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EXISTENCE OF SOLUTIONS FOR SEMILINEAR FRACTIONAL INTEGRO-DIFFERENTIAL EQUATIONS WITH NONLOCAL CONDITIONS

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Abstract. In this work, we consider the existence of solutions for semilinear fractional integro-differential equations with nonlocal conditions. A new set of sufficient conditions proving existence of solutions are derived by using the fractional calculus, α-resolvent operators theory, measure of noncompactness and some fixed point theorems. In the end, an example is provided to illustrate the applicability of our results.

Key Words and Phrases: Fractional integro-differential equation, α-resolvent operator, existence, nonlocal condition, fixed point.
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1. Introduction

In this papaer, we consider the existence of solutions for semilinear fractional integro-differential equations with nonlocal conditions of the form

$$\begin{cases} D_t^\alpha x(t) = Ax(t) + \displaystyle\int_0^t B(t-s)x(s)ds + G(t, x(h(t))),\ t \in [0,T], \\ x(0) + g(x) = x_0, \end{cases} \tag{1.1}$$

where $x(\cdot)$ is the state variable taking values in a Banach space X. A, $(B(t))_{t\geq 0}$ are closed linear operators defined on a common domain which is dense in X, and $D_t^\alpha v(t)$ represents the Caputo derivative of v for $\alpha \in (0,1)$ defined by

$$D_t^\alpha v(t) := \int_0^t \psi_{1-\alpha}(t-s)v'(s)ds,$$

where $\psi_{1-\alpha}$ is the Gelfand-Shilov function $\psi_\beta(t) := \frac{t^{\beta-1}}{\Gamma(\beta)}$, $t > 0$, with $\beta = 1-\alpha$. The function $h(\cdot) : [0,T] \to [0,T]$ is continuous and satisfies $0 \leq h(t) \leq t$, which is regarded as delay function. G and g are given functions to be specified later.

Fractional differential equations have attracted the attention of many researchers, because of its wide applicability in sciences and engineering such as material sciences, mechanics, population dynamics, economics, chemical technology and so on.

In recent years, there has been a significant development in ordinary and partial differential equations involving fractional derivatives, see the monographs [23, 28, 32]. Up to now one of the important way to deal with fractional partial differential equations is to transform them into abstract fractional differential equations in Banach space. For example, an existence result for semilinear fractional differential equations with infinite delay and non-instantaneous impulses in a Banach space was studied by Benchohra et al. [6], using the technique of measures of noncompactness and Mönch's fixed point theorem. El-Sayed and Herzallah [16] investigated the existence, continuation, maximal regularity and some other properties of the non-homogeneous fractional order evolution equations with Captuo fractional derivative. In [24], Kumar and Sukavanam considered the approximate controllability of mild solutions for a class of semilinear delay control systems of fractional order by applying contraction principle and the Schauder fixed point theorem. The existence of mild solutions for a class of impulsive fractional partial semilinear differential equations was discussed by Shu et al. [36] with the aid of Leray Schauder Alternative fixed point theorem. Wang and Zhou [41] researched sufficient conditions for the complete controllability of fractional evolution systems with the help of the fractional calculus, properties of characteristic solution operators and fixed point technique. Utilizing the theory of fractional calculus and Schauder fixed point theorem, Zhou et al. [46] derived sufficient conditions for the existence and attractivity of solutions for fractional evolution equations with Riemann-Liouville fractional derivative. For more relative works, see [5, 15, 22, 25, 35, 39].

Meanwhile, the existence and other quantitative and qualitative properties of solutions for abstract fractional integro-differential equations have attracted more and more attention of some authors. Agarwal et al. [1] proved the existence results of solutions of the following fractional integro-differential evolution equations in Banach space X

$$\begin{cases} D_t^\gamma u(t) = Au(t) + \displaystyle\int_0^t B(t-s)u(s)ds + f(t),\ t \in [0,T], \\ u(0) = u_0, \end{cases} \tag{1.2}$$

where D_t^γ is the Caputo fractional derivative of order $\gamma \in (1,2)$ and $f : [0,T] \to X$ is a continuous function. The authors obtained the existence and regularity of solutions of (1.2) via the γ-resolvent operators associated to the following linear homogeneous equation

$$\begin{cases} D_t^\gamma u(t) = Au(t) + \displaystyle\int_0^t B(t-s)u(s)ds,\ t \geq 0, \\ u(0) = u_0. \end{cases}$$

Based on the important work, several papers have addressed the issue of existence results and controllability results for fractional integro-differential through applying the theory of γ-resolvent operators [2, 9, 27, 29, 33, 37, 42] and the references therein. Especially, Dos Santos in [17] has established the existence and qualitative properties of an α-resolvent operator for abstract fractional integro-differential equations

$$\begin{cases} D_t^\alpha x(t) = Ax(t) + \int_0^t B(t-s)x(s)ds, \ t \geq 0, \\ x(0) = x_0, \end{cases}$$

where D_t^α represents the Caputo fractional derivative of order $\alpha \in (0,1)$, A and $B(t)$, $t \geq 0$ are the same operators as in (1.1), and the existence and uniqueness of mild solutions for semi-linear fractional integro-differential equations

$$\begin{cases} D_t^\alpha x(t) = Ax(t) + \int_0^t B(t-s)x(s)ds + f(t, x(t)), \ t \in [0,T], \\ x(0) = x_0, \end{cases} \tag{1.3}$$

were investigated by using the Banach fixed point theorem and α-resolvent operators.

On the other hand, nonlocal initial conditions can be applied in physics with better effect than the classical initial condition $x(0) = x_0$. In 1991, the nonlocal Cauchy problem of evolution equations was studied by Byszewski [8] and the importance of nonlocal conditions in different fields has been discussed in [8] and [13]. In the few past years, some authors have been devoted to research the existence and regularity, periodicity and controllability of solutions for (integro) differential evolution equations with nonlocal conditions. Among others, we refer to Balachandran and Trujillo [3], Brindle and N'Guérékata [7], Chang and Liu [10], Ding et al [14], Ezzinbi et al [19], Ntouyas and Tsamatos [30], Pinaud and Henríquez [31], Vrabie [38], Wang et al [40], Zhang et al [44] and Zhu and Fu [45].

To the best of author's knowledge, there is no work reported on Eq. (1.3) with nonlocal conditions and delays. To close the gap, motivated by the above works, the aim of this paper is to study the existence of solutions for semilinear fractional integro-differential equations with nonlocal conditions (1.1) by utilizing Schauder's fixed point theorem and Sadovskii's fixed point theorem, respectively. The main contributions as follows:

(1) We introduce a resolvent identity $R(\lambda_0^\alpha, A) := (\lambda_0^\alpha I - A)^{-1}$ $(0 < \alpha < 1)$, and we certify that if $R(\lambda_0^\alpha, A)$ is compact for some $\lambda_0^\alpha \in \rho(A)$ ($\rho(A)$ is defined in Section 2), then the solution operators are compact (see Lemma 2.7). Subsequently, applying this crucial Lemma and Schauder's fixed point theorem, we can get the desire result.

(2) We just require that the nonlinear term $G(\cdot,\cdot)$ satisfies Carathéodory or noncompactness measure condition, other than Lipschitz condition as in [17].

