = |g(r^*, f(p^*))|$$

and

$$\frac{1}{1+\lambda+\mu}|g^*(p_{n_k}, f(p_{n_k}))| \leq |g(p_{n_k}, p^*)|$$

As f is a topologically Berinde weak proximal contraction w.r.t g, we get

$$\begin{aligned}|g(p_{n_{k+1}}, r^*)| &\leq \lambda|g(p_{n_k}, p^*)| + \mu|g^*(p^*, f(p_{n_k}))| \\ &\leq \lambda|g(p_{n_k}, p^*)| + \mu[|g(f(p_{n_k}), q^*)| + |g^*(p^*, q^*)|] \\ &\leq \lambda|g(p_{n_k}, p^*)| + \mu|g(f(p_{n_k}), q^*)|.\end{aligned}$$

Now by $k \to \infty$, we get $\lim_{k\to\infty} |g(p_{n_{k+1}}, r^*)| = 0$ i.e. $r^* = p^*$. Therefore p^* is a best proximity point of f.

Now, suppose the mapping f has two best proximity points p^* and p^{**}. So we have

$$|g(p^*, f(p^*))| = D_g(A, B)$$

and

$$|g(p^{**}, f(p^{**}))| = D_g(A, B)$$

and

$$\frac{1}{1 + \lambda + \mu}|g^*(p^*, f(p^*))| \leq |g(p^*, p^{**})|.$$

As f is topologically Berinde weak proximal contraction, so we have,

$$\begin{aligned}|g(p^*, p^{**})| \leq \lambda|g(p^*, p^{**})| + \mu|g^*(f(p^*), p^{**})| &\leq \lambda|g(p^*, p^{**})| + \mu|g(p^*, p^{**})| \\ &\implies (1 - \lambda - \mu)|g(p^*, p^{**})| \leq 0 \\ &\implies |g(p^*, p^{**})| = 0 \text{ [since } (1 - \lambda - \mu) > 0] \\ &\implies p^* = p^{**} \text{ [since } g(x, y) = 0 \Rightarrow x = y].\end{aligned}$$

So the best proximity point is unique.

Now we provide an example to validate Theorem 2.12.

Example 2.13. Consider $\mathbb{R}^2$ with the usual topology and $X = \{0\} \times \mathbb{R}$ with the subspace topology. Let $g : X \times X \to \mathbb{R}$ be defined by

$$g((x, y), (u, v)) = y - v.$$

Then g is a continuous function. Suppose

$$A = \{(0, 1), (0, 2), (0, 5)\} \text{ and } B = \{(0, 0), (0, 3), (0, 4), (0, 6)\}.$$

Then $D_g(A, B) = 1$ $A_g = A$ and $B_g = B$. Let $f : A \to B$ be a mapping defined by $f((0, 1)) = f((0, 2)) = (0, 3)$ and $f((0, 5)) = (0, 6)$.

Case 1. $u = (0, 5)$, $v = (0, 2)$, $x = (0, 5)$, $y = (0, 2)$. We have

$$\frac{1}{1 + 0.5 + 2}|g^*(x, f(x))| = \frac{1}{3.5} \times 0 \leq |g(x, y)|$$

$$\implies |g(u, v)| = 3 \leq \frac{1}{2}|g(x, y)| + 2|g^*(f(x), y)|.$$

Case 2. $u = (0, 2), v = (0, 5), x = (0, 2), y = (0, 5)$. We have

$$\frac{1}{1 + 0.5 + 2}|g^*(x, f(x))| = \frac{1}{3.5} \times 0 \leq |g(x, y)|$$

$$\implies |g(u, v)| = 3 \leq \frac{1}{2}|g(x, y)| + 2|g^*(f(x), y)|.$$

Case 3. $u = (0,5), v = (0,2), x = (0,5), y = (0,1)$. We have

$$\frac{1}{1+0.5+2}|g^*(x,f(x))| = \frac{2}{3.5} \times 0 \leq |g(x,y)|$$

$$\implies |g(u,v)| = 3 \leq \frac{1}{2}|g(x,y)| + 2|g^*(f(x),y)|.$$

Case 4. $u = (0,2), v = (0,5), x = (0,1), y = (0,5)$. We have

$$\frac{1}{1+0.5+2}|g^*(x,f(x))| = \frac{1}{3.5} \times 1 \leq |g(x,y)|$$

$$\implies |g(u,v)| = 3 \leq \frac{1}{2}|g(x,y)| + 2|g^*(f(x),y)|.$$

Therefore, f is a topologically Berinde weak proximal contraction with respect to g with $\lambda = \frac{1}{2}$ and $\mu = 2$ and satisfies all conditions of the Theorem 2.12. Here $p^* = (0,2)$ and $p^{**} = (0,5)$ are best proximity points of f. We observe that here $(1-\lambda-\mu) < 0$ and best proximity point of f is not unique. So this example not only just validate Theorem 2.12, but also shows that if the condition $(1-\lambda-\mu) > 0$ is violated, then the mapping f may have more than one best proximity point.

