Fixed Point Theory, 24(2023), No. 2, 701-720
DOI: 10.24193/fpt-ro.2023.2.16
http://www.math.ubbcluj.ro/~nodeacj/sfptcj.html

VISCOSITY APPROXIMATION FOR SPLIT MONOTONE VARIATIONAL INCLUSIONS AND FIXED POINT PROBLEM

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Abstract. In this work, we investigate an iterative method based on viscosity approximation method to approximate a common solution of split monotone variational inclusion problem and fixed point problem for a nonexpansive mapping in the frame work of real Hilbert spaces. Further, strong convergence theorem is proved by the sequences generated by the proposed iterative method under some mild conditions, which is the unique solution of the variational inequality problem. Furthermore, we provide some numerical experiments to support our main result. The results and method presented in this work may be treated as an improvement, extension and refinement of some corresponding ones in the literature.
Key Words and Phrases: Split monotone variational inclusion problem, nonexpansive mapping, fixed-point problem, iterative method.
2020 Mathematics Subject Classification: 65K15, 47J25, 47H10, 65J15, 90C33.

1. Introduction

Variational inclusions problem has emerged as an interesting branch of applicable mathematics and it is also being used as a models in many mathematical programming to study a large number problems arising in economics, operations research, transportation problems, optimization problems and engineering sciences, see, for example [1, 12, 14] and references therein. They have been extended and generalized the variational inclusion problem in different directions by using noval and innovative ideas both for their own sake and for their applications.

In recent years, much attention has been received by variational inclusion problems to develop the efficient and implementable numerical techniques including the projection methods and its variant forms, Wiener-Hopf equation, linear approximation, auxiliary principle technique, proximal point algorithm and descent framework for solving variational inequality problems and its related problems. It is well known that the projection methods and its variant forms; and Wiener-Hopf equations technique can not suggest and analyze iterative methods for solving the variational inequality problems due to the presence of nonlinear term.

This fact motivated to develop another numerical technique, which involve the use of resolvent operator associated with maximal monotone operator. By using this technique, one shows that the monotone variational inclusion problem are equivalent to the fixed point problem. this alternative fixed point formulation was used to develop numerical methods for solving the various classes of variational inclusion problems and its related problems, see [27, 28] and references therein.

Maximal monotone operators were first introduced in [23] and [41], and can be seen as a two-way generalization: a nonlinear generalization of linear endomorphisms with positive semi-definite matrices, and a multidimensional generalization of nondecreasing functions of a real variable; that is, of derivatives of convex and differentiable functions. Thus, not surprisingly, the main example of this kind of operator in a Banach space is the Fréchet derivative of a smooth convex function, or, in the set-valued realm, the subdifferential of an arbitrary lower semi-continuous convex function. Monotone operators are the key ingredients of monotone variational inequalities and monotone variational inclusions, which extend to the realm of set-valued mappings the constrained convex minimization problem. More details can be found in [14, 3, 42] and references therein.

Proximal mapping and resolvent operator techniques are played important role to compute the approximate solution of generalized variational and monotone variational inequalities, and generalized variational and quasi-variational inclusions. Rockafellar [34] and Martinet [22] used the resolvent operator associated with maximal monotone operators for solving the variational inclusion $0 \in M(x)$, where M is a maximal monotone operator on a Hilbert space. The main difficulty with this method is that the operator $I + \rho M$ may be hard to invert. One alternative of the previous difficulty is to decompose the given operator into the sum of two maximal monotone operators whose resolvent are easy to evaluate than the resolvent of the original operator, such a method is known as operator splitting method. The operator splitting method has been studied and generalized by many authors, see for instance [31, 39] and references therein.

Monotone operator theory is a fascinating field of research in nonlinear functional analysis and found valuable applications in the field of convex optimization, subgradients, partial differential equations, variational inequalities, signal and image processing, evolution equations and inclusions; see, for instance, [2, 11, 13, 34] and the references cited therein. It is noted that the convex optimization problem can be translated into finding a zero of a maximal monotone operator defined on a Hilbert space. On the other hand, the problem of finding a zero of the sum of two (maximal-) monotone operators is of fundamental importance in convex optimization and variational analysis [19, 38]. The forwardbackward algorithm is prominent among various splitting algorithms to find a zero of the sum of two maximal monotone operators [19]. The class of splitting algorithms has parallel computing architectures and thus reducing the complexity of the problems under consideration. On the other hand, the forwardbackward algorithm efficiently tackle the situation for smooth and/or non-smooth functions. It is worth mentioning that the forwardbackward algorithm has been modified by employing the heavy ball method [32] for convex optimization problems. For related work, see [29, 21, 20].

A viscosity approximation method is a well known iterative method which is used to approximate a fixed point for a nonexpansive mappings. In 2000, Moudafi [24] introduced and studied viscosity approximation method by combining nonexpansive mapping and a contraction mapping. Further, he proved that fixed point of a nonexpansive mapping solves variational inequality problem, Further, Xu [40] developed an iterative method based on viscosity approximation method to find the zero of maximal monotone operators in Banach space. Recently, Kazmi and Rizvi [18] developed an implicit iterative method based on viscosity approximation method to find common solution of split equilibrium problem and fixed points of nonexpansive semigroup. For related work, see [16, 33].

It is worth mentioning that if the nonlinear term involving in the variational inequality problem is the indicator function of a closed and convex set in a Hilbert space, then the resolvent operator is equal to the projection operator. Consequently the resolvent operators are equivalent to the Wiener-Hopf or normal equations, which were introduced and studied by Shi [36] and Robinson [35] in the relation with the classical variational inequality problems.

Recently, Moudafi [26] extended and generalized monotone variational inclusion problem into a new direction by considering two monotone variational inclusion problem in two different Hilbert spaces, which is governed by a bounded linear operator, such problem is known as split monotone variational inclusion problem. By using the averaged operators technique and resolvent operators associated with maximal monotone operators, for solving split monotone variational inclusion problem.

The computation of fixed points is important in the study of many problems including inverse problems. For instance, it is not hard to show that the split monotone variational inclusion problem can both be formulated as a problem of finding fixed points of certain operators. Construction of fixed points of nonexpansive mappings is an important subject in nonlinear operator theory and its applications; in particular, in image recovery and signal processing and in transition operators for initial valued problems of differential inclusions (see, for example, [26, 6, 17]).

2. Preliminaries

Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be real Hilbert spaces with scaler product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$. Let K_1 and K_2 are nonempty, closed and convex subsets of $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively.

Definition 2.1. A mapping $S : \mathcal{H}_1 \to \mathcal{H}_1$ is said to be

(i) *monotone*, if

$$\langle Sx - Sy, x - y \rangle \geq 0, \ \forall x, y \in \mathcal{H}_1;$$

(ii) *r-strongly monotone*, if there exists a constant $r > 0$ such that

$$\langle Sx - Sy, x - y \rangle \ \geq r\|x - y\|^2, \ \forall x, y \in \mathcal{H}_1;$$

(iii) *τ-inverse strongly monotone*, if there exists a constant $\tau > 0$ such that

$$\langle Sx - Sy, x - y \rangle \ \geq \tau\|Sx - Sy\|^2, \ \forall x, y \in \mathcal{H}_1;$$

(iv) *firmly nonexpansive*, if

$$\langle Sx - Sy, x - y \rangle \ \geq \|Sx - Sy\|^2, \ \forall x, y \in \mathcal{H}_1.$$

(v) *nonexpansive*, if

$$\|Sx - Sy\| \leq \|x - y\|, \ \forall x, y \in \mathcal{H}_1.$$

The *fixed point problem* (in short, FPP) for a nonexpansive mapping $S : \mathcal{H}_1 \to \mathcal{H}_1$ is to find $x \in \mathcal{H}_1$ such that

$$Sx = x. \tag{2.1}$$

The solution set of FPP(2.1) is denoted by Fix(S). It is well known that if Fix(S) $\neq \emptyset$, Fix(S) is closed and convex.