(3) We observed that semilinear fractional integro-differential equations have been extensively studied in recent years utilizing various fixed point theorems when the corresponding solution operators are compact, see for instance [9, 33, 42]. In this paper, however, we can use measure of noncompactness to remove the assumption for compactness of the solution operators (see Theorem 3.2).

The rest of this paper is organized as follows: In Section 2, we state some definitions, lemmas, notations and necessary preliminaries on α-resolvent operators, Kuratowski measure of noncompactness and fixed point theorem. In Section 3, we discuss the existence of mild solutions of Eq. (1.1). In Section 4, we provide an example to illustrate the applications of the obtained results.

2. Preliminarie

Let X be a Banach space with norm $\|\cdot\|$. Throughout this paper, for a closed linear operator $A:\mathcal{D}(A)\subseteq X\to X$, the notation Y represents the Banach space $(\mathcal{D}(A),\|\cdot\|_1)$ with the graph norm $\|x\|_1=\|Ax\|+\|x\|$, for $x\in\mathcal{D}(A)$, $\rho(A)$ denotes the resolvent set of A, and $R(\lambda,A)=(\lambda I-A)^{-1}$ is the resolvent operator of A for $\lambda\in\rho(A)$. We denote by $\mathscr{L}(Y,X)$ the Banach space of bounded linear operators from Y into X endowed with norm $\|\cdot\|_{1.0}$ and $X_0=X$, and abbreviate it to $\mathscr{L}(X)$ when $Y=X$. Hereafter by $C([0,T];X)$ we denote the Banach space of continuous functions from $[0,T]$ to X with the norm

$$\|x\|_C=\sup_{0\le t\le T}\|x(t)\|,\ x\in C([0,T];X).$$

Furthermore, we denote by $C([0,\infty);X)$ the space of continuous functions defined on $[0,\infty)$ into X, and we define the space $C^\alpha((0,\infty);X)$, by

$$C^\alpha((0,\infty);X):=\{x\in C((0,\infty);X):D_t^\alpha x\in C((0,\infty);X)\}.$$

Now, we present the basic theory of α-resolvent operators, which appeared in [17].

Definition 2.1 A one-parameter family of bounded linear operators $(\mathscr{R}_\alpha(t))_{t\ge0}$ on X is called an α-resolvent operator for

$$\begin{cases} D_t^\alpha x(t)=Ax(t)+\displaystyle\int_0^t B(t-s)x(s)ds,\ t\ge0,\\ x(0)=x_0\in X.\end{cases}\tag{2.1}$$

If the following conditions are verified:

(a) The function $\mathscr{R}_\alpha(\cdot):[0,\infty)\to\mathscr{L}(X)$ is strongly continuous, $\mathscr{R}_\alpha(0)x=x$ for all $x\in X$ and $\alpha\in(0,1)$.

(b) For $x\in\mathcal{D}(A)$, $\mathscr{R}_\alpha(\cdot)x\in C([0,\infty);Y)\cap C^\alpha((0,\infty);X)$, and

$$\begin{aligned} D_t^\alpha\mathscr{R}_\alpha(t)x&=A\mathscr{R}_\alpha(t)x+\int_0^t B(t-s)\mathscr{R}_\alpha(s)xds,\\ &=\mathscr{R}_\alpha(t)Ax+\int_0^t\mathscr{R}_\alpha(t-s)B(s)xds,\end{aligned}$$

for each $t\ge0$.

In what follows, we always suppose that the following conditions are verified:

(V_1) The operator $A:\mathcal{D}(A)\subseteq X\to X$ is a closed linear operator with $\mathcal{D}(A)$ dense in X, for some $\phi\in(\frac{\pi}{2},\pi)$ there is a positive constant $C_0=C_0(\phi)$ such that $\lambda\in\rho(A)$ for each

$$\Sigma_{0,\phi}=\{\lambda\in\mathbb{C}:|\arg(\lambda)|<\phi\}\subset\rho(A)$$

and $\|R(\lambda,A)\|\le C_0|\lambda|^{-1}$ for all $\lambda\in\Sigma_{0,\phi}$.

(V_2) For all $t \geq 0$, $B(t) : \mathcal{D}(B(t)) \subseteq X \to X$ is a closed linear operator, $\mathcal{D}(A) \subseteq \mathcal{D}(B(t))$ and $B(\cdot)x$ is strongly measurable on $(0, \infty)$ for each $x \in \mathcal{D}(A)$. There exists $b(\cdot) \in L^1_{loc}(\mathbb{R}^+)$ such that $\hat{b}(\lambda)$ exists for $Re\lambda > 0$ and $\|B(t)x\| \leq b(t)\|x\|_1$ for all $t > 0$ and $x \in \mathcal{D}(A)$. Moreover, the operator valued function $\hat{B} : \Sigma_{0,\pi/2} \to \mathscr{L}(Y, X)$ has an analytical extension (still denoted by $\hat{B}$) to $\Sigma_{0,\phi}$ such that $\|\hat{B}(\lambda)x\| \leq \|\hat{B}(\lambda)\|\|x\|_1$ for all $x \in \mathcal{D}(A)$, and $\|\hat{B}(\lambda)\| = O(|\lambda|^{-1})$, as $|\lambda| \to \infty$. Where $\hat{B}(\lambda)$ represents the Laplace transform of $B(t)$.

(V_3) There exists a subspace $E \subseteq \mathcal{D}(A)$ dense in Y and positive constant C_1, such that $A(E) \subseteq \mathcal{D}(A)$, $\hat{B}(\lambda)(E) \subseteq \mathcal{D}(A)$ and $\|A\hat{B}(\lambda)x\| \leq C_1\|x\|$ for every $x \in E$ and $\lambda \in \Sigma_{0,\phi}$.

In the sequel, for $r > 0$ and $\theta \in (\frac{\pi}{2}, \phi)$, we let

$$\Sigma_{r,\theta} = \{\lambda \in \mathbb{C} : |\lambda| \geq r, \text{ and } |\arg(\lambda)| < \theta\},$$

and we consider the paths

$$\begin{aligned}
\Gamma^1_{r,\theta} &= \{te^{i\theta} : t \geq r\}, \\
\Gamma^2_{r,\theta} &= \{re^{i\xi} : -\theta \leq \xi \leq \theta\}, \\
\Gamma^3_{r,\theta} &= \{te^{-i\theta} : t \geq r\},
\end{aligned}$$

with $\Gamma_{r,\theta} = \bigcup_{i=1}^3 \Gamma^i_{r,\theta}$ oriented counterclockwise. Moreover, $\rho(F_\alpha)$ and $\rho(G_\alpha)$ are the sets

$$\rho(F_\alpha) = \{\lambda \in \mathbb{C} : F_\alpha(\lambda) := (\lambda^\alpha I - A - \hat{B}(\lambda))^{-1} \in \mathscr{L}(X)\}$$

and

$$\rho(G_\alpha) = \{\lambda \in \mathbb{C} : G_\alpha(\lambda) := \lambda^{\alpha-1}(\lambda^\alpha I - A - \hat{B}(\lambda))^{-1} \in \mathscr{L}(X)\}.$$

Lemma 2.1 ([17, Lemma 1]) *Suppose that condition* (V_1) *holds, then* $\lambda^\alpha \in \rho(A)$ *for each* $\lambda \in \Sigma_{0,\phi}$ *and there exists* $N_0 = N_0(\phi)$ *such that*

$$\|R(\lambda^\alpha, A)\| \leq \frac{N_0}{|\lambda|^\alpha},$$

for all $\lambda \in \Sigma_{0,\phi}$.