2.2. **Best proximity point theorem for topologically proximal weakly contraction.**

In this paper, we use the symbol Θ to denote the class of mappings $\psi : [0,\infty) \to [0,\infty)$ such that ψ is continuous, non-decreasing, $\psi(t) > 0$ for all $t \in (0,\infty)$, $\psi(0) = 0$ and $\lim_{s\to\infty} \psi(s) = \infty$. If we take $\psi(t) = t^2,\ t \in [0,\infty)$ then $\psi \in \Theta$. So $\Theta \neq \phi$.

In the year 2013, Gabeleh [2] introduced the notion of a proximal weakly contractive mapping defined on a non-empty subset of a metric space as follows:

Definition 2.14. [2] Let (A,B) be a pair of non-empty subsets of a metric space (X,d). A mapping $T : A \to B$ is said to be proximal weakly contractive if there exists $\psi \in \Theta$ such that $d\Big(u_1, T(x_1)\Big) = D(A,B)$ and $d\Big(u_2, T(x_2)\Big) = D(A,B)$

$$\implies d(u_1,u_2) \leq d(x_1,x_2) - \psi\Big(d(x_1,x_2)\Big) \text{ for all } u_1,u_2,x_1,x_2 \in A.$$

Now we like to introduce the notion of topologically weakly contractive and topologically proximal weakly contractive mapping in a topological space X as follows:

Definition 2.15. Let (A,B) be a pair of non-empty subsets of a topological space X and $g : X \times X \to \mathbb{R}$ be a continuous function. A mapping $T : A \to B$ is said to be topologically weakly contractive with respect to g if there exists $\psi \in \Theta$ such that

$$|g\left(T(x_1),T(x_2)\right)| \leq |g(x_1,x_2)| - \psi\left(|g(x_1,x_2)|\right)$$

for all $x_1, x_2 \in A$.

Definition 2.16. Let (A,B) be a pair of non-empty subsets of a topological space X and $g : X \times X \to \mathbb{R}$ be a continuous function. A mapping $T : A \to B$ is said to be topologically proximal weakly contractive with respect to g if there exists $\psi \in \Theta$ such that

$$\Big|g(u_1, T(x_1))\Big| = D_g(A, B) = \Big|g(u_2, T(x_2))\Big|$$
$$\Longrightarrow \Big|g(u_1, u_2)\Big| \le \Big|g(x_1, x_2)\Big| - \psi\Big(|g(x_1, x_2)|\Big)$$

for all $u_1, u_2, x_1, x_2 \in A$.

Note 2.17. If the topological space X is metrizable with respect to a metric d then in Definition 2.16, by taking $g = d$ and $\psi(t) = (1 - \alpha)t,$ for $t \in [0, \infty)$ and $\alpha \in (0, 1)$, we will get the notion of proximal contraction for standard metric spaces introduced by Basha in [1] and if we take $g = d$, we will get the notion of proximal weakly contractive mapping for standard metric spaces introduced by Gabeleh in [2].

In Definition 2.16, we mention that the mapping f is topologically proximal weakly contractive with respect to the continuous mapping g and it is important. In our upcoming example, we show that there exist two subsets A and B of a topological space X and a mapping $T : A \to B$ such that T is topologically proximal weakly contractive with respect to a continuous function g but is not topologically proximal weakly contractive with respect to another continuous function h.

Example 2.18. Consider $\mathbb{R}^2$ with the usual topology. Let $A = \{1\} \times [0, 1]$ and $B = [0, 1] \times \{1\}$ and $T : A \to B$ be defined by $T(1, y) = (y, 1)$. Let $g : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ be defined by $g\Big((x, y), (u, v)\Big) = yv$. Then g is a continuous function. Now, we show that T is topologically proximal weakly contractive with respect to g. Let us define $\psi : [0, \infty) \to [0, \infty)$ by $\psi(t) = t^2,\ t \in [0, \infty)$. Then $\psi \in \Theta$. It is clear that $D_g(A, B) = 0$. Now let $x_1 = (1, p_1), x_2 = (1, p_2), u_1 = (1, y_1), u_2 = (1, y_2) \in A$ and $\Big|g(x_1, T(u_1))\Big| = 0$ and $\Big|g(x_2, T(u_2))\Big| = 0$. So $\Big|g((1, p_1), (y_1, 1))\Big| = 0$ which follows that $p_1 = 0$. Similarly, from the second equation, we get $p_2 = 0$. Now, $\Big|g(x_1, x_2)\Big| = p_1p_2 = 0$. On the other hand

$$\Big|g(u_1, u_2)\Big| - \psi\Big(|g(u_1, u_2)|\Big) = y_1y_2 - (y_1y_2)^2 \ge 0.$$

This shows that T is topologically proximal weakly contractive with respect to g.

On the other hand, let $h : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ be defined by $h\Big((x, y), (u, v)\Big) = xu$. It can be seen that $D_h(A, B) = 0$. Let $x_1 = (1, \frac{1}{2}), x_2 = (1, \frac{1}{4}), u_1 = (1, 0), u_2 = (1, 0) \in A$ and $\Big|h(x_1, T(u_1))\Big| = 0$ and $\Big|h(x_2, T(u_2))\Big| = 0$. But,

$$1 = \Big|h(x_1, x_2)\Big| > \Big|h(u_1, u_2)\Big| - \psi\Big(|h(u_1, u_2)|\Big) = 0.$$

This shows that T is not topologically proximal weakly contractive with respect to h.

