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CONVERGENCE OF S-ITERATION FOR NONLINEAR OPERATORS

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Abstract. In this paper the Halpern type S-iteration process introduced by D. R. Sahu is further analyzed for different class of nonlinear nonexpansive mappings in uniformly convex Banach space.
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1. Introduction

Let K be a nonempty subset of a real normed linear space E. A mapping T from K to E is called nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$. Let H be a Hilbert space with inner product $\langle ., . \rangle$ and norm $\|.\|$, respectively. Let C be a nonempty, closed and convex subset of H and $A : C \to H$ a nonlinear operator.

It is well known that the sequence $\{T^n x\}$ of iterates of nonexpansive operator T at a point $x \in C$ may in general, not behave well. This means that it may not converge in weak topology. Further we use Krasnoselskij Mann (KM) iteration method [[2], [4]] that produces a sequence $\{x_n\}$ via the recursive manner:

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T x_n \quad for\ all \quad n \in \mathbb{N},$$

where the initial guess $x_1 \in C$ is chosen arbitrarily and $\{\alpha_n\}$ is a real sequence in [0,1]. It is worth noting that the KM iteration process is well known for finding fixed points of nonexpansive operators.

In 2009, Agarwal *et al.* [1] have defined the S-iteration process as follows: Let E be a normed linear space, C a nonempty convex subset of E and $T : C \to C$ an operator. Then, for arbitrary $x_1 \in C$, the S-iteration process is defined by

$$(S) \qquad \begin{cases} x_{n+1} &= (1 - \alpha_n)Tx_n + \alpha_n T y_n, \\ y_n &= (1 - \beta_n)x_n + \beta_n T x_n, \quad n \in \mathbb{N}, \end{cases}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences in (0,1) satisfying the condition:

$$\sum_{n=1}^{\infty} \alpha_n \beta_n (1-\beta_n) = \infty.$$

It has also been shown that S-iteration process is faster than the Picard iteration process for contraction operators.

In 1997, Browder [2] and Halpern [5] proved independently the strong convergence of the path $\{x_t = tu + (1-t)Tx_t : t \in (0,1)\}$ as $t \to 0$ for nonexpansive operator T in a Hilbert space H. After Browder's result, such a problem has been investigated by several authors for example, Singh and Watson [8], Marino and Trombetta [6] and others. In 1998, W. Takahashi and G. E. Kim [9] defined contraction S_t and U_t from C into X by

$$S_t x = tPTx + (1-t)u \quad for\ all\ x \in C,$$

and

$$U_t x = P(tTx + (1-t)u) \quad for\ all\ x \in C,$$

where C is a closed and convex subset of a reflexive Banach space X and P is a sunny nonexpansive retraction from X onto C.

The purpose of this paper is to further analyze Halpern type S-iteration process for nonlinear operator which generalizes Theorem 5.4 [7] for nonself mapping.

2. Preliminaries

Let X be a Banach space and S_X denote the unit sphere $S_X = \{x \in X : \|x\| = 1\}$. A Banach space X is said to be strictly convex if

$$x, y \in S_X \text{ with } x \neq y \implies \|(1-\lambda)x + \lambda y\| < 1 \text{ for all } \lambda \in (0,1).$$

Recall that a Banach space X is said to be smooth provided the limit

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t}$$

exists for each $x, y \in S_X$. In this case, the norm of X is said to be Gateaux differentiable. It is said to be uniformly Gateaux differentiable if for each $y \in S_X$ this limit is attained uniformly for $x \in S_X$. It is well known that every uniformly smooth space (e.g. L_p space $(1 < p < \infty)$) has uniformly Gateaux differentiable norm.

Let E be an arbitrary real normed linear space with dual space E^*. We denote J the normalized duality mapping from E into 2^{E^*} defined by

$$J(x) = \left\{ f^* \in E^* : \langle x, f^* \rangle = \|x\|^2 = \|f^*\|^2 \right\}, \quad x \in E,$$

where $\langle ., . \rangle$ denotes the generalized duality pairing. Then for each $x, y \in E$, there exists $j(x+y) \in J(x+y)$ such that

$$\|x+y\|^2 \leq \|x\|^2 + 2 \langle y, j(x+y) \rangle .$$

It is well known that J is single-valued iff X is smooth. It is also known that if X has a uniformly Gateaux differentiable norm, J is uniformly continuous on bounded sets when X has its strong topology while X^* has its $weak^*$ topology.