Lemma 2.2 ([17, Lemma 2]) *There exists* $r_1 > 0$ *such that* $\Sigma_{r_1,\phi} \subseteq \rho(F_\alpha)$ *and the function* $F_\alpha : \Sigma_{r_1,\phi} \to \mathscr{L}(X)$ *is analytic. Moreover,*

$$F_\alpha(\lambda) = R(\lambda^\alpha, A)[I - \hat{B}(\lambda)R(\lambda^\alpha, A)]^{-1}, \tag{2.2}$$

and there exists constants N_i, *for* $i = 1, 2, 3$, *such that*

$$\|F_\alpha(\lambda)\| \leq \frac{N_1}{|\lambda|^\alpha},$$

$$\|AF_\alpha(\lambda)x\| \leq \frac{N_2}{|\lambda|^\alpha}\|x\|_1, \; x \in \mathcal{D}(A),$$

$$\|AF_\alpha(\lambda)\| \leq N_3,$$

for every $\lambda \in \Sigma_{r_1,\phi}$.

Lemma 2.3 ([17, Lemma 3]) *There exists $r_1 > 0$ such that $\Sigma_{r_1,\vartheta} \subseteq \rho(G_\alpha)$ and the function $G_\alpha : \Sigma_{r_1,\vartheta} \to \mathscr{L}(X)$ is analytic. Moreover,*

$$G_\alpha(\lambda) = \lambda^{\alpha-1}F_\alpha(\lambda) = \lambda^{\alpha-1}R(\lambda^\alpha, A)[I - \hat{B}(\lambda)R(\lambda^\alpha, A)]^{-1}, \tag{2.3}$$

and there exists constants N_i, for $i = 4, 5, 6$, such that

$$\|G_\alpha(\lambda)\| \leq \frac{N_4}{|\lambda|},$$

$$\|AG_\alpha(\lambda)x\| \leq \frac{N_5}{|\lambda}\|x\|_1,\ x \in \mathcal{D}(A),$$

$$\|AG_\alpha(\lambda)\| \leq \frac{N_6}{|\lambda|^{1-\alpha}},$$

for every $\lambda \in \Sigma_{r_1,\vartheta}$.

Definition 2.2 ([17, Definition 2]) We define the operator family $(\mathscr{R}_\alpha(t))_{t\geq 0}$ by

$$\mathscr{R}_\alpha(t) = \frac{1}{2\pi i}\int_{\Gamma_{r,\theta}} e^{\lambda t}G_\alpha(\lambda)d\lambda,\ t \geq 0, \tag{2.4}$$

and the auxiliary resolvent operator family $(\mathscr{S}_\alpha(t))_{t\geq 0}$ by

$$\mathscr{S}_\alpha(t) = \frac{t^{1-\alpha}}{2\pi i}\int_{\Gamma_{r,\theta}} e^{\lambda t}F_\alpha(\lambda)d\lambda,\ t \geq 0. \tag{2.5}$$

Lemma 2.4 ([17, Theorem 3]) *Assume that conditions (V_1)-(V_3) are fulfilled. Then the function $\mathscr{R}_\alpha(\cdot)$ is an α-resolvent operator for Eq. (2.1).*

The following lemmas give some properties of $(\mathscr{R}_\alpha(t))_{t\geq 0}$ and $(\mathscr{S}_\alpha(t))_{t\geq 0}$.

Lemma 2.5 ([17, Theorem 1]) *The operator function $\mathscr{R}_\alpha(\cdot)$ is:*

(i) *exponentially bounded in $\mathscr{L}(X)$;*
(ii) *exponentially bounded in $\mathscr{L}(Y)$;*
(iii) *strongly continuous on $[0, \infty)$;*
(iv) *strongly continuous on $[0, \infty)$ in $\mathscr{L}(Y)$;*
(v) *uniformly continuous on $(0, \infty)$.*

Lemma 2.6 ([17, Theorem 2]) *The operator function $t \to t^{\alpha-1}\mathscr{S}_\alpha(t)$ is:*

(i) *exponentially bounded in $\mathscr{L}(X)$;*
(ii) *strongly continuous on $(0, \infty)$;*
(iii) *uniformly continuous on $(0, \infty)$.*

In order to obtain our existence results, we establish the following Lemma.

Lemma 2.7 *If $R(\lambda_0^\alpha, A)$ $(0 < \alpha < 1)$ is compact for some $\lambda_0^\alpha \in \rho(A)$, then $\mathscr{R}_\alpha(t)$ and $t^{\alpha-1}\mathscr{S}_\alpha(t)$ are compact for all $t > 0$.*

Proof. We only state the sketch of the proof since the method is very similar to [18, Lemma 2.12]. It follows from (2.2) that $F_\alpha(\lambda)$ is compact for all $\lambda \in \sum_{r_1,\phi}$, therefore, by (2.5), we know that $t^{\alpha-1}\mathscr{S}_\alpha(t) = \frac{1}{2\pi i}\int_{\Gamma_{r,\theta}} e^{\lambda t}F_\alpha(\lambda)d\lambda$ is compact for all $t > 0$. Similarly, from (2.3) and (2.4), we conclude that $\mathscr{R}_\alpha(t)$ is also compact for all $t > 0$.

Next we turn to state the definition for Kuratowski measure of noncompactness, which will be used in the proof of our main results.

Definition 2.3 ([4]) The Kuratowski measure of noncompactness $\chi(\cdot)$ defined on bounded set S of Banach space Z is

$$\chi(S) := \inf\left\{d > 0 : S = \cup_{k=1}^{n} S_k \text{ and } \operatorname{diam}(S_k) \leq d \text{ for } k = 1, 2, \cdots, n\right\}.$$

Let us recall the basic properties of Kuratowski measure of noncompactness.

Lemma 2.8 ([4, 12]) *Let Z be a Banach space, and $U, V \subset Z$ be bounded, then the following properties hold:*

(i) $\chi(U) \leq \chi(V)$ *if* $U \subset V$;

(ii) $\chi(U) = 0$ *if and only if* $\overline{U}$ *is compact, where* $\overline{U}$ *means the closure hull of* U;

(iii) $\chi(U) = \chi(\overline{U}) = \chi(conv\ U)$, *where conv* U *means the convex hull of* U;

(iv) $\chi(aU) = |a|\chi(U)$, *where* $a \in \mathbb{R}$;

(v) $\chi(U \cup V) = \max\{\chi(U), \chi(V)\}$;

(vi) $\chi(U + V) \leq \chi(U) + \chi(V)$, *where* $U + V = \{z \mid z = u + v,\ u \in U, v \in V\}$;

(vii) *If the map* $Q : \mathcal{D}(Q) \subset Z \to W$ *is Lipschits continuous with constant* L, *then* $\chi(Q(V)) \leq L\chi(V)$ *for any bounded subset* $V \subset \mathcal{D}(Q)$, *where* W *is another Banach space.*

In what follows, we denote by $\eta(\cdot)$ the Kuratowski measure of noncompactness on the bounded set of Banach space X and denote by $\eta_C(\cdot)$ the Kuratowski measure of noncompactness on the bounded set of $C([0, T]; X)$. The following lemmas are needed in our argument.