In our next example, we show that the notion of topologically proximal weakly contractive with respect to a continuous function is indeed more general than the notion of proximal weakly contractive introduced by Gabeleh in [2]. We show that, there exists a topological space X with a real-valued continuous function g, two non-empty disjoint subsets A, B of X and a function $Q : A \to B$ such that Q is topologically proximal weakly contractive w.r.t g, but if the topological space is metrizable with respect to a metric d, then Q is not proximal weakly contractive w.r.t the metric d.

Example 2.19. Consider $\mathbb{R}$ with the usual topology. Let $g : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be defined by

$$g(x, y) = x^2 - y^2,\ x, y \in \mathbb{R}.$$

Then g is a continuous function. Let $A = \{0, 1, 2, 3, 5\}$ and $B = \{-1, -2, -3, 4\}$ and $Q : A \to B$ be defined by $Q(0) = Q(2) = 4,\ Q(1) = Q(3) = -1,\ Q(5) = -2$. Then, it can be seen that $D_g(A, B) = 0$. Let $\psi : [0, \infty) \to [0, \infty)$ be defined by

$$\psi(t) = \frac{t}{2},\ t \in [0, \infty).$$

Then $\psi \in \Theta$. Now

$$\Big|g\Big(1, Q(3)\Big)\Big| = D_g(A, B)$$

and

$$\Big|g\Big(2, Q(5)\Big)\Big| = D_g(A, B).$$

Then,

$$\Big|g\Big(1, 2\Big)\Big| \leq \Big|g\Big(3, 5\Big)\Big| - \psi\Big(\Big|g\Big(3, 5\Big)\Big|\Big).$$

Also,

$$\Big|g\Big(1, Q(1)\Big)\Big| = D_g(A, B)$$

and

$$\Big|g\Big(2, Q(5)\Big)\Big| = D_g(A, B).$$

Then,

$$\Big|g\Big(1, 2\Big)\Big| \leq \Big|g\Big(1, 5\Big)\Big| - \psi\Big(\Big|g\Big(1, 5\Big)\Big|\Big).$$

This shows that Q is topologically proximal weakly contractive w.r.t g. Now let d denotes the usual metric on $\mathbb{R}$. Then $D(A, B) = \inf\{d(x, y) : x \in A, y \in B\} = 1$. Now

$$d\Big(0, Q(1)\Big) = D(A, B)$$

and

$$d\Big(3, Q(0)\Big) = D(A, B).$$

But

$$d(0, 3) > d(1, 0) - \psi\Big(d(1, 0)\Big).$$

So, Q is not proximal weakly contractive with respect to the usual metric on $\mathbb{R}$.

As in Note 2.17, we mentioned that if the topological space X is metrizable with respect to a metric d then by taking $g = d$ and $\psi(t) = (1 - \alpha)t,$ for $t \in [0, \infty)$ and $\alpha \in (0, 1)$, we will get the notion of proximal contraction for standard metric spaces. So, the notion of topologically proximal weakly contractive is more general than proximal contractions. But in next example we show that the class of all topologically proximal weakly contractive mappings are different from the class of all proximal contractions.

Example 2.20. Consider $\mathbb{R}^2$ with the usual topology and

$$A = \{1\} \times [-1, 0],\ B = \{1\} \times [0, 1].$$

Define $T : A \to B$ by

$$T(1, x) = (1, -\frac{x}{2}),\ (1, x) \in A.$$

Let $\psi : [0, \infty) \to [0, \infty)$ be defined by

$$\psi(t) = \frac{t}{2};\ t \in [0, \infty).$$

Then $\psi \in \Theta$. Now, it can be seen that, T is a proximal contraction with respect to the standard metric d on $\mathbb{R}^2$.

Let $g : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$g\left((x, y), (u, v)\right) = yv,\ (x, y), (u, v) \in \mathbb{R}^2.$$

Then g is a continuous function on $\mathbb{R}^2 \times \mathbb{R}^2$ and $D_g(A, B) = 0$. Let $x_1 = (1, -\frac{1}{4})$, $x_2 = (1, -\frac{1}{2})$, $u_1 = (1, 0)$, $u_2 = (1, 0)$. So we have $|g\left(x_1, T(u_1)\right)| = 0$ and $|g\left(x_2, T(u_2)\right)| = 0$. But

$$\frac{1}{8} = |g(x_1, x_2)| > |g(u_1, u_2)| - \psi\left(|g(u_1, u_2)|\right) = 0.$$

This shows that T is not topologically proximal weakly contractive with respect to g.