Next, we consider the following *split monotone variational inclusion problem* (in short, $\mathrm{S_P}$MVIP): Find $\bar{x} \in \mathcal{H}_1$ such that

$$0 \in M_1(\bar{x}) + f(\bar{x}), \tag{2.2}$$

and such that

$$\bar{y} = B\bar{x} \in \mathcal{H}_2 \text{ solves } 0 \in M_2(\bar{y}) + g(\bar{y}), \tag{2.3}$$

where $M_1 : \mathcal{H}_1 \to 2^{\mathcal{H}_1}$ and $M_2 : \mathcal{H}_2 \to 2^{\mathcal{H}_2}$ are multi-valued maximal monotone mappings and $f : \mathcal{H}_1 \to \mathcal{H}_1$ and $g : \mathcal{H}_2 \to \mathcal{H}_2$ are nonlinear mappings and $B : \mathcal{H}_1 \to \mathcal{H}_2$ be a bounded linear operator.

$\mathrm{S_P}$MVIP(2.2)-(2.3) has been introduced and studied by Moudafi [26]. Further, (2.2) and (2.3) are the monotone variational inclusion problems in two different spaces, we denote their solution sets by Sol(MVIP(2.2)) and Sol(MVIP(2.3)) respectively. The solution set of $\mathrm{S_P}$MVIP(2.2)-(2.3) is denoted by Sol($\mathrm{S_P}$MVIP(2.2)-(2.3))$= \{\bar{x} \in \mathcal{H}_1 : \bar{x} \in \text{Sol(MVIP(2.2))} : \text{and } B\bar{x} \in \text{Sol(MVIP(2.3))}\}$.

In 2011, Moudafi [26] introduced and studied an iterative method to find the solution of $\mathrm{S_P}$MVIP(2.2)-(2.3), under the certain conditions, he obtained weak convergence theorem. If $f \equiv 0$ and $g \equiv 0$ then $\mathrm{S_P}$MVIP(2.2)-(2.3) reduces to the following *split variational inclusion problem* (in short, $\mathrm{S_P}$VIP) or *split null point problem* (in short, $\mathrm{S_P}$NPP): Find $\bar{x} \in \mathcal{H}_1$ such that

$$0 \in M_1(\bar{x}), \tag{2.4}$$

and such that

$$\bar{y} = B\bar{x} \in \mathcal{H}_2 \text{ solves } 0 \in M_2(\bar{y}). \tag{2.5}$$

The solution set of $\mathrm{S_P}$VIP(2.4)-(2.5) is denoted by Sol($\mathrm{S_P}$VIP(2.4)-(2.5))$= \{\bar{x} \in \mathcal{H}_1 : \bar{x} \in \text{Sol(VIP(2.4))} : \text{and } B\bar{x} \in \text{Sol(VIP(2.5))}\}$.

In 2012, Byrne *et al.* [6] studied the weak and strong convergence of the following iterative method for $\mathrm{S_P}$NPP(2.4)-(2.5): For given $x_1 \in \mathcal{H}_1$, compute iterative sequence $\{x_n\}$ generated by the following scheme:

$$x_{n+1} = J_\rho^{M_1}(x_n + \delta B^*(J_\rho^{M_2} - I)Bx_n),$$

for $\rho > 0$.

In 2014, Kazmi and Rizvi [17] introduced and studied an iterative method, based on viscosity approximation method to approximate a common solution of $\mathrm{S_P}$VIP(2.4)-(2.5) and fixed point problem of a nonexpansive mapping in the framework of real Hilbert spaces.

$$\begin{cases} u_n = J_\rho^{M_1}(x_n + \delta B^*(J_\rho^{M_2} - I)Bx_n); \\ x_{n+1} = \alpha_n Q(x_n) + (1 - \alpha_n)Su_n, \end{cases}$$

where Q is a contraction mapping and $\rho > 0$.

Very recently, Sitthithakerngkiet *et al.* [37] extended and generalized the work of Byrne *et al.* [6] and Kazmi and Rizvi [17] for $\mathrm{S_P}$VIP(2.4)-(2.5). They obtained a strong convergence theorem to approximate the common solution of $\mathrm{S_P}$VIP(2.4)-(2.5) and FPP(2.1).

$\mathrm{S_P}$MVIP(2.2)-(2.3) includes as special cases, the split variational inequality problem, split zero problems and split feasibility problems, see for details [25, 9, 8, 7]. This problem has received much attention due to its various applications in the modeling of many inverse problems arising for phase retrieval and other real-world problems; viz, in sensor networks in computerized tomography and data compression; see [5, 4, 10] and references quoted therein.

Remark 2.1.
(i) Byrne *et al.* [6], Kazmi and Rizvi [17] and Sitthithakerngkiet *et al.* [37] remarked that $\mathrm{S_P}$VIP(2.4)-(2.5) needs further research affords.
(ii) Byrne *et al.* [6] obtain the solution of $\mathrm{S_P}$VIP(2.4)-(2.5), while Kazmi and Rizvi [17] and Sitthithakerngkiet *et al.* [37] obtain the common solution of $\mathrm{S_P}$VIP(2.4)-(2.5) and FPP(2.1).

Open Question: Could we obtain the common solution of $\mathrm{S_P}$MVIP(2.2)-(2.3) and FPP(2.1)?

Therefore, the main objective of this work is to investigate and analyze an iterative method based on viscosity approximation method to approximate the common element of set of solution of $\mathrm{S_P}$MVIP(2.2)-(2.3) and FPP(2.1). Further, we prove that the sequences generated by the iterative scheme converges strongly to a common solution of $\mathrm{S_P}$MVIP(2.2)-(2.3) and FPP(2.1). Finally, we provide some numerical experiments to support our main result. The results and method presented in this paper generalize the corresponding results of Byrne *et al.* [6], Kazmi and Rizvi [17] and Sitthithakerngkiet *et al.* [37].

We recall some concepts and results which are needed in sequel.

Lemma 2.1. [2] *Assume that S be a nonexpansive self mapping of a closed and convex subset K_1 of a Hilbert space $\mathcal{H}_1$. If S has a fixed point, then $I - S$ is demiclosed, i.e., whenever $\{x_n\}$ is a sequence in K_1 converging weakly to some $x \in K_1$ and the sequence $\{(I-S)x_n\}$ converges strongly to some y, it follows that $(I-S)x = y$.*

Lemma 2.2. [2] *In a real Hilbert space the following hold:*
(i) *For all $x, y \in \mathcal{H}_1$ and $\lambda \in [0,1]$, then*
$$\|\lambda x + (1-\lambda)y\|^2 = \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)\|x-y\|^2. \tag{2.6}$$
(ii) *Opial's condition* [30]*: For any sequence $\{x_n\}$ with $x_n \rightharpoonup x$ the inequality*
$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\| \tag{2.7}$$
holds for every $y \in \mathcal{H}_1$ with $y \neq x$.

Definition 2.2. A multi-valued mapping $M_1 : \mathcal{H}_1 \to 2^{\mathcal{H}_1}$ is called *monotone* if for all $x, y \in \mathcal{H}_1$, $u \in M_1 x$ and $v \in M_1 y$ such that
$$\langle x - y, u - v\rangle \geq 0.$$

Definition 2.3. A monotone mapping $M_1 : \mathcal{H}_1 \to 2^{\mathcal{H}_1}$ is *maximal* if the Graph(M_1) is not properly contained in the graph of any other monotone mapping.

It is known that a monotone mapping M_1 is maximal if and only if for $(x,u) \in \mathcal{H}_1 \times \mathcal{H}_1$, $\langle x-y, u-v\rangle \geq 0$, for every $(y,v) \in \text{Graph}(M_1)$ implies that $u \in M_1 x$.

Definition 2.4. Let $M_1 : \mathcal{H}_1 \to 2^{\mathcal{H}_1}$ be a multi-valued maximal monotone mapping. Then, the resolvent mapping $J_\rho^{M_1} : \mathcal{H}_1 \to \mathcal{H}_1$ associated with M_1, is defined by

$$J_\rho^{M_1}(x) := (I + \rho M_1)^{-1}(x),\ \forall x \in \mathcal{H}_1,$$

for some $\rho > 0$.