Lemma 2.9 ([26, Lemma 2.1]) *Let* $D \subset C([0, T]; X)$ *be bounded and equicontinuous, then* $\overline{co}\ D \subset C([0, T]; X)$ *is also bounded and equicontinuous (where* $\overline{co}\ D$ *denotes the closed convex hull of* D*).*

Lemma 2.10 ([11, Lemma 2.5]) *Let* $D \subset X$ *be a bounded set, then there exists a countable set* $D_0 \subset D$, *such that* $\eta(D) \leq 2\eta(D_0)$.

Lemma 2.11 ([21, Theorem 2.1]) *If* $\{x_m\}_{m=1}^{\infty} \subset L^1([0, T]; X)$ *is uniformly integrable, i.e., there exists a function* $\psi \in L^1([0, T]; \mathbb{R}^+)$ *such that*

$$\|x_m(t)\| \leq \psi(t),\ a.e.\ t \in [0, T],\ m = 1, 2, \cdots.$$

Then the function $t \to \eta\left(\{x_m(t)\}_{m=1}^{\infty}\right)$ *is measurable and*

$$\eta\left(\left\{\int_0^t x_m(s)ds\right\}_{m=1}^{\infty}\right) \leq 2\int_0^t \eta\left(\{x_m(s)\}_{m=1}^{\infty}\right) ds.$$

Lemma 2.12 ([4]) *Let* $D \subset C([0, T]; X)$ *be a bounded set. Then* $\eta(D(t)) \leq \eta_C(D)$ *for all* $t \in [0, T]$, *where* $D(t) = \{x(t) : x \in D\}$. *Moreover, if* D *is equicontinuous on* $t \in [0, T]$, *Then* $\eta(D(t))$ *is continuous on* $[0, T]$, *and* $\eta_C(D) = \max_{t \in [0,T]} \eta(D(t))$.

Lemma 2.13 ([20, P_{111}]) *For* $q > 1$ *and* $0 < a \leq b$, *we have*

$$(b - a)^q \leq b^q - a^q.$$

Lemma 2.14 ([19, Lemma 12]) *Let* $\{R_m\}_{m \geq 1}$ *be a sequence of bounded linear maps on* X *converging pointwise to* $R \in \mathscr{L}(X)$. *Then for any compact set* K *in* X, R_m *converges to* R *uniformly in* K, *namely,*

$$\sup_{x \in K} \|R_m x - Rx\| \to 0,\ as\ m \to \infty.$$

Lemma 2.15 ([34] Sadovskii's fixed point theorem) *Assume that Φ is a condensing operator on a Banach space X, i.e., Φ is continuous and $\eta(\Phi(D)) < \eta(D)$ for every bounded subset D of X with $\eta(D) > 0$. If $\Phi(B) \subseteq B$ for a convex, closed and bounded subset B of X, then Φ has at least one fixed point in B.*

3. Main results

In this section, we discuss the existence of mild solutions for the equation (1.1) by using Schauder's and Sadovskii's fixed point theorem, respectively. In what follows, we assume that there exist positive constants M_i, $i = 1, 2.$ such that $\|\mathscr{R}_\alpha(t)\| \leq M_1$ and $\|\mathscr{S}_\alpha(t)\| \leq M_2$ for every $t \in [0, T]$. Inspired by [17, Definition 3], we give the following concept of mild solutions of Eq. (1.1).

Definition 3.1 A function $x(\cdot) \in C([0, T]; X)$ is said to be a mild solution of Eq. (1.1), if it verifies

$$x(t) = \mathscr{R}_\alpha(t)\,[x_0 - g(x)] + \int_0^t (t-s)^{\alpha-1}\mathscr{S}_\alpha(t-s)G(s, x(h(s)))ds,\ \ for\ t \in [0, T].$$

To guarantee the existence of mild solutions, we impose the following restrictions on Eq. (1.1).

(H_1) $R(\lambda_0^\alpha, A)$ is a compact operator for some $\lambda_0^\alpha \in \rho(A)$.

(H_2) The function $G : [0, T] \times X \to X$ satisfies the following conditions:

(i) For each $t \in [0, T]$, the function $G(t, \cdot) : X \to X$ is continuous and for each $x \in X$ the function $G(\cdot, x) : [0, T] \to X$ is strongly measurable;

(ii) There is a constant $\varphi > 0$, a function $\zeta \in L^p([0, T]; [0, \infty))$ $\left(p > \frac{1}{\alpha} > 1\right)$ and a nondecreasing continuous function $\Omega : [0, \infty) \to (0, \infty)$ such that for each $t \in [0, T]$ and $x \in X$,

$$\|G(t, x)\| \leq \zeta(t)\Omega(\|x\|) \text{ and } \liminf_{l\to+\infty} \frac{\Omega(l)}{l} := \varphi < +\infty.$$

(H_3) $g : C([0, T]; X) \to X$ is continuous and satisfies the following conditions:

(i) g is a compact map;

(ii) There is a constant $\delta > 0$, and a continuous nondecreasing function $\Lambda : [0, \infty) \to (0, \infty)$ such that for any $u \in C([0, T]; X)$,

$$\|g(u)\| \leq \Lambda(\|u\|_C) \text{ and } \liminf_{l\to+\infty} \frac{\Lambda(l)}{l} := \delta < +\infty.$$

For any constant $l > 0$, let $B_l = \{x \in C([0, T]; X) : \|x\|_C \leq l\}$, it is clearly a bounded, closed and convex subset in $C([0, T]; X)$.

We now establish the first existence result.

Theorem 3.1 *Suppose that the conditions $(H_1)-(H_3)$ hold, then the nonlocal Cauchy problem (1.1) has a mild solution provided that*

$$M_1\delta + M_2\varphi\left(\frac{p-1}{p\alpha-1}\right)^{\frac{p-1}{p}} T^{\alpha-\frac{1}{p}}\|\zeta\|_{L^p([0,T];[0,\infty))} < 1. \tag{3.1}$$

Proof. The operator $\Phi : C([0,T];X) \to C([0,T];X)$ defined by

$$(\Phi x)(t) = \mathscr{R}_\alpha(t)[x_0 - g(x)] + \int_0^t (t-s)^{\alpha-1}\mathscr{S}_\alpha(t-s)G(s,x(h(s)))ds. \tag{3.2}$$

Since by the condition $(H_2)(ii)$, we have

$$\begin{aligned}
&\|(t-s)^{\alpha-1}\mathscr{S}_\alpha(t-s)G(s,x(h(s)))\| \\
&\qquad \le\ (t-s)^{\alpha-1}\|\mathscr{S}_\alpha(t-s)\|\|G(s,x(h(s)))\| \\
&\qquad \le\ (t-s)^{\alpha-1}M_2\zeta(s)\Omega(\|x\|_C),\ \text{for } x \in C([0,T];X),
\end{aligned}$$

then from Bochner's theorem ([47, Lemma 2.2]), it follows that

$$(t-s)^{\alpha-1}\mathscr{S}_\alpha(t-s)G(s,x(h(s)))$$

is Bochner's integrable with respect to $s \in [0,t]$ for all $t \in [0,T]$. So the operator Φ is well defined on $C([0,T];X)$. In the following, we prove that the operator Φ satisfies the conditions of the well-known Schauder's fixed point theorem, and hence Φ has a fixed point, which is a mild solution of Eq. (1.1). For the sake of convenience, the proof will be given in several steps.