Now we recall the notion of P-property and topological P-property from [2, 7] as follows:

Definition 2.21. [2] Let (A, B) be a pair of non-empty subsets of a metric space (M, d). The pair (A, B) is said to have the P-property if $d(u_1, x_1) = D(A, B)$ and $d(u_2, x_2) = D(A, B) \implies d(u_1, u_2) = d(x_1, x_2)$ for all $u_1, u_2 \in A_0$ and $x_1, x_2 \in B_0$ where,

$$A_0 = \{x \in A : d(x, y) = D(A, B) \text{ for some } y \in B\},$$
$$B_0 = \{y \in B : d(x, y) = D(A, B) \text{ for some } x \in A\}.$$

Definition 2.22. [7] Let (A, B) be a pair of non-empty subsets of a topological space X and $g : X \times X \to \mathbb{R}$ be a continuous function. The pair (A, B) is said to have topological P-property with respect to g if

$$\Big|g((u_1, x_1))\Big| = D_g(A, B) = \Big|g((u_2, x_2))\Big|$$
$$\implies \Big|g((u_1, u_2))\Big| = \Big|g((x_1, x_2))\Big|$$

for all $u_1, u_2 \in A_g$ and $x_1, x_2 \in B_g$.

In upcoming example, we show that the notion of topological P-property with respect to a continuous function is more general than the notion of P-property for metric spaces.

Example 2.23. Consider $\mathbb{R}$ with the usual topology. Let $g : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be defined by

$$g(x, y) = x^2 - y^2,\ x, y \in \mathbb{R}.$$

Then g is a continuous function. Let $A = \{0, 1, 2, 3, 5\}$ and $B = \{-1, -2, 4\}$. Then

$$D(A, B) = \inf\{d(x, y) : x \in A, y \in B\} = 1$$

and

$$D_g(A, B) = \inf\Big\{|g(x, y)| : x \in A, y \in B\Big\} = 0.$$

Now, we show that the pair (A, B) does not have the P-property with respect to the standard metric d on $\mathbb{R}$ but the pair (A, B) have topological P-property with respect to g. Now

$$d(0, -1) = D(A, B)$$

and

$$d(3, 4) = D(A, B).$$

But $d(0, 3) \neq d(-1, 4)$. On the other hand, the relation

$$\Big|g(u_1, x_1)\Big| = D_g(A, B)$$

and

$$\Big|g(u_2, x_2)\Big| = D_g(A, B)$$

hold only if $u_1 = 1, x_1 = -1$ and $u_2 = 2, x_2 = -2$.
In this case, $\Big|g(u_1, u_2)\Big| = \Big|g(x_1, x_2)\Big|$.

Now we present a best proximity point theorem for topologically proximal weakly contractive mappings as follows.

Theorem 2.24. *Let X be a g-complete topological space where $g : X \times X \to \mathbb{R}$ is a continuous function such that $g(x, y) = 0 \implies x = y$, $|g(x, y)| = |g(y, x)|$ and $|g(x, z)| \leq |g(x, y)| + |g(y, z)|$ for all $x, y, z \in X$. Let (A, B) be a pair of non-empty g-closed subsets of X such that the pair (A, B) have topological P-property and A_g is non-empty. Let $T : A \to B$ be topologically proximal weakly contractive mapping w.r.t g such that $T(A_g) \subseteq B_g$. Then there exists a unique best proximity point $p^* \in A_g$ of T.*

Proof. Let $p_0 \in A_g$. Since $T(A_g) \subseteq B_g \implies T(p_0) \in B_g$. So there exists $p_1 \in A_g$ such that $\Big|g(p_1, T(p_0))\Big| = D_g(A, B)$. Now $T(p_1) \in B_g$ so, there exists $p_2 \in A_g$ such that $\Big|g(p_2, T(p_1))\Big| = D_g(A, B)$. Continuing in this way we get a sequence $\{p_n\} \subset A_g$ such that $\Big|g(p_{n+1}, T(p_n))\Big| = D_g(A, B)$, for all $n \geq 0$. Now

$$\Big|g(p_{n+1}, T(p_n))\Big| = D_g(A, B)$$

and

$$\Big|g(p_n, T(p_{n-1}))\Big| = D_g(A, B).$$

As T is topologically proximal weakly contractive with respect to g,so there exists $\psi \in \Theta$ such that

$$\Big|g(p_{n+1}, p_n)| \leq \Big|g(p_n, p_{n-1})\Big| - \psi\Big(|g(p_n, p_{n-1})|\Big)$$

$$\Longrightarrow \Big|g(p_{n+1}, p_n)| \leq \Big|g(p_n, p_{n-1})\Big|.$$

So, the sequence $\Big\{|g(p_{n+1}, p_n)|\Big\}$ is a decreasing sequence and let $|g(p_{n+1}, p_n)| \to q$ as $n \to \infty$. By the continuity of ψ and g we can show that $q = 0$. Since $\Big|g(x_{n+1}, x_n)\Big| \to 0$ as $n \to \infty$ so for a preassigned $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that

$$\Big|g(x_{m+1}, x_m)\Big| < \min\Big\{\frac{\varepsilon}{2}, \psi(\frac{\varepsilon}{2})\Big\} \text{ for all } m \geq n_0.$$

Now, from the selection of the sequence $\{p_n\}$ we have,

$$\begin{aligned} \Big|g(p_{m+1}, T(p_m))\Big| &= D_g(A, B), \\ \Big|g(p_m, T(p_{m-1}))\Big| &= D_g(A, B). \end{aligned} \tag{2.1}$$