Remark 2.2. [2]

(i) We note that for all $\rho > 0$ the resolvent operator $J_\rho^{M_1}$ is single-valued, nonexpansive and firmly nonexpansive;

(ii) If we take $M_1 = \partial I_{K_1}$, the subdifferential of the indicator function I_{K_1} of K_1, where I_{K_1} is defined by

$$I_{K_1}(x) = \begin{cases} 0 & x \in K_1 \\ +\infty & x \notin K_1, \end{cases}$$

then $y = J_\rho^{I_{K_1}}(x) = (I + \rho I_{K_1})^{-1}(x) \Longleftrightarrow y = P_{K_1}(x)$, see for details [34].

The following Lemma is the fixed point formulation of S_PMVIP(2.2)-(2.3) and is followed by the definition of resolvent mapping $J_\rho^{M_1}$:

Lemma 2.3. S_PMVIP(2.2)-(2.3) *is equivalent to find* $\bar{x} \in \mathcal{H}_1$ *such that* $\bar{y} = B\bar{x} \in \mathcal{H}_2$,

$$\bar{x} = J_\rho^{M_1}(I - \rho_n f)\bar{x} \text{ and } \bar{y} = J_\rho^{M_2}(I - \rho_n g)\bar{y},\ \text{for some } \rho > 0.$$

Lemma 2.4. [40]. *Let* $\{a_n\}$ *be a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1 - \beta_n)a_n + \delta_n,\ n \geq 0,$$

where $\{\beta_n\}$ *is a sequence in* $(0,1)$ *and* $\{\delta_n\}$ *is a sequence in* $\mathbb{R}$ *such that*

(i) $\sum\limits_{n=1}^{\infty} \beta_n = \infty$;

(ii) $\limsup\limits_{n\to\infty} \dfrac{\delta_n}{\beta_n} \leq 0$ *or* $\sum\limits_{n=1}^{\infty} |\delta_n| < \infty$.

Then $\lim\limits_{n\to\infty} a_n = 0$.

3. Iterative Algorithm

In this section, we prove a strong convergence theorem based on the proposed iterative method to approximate the solution of S_PMVIP(1.2)-(1.3) and FPP(2.1).

Theorem 3.1. *Let* $\mathcal{H}_1$ *and* $\mathcal{H}_2$ *are real Hilbert spaces and* $B : \mathcal{H}_1 \to \mathcal{H}_2$ *be a bounded linear operator. Let* $M_1 : \mathcal{H}_1 \to 2^{\mathcal{H}_1}$, $M_2 : \mathcal{H}_2 \to 2^{\mathcal{H}_2}$ *are the multi-valued maximal monotone mappings, let the mapping* $f : \mathcal{H}_1 \to \mathcal{H}_1$ *and* $g : \mathcal{H}_2 \to \mathcal{H}_2$ *are* θ_1, θ_2*-inverse strongly monotone and* $Q : \mathcal{H}_1 \to \mathcal{H}_1$ *be a contraction mapping with constant* $\alpha \in (0,1)$. *Let* $S : \mathcal{H}_1 \to \mathcal{H}_1$ *be a nonexpansive mapping such that*

$$\Omega = \text{Fix}(S) \cap \text{Sol}(\text{S}_\text{P}\text{MVIP}(2.2) - (2.3)) \neq \emptyset.$$

Let the iterative sequences $\{x_n\}$, $\{y_n\}$, $\{v_n\}$ and $\{z_n\}$ are generated by the following iterative schemes:

$$\begin{cases} x_1 = x \in \mathcal{H}_1, \\ y_n = J_{\rho_n}^{M_1}(x_n - \rho_n f x_n), \\ v_n = J_{\rho_n}^{M_2}(I - \rho_n g)By_n, \\ z_n = P_{K_1}[y_n + \delta B^*(v_n - By_n)], \\ x_{n+1} = \alpha_n Q(x_n) + (1-\alpha_n)Sz_n, \end{cases} \tag{3.1}$$

where $\delta \in \left(0, \frac{1}{\|B\|^2}\right)$, $\{\rho_n\}$ and $\{\alpha_n\}$ are the sequences in $(0,1)$ and satisfying the following conditions

(i) $\lim\limits_{n\to\infty} \alpha_n = 0$, $\sum\limits_{n=0}^{\infty} \alpha_n = \infty$, *and*

$$\sum_{n=1}^{\infty} |\alpha_n - \alpha_{n-1}| < \infty;$$

(ii) $\liminf\limits_{n\to\infty} \rho_n > 0$, $\sum\limits_{n=0}^{\infty} \rho_n = \infty$, *and*

$$\sum_{n=1}^{\infty} |\rho_n - \rho_{n-1}| < \infty.$$

Then the sequences $\{x_n\}$ and $\{y_n\}$ converges strongly to $z \in \Omega$, where $z = P_\Omega Q(z)$.

Proof. Let $p \in \Omega$ then $p \in \mathrm{Sol}(\mathrm{S_P MVIP}(2.2)\text{-}(2.3))$ and hence

$$p = J_{\rho_n}^{M_1}(x_n - \rho_n f x_n)$$

and

$$Bp = J_{\rho_n}^{M_2}(I - \rho_n g)Bp.$$

We compute

$$\begin{aligned} \|y_n - p\|^2 &= \|J_{\rho_n}^{M_1}(x_n - \rho_n f x_n) - J_{\rho_n}^{M_1}(p - \rho_n f p)\|^2 \\ &\leq \|(x_n - p) - \rho_n(f x_n - f p)\|^2 \\ &= \|x_n - p\|^2 + \rho_n^2\|f x_n - f p\|^2 + 2\rho\langle x_n - p, f x_n - f p\rangle \\ &\leq \|x_n - p\|^2 - \rho_n(2\theta_1 - \rho_n)\|f x_n - f p\|^2 \qquad (3.2) \\ &\leq \|x_n - p\|^2. \qquad (3.3) \end{aligned}$$

Next, we compute

$$\begin{aligned} \|v_n - Bp\|^2 &= \|J_{\rho_n}^{M_2}(I - \rho_n g)By_n - J_{\rho_n}^{M_2}(I - \rho g)Bp\|^2 \\ &\leq \|(By_n - Bp) - \delta(gBy_n - gBp)\|^2 \\ &= \|By_n - Bp\|^2 + \delta^2\|gBy_n - gBp\|^2 + 2\delta_n\langle By_n - Bp, gBy_n - gBp\rangle \\ &\leq \|By_n - Bp\|^2 - \delta(2\theta_2 - \delta)\|gBy_n - gBp\|^2 \qquad (3.4) \\ &\leq \|By_n - Bp\|^2. \qquad (3.5) \end{aligned}$$

Next, we compute

$$\begin{aligned}
\|z_n - p\|^2 &= \|P_{K_1}[y_n + \delta B^*(v_n - By_n)] - p\|^2 \\
&\le \|y_n + \delta B^*(v_n - By_n) - p)\|^2 \\
&= \|y_n - p\|^2 + \|\delta B^*(v_n - By_n)\|^2 + 2\delta\langle y_n - p, B^*(v_n - By_n)\rangle \\
&\le \|y_n - p\|^2 + \delta^2\|B^*\|^2\|v_n - By_n\|^2 \\
&+ 2\delta\langle B(y_n - p) + (v_n - By_n) - (v_n - By_n), (v_n - By_n)\rangle \\
&= \|y_n - p\|^2 + \delta^2\|B^*\|^2\|v_n - By_n\|^2 + 2\delta\Big[\frac{1}{2}\|v_n - B\bar{x}\|^2 + \frac{1}{2}\|v_n - By_n\|^2 \\
&\quad -\|v_n - Bp\|^2 - \|v_n - By_n\|^2\Big] \\
&= \|y_n - p\|^2 - \delta(1 - \delta\|B^*\|^2)\|v_n - By_n\|^2 && (3.6) \\
&\le \|y_n - p\|^2 \le \|x_n - p\|^2. && (3.7)
\end{aligned}$$