Step 1. We claim that there is a number $l_0 > 0$ such that $\Phi(B_{l_0}) \subseteq B_{l_0}$.

In fact, if this is not true, then for each $l > 0$, there exist $x_l(\cdot) \in B_l$ and $t_l \in [0,T]$, such that $\|(\Phi x_l)(t_l)\| > l$. On the other hand, however, by $(H_2)(ii)$, $(H_3)(ii)$ and Hölder inequality, we have

$$\begin{aligned}
l &< \|(\Phi x_l)(t_l)\| \\
&\le M_1\left[\|x_0\| + \|g(x_l)\|\right] + \int_0^{t_l}(t_l-s)^{\alpha-1}\left\|\mathscr{S}_\alpha(t_l-s)\right\|\left\|G(s,x_l(h(s)))\right\|ds \\
&\le M_1\left[\|x_0\| + \Lambda(\|x_l\|_C)\right] + \int_0^{t_l}(t_l-s)^{\alpha-1}M_2\zeta(s)\Omega(\|x_l(h(s))\|)ds \\
&\le M_1\left[\|x_0\| + \Lambda(l)\right] + M_2\Omega(l)\left(\int_0^{t_l}(t_l-s)^{\frac{p(\alpha-1)}{p-1}}ds\right)^{\frac{p-1}{p}}\left(\int_0^{t_l}\zeta^p(s)ds\right)^{\frac{1}{p}} \\
&\le M_1[\|x_0\| + \Lambda(l)] + M_2\Omega(l)\left(\frac{p-1}{p\alpha-1}\right)^{\frac{p-1}{p}}T^{\alpha-\frac{1}{P}}\|\zeta\|_{L^p([0,T];[0,\infty))}.
\end{aligned}$$

Dividing on both sides by the l and taking the lower limit as $l \to +\infty$, we get

$$1 \le M_1\delta + M_2\varphi\left(\frac{p-1}{p\alpha-1}\right)^{\frac{p-1}{p}}T^{\alpha-\frac{1}{P}}\|\zeta\|_{L^p([0,T];[0,\infty))}.$$

This contradicts (3.1). Hence, there is a positive number l_0 such that $\Phi(B_{l_0}) \subseteq B_{l_0}$.

Step 2. We show that Φ is continuous on B_{l_0}.

Let $\{x_n\} \subseteq B_{l_0}$ with $x_n \to x$ in B_{l_0}, then by the continuity of g and G, we get that

$$g(x_n) \to g(x), \qquad n \to \infty,$$

and

$$G(s,x_n(h(s))) \to G(s,x(h(s))), \qquad n \to +\infty.$$

Applying condition $(H_2)(ii)$, we have

$$(t-s)^{\alpha-1}\left\|G(s,x_n(h(s)))-G(s,x(h(s)))\right\| \le (t-s)^{\alpha-1}2\zeta(s)\Omega(l_0),\ \text{for } s\in[0,T].$$

Then, using the Lebesgue dominated convergence theorem, we obtain that

$$\begin{aligned}\|\Phi x_n-\Phi x\|_C \le \sup_{0\le t\le T}&\Bigg(M_1\left\|g(x_n)-g(x)\right\|\\ &+M_2\int_0^t (t-s)^{\alpha-1}\left\|G(s,x_n(h(s)))-G(s,x(h(s)))\right\|ds\Bigg)\\ &\to 0,\ \text{as } n\to\infty,\end{aligned}$$

which implies that Φ is continuous on B_{l_0}.

Step 3. We certify that Φ is compact.

(i) $\Phi(B_{l_0})=\{\Phi x : x\in B_{l_0}\}$ is a family of equicontinuous functions.

We let $0<t_1<t_2\le T$ and $\varepsilon>0$ be small enough, then, for any $x\in B_{l_0}$,

$$\begin{aligned}&\|(\Phi x)(t_2)-(\Phi x)(t_1)\|\\ \le\ &\|(\mathscr{R}_\alpha(t_2)-\mathscr{R}_\alpha(t_1))\left[x_0-g(x)\right]\|\\ &+\left\|\int_0^{t_1-\varepsilon}\left[(t_2-s)^{\alpha-1}\mathscr{S}_\alpha(t_2-s)-(t_1-s)^{\alpha-1}\mathscr{S}_\alpha(t_1-s)\right]G(s,x(h(s)))ds\right\|\\ &+\left\|\int_{t_1-\varepsilon}^{t_1}\left[(t_2-s)^{\alpha-1}\mathscr{S}_\alpha(t_2-s)-(t_1-s)^{\alpha-1}\mathscr{S}_\alpha(t_1-s)\right]G(s,x(h(s)))ds\right\|\\ &+\left\|\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}\mathscr{S}_\alpha(t_2-s)G(s,x(h(s)))ds\right\|\\ =:\ &I_1+I_2+I_3+I_4,\end{aligned}$$

where

$$\begin{aligned}I_1=\ &\|(\mathscr{R}_\alpha(t_2)-\mathscr{R}_\alpha(t_1))\left[x_0-g(x)\right]\|,\\ I_2=\ &\left\|\int_0^{t_1-\varepsilon}\left[(t_2-s)^{\alpha-1}\mathscr{S}_\alpha(t_2-s)-(t_1-s)^{\alpha-1}\mathscr{S}_\alpha(t_1-s)\right]G(s,x(h(s)))ds\right\|,\\ I_3=\ &\left\|\int_{t_1-\varepsilon}^{t_1}\left[(t_2-s)^{\alpha-1}\mathscr{S}_\alpha(t_2-s)-(t_1-s)^{\alpha-1}\mathscr{S}_\alpha(t_1-s)\right]G(s,x(h(s)))ds\right\|,\\ I_4=\ &\left\|\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}\mathscr{S}_\alpha(t_2-s)G(s,x(h(s)))ds\right\|.\end{aligned}$$

Now, we only need to prove that $I_i\to 0$ independent of $x\in B_{l_0}$ as $t_2-t_1\to 0$ for $i=1,2,3,4$.