Since the pair (A, B) have the topological P-property, so from equation 2.1, we have

$$\Big|g(p_{m+1}, p_m)\Big| = \Big|g(T(p_m), T(p_{m-1}))\Big|.$$

Also,

$$\begin{aligned} \Big|g(p_{m+2}, T(p_{m+1}))\Big| &= D_g(A, B), \\ \Big|g(p_{m+1}, T(p_m))\Big| &= D_g(A, B). \end{aligned} \tag{2.2}$$

As, T is topologically proximal weakly contractive with respect to g, so, we have from equation 2.2 that

$$\begin{aligned} \Big|g(p_{m+2}, p_{m+1})\Big| &\leq \Big|g(p_{m+1}, p_m)\Big| - \psi\Big(|g(p_{m+1}, p_m)|\Big) \\ &\leq \Big|g(p_{m+1}, p_m)\Big|. \end{aligned} \tag{2.3}$$

Now from equation 2.3 we have

$$\begin{aligned} \Big|g(T(p_{m+1}), T(p_{m-1}))\Big| &\leq \Big|g(T(p_{m+1}), T(p_m))\Big| + \Big|g(T(p_m), T(p_{m-1}))\Big|, \\ &= \Big|g(p_{m+2}, p_{m+1})\Big| + \Big|g(p_{m+1}, p_m)\Big|, \\ &\leq 2\Big|g(p_{m+1}, p_m)\Big|, \\ &< \varepsilon. \end{aligned} \tag{2.4}$$

This is true for all $m \geq n_0$. Now by using the topological P-property we have $\Big|g(p_{m+2}, p_m)\Big| = \Big|g(T(p_{m+1}), T(p_{m-1}))\Big| < \varepsilon$ for all $m \geq n_0$.
Now we have

$$\begin{aligned} \Big|g(p_{m+3}, T(p_{m+2}))\Big| &= D_g(A, B), \\ \Big|g(p_{m+1}, T(p_m))\Big| &= D_g(A, B). \end{aligned} \tag{2.5}$$

As T is topologically proximal weakly contractive with respect to g, so we have from equation 2.5 that

$$\begin{aligned} \Big|g(p_{m+3}, p_{m+1})\Big| &\leq \Big|g(p_{m+2}, p_m)\Big| - \psi\Big(|g(p_{m+2}, p_m)|\Big) \\ &\leq \Big|g(p_{m+2}, p_m)\Big|. \end{aligned} \tag{2.6}$$

Now from equation 2.6, if $\Big|g(p_{m+2}, p_m)\Big| < \frac{\varepsilon}{2}$ then we have for all $m \geq n_0$,

$$\begin{aligned} \Big|g(T(p_{m+2}), T(p_{m-1}))\Big| &\leq \Big|g(T(p_{m+2}), T(p_m))\Big| + \Big|g(T(p_m), T(p_{m-1}))\Big|, \\ &= \Big|g(p_{m+3}, p_{m+1})\Big| + \Big|g(p_{m+1}, p_m)\Big|, \\ &\leq \Big|g(p_{m+2}, p_m)\Big| + \Big|g(p_{m+1}, p_m)\Big|, \\ &< \varepsilon. \end{aligned} \tag{2.7}$$

If for all $m \geq n_0$, $\varepsilon > \Big|g(p_{m+2}, p_m)\Big| \geq \frac{\varepsilon}{2}$ then $\psi\Big(\Big|g(p_{m+2}, p_m)\Big|\Big) \geq \psi\Big(\frac{\varepsilon}{2}\Big)$ as $\psi \in \Theta$, then from equation 2.6 and 2.7, we have for all $m \geq n_0$,

$$\begin{aligned} \Big|g(T(p_{m+2}), T(p_{m-1}))\Big| &\leq \Big|g(T(p_{m+2}), T(p_m))\Big| + \Big|g(T(p_m), T(p_{m-1}))\Big|, \\ &= \Big|g(p_{m+3}, p_{m+1})\Big| + \Big|g(p_{m+1}, p_m)\Big|, \\ &\leq \Big|g(p_{m+1}, p_m)\Big| + \Big|g(p_{m+2}, p_m)\Big| - \psi\Big(|g(p_{m+2}, p_m)|\Big), \\ &< \psi\Big(\frac{\varepsilon}{2}\Big) + \varepsilon - \psi\Big(\frac{\varepsilon}{2}\Big), \\ &= \varepsilon. \end{aligned} \tag{2.8}$$

So $\Big|g(p_{m+3}, p_m)\Big| = \Big|g(T(p_{m+2}), T(p_{m-1}))\Big| < \varepsilon$ for all $m \geq n_0$. Similarly we can show that $\Big|g(p_{m+q}, p_m)\Big| < \varepsilon$ for all $m \geq n_0$ and $q \in \mathbb{N}$. This shows that the sequence $\{p_n\}$ is a g-Cauchy sequence in A_g. Since X is g-complete, the pair (A, B) is g-closed and have the topological P-property so, it can be easily seen that A_g is g-closed. Since A_g is g-closed, so, let $\{p_n\}$ is g-convergent to $p^* \in A_g$. As $T(p^*) \in B_g$ so there exists $p^{**} \in A_g$ such that $\Big|g(p^{**}, T(p^*))\Big| = D_g(A, B)$. Also we have $\Big|g(p_{n+1}, T(p_n))\Big| = D_g(A, B)$. As T is topologically proximal weakly contractive with respect to g, so we have