A closed convex subset C of Banach space X is said to have a normal structure if for each bounded closed convex subset K of C, which contains at least two points, there exists an element of K, which is not a diameteral point of K. It is well known that a closed convex subset of a uniformly convex Banach space has normal structure and a compact convex subset of Banach space has normal structure.

A subset C of a Banach space X is called a retract of X, if there exists a continuous mapping P from X onto C such that $Px = x$ for all $x \in C$. We call such P a retraction of X onto C. It follows that if a mapping P is a retraction, then $Py = y$ for all y in the range of P. A retraction P is said to be sunny if $P(Px + t(x - Px)) = Px$ for each $x \in X$ and $t \geq 0$. If a sunny retraction P is also nonexpansive, then C is said to be sunny nonexpansive retract of X.

The existence of fixed point for nonself nonexpansive mapping is given by the following lemmas:

Lemma 2.1. ([1], Theorem 5.2.26) *Let C be a nonempty closed convex bounded subset of a uniformly convex Banach space X and $T : C \to X$ a weakly inward nonexpansive mapping. Then T has a fixed point in C.*

The following theorem shows the convergence of path for nonself nonexpansive mapping.

Lemma 2.2. ([9], Theorem 3) *Let X be a reflexive Banach space with uniformly Gateaux differentiable norm. Let C be a nonempty closed convex subset of X which has normal structure and let $T : C \to X$ be a nonexpansive nonself mapping satisfying the weak inwardness condition. Suppose that C is sunny nonexpansive retract of X and for some $u \in C$ and each $t \in (0, 1)$, $y_t \in C$ is a (unique) fixed point of contraction U_t defined by*

$$U_t x = P(tTx + (1 - t)u),$$

where P is a sunny nonexpansive retraction of X onto C. Then T has a fixed point iff $\{y_t\}$ remains bounded as $t \to 1$. In this case, $\{y_t\}$ converges strongly as $t \to 1$ to a fixed point of T.

3. Strong convergence of Halpern type S-iteration process

Motivated by the works of D. R. Sahu [7], we propose the following algorithms for nonself mapping to further analyze Halpern type S-iteration process for nonlinear operator. We generalize Theorem from self mapping to nonself mapping.

Algorithm 3.1. Let C be a nonempty closed convex subset of Banach space X and $T : C \to X$ an operator. Given $u, x_1 \in C$, a sequence $\{x_n\}$ in C is constructed as follows:

$$\begin{aligned} x_{n+1} &= P\left[(1 - \alpha_n)Tx_n + \alpha_n Ty_n\right], \\ y_n &= (1 - \beta_n)x_n + \beta_n u, \quad n \in \mathbb{N}, \end{aligned} \tag{3.1}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are two sequences in (0,1] satisfying the following condition:

(C1) $\lim_{n\to\infty} \beta_n = 0$, $\lim_{n\to\infty} \frac{\alpha_n}{\alpha_{n+1}} = \lim_{n\to\infty} \frac{\beta_n}{\beta_{n+1}} = 1$ and $\sum_{n=1}^{\infty} \alpha_n \beta_n = \infty$,

where P is a sunny nonexpansive retraction of X.

Algorithm 3.2. Let C be a nonempty closed convex subset of a Banach space X and $T : C \to X$ an operator. Given $u, x_1 \in C$, a sequence $\{x_n\}$ in C is constructed as follows:

$$\begin{aligned} x_{n+1} &= P\left[(1-\lambda)Tx_n + \lambda Ty_n\right] \\ y_n &= (1-\beta_n)x_n + \beta_n u, \quad n \in \mathbb{N}, \end{aligned} \tag{3.2}$$

where $\lambda \in (0, 1]$ and $\{\beta_n\}$ is sequence in (0,1] satisfying the following condition:

(C2) $\lim\limits_{n\to\infty} \beta_n = 0$, $\lim\limits_{n\to\infty} \dfrac{\beta_n}{\beta_{n+1}} = 1$ and $\sum\limits_{n=1}^{\infty} \beta_n = \infty$,

where P is a sunny nonexpansive retraction of X.

Lemma 3.1. *Let X be a smooth Banach space. Then*

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, J(x+y)\rangle \ \ for\ all\ x, y \in X.$$

Lemma 3.2. *Let $\{a_n\}$ be a sequence of nonnegative real numbers satisfying:*

$$a_{n+1} \leq (1-t_n)a_n + t_n b_n \ \ for\ all\ n \in \mathbb{N},$$

where $\{b_n\}$ and $\{t_n\}$ are sequences of real numbers which satisfy the conditions:

(i) $\{t_n\} \subset [0, 1]$ and $\sum\limits_{n=1}^{\infty} t_n = \infty$ and

(ii) $\limsup\limits_{n\to\infty} b_n = 0$.