For I_1, by Lemma 2.5(v) and $(H_3)(ii)$, we obtain

$$\begin{aligned}I_1\le\ &\|\mathscr{R}_\alpha(t_2)-\mathscr{R}_\alpha(t_1)\|\left(\|x_0-g(x)\|\right)\\ \le\ &\|\mathscr{R}_\alpha(t_2)-\mathscr{R}_\alpha(t_1)\|\left(\|x_0\|+\Lambda(l_0)\right)\\ &\to 0,\ \text{as } t_2-t_1\to 0.\end{aligned}$$

For I_2, by Lemma 2.6(iii) and $(H_2)(ii)$, we get

$$\begin{aligned} I_2 \leq & \int_0^{t_1-\varepsilon} \left\|(t_2-s)^{\alpha-1}\mathscr{S}_\alpha(t_2-s)-(t_1-s)^{\alpha-1}\mathscr{S}_\alpha(t_1-s)\right\| \|G(s,x(h(s)))\| \, ds \\ \leq & \sup_{s\in[0,t_1-\varepsilon]} \left\|(t_2-s)^{\alpha-1}\mathscr{S}_\alpha(t_2-s)-(t_1-s)^{\alpha-1}\mathscr{S}_\alpha(t_1-s)\right\| \int_0^{t_1-\varepsilon} \zeta(s)\Omega(l_0)ds \\ & \to 0, \text{ as } t_2-t_1\to 0. \end{aligned}$$

For I_3, we have

$$\begin{aligned} I_3 \leq & \int_{t_1-\varepsilon}^{t_1} \left\|(t_2-s)^{\alpha-1}\mathscr{S}_\alpha(t_2-s)-(t_1-s)^{\alpha-1}\mathscr{S}_\alpha(t_1-s)\right\| \|G(s,x(h(s)))\| ds \\ \leq & \int_{t_1-\varepsilon}^{t_1} \left\|(t_2-s)^{\alpha-1}\mathscr{S}_\alpha(t_2-s)-(t_2-s)^{\alpha-1}\mathscr{S}_\alpha(t_1-s)\right\| \|G(s,x(h(s)))\| ds \\ & + \int_{t_1-\varepsilon}^{t_1} \left\|(t_2-s)^{\alpha-1}\mathscr{S}_\alpha(t_1-s)-(t_1-s)^{\alpha-1}\mathscr{S}_\alpha(t_1-s)\right\| \|G(s,x(h(s)))\| ds \\ := & I_3^1 + I_3^2. \end{aligned}$$

Using $(H_2)(ii)$ and Hölder inequality, we find

$$\begin{aligned} I_3^1 \leq & \int_{t_1-\varepsilon}^{t_1} (t_2-s)^{\alpha-1} \|\mathscr{S}_\alpha(t_2-s)-\mathscr{S}_\alpha(t_1-s)\| \|G(s,x(h(s)))\| ds \\ \leq & \int_{t_1-\varepsilon}^{t_1} (t_2-s)^{\alpha-1} 2M_2\zeta(s)\Omega(l_0)ds \\ \leq & 2M_2\Omega(l_0)\left(\int_{t_1-\varepsilon}^{t_1}(t_2-s)^{\frac{p(\alpha-1)}{p-1}}ds\right)^{\frac{p-1}{p}} \left(\int_{t_1-\varepsilon}^{t_1}\zeta^p(s)ds\right)^{\frac{1}{p}} \\ \leq & 2M_2\Omega(l_0)\left(\frac{p-1}{p\alpha-1}\right)^{\frac{p-1}{p}} \left((t_2-t_1+\varepsilon)^{\frac{p\alpha-1}{p-1}}-(t_2-t_1)^{\frac{p\alpha-1}{p-1}}\right)^{\frac{p-1}{p}} \|\zeta\|_{L^p([0,T];[0,\infty))} \\ & \to 0, \text{ as } t_2-t_1\to 0 \text{ and } \varepsilon\to 0. \end{aligned}$$

And, analogously, it follows immediately that

$$\begin{aligned} I_3^2 \leq & \int_{t_1-\varepsilon}^{t_1} \left((t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}\right) \|\mathscr{S}_\alpha(t_1-s)\|\|G(s,x(h(s)))\| ds \\ \leq & \int_{t_1-\varepsilon}^{t_1} \left((t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}\right) M_2\zeta(s)\Omega(l_0)ds \\ \leq & M_2\Omega(l_0)\left(\int_{t_1-\varepsilon}^{t_1}\left((t_1-s)^{\alpha-1}-(t_2-s)^{\alpha-1}\right)^{\frac{p}{p-1}}ds\right)^{\frac{p-1}{p}} \|\zeta\|_{L^p([0,T];[0,\infty))}. \end{aligned} \quad (3.3)$$

Since $\frac{p}{p-1} > 1$, then by Lemma 2.13 and (3.3), we see

$$\begin{aligned} I_3^2 &\le M_2\Omega(l_0)\left(\int_{t_1-\varepsilon}^{t_1}\left((t_1-s)^{\frac{p(\alpha-1)}{p-1}}-(t_2-s)^{\frac{p(\alpha-1)}{p-1}}\right)ds\right)^{\frac{p-1}{p}}\|\zeta\|_{L^p([0,T];[0,\infty))} \\ &= M_2\Omega(l_0)\left[\left(\frac{p-1}{p\alpha-1}\right)^{\frac{p-1}{p}}\left(\varepsilon^{\frac{p\alpha-1}{p-1}}+(t_2-t_1)^{\frac{p\alpha-1}{p-1}}-(t_2-t_1+\varepsilon)^{\frac{p\alpha-1}{p-1}}\right)^{\frac{p-1}{p}}\right] \\ &\quad \|\zeta\|_{L^p([0,T];[0,\infty))} \\ &\quad \to 0,\ \text{ as } t_2-t_1\to 0 \text{ and } \varepsilon\to 0. \end{aligned}$$

For I_4, by simple calculations, we have

$$\begin{aligned} I_4 &\le M_2\Omega(l_0)\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}\zeta(s)ds \\ &\le M_2\Omega(l_0)\left(\int_{t_1}^{t_2}(t_2-s)^{\frac{p(\alpha-1)}{p-1}}ds\right)^{\frac{p-1}{p}}\|\zeta\|_{L^p([0,T];[0,\infty))} \\ &= M_2\Omega(l_0)\left(\frac{p-1}{p\alpha-1}\right)^{\frac{p-1}{p}}(t_2-t_1)^{\alpha-\frac{1}{p}}\|\zeta\|_{L^p([0,T];[0,\infty))} \\ &\quad \to 0,\ \text{ as } t_2-t_1\to 0. \end{aligned}$$

Then, from above estimates, we can get

$$\|(\Phi x)(t_2)-(\Phi x)(t_1)\|\to 0,\ \text{ as } t_2-t_1\to 0.$$

Moreover, we conclude that the functions Φx, $x\in B_{l_0}$ are equicontinuous at $t=0$. In fact, from Lemma 2.5(iii), Lemma 2.14 and the compactness of $\overline{g(B_{l_0})}$, we have

$$\begin{aligned} \|(\Phi x)(t)-(\Phi x)(0)\| \le\ & \|\mathscr{R}_\alpha(t)\left[x_0-g(x)\right]-\left[x_0-g(x)\right]\| \\ & +\int_0^t (t-s)^{\alpha-1}\|\mathscr{S}_\alpha(t-s)\|\|G(s,x(h(s)))\|ds \\ \le\ & \|\mathscr{R}_\alpha(t)x_0-x_0\|+\|\mathscr{R}_\alpha(t)g(x)-g(x)\| \\ & +\int_0^t (t-s)^{\alpha-1}M_2\zeta(s)\Omega(l_0)ds \\ \le\ & \|\mathscr{R}_\alpha(t)x_0-x_0\|+\sup_{y\in\overline{g(B_{l_0})}}\|\mathscr{R}_\alpha(t)y-y\| \\ & +M_2\Omega(l_0)\left(\frac{p-1}{p\alpha-1}\right)^{\frac{p-1}{p}}t^{\alpha-\frac{1}{p}}\|\zeta\|_{L^p([0,T];[0,\infty))} \\ & \to 0,\ \text{ as } t\to 0. \end{aligned}$$

Therefore, the operator Φ maps B_{l_0} into a family of equicontinuous functions.