$$\Big|g(p_{n+1}, p^{**})\Big| \leq \Big|g(p_n, p^*)\Big| - \psi\Big(|g(p_n, p^*)|\Big)$$

$$\implies \Big|g(p_{n+1}, p^{**})\Big| \leq \Big|g(p_n, p^*)\Big|$$

$$\implies \Big|g(p_{n+1}, p^{**})\Big| \to 0 \text{ as } n \to \infty.$$

Now from lemma [10, Lemma 2.4.], since the limit is unique so, we have $p^* = p^{**}$. So, $\Big|g(p^*, T(p^*))\Big| = D_g(A, B)$. Hence p^* is a best proximity point of T. Now suppose

p^*, q^* are two best proximity points of the mapping T. So,

$$\Big|g(p^*, T(p^*))\Big| = D_g(A, B)$$

and

$$\Big|g(q^*, T(q^*))\Big| = D_g(A, B).$$

As T is topologically proximal weakly contractive with respect to g**,** so we have

$$\begin{aligned}\Big|g(p^*, q^*)\Big| &\leq \Big|g(p^*, q^*)\Big| - \psi\Big(\Big|g(p^*, q^*)\Big|\Big)\\ &\Longrightarrow \psi\Big(\Big|g(p^*, q^*)\Big|\Big) = 0\\ &\Longrightarrow \Big|g(p^*, q^*)\Big| = 0\\ &\Longrightarrow p^* = q^*.\end{aligned}$$

So T has a unique best proximity point in A_g. This completes the proof.

Example 2.25. Consider $\mathbb{R}^2$ with the usual topology and $X = \{1\} \times [-1, 1]$ with the subspace topology. Let $g : X \times X \to \mathbb{R}$ be defined by

$$g\Big((x, y), (u, v)\Big) = y - v,\ (x, y), (u, v) \in X.$$

Then g is a continuous function on $X \times X$. We will show that X is g-complete. Let $\{(1, x_n)\}$ be a g-Cauchy sequence in X. So

$$\begin{aligned}&\Big|g\Big((1, x_n), (1, x_m)\Big)\Big| \to 0 \text{ as } n, m \to \infty\\ &\Longrightarrow |x_n - x_m| \to 0 \text{ as } n, m \to \infty.\end{aligned}$$

So, the sequence $\{x_n\}$ is a Cauchy sequence in $[-1, 1]$. Since $[-1, 1]$ is complete, so let $x_n \to p \in [-1, 1]$ as $n \to \infty$. Now,

$$\Big|g\Big((1, x_n), (1, p)\Big)\Big| = |x_n - p| \to 0 \text{ as } n \to \infty.$$

So, the sequence $\{(1, x_n)\}$ is g-convergent to $(1, p) \in X$. This implies X is g-complete. Now let $A = \{1\} \times [-1, 0]$ and $B = \{1\} \times [0, 1]$. Then $D_g(A, B) = 0$. Let $(1, x) \in A_g$. Then there exists $(1, y) \in B$ such that $|g((1, x), (1, y))| = 0$. So $|x - y| = 0$. This is satisfied only by $x = 0$. This shows that $A_g = \{(1, 0)\}$. Also, $B_g = \{(1, 0)\}$. So, the pair (A, B) is g-closed, have the topological P-property and A_g is non-empty. Now it can be seen that the function g is satisfied all the conditions of Theorem 2.24.

Define $f : A \to B$ by

$$f(1, x) = (1, -\frac{x}{2}),\ (1, x) \in A.$$

So, $f(1, 0) = (1, 0) \Longrightarrow f(A_g) \subseteq B_g$. Let $\psi : [0, \infty) \to [0, \infty)$ be defined by

$$\psi(s) = \frac{s}{2} \text{ for all } s \in [0, \infty).$$

It can be easily seen that $\psi \in \Theta$ and the mapping f is topologically proximal weakly contractive with respect to g. So, all the conditions of Theorem 2.24 are satisfied. So, by the Theorem 2.24 the mapping f has unique best proximity point in A_g. Here $p^* = (1, 0) \in A_g$ is the best proximity point of f.

In the year 2013, M. Gabeleh [3] proved that the best proximity point theorem for proximal weakly contractive mapping can be deduced from a fixed point theorem. Next, we are going to introduce the notion of g-isometry in topological spaces. It is more general than isometry in metric spaces. We also give an example supporting this. We use g-isometry together with topological P-property, for proving the best proximity point theorem for topologically proximal weakly contractive mapping deduced from some fixed point theorems on arbitrary topological spaces given by Liepins [6].

Definition 2.26. Let X be a topological space and $g : X \times X \to \mathbb{R}$ be a continuous function. The mapping $T : A \to B$ is said to be a g-isometry if

$$|g(Tx, Ty)| = |g(x, y)| \text{ for all } x, y \in A.$$

The next example shows that g-isometry is not in general an isometry.

