SYNCHRONOUS r-CYCLIC CONTRACTIONS ON METRIC SPACES

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Abstract. Motivated by the existence of cyclic phenomena in which some characteristics are mapped into corresponding ones over more than one phase, we introduce the r-cyclic operators with respect to a covering of a metric space and investigate their behavior. We study the convergence of the Picard iteration to a fixed point of such an operator under a Banach type generalized contraction condition. The obtained results may have interesting practical applications in various research areas.
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1. Introduction

In [9] an interesting type of contractive condition for operators defined on metric spaces was suggested. The authors investigated there corresponding generalizations of important fixed point results, such as Banach's, Edelstein's or Caristi's fixed point theorems. They developed this way a technique which was used in the same paper in proving a known fixed point result for nonexpansive mappings.

This paper gave rise to a series of papers on what we are now used to call *cyclic operators*. To our best knowledge, the notion of *cyclic representation* first appeared in the short but interesting paper [20], which was also inspired by [9].

Since then many classes of cyclic operators have been studied, generally by extending known classes of generalized contractions on metric or Banach spaces (see for example [1], [4]-[8], [11]-[14], [18], [22], for a very short list of them).

There are several other research areas where different types of cyclic phenomena are observed and studied. Of course, only some of these phenomena are likely to be approached by means of the instruments offered by our research. Here we mention only a few of these inspiring topics, where our results could be refined and adapted in order to count as instruments in further studies and for developing new models: different types of oscillators (chemical, physical and others), social cycles, business cycles, politico-economic cycles, financial cycles, ecosystems cycles, cycles in astrophysics and so on (see for example [2], [3], [10]).

Inspired by these phenomena, we aim to introduce and study in the present paper cyclic contractive type operators corresponding to phenomena in which some characteristics do not appear in each and every phase of a cycle, but they appear with a certain periodicity (of r phases). Besides new definitions and results that are given in the following, several remarks will complete the approach, mentioning aspects that can be relevant in applications.

These new r-cyclic operators are an answer to natural problems and they prove to complete the family of cyclic type operators studied in the previous literature. They open this way a very wide space for further research, in order to extend the already existing results and theories referring to cyclic operators, if possible, in the case of r-cyclic operators.

The *Introduction* is followed by five sections:

2. *Preliminaries*
3. *Nuances of a definition. r-Cyclic operators*
4. *Properties of r-cyclic operators*
5. *Synchronous r-cyclic contractions*
6. *Instead of a conclusion. The cyclic operators that have been missing*

This first study of r-cyclic contractions does not exhaust the topic, as there are more questions to be answered and problems to be solved.

2. Preliminaries

A first (to our best knowledge) cyclic-type generalization of a classical fixed point result is Theorem 1.3 in [9], which we recall below in its original notation.

Theorem 2.1 ([9])**.** *Let $\{A_i\}_{i=1}^p$ be nonempty closed subsets of a complete metric space and suppose $F : \bigcup_{i=1}^p A_i \to \bigcup_{i=1}^p A_i$ satisfies the following conditions (where $A_{p+1} = A_1$):*

(1) $f(A_i) \subseteq A_{i+1}$, *for* $1 \le i \le p$;
(2) $\exists k \in (0,1)$ *such that* $d(F(x), F(y)) \le k \cdot d(x,y)$, $\forall x \in A_i, y \in A_{i+1}$, *for* $1 \le i \le p$.

Then F has a unique fixed point.

The "secret" behind this result is that the Picard iteration associated to F has infinitely many terms in each set A_i, $1 \le i \le p$, and it is also a Cauchy sequence due to the contraction condition, these two arguments together with the completeness of the space and the closeness of $A_i, i = \overline{1, p}$ leading to the conclusion.

In [20] the *cyclic representation* relative to an operator $f : X \to X$ is defined as being $\bigcup_{i=1}^m X_i$ such that X_i are all nonempty ($1 \le i \le m$) and besides $f(X_1) \subseteq X_2, f(X_2) \subseteq X_3, \ldots, f(X_{m-1}) \subseteq X_m, f(X_m) \subseteq X_1$. Some examples of cyclic representations are given in [20]. Other interesting examples of cyclic operators are to be found in the paper [5].

The study of cyclic operators is still having an impressive development, judging after the number of papers in which the existence of fixed points or best proximity points is investigated, for various classes of operators and in various framework spaces.

This is why we found it interesting to revisit the papers that stood as starting point for this rich literature, to reconsider some notations and to reformulate some basic notions, having in view the phenomena they should be able to speak about, in mathematical language. This apparently trivial approach can lead to surprisingly interesting observations.