Next, we compute

$$\begin{aligned}
\|x_{n+1} - p\| &= \|\alpha_n Q(x_n) + (1 - \alpha_n)Sz_n - p\| \\
&\le \alpha_n\|Q(x_n) - p\| + (1 - \alpha_n)\|z_n - p\| \\
&\le \alpha_n\big[\|Q(x_n) - Q(p)\| + \|Q(p) - p\|\big] + (1 - \alpha_n)\|z_n - p\| \\
&\le \alpha_n\alpha\|x_n - p\| + \alpha_n\|Q(p) - p\| + (1 - \alpha_n)\|x_n - p\| \\
&\le [1 - \alpha_n(1 - \alpha)]\|x_n - p\| + \alpha_n\|Q(p) - p\| \\
&\le \max\Big\{\|x_n - p\|, \frac{\|Q(p) - p\|}{1 - \alpha}\Big\} \\
&\ \ \vdots \\
&\le \max\Big\{\|x_0 - p\|, \frac{\|Q(p) - p\|}{1 - \alpha}\Big\}. && (3.8)
\end{aligned}$$

Hence $\{x_n\}$ is bounded and consequently, we deduce that $\{y_n\}$, $\{v_n\}$, $\{z_n\}$, $\{Q(x_n)\}$ and $\{Sz_n\}$ are bounded.

Next, we show that the sequence $\{x_n\}$ is asymptotically regular, i.e.,

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0.$$

It follows from (3.1) that

$$\begin{aligned}
\|y_n - y_{n-1}\| &\le \|J^{M_1}_{\rho_n}(x_n - \rho_n f x_n) - J^{M_1}_{\rho_n}(x_{n-1} - \rho_n f x_{n-1})\| \\
&\quad +\|J^{M_1}_{\rho_n}(x_{n-1} - \rho_n f x_{n-1})\| - J^{M_1}_{\rho_{n-1}}(x_{n-1} - \rho_{n-1} f x_{n-1})\| \\
&\le \|(x_n - x_{n-1}) - \rho_n(f x_n - f x_{n-1}) + (\rho_n - \rho_{n-1}) f x_{n-1}\| \\
&\le \|x_n - x_{n-1}\| + |\rho_n - \rho_{n-1}|\|f x_{n-1}\|. && (3.9)
\end{aligned}$$

Similarly

$$\begin{aligned}
\|v_n - v_{n-1}\| &\le \|J^{M_2}_{\rho_n}(gBy_n - \rho_n gBy_n) - J^{M_2}_{\rho_n}(gBy_{n-1} - \rho_n gBy_{n-1})\| \\
&\quad +\|J^{M_2}_{\rho_n}(gBy_{n-1} - \rho_n gBy_{n-1})\| - J^{M_2}_{\rho_{n-1}}(gBy_{n-1} - \rho_{n-1} gBy_{n-1})\| \\
&\le \|(gBy_n - gBy_{n-1}) - \rho_n(gBy_n - gBy_{n-1}) + (\rho_n - \rho_{n-1})gBy_{n-1}\| \\
&\le \|gBy_n - gBy_{n-1}\| + |\rho_n - \rho_{n-1}|\|gBy_{n-1}\|. && (3.10)
\end{aligned}$$

Next, we compute

$$
\begin{aligned}
\|z_n - z_{n-1}\|^2 &\leq \|y_n + \delta B^*(v_n - By_n) - y_{n-1} + \delta B^*(v_{n-1} - By_{n-1})\|^2 \\
&\leq \|y_n - y_{n-1}\|^2 + \|\delta B^*\big((v_n - By_n) - (v_{n-1} - By_{n-1})\big)\|^2 \\
&\quad +2\delta\langle y_n - y_{n-1}, B^*\big((v_n - By_n) - (v_{n-1} - By_{n-1})\big)\rangle \\
&\leq \|y_n - y_{n-1}\|^2 + \delta\|B^*\|^2 + \|\big((v_n - By_n) - (v_{n-1} - By_{n-1})\big)\|^2 \\
&\quad +2\delta\langle By_n - By_{n-1} + v_n - By_n - (v_{n-1} - By_{n-1}), v_n \\
&\quad -By_n - (v_{n-1} - By_{n-1})\rangle \\
&\quad -2\delta\langle v_n - By_n - (v_{n-1} - By_{n-1}), v_n - By_n - (v_{n-1} - By_{n-1})\rangle \\
&= \|y_n - y_{n-1}\|^2 + \delta^2\|B^*\|^2 + \|v_n - By_n - (v_{n-1} - By_{n-1})\|^2 \\
&\quad +2\delta\big[\frac{1}{2}\|v_n - v_{n-1}\|^2 \\
&\quad +\frac{1}{2}\|v_n - By_n - (v_{n-1} - By_{n-1})\|^2 - \frac{1}{2}\|By_n - By_{n-1}\|^2\big] \\
&\quad -2\delta\|v_n - By_n - (v_{n-1} - By_{n-1})\|^2 \\
&= \|y_n - y_{n-1}\|^2 + \delta(1 - \delta\|B^*\|^2)\|v_n - By_n - (v_{n-1} - By_{n-1})\|^2 \\
&\quad +\delta\big[\|v_n - v_{n-1}\|^2 - \|By_n - By_{n-1}\|^2\big] \\
&= \|y_n - y_{n-1}\|^2 + \delta(1 - \delta\|B^*\|^2)\|v_n - By_n - (v_{n-1} - By_{n-1})\|^2 \\
&\quad +\delta|\rho_n - \rho_{n-1}|\big(\|v_n - v_{n-1}\|^2 - \|By_n - By_{n-1}\|^2\big)\|gBy_{n-1}\| \\
&\leq \|x_n - x_{n-1}\|^2 + |\rho_n - \rho_{n-1}|\Big[\|fx_{n-1}\|\|x_n - x_{n-1}\| \\
&\quad +|\rho_n - \rho_{n-1}|\|fx_{n-1}\|\Big] \\
&\quad -\delta(1 - \delta\|B^*\|^2)\|v_n - By_n - (v_{n-1} - By_{n-1})\|^2 \\
&\quad +\delta|\rho_n - \rho_{n-1}|\big[\|v_n - v_{n-1}\|^2 - \|By_n - By_{n-1}\|^2\big]\|gBy_{n-1}\| \\
&\leq \|x_n - x_{n-1}\|^2 + |\rho_n - \rho_{n-1}|c_1, \\
&\leq \big|\|x_n - x_{n-1}\|^2 + |\rho_n - \rho_{n-1}|c_1\big|, \qquad (3.11)
\end{aligned}
$$

where c_1 is a constant such that

$$
\begin{aligned}
\Big[\|x_n - x_{n-1}\| + |\rho_n - \rho_{n-1}|\Big]\|fx_{n-1}\| + \delta|\rho_n - \rho_{n-1}|\Big[\|v_n - v_{n-1}\|^2 \\
- \|By_n - By_{n-1}\|^2\Big]\|gBy_{n-1}\| \leq c_1. \qquad (3.12)
\end{aligned}
$$

On taking the square root from both sides in (3.11), we obtain

$$\|z_n - z_{n-1}\| \leq \|x_n - x_{n-1}\| + \sqrt{|\rho_n - \rho_{n-1}|c_1}. \qquad (3.13)$$

Next, we have

$$\begin{aligned}\|x_{n+1}-x_n\| &= \|\alpha_n Q(x_n)+(1-\alpha_n)Sz_n-[\alpha_{n-1}Q(x_{n-1})+(1-\alpha_{n-1})Sz_{n-1}]\|\\ &= \|\alpha_n Q(x_n)-\alpha_n Q(x_{n-1})+\alpha_n Q(x_{n-1})-\alpha_{n-1}Q(x_{n-1})+(1-\alpha_n)Sz_n\\ &\quad -(1-\alpha_n)Sz_{n-1}+(1-\alpha_n)Sz_{n-1}-(1-\alpha_{n-1})Sz_{n-1}\|\\ &\le \alpha_n\alpha\|x_n-x_{n-1}\|+(1-\alpha_n)\|Sz_n-Sz_{n-1}\|+2|\alpha_n-\alpha_{n-1}|c_2\\ &\le \alpha_n\alpha\|x_n-x_{n-1}\|+(1-\alpha_n)\|z_n-z_{n-1}\|+2|\alpha_n-\alpha_{n-1}|c_2, \qquad (3.14)\end{aligned}$$

where $c_2 := \sup\{\|f(x_n)\|+\|Su_n\| : n\in\mathbb{N}\}$.
It follows from (3.13) and (3.14) that

$$\|x_{n+1}-x_n\| \le (1-\alpha_n(1-\alpha))\|x_n-x_{n-1}\|+2|\alpha_n-\alpha_{n-1}|c_2+\sqrt{|\rho_n-\rho_{n-1}|c_1}.$$