Example 2.27. Consider $X = \mathbb{R}^2$ with usual topology. Let $g : X \times X \to \mathbb{R}$ be defined by $|g((x, u), (y, v))| = uv$ for all $(x, u),\ (y, v) \in \mathbb{R}^2$. Let $f : A \to B$ be defined by $f(x, y) = (\frac{x}{2}, y)$ for all $(x, y) \in A$. Therefore

$$|g(f(x, u), f(y, v))| = \left|g\left(\left(\frac{x}{2}, u\right), \left(\frac{y}{2}, v\right)\right)\right| = uv = |g((x, u), (y, v))|$$

for all $(x, u),\ (y, v) \in \mathbb{R}^2$. So, f is a g-isometry.

Now,

$$d((x, u), (y, v)) = \sqrt{(x - y)^2 + (u - v)^2}.$$

Therefore

$$d(f(x, u), f(y, v)) = d\left(\left(\frac{x}{2}, u\right), \left(\frac{y}{2}, v\right)\right) = \sqrt{\left(\frac{x}{2} - \frac{y}{2}\right)^2 + (u - v)^2}.$$

So, $d(f(x, u), f(y, v)) \neq d((x, u), (y, v))$. Therefore, f is not an isometry with respect to usual metric d.

Next, we show that the best proximity point theorem for topologically proximal weakly contractive mapping can be deduced from fixed point theorems [6, Theorem 1] and [8, Theorem 3.9].

Here we use the notation $lp\ O(f, x) = \cap\{cl\{f^m(x) : m \geq n; n \in \mathbb{N}\}\}$; where we use the symbol $'cl'$ to denote the closure of a set.

We recall the following theorems:

Theorem 2.28. [6] *Let X be a topological space, and let f be a continuous selfmap of X. Suppose there exists a continuous mapping $g : X \times X \to \mathbb{R}$ satisfying $x \neq y \Rightarrow |g(f(x), f(y))| \leq |g(x, y)|$ for each $x, y \in X$. For each $z \in X$ then $lp\ O(z, f)$ is empty or consists of a single point, which is the unique fixed point of f.*

Theorem 2.29. [8] *Let X be a topological space, $f : X \to X$ be a continuous mapping and for some continuous $F : X \times X \to \mathbb{R}$ (with $F(x,y) = 0$ when $x = y$),*

$$F(f(x), f(y)) < max\{F(x,y), [max\{F(x, f(x)), F(y, f(y))\} + \lambda min\{|F(x, f(y))|, F(f(x), y)\}]\}$$

for all $x, y \in X$, $x \neq y$; where $\lambda \geq 0$. If there exists a point $x_0 \in X$ whose sequence of iterates $\{f^n(x_0)\}$ contains an convergent subsequence $\{f^{n_i}(x_0)\}$, then $a = \lim_{i\to\infty} f^{n_i}(x_0) \in X$ is a fixed point of f. If $\lambda = 0$, then f has a unique fixed point.

Theorem 2.30. *Let X be a topological space and $g : X \times X \to \mathbb{R}$ be a continuous mapping such that $|g(x,y)| = 0 \Leftrightarrow x = y$. Let (A,B) be a pair of non-empty subsets of X such that A_g is non-empty. Let $T : A \to B$ be topologically proximal weakly contractive mapping w.r.t g such that $T(A_g) \subset B_g$. Then we have the following:*
(1) If (A,B) has the topological P-property then there exists a bijective g-isometry $f : A_g \to B_g$ s.t. $|g(x, f(x))| = D_g(A,B)$.
(2) Further, if $f^{-1}T : A_g \to A_g$ is continuous, then the existence of best proximity point of the mapping T implies its uniqueness.
(3) Moreover, let $Im(g) \subset \mathbb{R}_+$. If there exists a point $z \in A_g$ such that the sequence of iterates $\{f^{-1}T\}^n(z)$ contains a convergent subsequence $\{f^{-1}T\}^{n_k}(z)$, then T has an unique best proximity point $p = \lim_{k\to\infty}\{f^{-1}T\}^{n_k}(z)$.

Proof. (1) Let $x \in A_g$. Then by definition there exists $y \in B_g$ such that

$$|g(x,y)| = D_g(A,B).$$

Now we define $f : A_g \to B_g$ by $f(x) = y$.

Let $x,\ x' \in A_g$. Then $|g(x, f(x))| = D_g(A,B) = |g(x', f(x'))|$. If $x = x'$ then by topological P-property of (A,B), we have $f(x) = f(x')$. This implies that f is well defined. If $f(x) = f(x')$, again by topological P-property $x = x'$. Therefore f is injective.

Next, we show that f is surjective. Let $y \in B_g$ then there exists $x \in A_g$ s.t $|g(x,y)| = D_g(A,B)$. Also, by definition of f, we have $|g(x, f(x))| = D_g(A,B)$. Therefore, topological P-property of (A,B) gives $f(x) = y$. Therefore f is a bijection. Let $x_1,\ x_2 \in A_g$. Then

$$|g(x_1, f(x_1))| = D_g(A,B) = |g(x_2, f(x_2))|.$$

By topological P-property we have that $|g(x_1,x_2)| = |g(f(x_1), f(x_2))|$. Therefore, f is an bijective g-isometry.