Then $\lim\limits_{n\to\infty} a_n = 0$.

Proposition 3.3. *Let C be a nonempty closed convex subset of a Banach space X and $T : C \to X$ a nonexpansive operator such that $F(T) \neq \emptyset$. For given $u, x_1 \in C$, let $\{x_n\}$ be a sequence in C generated by Algorithm 3.1. Then we have $\{x_n\}$ and $\{y_n\}$ are bounded.*

Proof. Suppose $p \in F(T)$. From Algorithm 3.1, we have

$$\|y_n - p\| \leq (1-\beta_n)\|x_n - p\| + \beta_n \|u - p\|. \tag{3.3}$$

Since

$$\begin{aligned} \|x_{n+1} - p\| &= \|P\left[(1-\alpha_n)Tx_n + \alpha_n Ty_n\right] - Pp\| \\ &\leq \|(1-\alpha_n)(Tx_n - p) + \alpha_n(Ty_n - p)\| \\ &\leq (1-\alpha_n)\|Tx_n - p\| + \alpha_n \|Ty_n - p\| \\ &\leq (1-\alpha_n)\|x_n - p\| + \alpha_n \|y_n - p\| \\ &\leq (1-\alpha_n)\|x_n - p\| + \alpha_n\left[(1-\beta_n)\|x_n - p\| + \beta_n\|u - p\|\right] \\ &= (1-\alpha_n)\|x_n - p\| + (\alpha_n - \alpha_n\beta_n)\|x_n - p\| + \alpha_n\beta_n\|u - p\| \\ &= (1-\alpha_n\beta_n)\|x_n - p\| + \alpha_n\beta_n\|u - p\| \\ &\leq \max\{\|x_n - p\|, \|u - p\|\} \\ &\ \ \vdots \\ &\leq \max\{\|x_1 - p\|, \|u - p\|\}. \end{aligned} \tag{3.4}$$

Thus, $\{x_n\}$ is bounded and hence from (3.3), $\{y_n\}$ is bounded.

Theorem 3.4. *Let C be a nonempty closed convex subset of a Banach space X and $T : C \to X$ a nonexpansive operator such that $F(T) \neq \emptyset$. For given $u, x_1 \in C$, let $\{x_n\}$ be a sequence in C generated by Algorithm 3.1. Then* $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$.

Proof. Observe that

$$\begin{aligned}
\|x_{n+1} - x_n\| &= \Big\| P\left[(1-\alpha_n)Tx_n + \alpha_n Ty_n\right] \\
&\quad - P\left[(1-\alpha_{n-1})Tx_{n-1} + \alpha_{n-1}Ty_{n-1}\right] \Big\| \\
&\leq \Big\| \left[(1-\alpha_n)Tx_n + \alpha_n Ty_n\right] \\
&\quad - \left[(1-\alpha_{n-1})Tx_{n-1} + \alpha_{n-1}Ty_{n-1}\right] \Big\| \\
&= \Big\| (1-\alpha_n)Tx_n - (1-\alpha_n)Tx_{n-1} + (1-\alpha_n)Tx_{n-1} + \alpha_n Ty_n \\
&\quad - \alpha_n Ty_{n-1} + \alpha_n Ty_{n-1} - (1-\alpha_{n-1})Tx_{n-1} - \alpha_{n-1}Ty_{n-1} \Big\| \\
&\leq (1-\alpha_n)\|x_n - x_{n-1}\| + \alpha_n \|y_n - y_{n-1}\| \\
&\quad + |\alpha_n - \alpha_{n-1}| \, \|x_{n-1} - y_{n-1}\| . \qquad (3.5)
\end{aligned}$$