By applying Lemma 2.4 with

$$\beta_n := \alpha_n(1-\alpha)$$

and

$$\delta_n := 2|\alpha_n-\alpha_{n-1}|c_2+\sqrt{|\rho_n-\rho_{n-1}|c_1},$$

we obtain

$$\lim_{n\to\infty}\|x_{n+1}-x_n\|=0. \qquad (3.15)$$

Since, we have

$$\begin{aligned} x_{n+1}-x_n &= \alpha_n Q(x_n)+(1-\alpha_n)Sz_n-x_n\\ &= \alpha_n(Q(x_n)-x_n)+(1-\alpha_n)(Sz_n-x_n). \qquad (3.16)\end{aligned}$$

Then, we obtain

$$(1-\alpha_n)\|Sz_n-x_n\| \le \|x_{n+1}-x_n\|+\alpha_n\|Q(x_n)-x_n\|.$$

Since $\|x_{n+1}-x_n\|\to 0$ and $\alpha_n\to 0$ as $n\to\infty$, we obtain

$$\lim_{n\to\infty}\|Sz_n-x_n\|=0. \qquad (3.17)$$

Next, we show that $\lim_{n\to\infty}\|x_n-y_n\|=0$. It follows from (3.1) that

$$\begin{aligned}\|x_{n+1}-p\|^2 &\le \alpha_n\|Q(x_n)-p\|^2+(1-\alpha_n)\|Sz_n-p\|^2 \qquad (3.18)\\ &\le \alpha_n\|Q(x_n)-p\|^2+(1-\alpha_n)\|y_n-p\|^2\\ &\le \alpha_n\|Q(x_n)-p\|^2+(1-\alpha_n)\{\|x_n-p\|^2+\rho_n(\rho_n-2\theta_1)\|fu_n-fp\|^2\}\\ &\le \alpha_n\|Q(x_n)-p\|^2+\|x_n-p\|^2+\rho_n(\rho_n-2\theta_1)\|fu_n-fp\|^2,\end{aligned}$$

which yields

$$\begin{aligned}\rho_n(\rho_n-2\theta_1)\|fu_n-fp\|^2 &\le \alpha_n\|Q(x_n)-p\|^2+\|x_n-p\|^2-\|x_{n+1}-p\|^2\\ &\le \alpha_n\|Q(x_n)-p\|^2+(\|x_n-p\|+\|x_{n+1}-p\|)\|x_n-x_{n+1}\|.\end{aligned}$$

Since $\|x_{n+1}-x_n\|\to 0$ and $\alpha_n\to 0$ as $n\to\infty$, we obtain

$$\lim_{n\to\infty}\|fx_n-fp\|=0. \qquad (3.19)$$

Further, we observe that

$$\begin{aligned}\|y_n - p\|^2 &= \|J_{\rho_n}^{M_1}(x_n - \rho_n f x_n) - J_{\rho_n}^{M_1}(p - \rho_n f p)\|^2 \\ &\leq \langle y_n - p, (x_n - \rho_n f x_n) - (p - \rho_n f p)\rangle \\ &\leq \frac{1}{2}\Big\{\|y_n - p\|^2 + \|(x_n - \rho_n f x_n) - (p - \rho_n f p)\|^2 \\ &\quad -\|(y_n - x_n) + \rho_n(f x_n - f p)\|^2\Big\} \\ &\leq \frac{1}{2}\Big\{\|y_n - p\|^2 + \|x_n - p\|^2 - \|y_n - x_n + \rho_n(f x_n - f p)\|^2\Big\}.\end{aligned}$$

Hence,

$$\begin{aligned}\|y_n - p\|^2 &\leq \|x_n - p\|^2 - \|y_n - x_n\|^2 - \rho_n^2\|f x_n - f p\|^2 + 2\rho_n\langle y_n - x_n, f x_n - f p\rangle \\ &\leq \|x_n - p\|^2 - \|y_n - x_n\|^2 + 2\rho_n\|y_n - x_n\|\|f x_n - f p\|.\end{aligned}$$

It follows that

$$\begin{aligned}\|x_{n+1} - p\|^2 &\leq \alpha_n\|Q(x_n) - p\|^2 + (1 - \alpha_n)\|S z_n - p\|^2 \\ &\leq \alpha_n\|Q(x_n) - p\|^2 + (1 - \alpha_n)\|y_n - p\|^2 \\ &\leq \alpha_n\|Q(x_n) - p\|^2 + +(1 - \alpha_n)[\|x_n - p\|^2 - \|y_n - x_n\|^2 \\ &\quad +2\rho_n\|y_n - x_n\|\|f x_n - f p\|] \\ &\leq \alpha_n\|Q(x_n) - p\|^2 + \|x_n - p\|^2 - \|y_n - x_n\|^2 \\ &\quad +2\rho_n\|y_n - x_n\|\|f x_n - f p\|.\end{aligned}$$

Therefore, we obtain

$$\begin{aligned}\|y_n - x_n\|^2 &\leq \alpha_n\|Q(x_n) - p\|^2 + \|x_n - p\|^2 - \|x_{n+1} - p\|^2 \\ &\quad +2\rho_n\|y_n - x_n\|\|f x_n - f p\| \\ &\leq \alpha_n\|Q(x_n) - p\|^2 + (\|x_n - p\| + \|x_{n+1} - p\|)\|x_n - x_{n+1}\| \\ &\quad +2\rho_n\|y_n - x_n\|\|f x_n - f p\|.\end{aligned}$$

Since $\|x_{n+1} - x_n\| \to 0$ and $\alpha_n \to 0$ as $n \to \infty$ and $\lim_{n\to\infty}\|f x_n - f p\| = 0$, we obtain

$$\lim_{n\to\infty}\|y_n - x_n\| = 0. \tag{3.20}$$

Similarly again, it follows from (3.2) and (3.18) that

$$\begin{aligned}\|x_{n+1} - p\|^2 &\leq \alpha_n\|Q(x_n) - p\|^2 + (1 - \alpha_n)\|S z_n - p\|^2 \\ &\leq \alpha_n\|Q(x_n) - p\|^2 + (1 - \alpha_n)\|z_n - p\|^2 \\ &\leq \alpha_n\|Q(x_n) - p\|^2 + (1 - \alpha_n)\big[\|y_n - p\|^2 - \delta(1 - \delta\|B^*\|^2)\|v_n - B y_n\|^2\big] \\ &\leq \alpha_n\|Q(x_n) - p\|^2 + \|x_n - p\|^2 - \delta(1 - \delta\|B^*\|^2)\|v_n - B y_n\|^2.\end{aligned}$$

Therefore, we obtain

$$\begin{aligned}\delta(1 - \delta\|B^*\|^2)\|v_n - B y_n\|^2 &\leq \alpha_n\|Q(x_n) - p\|^2 + \|x_n - p\|^2 - \|x_{n+1} - p\|^2 \\ &\leq \alpha_n\|Q(x_n) - p\|^2 + (\|x_n - p\| + \|x_{n+1} - p\|)\|x_n - x_{n+1}\|.\end{aligned}$$