(2) Since T is topologically proximal weakly contractive mapping w.r.t g, so, we have,$|g(u_1, T(x_1))| = D_g(A,B)$ and $|g(u_2, T(x_2))| = D_g(A,B)$
$\Longrightarrow |g(u_1,u_2)| \leq |g(x_1,x_2)| - \psi(|g(x_1,x_2)|)$ for all $u_1,\ u_2,\ x_1,\ x_2 \in A$ and $\psi \in \Theta$. By topological P-property, $|g(u_1,u_2)| = |g(T(x_1), T(x_2))|$. So,

$$|g(T(x_1), T(x_2))| \leq |g(x_1,x_2)| - \psi(|g(x_1,x_2)|)$$

for all $x_1,\ x_2 \in A_g$ and $\psi \in \Theta$. Therefore, it is sufficient to prove the theorem for topologically weakly contractive mapping T on A_g. Now we have $f^{-1}T : A_g \to A_g$ is continuous. Since f is bijective, then,

$$\left|g\left(f^{-1}T(x), f^{-1}T(y)\right)\right| = |g\left(T(x), T(y)\right)| \leq |g(x,y)| - \psi\left(|g(x,y)|\right) < |g(x,y)|$$

for all $x,\ y \in A_g$ with $x \neq y$.

Then for each $x \in A_g$, $lp\ O(f^{-1}T, x)$ is empty or consists of a single point which is the unique fixed point of $f^{-1}T$ by Theorem 1 of [6]. Let p be a fixed point of $f^{-1}T$, then $f^{-1}T(p) = p \Longrightarrow f(p) = T(p)$. Therefore,

$$|g\left(p, T(p)\right)| = |g\left(p, f(p)\right)| = D_g(A, B).$$

Therefore p is the unique best proximity point of T.
(3) Since, $Im(g) \subset \mathbb{R}_+$ and T is topologically weakly contractive mapping on A_g w.r.t g then by the previous result, we get,

$$g\left(T(x_1), T(x_2)\right) \leq g(x_1, x_2) - \psi\left(g(x_1, x_2)\right) < g(x_1, x_2)$$

for all $x_1,\ x_2 \in A_g$ and $\psi \in \Theta$.

Therefore we can write,

$$\begin{aligned} g\left(T(x_1), T(x_2)\right) &< g(x_1, x_2) \\ &\leq \max\{g(x_1, x_2), (\max\{g\left(x_1, T(x_1)\right), g\left(x_2, T(x_2)\right)\} \\ &\quad +\lambda \min\{|g\left(x_1, T(x_2)\right)|, g\left(T(x_1), x_2\right)\})\} \end{aligned}$$

for all $x_1,\ x_2 \in A_g$ with $\lambda = 0$. Then by condition (3) and [8, Theorem 3.9], we get $f^{-1}T$ has a unique fixed point. In a similar manner, it can be shown that T has a unique best proximity point in A_g.

Now we present an example to validate Theorem 2.30.

Example 2.31. Consider $\mathbb{R}^2$ with the usual topology and $X = [0, 100]$ with subspace topology. Let $g : X \times X \to \mathbb{R}$ be defined by

$$g\left((x, y)\right) = \sqrt{y} - \sqrt{x},\ (x, y) \in X.$$

Then g is a continuous function on $X \times X$. Now, let $A = \{0, 9, 50\}$ and $B = \{1, 16\}$. Then $D_g(A, B) = 1$ and $A_g = \{0, 9\}$. Also, $B_g = \{1, 16\}$. So, A_g is non-empty. Again, $|g(0, 1)| = 1 = |g(9, 16)|$ and $|g(0, 9)| = 3 = |g(1, 16)|$. This shows that (A, B) satisfies topological P property with respect to g. Now define $f : A_g \to B_g$ by $f(0) = 1$ and $f(9) = 16$. Therefore, $|g(f(0), f(9))| = |g(1, 16)| = 3 = |g(0, 9)|$. So, f is an bijective g-isometry. Next, define $T : A \to B$ by $T(0) = 16; T(9) = 16; T(50) = 1$. So $T(A_g) \subseteq B_g$.

$|g\left(0, T(50)\right)| = 1 = |g\left(9, T(0)\right)| \Longrightarrow |g(0, 9)| = 3 \leq |g(50, 0)| - \psi\left(|g(50, 0)|\right)$
$|g\left(0, T(50)\right)| = 1 = |g\left(9, T(9)\right)| \Longrightarrow |g(0, 9)| = 3 \leq |g(50, 9)| - \psi\left(|g(50, 9)|\right)$
where $\psi(t) = \frac{1}{50}t$.

Therefore T is a topologically proximal weakly contractive mapping with respect to g. Also A_g is compact. So, the mapping T has a unique best proximity point in A_g. Here $p = 9 \in A_g$ is a best proximity point of T and the best proximity point is unique.
Now if we consider d as a standard metric on $\mathbb{R}$, then $A_0 = \{0\}$ and $B_0 = \{1\}$. But

$T(A_0) = \{16\} \subsetneq B_0$. So it does not satisfy the hypothesis of Theorem 2.30. Also, it is clear that the mapping T has no best proximity point w.r.t d.

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