Now,

$$\begin{aligned}
\|y_n - y_{n-1}\| &= \|[(1-\beta_n)x_n + \beta_n u] - [(1-\beta_{n-1})x_{n-1} + \beta_{n-1}u]\| \\
&= \Big\| (1-\beta_n)x_n - (1-\beta_n)x_{n-1} + (1-\beta_n)x_{n-1} \\
&\quad - (1-\beta_{n-1})x_{n-1} + (\beta_n - \beta_{n-1})u \Big\| \\
&\leq (1-\beta_n)\|x_n - x_{n-1}\| + |\beta_n - \beta_{n-1}| \left(\|u\| + \|x_{n-1}\|\right) \\
&\leq (1-\beta_n)\|x_n - x_{n-1}\| + |\beta_n - \beta_{n-1}| K_1 \qquad (3.6)
\end{aligned}$$

for some constant $K_1 > 0$. From proposition 3.3, $\{x_n\}$ is bounded, so

$$\begin{aligned}
\|x_{n-1} - y_{n-1}\| &= \beta_{n-1}\|x_{n-1} - u\| \\
&\leq \beta_{n-1}\left(\|x_{n-1}\| + \|u\|\right) \\
&\leq \beta_{n-1}K_1 \qquad (3.7)
\end{aligned}$$

for some constant $K_2 > 0$. Using (3.6) and (3.7) in (3.5), we have

$$\begin{aligned}
\|x_{n+1} - x_n\| &\leq (1-\alpha_n)\|x_n - x_{n-1}\| + \alpha_n \Big[(1-\beta_n)\|x_n - x_{n-1}\| \\
&\quad + |\beta_n - \beta_{n-1}| K_1 \Big] + |\alpha_n - \alpha_{n-1}| \, \|x_{n-1} - y_{n-1}\| \\
&\leq (1-\alpha_n\beta_n)\|x_n - x_{n-1}\| + \alpha_n |\beta_n - \beta_{n-1}| K_1 \\
&\quad + |\alpha_n - \alpha_{n-1}| \beta_{n-1}K_2
\end{aligned}$$

$$\begin{aligned}
&= (1-\alpha_n\beta_n)\|x_n - x_{n-1}\| + \alpha_n\beta_n\left|1-\frac{\beta_{n-1}}{\beta_n}\right|K_1 \\
&\quad + \alpha_n\beta_{n-1}\left|1-\frac{\alpha_{n-1}}{\alpha_n}\right|K_2 \\
&= (1-\alpha_n\beta_n)\|x_n - x_{n-1}\| + \alpha_n\beta_n\left|1-\frac{\beta_{n-1}}{\beta_n}\right|K_1 \\
&\quad + \alpha_n\beta_{n-1}\left|1-\frac{\alpha_{n-1}}{\alpha_n}\right|K_2 \\
&= (1-\alpha_n\beta_n)\|x_n - x_{n-1}\| \\
&\quad + \alpha_n\beta_n\left[\left|1-\frac{\beta_{n-1}}{\beta_n}\right|K_1 + \frac{\beta_{n-1}}{\beta_n}\left|1-\frac{\alpha_{n-1}}{\alpha_n}\right|K_2\right] \qquad (3.8)
\end{aligned}$$

Using condition (C1), we have

$$\lim_{n\to\infty}\left|1-\frac{\alpha_{n-1}}{\alpha_n}\right| = 0 \ and \ \lim_{n\to\infty}\left|1-\frac{\beta_{n-1}}{\beta_n}\right| = 0.$$

Using Lemma 3.2, we have

$$\lim_{n\to\infty}\|x_{n+1}-x_n\| = 0.$$

Hence

$$\begin{aligned}
\|x_n - Tx_n\| &= \|x_n - PTx_n\| \\
&= \|x_n - x_{n+1} + x_{n+1} - PTx_n\| \\
&\le \|x_n - x_{n+1}\| + \|x_{n+1} - PTx_n\|. \qquad (3.9)
\end{aligned}$$

Consider,

$$\begin{aligned}
\|x_{n+1} - PTx_n\| &= \|P[(1-\alpha_n)Tx_n + \alpha_n Ty_n] - PTx_n\| \\
&\le \|(1-\alpha_n)Tx_n + \alpha_n Ty_n - Tx_n\| \\
&= \|\alpha_n(Ty_n - Tx_n)\| \\
&= \alpha_n\|Ty_n - Tx_n\| \\
&\le \alpha_n\|y_n - x_n\| \\
&= \alpha_n\|\beta_n(x_n - u)\| \\
&= \alpha_n\beta_n\|x_n - u\| \to 0 \ as \ n\to\infty.
\end{aligned}$$