Since $\delta(1-\delta\|B^*\|^2)>0$, $\|x_{n+1}-x_n\|\to 0$ and $\alpha_n\to 0$ as $n\to\infty$, we obtain

$$\lim_{n\to\infty}\|v_n-By_n\|=0. \tag{3.21}$$

It follows from (3.4) and (3.21) that

$$\begin{aligned}\rho_n(2\theta_2-\rho_n)\|gBy_n-gBp\|^2 &\le \|By_n-Bp\|^2-\|v_n-Bp\|^2\\ &=(\|By_n-Bp\|+\|v_n-Bp\|)(\|By_n-Bp\|-\|v_n-Bp\|)\\ &\le(\|By_n-Bp\|+\|v_n-Bp\|)\|v_n-By_n\|.\end{aligned}$$

Since $\rho_n(2\theta_2-\rho_n)>0$, $\|v_n-By_n\|\to 0$ as $n\to\infty$, we obtain

$$\lim_{n\to\infty}\|gBy_n-gBp\|=0. \tag{3.22}$$

Next, we compute

$$\begin{aligned}\|z_n-p\|^2 &= \|P_{K_1}[y_n+\delta B^*(v_n-By_n)]-p)\|^2\\ &\le\langle y_n+\delta B^*(v_n-By_n)-p, z_n-p\rangle\\ &=\frac{1}{2}[\|(y_n-p)+\delta B^*(v_n-By_n)\|^2+\|z_n-p\|^2\\ &-\|y_n+\delta B^*(v_n-By_n)-p-z_n+p\|^2]\\ &=\frac{1}{2}\big[\|y_n-p\|^2+\|z_n-p\|^2+\|+\delta B^*(v_n-By_n)\|^2\\ &+2\delta\langle By_n-Bp, v_n-By_n\rangle\\ &-\|(y_n-z_n)+\delta B^*(v_n-By_n)\|^2\big]\\ &\le\frac{1}{2}\big[\|y_n-p\|^2+\|z_n-p\|^2+\|\delta B^*(v_n-By_n)\|^2\\ &+2\delta\|By_n-Bp\|\|v_n-By_n\|\\ &-\|y_n-z_n\|^2-\|\delta B^*(v_n-By_n)\|^2+\|z_n-p\|^2\\ &+2\delta\langle y_n-z_n, B^*(v_n-By_n)\rangle],\end{aligned}$$

which in turns yields

$$\begin{aligned}\|z_n-p\|^2 &\le \|y_n-p\|^2-\|y_n-z_n\|^2+2\delta\|By_n-Bp\|\|v_n-By_n\|\\ &+2\delta\|y_n-z_n\|\|B^*\|\|v_n-By_n\|\\ &\le\|y_n-p\|^2-\|y_n-z_n\|^2+2\delta\|v_n\\ &-By_n\|(\|By_n-Bp\|+\|B^*\|\|y_n-z_n\|).\end{aligned} \tag{3.23}$$

It follows from (3.18), (3.21) and (3.24) that

$$\begin{aligned}\|x_{n+1}-p\|^2 &\le \alpha_n\|Q(x_n)-p\|^2+(1-\alpha_n)\|Sz_n-p\|^2\\ &\le \alpha_n\|Q(x_n)-p\|^2++(1-\alpha_n)\|z_n-p\|^2\\ &\le \alpha_n\|Q(x_n)-p\|^2++(1-\alpha_n)\big[\|y_n-p\|^2-\|y_n-z_n\|^2\\ &\quad +2\delta\|v_n-By_n\|(\|By_n-Bp\|+\|B^*\|\|y_n-z_n\|)\big]\\ &\le \alpha_n\|Q(x_n)-p\|^2+\|x_n-p\|^2-\|y_n-z_n\|^2\\ &\quad +2\delta\|v_n-By_n\|(\|By_n-Bp\|+\|B^*\|\|y_n-z_n\|),\end{aligned}$$

which implies that

$$\begin{aligned}\|y_n - z_n\|^2 &\leq \alpha_n\|Q(x_n) - p\|^2 + \|x_n - p\|^2 - \|x_{n+1} - p\|^2 \\ &\quad +2\delta\big[\|v_n - By_n\|(\|By_n - Bp\| + \|B^*\|\|y_n - z_n\|)\big] \\ &\leq \alpha_n\|Q(x_n) - p\|^2 + (\|x_n - p\| + \|x_{n+1} - p\|)\|x_n - x_{n+1}\| \\ &\quad +2\delta\|v_n - By_n\|(\|By_n - B\bar{x}\| + \|B^*\|\|y_n - z_n\|). \qquad (3.24)\end{aligned}$$

Since the sequences $\{x_n\}$ and $\{u_n\}$ all are bounded and $\lim_{n\to\infty}\|u_n - x_n\| = 0$ and $\lim_{n\to\infty}\|v_n - By_n\| = 0$, therefore (3.24) implies that

$$\lim_{n\to\infty}\|y_n - z_n\| = 0. \qquad (3.25)$$

Now, we can write

$$\begin{aligned}\|z_n - x_n\| &\leq \|z_n - y_n\| + \|y_n - x_n\| \\ &\quad \to 0 \text{ as } n \to \infty. \qquad (3.26)\end{aligned}$$

and

$$\begin{aligned}\|Sz_n - z_n\| &\leq \|Sz_n - x_n\| + \|x_n - y_n\| + \|y_n - z_n\| \\ &\quad \to 0 \text{ as } n \to \infty. \qquad (3.27)\end{aligned}$$

Since $\{z_n\}$ is bounded, there exists a subsequence $\{z_{n_k}\}$ of $\{z_n\}$ such that $z_{n_k} \rightharpoonup \hat{x}$ say. Therefore, it follows from (3.27) that there also exists a subsequence $\{y_{n_k}\}$ of $\{y_n\}$ such that $y_{n_k} \rightharpoonup \hat{x}$.

We first show that $\hat{x} \in \mathrm{Fix}(S)$. On contrary, we assume that $\hat{x} \notin \mathrm{Fix}(S)$. From (3.26) and $x_{n_k} \rightharpoonup \hat{x}$, we have $z_{n_k} \rightharpoonup \hat{x}$. Since $S\hat{x} \neq \hat{x}$, then from Opial's condition (2.7) and (3.27), we have

$$\begin{aligned}\liminf_{k\to\infty}\|z_{n_k} - \hat{x}\| &< \liminf_{k\to\infty}\|z_{n_k} - S\hat{x}\| \\ &\leq \liminf_{k\to\infty}\big\{\|z_{n_k} - Sz_{n_k}\| + \|Sz_{n_k} - S\hat{x}\|\big\} \\ &\leq \liminf_{k\to\infty}\|z_{n_k} - \hat{x}\|,\end{aligned}$$

which is a contradiction. Thus, $\hat{x} \in \mathrm{Fix}(S)$. On the other hand

$$y_{n_k} = J_{\rho_n}^{M_1}(x_{n_k} - \rho_{n_k} f x_{n_k})$$

can be rewritten as

$$\frac{(x_{n_k} - y_{n_k}) - \rho_{n_k} f(x_{n_k})}{\rho_{n_k}} \in M_1 y_{n_k}. \qquad (3.28)$$

By passing to the limit $k \to \infty$ in (3.28) and by taking account (3.20), (3.25) and the fact that the graph of maximal monotone operator is weakly-strongly closed, we obtain $0 \in M_1(\hat{x}) + f(\hat{x})$, i.e., $\hat{x} \in \mathrm{Sol}(\mathrm{MVIP}(2.2))$. Furthermore, since $\{x_n\}$ and $\{y_n\}$ have the same asymptotical behavior, $\{Bx_n\}$ weakly converges to $B\hat{x}$. Again by (3.21) and the fact that the mapping $J_{\rho_n}^{M_2}(I - \rho_n g)$ is nonexpansive and Lemma 2.1, we obtain that $0 \in M_2(B\hat{x}) + g(B\hat{x})$, i.e., $B\hat{x} \in \mathrm{Sol}(\mathrm{MVIP}(2.3))$.