(ii) $\Phi(B_{l_0})=\{\Phi x : x\in B_{l_0}\}$ is obviously bounded.

(iii) For $t\in[0,T]$, the set $(\Phi B_{l_0})(t)=\{(\Phi x)(t) : x\in B_{l_0}\}$ is relatively compact in X.

Clearly, due to $(H_3)(i)$, $(\Phi B_{l_0})(0) = x_0 - g(x)$ is relatively compact in X. Let $t \in (0, T]$ be fixed, $0 < \varepsilon < t$, for $x \in B_{l_0}$, set

$$(\Phi_\varepsilon x)(t) = \mathscr{R}_\alpha(t)[x_0 - g(x)] + \int_0^{t-\epsilon} (t-s)^{\alpha-1} \mathscr{S}_\alpha(t-s) G(s, x(h(s)))ds.$$

As $\mathscr{R}_\alpha(t)$ and $t^{\alpha-1}\mathscr{S}_\alpha(t)$ are compact for $t > 0$ $\big($note condition (H_1) and Lemma 2.7$\big)$, we conclude that $(\Phi_\varepsilon B_{l_0})(t) = \{(\Phi_\varepsilon x)(t) : x \in B_{l_0}\}$ is relatively compact in X. Furthermore, for $x \in B_{l_0}$, we have that

$$\begin{aligned} \|(\Phi x)(t) - (\Phi_\varepsilon x)(t)\| &\leq \int_{t-\varepsilon}^t (t-s)^{\alpha-1} \|\mathscr{S}_\alpha(t-s)\| \|G(s, x(h(s)))\| ds \\ &\leq M_2 \Omega(l_0) \int_{t-\varepsilon}^t (t-s)^{\alpha-1} \zeta(s) ds \\ &\leq M_2 \Omega(l_0) \left(\frac{p-1}{p\alpha - 1} \right)^{\frac{p-1}{p}} \varepsilon^{\alpha - \frac{1}{p}} \|\zeta\|_{L^p([0,T];[0,\infty))}. \end{aligned}$$

Then, there are relatively compact sets arbitrary close to the set $(\Phi B_{l_0})(t)$. Hence the set $(\Phi B_{l_0})(t)$ is also relatively compact in X.

In view of steps 2 and 3 together with the infinite-dimensional version of Ascoli-Arzela theorem, we can deduce that Φ is a completely continuous map on B_{l_0}, and by Schauder's fixed point theorem there is a fixed point $x(\cdot)$ for Φ on B_{l_0}. Therefore, the nonlocal Cauchy problem (1.1) has a mild solution $x(\cdot)$ on $[0, T]$. The proof is finished.

The following existence result for the equation (1.1) is based on Sadovskii's fixed point theorem.

Theorem 3.2 *Assume that the conditions* (H_2), (H_3) *and* (3.1) *are fulfilled. Also the following condition holds:*

(H_4) *There exists a function* $\omega \in L^p([0,T];[0,\infty))$ $\big(p > \frac{1}{\alpha} > 1\big)$ *such that, for any bounded and countable set* $D \in X$ *and* $t \in [0, T]$,

$$\eta(G(t, D)) \leq \omega(t)\eta(D).$$

Then the nonlocal Cauchy problem (1.1) has at least one mild solution provided that

$$4M_2 \left(\frac{p-1}{p\alpha - 1} \right)^{\frac{p-1}{p}} T^{\alpha - \frac{1}{p}} \|\omega\|_{L^p([0,T];[0,\infty))} < 1. \tag{3.4}$$

Proof. Let the operator Φ on $C([0,T]; X)$ be defined by (3.2) in Theorem 3.1. We have already shown by steps 1, 2 and 3(i) that $\Phi : B_{l_0} \to B_{l_0}$ is continuous, and $\Phi(B_{l_0}) = \{\Phi x : \ x \in B_{l_0}\}$ is a family of equi-continuous functions.

Next, we prove that $\Phi : \Psi \to \Psi$ is a condensing operator, where $\Psi = \overline{co}\ \Phi(B_{l_0})$ and $\overline{co}\ \Phi(B_{l_0})$ means the closed convex hull of $\Phi(B_{l_0})$. By Lemma 2.9, we know that $\Psi \subset B_{l_0}$ is bounded and equicontinuous, and operator $\Phi : \Psi \to \Psi$ is continuous. For any $D \subset \Psi$, $\Phi(D)$ is bounded. Thus, by Lemma 2.10, there exists a countable set $D_0 = \{x_m\}_{m=1}^\infty \subset D$, such that

$$\eta_C (\Phi(D)) \leq 2\eta_C (\Phi(D_0)). \tag{3.5}$$

By the equicontinuity of D, we know that $D_0 \subset D$ is also equicontinuous. Then, applying Lemmas 2.8, 2.11, 2.12 and the conditions $(H_3)(i)$ and (H_4), we have that

$$\begin{aligned}
\eta\left(\Phi(D_0)(t)\right) &\leq \eta\left(\{\mathscr{R}_\alpha(t)[x_0 - g(x_m)]\}_{m=1}^{\infty}\right) \\
&\quad + \eta\left(\left\{\int_0^t (t-s)^{\alpha-1}\mathscr{S}_\alpha(t-s)G(s, x_m(h(s)))ds\right\}_{m=1}^{\infty}\right) \\
&\leq M_1\left[\eta(\{x_0\}) + \eta\left(\{g(x_m)\}_{m=1}^{\infty}\right)\right] \\
&\quad + \eta\left(\left\{\int_0^t (t-s)^{\alpha-1}\mathscr{S}_\alpha(t-s)G\left(s, x_m(h(s))\right)ds\right\}_{m=1}^{\infty}\right) \\
&= \eta\left(\left\{\int_0^t (t-s)^{\alpha-1}\mathscr{S}_\alpha(t-s)G\left(s, x_m(h(s))\right)ds\right\}_{m=1}^{\infty}\right) \\
&\leq 2\int_0^t (t-s)^{\alpha-1}M_2\eta\left(\{G\left(s, x_m(h(s))\right)\}_{m=1}^{\infty}\right)ds \\
&\leq 2M_2\int_0^t (t-s)^{\alpha-1}\omega(s)\eta\left(D_0(h(s))\right)ds \\
&\leq 2M_2\left(\int_0^t (t-s)^{\frac{p(\alpha-1)}{p-1}}ds\right)^{\frac{p-1}{p}}\|\omega\|_{L^p([0,T];[0,\infty))}\eta_C(D_0) \\
&\leq 2M_2\left(\frac{p-1}{p\alpha-1}\right)^{\frac{p-1}{p}}T^{\alpha-\frac{1}{P}}\|\omega\|_{L^p([0,T];[0,\infty))}\eta_C(D).
\end{aligned} \tag{3.6}$$