Hence (3.9) becomes

$$\|x_n - Tx_n\| \to 0 \ as \ n \to \infty.$$

Lemma 3.5. *Let X be a Banach space with a uniformly Gateaux differentiable norm, C a nonempty closed convex subset of X, Let $T : C \to X$ a nonexpansive mapping and $\{x_n\}$ is a bounded sequence in C such that $\lim_{n\to\infty}\|x_n - Tx_n\| = 0$. Suppose $\{z_t\}$ is a path in C defined by $z_t = P((1-t)Tz_t + tu)$, $t \in (0,1)$ such that $z_t \to z$ as $t \to 0^+$. Then*

$$\limsup_{n\to\infty}\langle u - z, J(x_n - z)\rangle \le 0.$$

Proof. Since $z_t - x_n = z_t - Px_n = P((1-t)Tz_t + tu) - Px_n$. Now, we consider

$$\begin{aligned}
\|z_t - x_n\|^2 &= \langle z_t - x_n, J(z_t - x_n)\rangle \\
&= \langle P((1-t)Tz_t + tu) - Px_n, J(z_t - x_n)\rangle \\
&\leq \Big\langle (1-t)(Tz_t - x_n) + t(u - x_n) - Px_n, \\
&\qquad J(z_t - x_n)\Big\rangle \\
&\leq (1-t)\langle Tz_t - x_n, J(z_t - x_n)\rangle \\
&\quad + t\langle u - x_n, J(z_t - x_n)\rangle \\
&= (1-t)\langle Tz_t - Tx_n + Tx_n - x_n, J(z_t - x_n)\rangle \\
&\quad + t\langle u - z_t + z_t - x_n, J(z_t - x_n)\rangle \\
&\leq (1-t)\|z_t - x_n\|^2 + (1-t)\Big\langle Tx_n - x_n, \\
&\qquad J(z_t - x_n)\Big\rangle + t\langle u - z_t, J(z_t - x_n)\rangle \\
&\quad + t\|z_t - x_n\|^2 \\
i.e.\quad t\langle u - z_t, J(x_n - z_t)\rangle &\leq (1-t)\,\|Tx_n - x_n\|\,\|z_t - x_n\| \\
\langle u - z_t, J(x_n - z_t)\rangle &\leq \frac{1}{t}\,\|Tx_n - x_n\|\,K_1
\end{aligned}$$

for some constant $K_1 > 0$. It follows that

$$\limsup_{n\to\infty}\langle u - z_t, J(x_n - z_t)\rangle \quad \leq \quad 0. \tag{3.10}$$

Further, since $z_t \to z$ as $t \to 0$. The set $\{z_t - x_n\}$ is bounded and the duality mapping J is norm to weak* uniformly continuous on bounded subsets of X, it follows that

$$\begin{aligned}
\Big|\langle (u-z), J(x_n - z)\rangle - \langle (u - z_t), J(x_n - z_t)\rangle\Big| &= \Big|\langle u - z, J(x_n - z) - J(x_n - z_t)\rangle \\
&\quad + \langle (u-z) - (u - z_t), J(x_n - z_t)\rangle\Big| \\
&\leq |\langle u - z, J(x_n - z) - J(x_n - z_t)\rangle| \\
&\quad + |\langle (u-z) - (u - z_t), J(x_n - z_t)\rangle| \\
&= |\langle u - z, J(x_n - z) - J(x_n - z_t)\rangle| \\
&\quad + \|(u-z) - (u - z_t)\|\|x_n - z_t\| \\
&\to 0 \quad as \quad t \to 0^+.
\end{aligned}$$

Let $\epsilon > 0$. Then there exists $\delta > 0$ such that

$$\langle u - z, J(x_n - z)\rangle < \langle u - z_t, J(x_n - z_t)\rangle + \epsilon,$$

for all $n \in \mathbb{N}$ and $t \in (0, \delta)$. We have

$$\begin{aligned}
\limsup_{n\to\infty}\langle u - z, J(x_n - z)\rangle &< \limsup_{n\to\infty}\langle u - z_t, J(x_n - z_t)\rangle + \epsilon \\
&\leq \epsilon.
\end{aligned}$$

Since ϵ is arbitrary, we obtain that

$$\limsup_{n\to\infty} \langle u - z, J(x_n - z)\rangle \leq 0.$$

Theorem 3.6. *Let X be a uniformly convex Banach space whose norm is uniformly Gateaux differentiable, C a nonempty closed convex subset of X and $T : C \to X$ a nonexpansive operator satisfying weakly inwardness condition with $F(T) \neq \emptyset$. For given $u, x_1 \in C$, let $\{x_n\}$ be a sequence in C generated by Algorithm 3.1. Then $\{x_n\}$ converges strongly to a fixed point of T.*