Next, we claim that $\limsup_{n\to\infty} \langle Q(z)-z, x_n - z\rangle \leq 0$, where $z = P_\Omega Q(z)$. Indeed, we have

$$\begin{aligned}\limsup_{n\to\infty} \langle Q(z)-z, x_n - z\rangle &= \limsup_{n\to\infty} \langle Q(z)-z, Sz_n - z\rangle \\ &\leq \limsup_{n\to\infty} \langle f(z)-z, Sz_n - z\rangle \\ &= \langle Q(z)-z, w - z\rangle \\ &\leq 0. \end{aligned} \tag{3.29}$$

Finally, we show that $x_n \to z$. Therefore, it follows from Lemma 2.4 that

$$\begin{aligned}\|x_{n+1}-z\|^2 &= \|\alpha_n Q(x_n) + (1-\alpha_n)Su_n - z\|^2 \\ &= \|\alpha_n(Q(x_n)-z) + (1-\alpha_n)(Su_n - z)\|^2 \\ &= \|\alpha_n(Q(x_n)-Q(p)) + \alpha_n(f(z)-z) + (1-\alpha_n)Su_n - z\|^2. \\ &\leq \|\alpha_n(Q(x_n)-Q(z)) + (1-\alpha_n)Su_n - z\|^2 \\ &\quad +2\alpha_n\langle Q(x_n)-z, x_{n+1}-z\rangle \\ &\leq \|\alpha_n\|Q(x_n)-Q(z)\|^2 + (1-\alpha_n)\|u_n - z\|^2 \\ &\quad +2\alpha_n\langle Q(x_n)-z, x_{n+1}-z\rangle \\ &\leq \alpha_n\alpha^2\|x_n - z)\|^2 + (1-\alpha_n)\|x_n - z\|^2 \\ &\quad +2\alpha_n\langle Q(x_n)-z, x_{n+1}-z\rangle \\ &\leq (1-(1-\alpha^2)\alpha_n)\|x_n - z)\|^2 + 2\alpha_n\langle Q(x_n)-z, x_{n+1}-z\rangle. \end{aligned}$$

Hence, we obtain

$$\|x_{n+1}-z\|^2 \leq (1-(1-\alpha^2)\alpha_n)\|x_n - z)\|^2 + 2\alpha_n\langle Q(x_n)-z, x_{n+1}-z\rangle.$$

Finally, by using (3.29) and Lemma 2.4, we deduce that $x_n \to z$ in norm as $n \to \infty$. Further it follows from $\|y_n - x_n\| \to 0$, $y_n \rightharpoonup \hat{x} \in \Omega$ and $x_n \to z$ as $n \to \infty$, that $z = \hat{x}$. This completes the proof. □

As a direct consequences of Theorem 3.1, we obtain the following result due to Kazmi and Rizvi [17] to approximate the common solution of $\mathrm{S_P}$VIP(2.4)-(2.5) and FPP(2.1). Take $f = g = 0$ and $P_{K_1} = I$ in Theorem 3.1 then the following Corollary is obtained.

Corollary 3.1. [17] *Let $\mathcal{H}_1$ and $\mathcal{H}_2$ are real Hilbert spaces and $B : \mathcal{H}_1 \to \mathcal{H}_2$ be a bounded linear operator. Let $M_1 : \mathcal{H}_1 \to 2^{\mathcal{H}_1}$, $M_2 : \mathcal{H}_2 \to 2^{\mathcal{H}_2}$ are the multi-valued maximal monotone mappings, and let $Q : \mathcal{H}_1 \to \mathcal{H}_1$ be a contraction mapping with constant $\alpha \in (0,1)$. Let $S : \mathcal{H}_1 \to \mathcal{H}_1$ be a nonexpansive mapping such that* Fix$(S) \cap$ Sol($\mathrm{S_P}$VIP(2.4)-(2.5)) $\neq \emptyset$. *Let the iterative sequences $\{x_n\}$, $\{y_n\}$, $\{v_n\}$ and $\{z_n\}$ are generated by the following iterative schemes:*

$$\begin{cases} x_1 = x \in \mathcal{H}_1, \\ y_n = J_\rho^{M_1}(x_n), \\ v_n = J_\rho^{M_2}(By_n), \\ z_n = P_{K_1}[y_n + \delta B^*(v_n - By_n)], \\ x_{n+1} = \alpha_n Q(x_n) + (1-\alpha_n)Sz_n, \end{cases}$$

where $\delta \in \left(0, \frac{1}{\|B\|^2}\right)$ *and* $\{\alpha_n\}$ *are the sequences in* $(0,1)$ *and satisfying the conditions (i) of Theorem* 3.1. *Then the sequences* $\{x_n\}$ *converges strongly to* $z \in \mathrm{Fix}(S) \cap \mathrm{Sol}(\mathrm{S_P VIP}(2.4)\text{-}(2.5))$, *where* $z = P_{\mathrm{Fix}(S)\cap \mathrm{Sol}(\mathrm{S_P VIP}(2.4)-(2.5))}Q(z)$.

The following Corollary is due to Byrne *et al.* [6] to approximate the solution of $\mathrm{S_P VIP}(2.4)\text{-}(2.5)$. Take $f = g = 0$, $P_{K_1} = S = I$ and $\alpha_n = 0$ in Theorem 3.1 then the following Corollary is obtained.

Corollary 3.2. [6] *Let* $\mathcal{H}_1$ *and* $\mathcal{H}_2$ *are real Hilbert spaces and* $B : \mathcal{H}_1 \to \mathcal{H}_2$ *be a bounded linear operator. Let* $M_1 : \mathcal{H}_1 \to 2^{\mathcal{H}_1}$, $M_2 : \mathcal{H}_2 \to 2^{\mathcal{H}_2}$ *are the multi-valued maximal monotone mappings such that* $\mathrm{Sol}(\mathrm{S_P VIP}(2.4)\text{-}(2.5)) \neq \emptyset$. *Let the iterative sequences* $\{x_n\}$ *be generated by the following iterative schemes:*

$$J_\rho^{M_1}(x_n + \delta B^*(J_\rho^{M_2} - I)Bx_n),$$

where $\delta \in \left(0, \frac{1}{\|B\|^2}\right)$.
Then the sequences $\{x_n\}$ *converges strongly to* $z \in \mathrm{Sol}(\mathrm{S_P VIP}(2.4)\text{-}(2.5))$.

Remark 3.1. Theorem 3.1 extends and generalize the approximation schemes of Byrne et al. [6] and the viscosity results of Kazmi and Rizvi [17] to a general iterative method for a split variational inclusion and Moudafi [26] for a split monotone variational inclusion, which includes the results of [6, 17, 26] as special cases.

4. Numerical Example

In this section, we provide a numerical example to support our main result.

Example 4.1. Let $\mathcal{H}_1 = \mathcal{H}_2 = \mathbb{R}$, the set of all real numbers. Let $K_1 = [0,1]$ and $K_2 = [-\infty, 0]$; $M_1(x) = 2x$ and $M_2(x) = 3x$, for all $x \in \mathbb{R}$. Let for each $x \in \mathbb{R}$, we define $Q(x) = \frac{1}{8}x$, $B(x) = 2x$ and $S(x) = x$ and let $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ are defined by $f(x) = g(x) = 0,\ \forall x \in \mathbb{R}$. Then it is easy to prove that the mappings M_1 and M_2 are maximal monotone; S is nonexpansive and B is a bounded linear operator with its adjoint B^* such that $\|B\| = \|B^*\| = 2$. The iterative sequences $\{x_n\}$, $\{y_n\}$, $\{v_n\}$, and $\{z_n\}$ are generated by (3.1) are then reduced to the following iterative schemes:

$$\begin{aligned} &x_1 = x \in \mathcal{H}_1,\\ &y_n = \tfrac{1}{3}(x_n),\\ &v_n = \tfrac{1}{4}(2y_n),\\ &z_n = \begin{cases} 0 \text{ if } x < 0,\\ 1 \text{ if } x > 0,\\ P_{K_1}[y_n + 0.2((v_n - 2y_n)],\ \text{Otherwise}, \end{cases} \\ &x_{n+1} = \tfrac{1}{8}\tfrac{sin(n)}{n}x_n + \left(1 - \tfrac{sin(n)}{n}\right)\tfrac{1}{6}x_n, \end{aligned} \tag{4.1}$$

where $\alpha_n = \frac{sin(n)}{n}$ and $\rho_n = 1$. Then $\{x_n\}$ converges strongly to $0 \in \Omega$.