Since $\Phi(D_0) \subset \Phi(B_{l_0})$ is bounded and equicontinuous, from Lemma 2.12, we obtain that

$$\eta_C\left(\Phi(D_0)\right) = \max_{t\in[0,T]}\eta\left(\Phi(D_0)(t)\right). \tag{3.7}$$

Hence, using (3.5), (3.6) and (3.7), we see that

$$\eta_C\left(\Phi(D)\right) \leq\ 4M_2\left(\frac{p-1}{p\alpha-1}\right)^{\frac{p-1}{p}}T^{\alpha-\frac{1}{P}}\|\omega\|_{L^p([0,T];[0,\infty))}\eta_C(D).$$

Due to (3.4), Φ is a condensing map on Ψ. It follows from Sadovskii's fixed point theorem that Φ has at least one fixed point $x(\cdot)$ in Ψ, which is a mild solution of the nonlocal problem (1.1). The proof is completed.

Remark 3.1 *Comparing Theorem 3.1 with Theorem 3.2, we notice that the compactness of the solution operators $\mathscr{R}_\alpha(t)$ and $t^{\alpha-1}\mathscr{S}_\alpha(t)$, $t > 0$ is not necessary in Theorem 3.2.*

4. Application

In this section, we apply the obtained abstract results to study fractional partial integro-differential equations with nonlocal problem:

$$\begin{cases} D_t^\alpha z(t,x) = \dfrac{\partial^2 z(t,x)}{\partial x^2} + \displaystyle\int_0^t e^{-\frac{(t-s)}{\tau}} \dfrac{\partial^2 z(s,x)}{\partial x^2} ds + c(t, z(t\cos t, x)), \\ 0 \le x \le \pi,\ 0 \le t \le 1, \\ z(t,0) = z(t,\pi) = 0,\ t \in [0,1], \\ z(0,x) + \displaystyle\int_0^\pi k(x,y)\sin(z(t,y))dy = z_0(x),\ 0 \le x \le \pi, \end{cases} \tag{4.1}$$

where D_t^α denotes the Caputo fractional partial derivative of order $\alpha \in (0,1)$, τ is a positive number and $z_0(x) \in X := L^2([0,\pi];\mathbb{R})$. The functions $c(\cdot,\cdot)$ and $k(\cdot,\cdot)$ will be described below.

To apply our results to the system (4.1), we first need to rewrite this system into the form of Eq. (1.1). To this end, let the operator $A : \mathcal{D}(A) \subseteq X \to X$ be defined by

$$Az = z''$$

with the domain

$$\mathcal{D}(A) = \{z(\cdot) \in X : z', z'' \in X, \text{ and } z(0) = z(\pi) = 0\}.$$

Then A generates a strongly continuous semigroup $(T(t))_{t\ge 0}$ which is compact, analytic, and self-adjoint. Furthermore, A has a discrete spectrum, the eigenvalues are $-n^2, n \in \mathbb{N}$, with the corresponding normalized eigenvectors $e_n(x) = \sqrt{\frac{2}{\pi}}\sin(nx)$, $n = 1, 2, \cdots$. Thus, A is sectorial of type and (V_1) is fulfilled. We also consider the operator $B(t) : \mathcal{D}(A) \to X$, $t \ge 0$, $B(t)x = e^{-\frac{t}{\tau}}Ax$ for $x \in \mathcal{D}(A)$. Furthermore, it is not difficult to see that the hypotheses (V_2) and (V_3) (in Section 2) are satisfied with $b(t) = e^{-\frac{t}{\tau}}$ and $E = C_0^\infty([0,\pi];\mathbb{R})$, where $C_0^\infty([0,\pi];\mathbb{R})$ is the space of infinitely differentiable functions from $[0,\pi]$ to $\mathbb{R}$ vanishing at 0 and π. Then by Lemma 2.4 the corresponding linear system of (4.1) has an α-resolvent operator $(\mathscr{R}_\alpha(t))_{t\ge 0}$.

We impose the following conditions on System (4.1):

(A_1) The function $c : [0,1] \times \mathbb{R} \to \mathbb{R}$ is continuous, and there exists a function $\mu \in L^p([0,1];[0,\infty))$ $(p > \frac{1}{\alpha} > 1)$ such that

$$|c(t,x_1) - c(t,x_2)| \le \mu(t)|x_1 - x_2|,$$

and

$$|c(t,x)| \le \mu(t)|x|,$$

for any $t \in [0,1]$, $x, x_1, x_2 \in \mathbb{R}$.

(A_2) $k(\cdot,\cdot) : [0,\pi] \times [0,\pi] \to \mathbb{R}$ is a measurable function satisfying

$$\sigma := \left(\int_0^\pi \int_0^\pi k^2(x,y)dydx\right)^{\frac{1}{2}} < +\infty.$$

Now, we take $u(t)(x) = z(t,x)$, and define the functions $G : [0,1] \times X \to X$, and $g : C([0,1]; X) \to X$, respectively, as

$$G(t,z)(x) = c(t, z(x)),\ z \in X,$$

and

$$g(u)(x) = \int_0^\pi k(x,y) \sin(u(t)(y))dy,\ u \in C([0,1]; X).$$

Let $h(t) = t\cos t$, then, with these notations, System (4.1) can be rewritten into the form of (1.1).

We next show that all conditions of Theorem 3.2 are fulfilled. Assumption (A_1) implies readily that the function G satisfies (H_2) and (H_4). As a matter of fact, for $t \in [0,1]$ and $z_1, z_2 \in X$, we get

$$\begin{aligned} \|G(t,z_1) - G(t,z_2)\| = &\left(\int_0^\pi |c(t,z_1(x)) - c(t,z_2(x))|^2 dx\right)^{\frac{1}{2}} \\ \leq &\left(\int_0^\pi \mu^2(t)|z_1(x) - z_2(x)|^2 dx\right)^{\frac{1}{2}} \\ \leq &\ \mu(t)\|z_1 - z_2\|. \end{aligned}$$

Then, from Lemma 2.8(vii), we obtain that $\omega(\cdot) := \mu(\cdot)$. Similarly, we can also get

$$\|G(t,z)\| \leq \mu(t)\|z\|,\ t \in [0,1],\ z \in X. \tag{4.2}$$

According to (4.2), it is easy to see that $\zeta(\cdot) := \mu(\cdot)$ and $\Omega(\|z\|) := \|z\|$ (note that $\varphi := \liminf_{l\to+\infty} \frac{\Omega(l)}{l} = 1$). On the other hand, the function g clearly satisfies the condition (H_3) with $\delta := \sigma$ (also see [43] for the compactness property). Consequently, by Theorem 3.2, the system (4.1) admits at least one mild solution on $[0,1]$ provided that

$$M_1\sigma + 4M_2\left(\frac{p-1}{p\alpha - 1}\right)^{\frac{p-1}{p}} \|\mu\|_{L^p([0,T];[0,\infty))} < 1.$$

Where M_1 and M_2 are defined in Section 3.

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