Proof. Using Lemma 2.2, we see that the path $\{z_t\}$ defined by

$$z_t = (1-t)Tz_t + tu, \quad for\ t \in (0,1)$$

converges strongly to $F(T)$ as $t \to 0^+$. Let $z = \lim_{t\to 0} z_t$, by Lemma 3.1, we get

$$\begin{aligned} \|y_n - z\|^2 &= \|(1-\beta_n)(x_n - z) + \beta_n(u - z)\|^2 \\ &\leq (1-\beta_n)\|x_n - z\|^2 + 2\beta_n \langle u - z, J(y_n - z)\rangle . \end{aligned} \tag{3.11}$$

Now, since X is uniformly convex, there exists a continuous strictly convex function $\phi : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ such that $\phi(0) = 0$ and

$$\|\lambda x + (1-\lambda)y\|^2 = \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)\phi(\|x - y\|) \tag{3.12}$$

for all $x, y \in X$ with $\|x\| \leq r$, $\|y\| \leq r$ and for all $\lambda \in [0,1]$ and for some $r > 0$. Choose $r > 0$ large enough so that $\|Tx_n - z\| \leq r$ for all $n \in \mathbb{N}$. From (3.12), we have

$$\begin{aligned} \|x_{n+1} - z\|^2 &= \|P[(1-\alpha_n)Tx_n + \alpha_n Ty_n] - Pz\| \\ &\leq \|(1-\alpha_n)(Tx_n - z) + \alpha_n(Ty_n - z)\|^2 \\ &\leq (1-\alpha_n)\|Tx_n - z\|^2 + \alpha_n\|Ty_n - z\|^2 \\ &\leq (1-\alpha_n)\|x_n - z\|^2 + \alpha_n\|y_n - z\|^2 \\ &\leq (1-\alpha_n\beta_n)\|x_n - z\|^2 + 2\alpha_n\beta_n \langle u - z, J(y_n - z)\rangle \\ &\leq (1-\alpha_n\beta_n)\|x_n - z\|^2 + \lambda_n\sigma_n, \end{aligned} \tag{3.13}$$

where $\lambda_n = \alpha_n\beta_n$ and $\sigma_n = \langle u - z, J(y_n - z)\rangle$. Since

$$\sum_{n=1}^{\infty} \lambda_n = \infty \text{ and } \limsup \sigma_n \leq 0$$

using Lemma 3.5. Hence we conclude from Lemma 3.2 that $\{x_n\}$ converges strongly to z.

Example 3.7. Let $X = R$ with absolute norm. Let $C = [-2, 2]$ and let $T : C \to X$ be defined by $Tx = -x$ a nonexpansive operator satisfying weakly inwardness condition and $F(T) = 0$. For $u = x_1 = 1/2$. let $\{x_n\}$ be a sequence in C generated by Algorithm 3.1 for $\alpha_n = 1$ and $\beta_n = 1/n$ satisfying condition (C1) then we can see that sequence $\{x_n\}$ converges strongly to 0, a fixed point of T.

Corollary 3.8. *Let X be a uniformly convex Banach space whose norm is uniformly Gateaux differentiable, C a nonempty closed convex subset of X and $T : C \to X$ a nonexpansive operator satisfying weakly inwardness condition with $F(T) \neq \emptyset$. For* *given* $u, x_1 \in C$, $\{x_n\}$ *be a sequence in* C *generated by Algorithm 3.2. Then sequence* $\{x_n\}$ *converges strongly to the fixed point of* T.

Proof. Let us take $\alpha_n = \lambda$ for all $n \in \mathbb{N}$ in Algorithm 3.1. Then Algorithm 3.2 with condition (C2) is same as Algorithm 3.1 with condition (C1). Hence it follows from Theorem 3.2.

Theorem 3.9. *Let* X *be a uniformly convex Banach space whose norm is uniformly Gateaux differentiable,* C *a nonempty closed convex subset of* X *and* $T : C \to X$ *a nonexpansive operator satisfying weakly inwardness condition with* $F(T) \neq \emptyset$. *For given* $u, x_1 \in C$, $\{x_n\}$ *be a sequence in* C *generated by*

$$x_{n+1} = PT\left[(1-\beta_n)x_n + \beta_n u\right], \quad n \in \mathbb{N},$$

where $\{\beta_n\}$ *is a sequence in (0,1] satisfying the condition (C2).Then* $\{x_n\}$ *converges strongly to the fixed point of* T.

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