Table 1: Numerical results for some initial points x(1) = -1, 0, 1, -10, 25

Iter. (n)	$x_n^{(1)}$	$x_n^{(2)}$	$x_n^{(3)}$	$x_n^{(4)}$	$x_n^{(5)}$
1.	-0.4472	0.0000	0.4472	-4.4472	11.1789
2.	-0.1423	0.0000	0.1423	-1.4229	3.5573
3.	-0.0259	0.0000	0.0259	-0.2595	0.6487
4.	-0.0027	0.0000	0.0027	-0.0269	0.0672
5.	-0.0003	0.0000	0.0003	-0.0028	0.0069
6.	0.0000	0.0000	0.0000	-0.0004	0.0010
7.	⋮	⋮	⋮	⋮	⋮
8.	0.0000	0.0000	0.0000	0.0000	0.0000
9.	0.0000	0.0000	0.0000	0.0000	0.0000
10.	0.0000	0.0000	0.0000	0.0000	0.0000

Setting $\|x_n - p\| < 10^{-4}$ as stop criterion, we obtain the numerical results of scheme (4.1) with different initial points x_1. in Table 1. The computations are performed by Matlab R2007a running on a PC Desktop Intel(R) Core(TM)i3-2330M, CPU @2.20 GHz, 790 MHz, 2 GB RAM. Next, by using Matlab 7.0, we study the convergence of $\{x_n\}$, for different initial values which shows that $\{x_n\}$ converges strongly to 0.

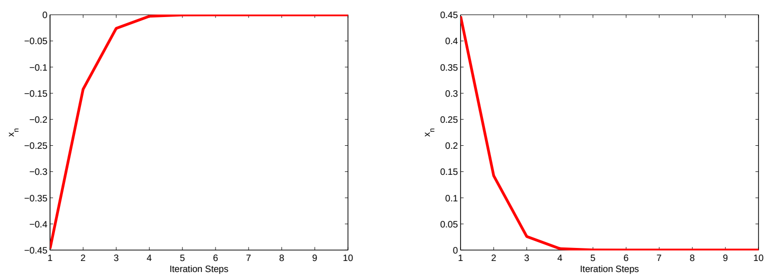

FIGURE 1. Convergence of $\{x_n\}$ for the initial values -1 and 1

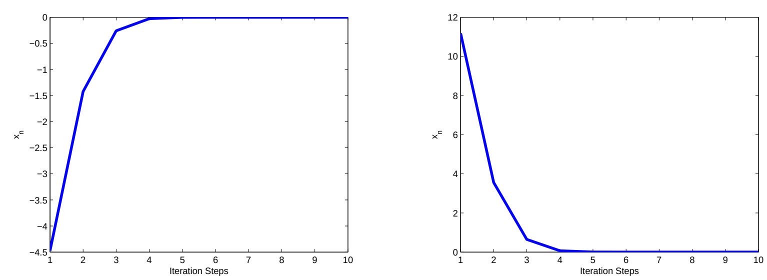

FIGURE 2. Convergence of $\{x_n\}$ for the initial values -10 and 25

Example 4.2. Let $\mathcal{H}_1 = \mathcal{H}_2 = \mathbb{R}^3$, the set of all real numbers. Let

$$K_1 = [0, +\infty) \times [0, +\infty) \times [0, +\infty).$$

Let the inner product $\langle \cdot, \cdot \rangle : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$ be defined by

$$\langle x, y \rangle = x \cdot y = x_1 y_1 + x_2 y_2 + x_3 y_3$$

and with respect to the usual norm $\|.\| : \mathbb{R}^{\to}\mathbb{R}$ be defined by

$$\|x\| = \sqrt{x_1^2 + x_2^2 + x_3^2}.$$

Let $M_1 : \mathbb{R}^3 \to \mathbb{R}^3$ and $M_2 : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by

$$M_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

and

$$M_2 = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 3 \end{bmatrix}.$$

Then we can define the resolvent operators $J_{\rho_1}^{M_1}$ and $J_{\rho_2}^{M_2}$ on $\mathbb{R}^3$ associated with M_1 and M_2 where $\rho_1, \rho_2 > 0$. Let

$$B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \in \mathbb{R}^3 \times \mathbb{R}^3$$

be a singular matrix operator and B^* be the adjoint of B. It is easy to calculate that

$$B^* = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

Let the mapping $S : \mathcal{H}_1 \to \mathcal{H}_1$ defined by

$$Sx = \Big(\frac{x_1}{10}, \frac{x_2}{10}, \frac{x_3}{10} \Big)$$

and let $Qx = \dfrac{x}{2}$ $(\forall x \in \mathbb{R}^3)$. Let

$$f(x) = \frac{1}{2}x, \ \forall x \in \mathbb{R}^3 \text{ and } g(x) = \frac{1}{3}x, \ \forall x \in \mathbb{R}^3.$$

Then it is easy to prove that the mappings M_1 and M_2 are maximal monotone and S is nonexpansive. The iterative sequences $\{x_n\}$, $\{y_n\}$, $\{v_n\}$, and $\{z_n\}$ are generated by (3.1) are then reduced to the following iterative schemes:

$$\begin{cases} x_1 = x \in \mathcal{H}_1, \\ y_n = J_{1/2}^{M_1}(I - \frac{1}{2}f)x_n, \\ v_n = J_{1/2}^{M_2}(I - \frac{1}{3}g) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} y_n, \\ z_n = P_{K_1}\left(y_n + \frac{1}{5} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} (v_n - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} y_n) \right), \\ x_{n+1} = \frac{1}{2}x_n + \frac{1}{10}\left(1 - \frac{1}{n}\right) z_n, \end{cases} \tag{4.2}$$

where $\alpha_n = \dfrac{1}{n}$ and $\delta = 0.2$. Then $\{x_n\}$ converges strongly to $\mathbf{0} = (0,0,0) \in \Omega$.

Setting $\|x_n - p\| < 10^{-4}$ as stop criterion, then we obtain the numerical results of scheme (4.1) with different initial points x_1. in Table 1. The computations are performed by Matlab R2007a running on a PC Desktop Intel(R) Core(TM)i3-2330M, CPU @2.20 GHz, 790 MHz, 2 GB RAM.

Next, by using the software Matlab 7.0, we study the convergence behavior of $\{x_n\}$, for different initial values which shows that $\{x_n\}$ converges strongly to $\mathbf{0} = (0,0,0)$.

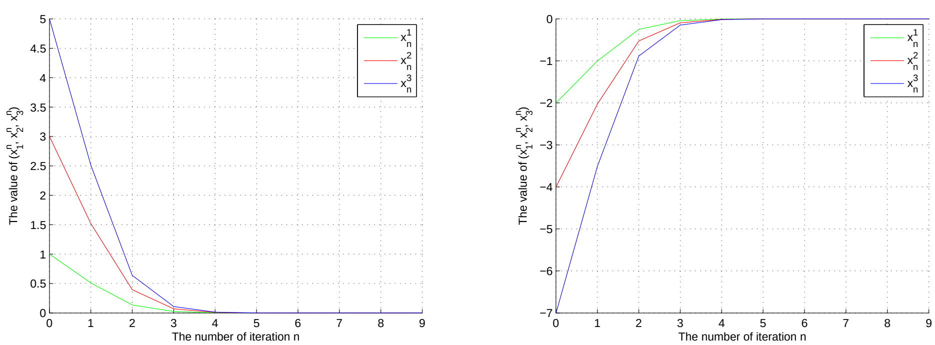

FIGURE 3. Convergence of $\{x_n\}$ for the initial values (1,3,5) and (-2,-4,-7)

Acknowledgement. This work was supported by the Deanship of Scientific Research (DSR), Saudi Electronic University, Riyadh. The author, therefore, gratefully acknowledge the DSR technical and financial support.

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Received: March 24, 2021; Accepted: May 24, 2022.