3. Nuances of a definition. r-Cyclic operators

In the rest of the paper we shall use the terminology dictated by the next definition, introduced in [21], which allows a more accurate language of cyclic operators and all related notions and results (instead of the initial *cyclic representation* terminology).

Definition 3.1 ([21])**.** Let X be a nonempty set and $f : X \to X$ an operator. If there exists a covering of $X = \bigcup_{i=1}^{m} X_i$, $m \geq 2$ such that

$$f(X_1) \subseteq X_2, f(X_2) \subseteq X_3, \ldots, f(X_{m-1}) \subseteq X_m, f(X_m) \subseteq X_1,$$

then:

i) $\bigcup_{i=1}^{m} X_i$ is called a cyclic covering of X w.r.t. f;

ii) f is called a cyclic operator w.r.t. the covering $\bigcup_{i=1}^{m} X_i$.

Remark 3.1. Note that in several papers (see for example [12]) the term p-cyclic is used to indicate an operator that is cyclic in terms of Definition 3.1, where p denotes the number of sets in the cyclic covering. This is still essentially different from what will be defined in the present paper as r-cyclic.

Remark 3.2. From the above definition one can see that i) and ii) are equivalent. Therefore in the sequel when one of them is mentioned the other one will be automatically assumed.

The following simple example, included also in [17], will lead to an important remark regarding the notation used in the above definition.

Example 3.1. Let $X = X_1 \cup X_2 \cup X_3$ be a cyclic covering of X w.r.t. the operator $f : X \to X$.

One can easily check that according to the definition $X_2 \cup X_1 \cup X_3$ is generally not a cyclic covering w.r.t. f, neither are $X_3 \cup X_2 \cup X_1$ or $X_1 \cup X_3 \cup X_2$.

Still $X_2 \cup X_3 \cup X_1$ and $X_3 \cup X_1 \cup X_2$ are cyclic coverings w.r.t. f.

Remark 3.3. It is now clear that the order in which the sets appear in the covering plays an important role, therefore none of them can change places. As the usual union of sets is commutative, we find that the notation $\bigcup_{i=1}^{m} X_i$ is not quite proper for a cyclic covering.

So we introduce the notation

$$\overset{m}{\underset{i=1}{\uplus}} X_i = X_1 \uplus X_2 \uplus \cdots \uplus X_m$$

to indicate a cyclic covering w.r.t. to an operator, which will actually say that each of the m cyclic permutations

$$X_1 \cup X_2 \cup X_3 \cup \cdots \cup X_{m-1} \cup X_m,$$
$$X_2 \cup X_3 \cup \cdots \cup X_{m-1} \cup X_m \cup X_1,$$
$$\vdots$$
$$X_m \cup X_1 \cup X_2 \cup \cdots \cup X_{m-1}$$

is a cyclic covering of X w.r.t. f, while generally any other permutation is not.

Now let us take a look at another simple but interesting example.

Example 3.2. If $\overset{m}{\underset{i=1}{\uplus}} X_i$ is a cyclic covering w.r.t. $f : X \to X$ and $k \geq 2$ is integer, then $f^k : X \to X$ is generally not a cyclic operator w.r.t. $\overset{m}{\underset{i=1}{\uplus}} X_i$, but $f^{km+1} : X \to X$ is a cyclic operator w.r.t. $\overset{m}{\underset{i=1}{\uplus}} X_i$.

Motivated by the above observations and by the existence of cyclic phenomena where some studied characteristics or parameters are mapped not in the next coming phase, but in a more "remote" one (e.g., not from parents to children, but from grandparents to grandchildren, so over two generations), we introduce the following definition:

Definition 3.2. Let X be a nonempty set, $f : X \to X$ an operator and $m \geq 2$, $1 \leq r \leq m$ integers. If there is a covering $X = \overset{m}{\underset{i=1}{\bigcup}} X_i$ such that

$$f(X_1) \subseteq X_{1+r}, f(X_2) \subseteq X_{2+r}, \ldots, f(X_m) \subseteq X_{m+r},$$

where for $p > m$ by X_p we mean $X_{p \mod m}$, then:

i) $\overset{m}{\underset{i=1}{\uplus}} X_i$ is called a r-cyclic covering of X w.r.t. f;

ii) f is called a r-cyclic operator w.r.t. the covering $\overset{m}{\underset{i=1}{\uplus}} X_i$.

Remark 3.4. One can see that a cyclic covering or cyclic operator in the sense of Definition 3.1 is a 1-cyclic covering, respectively 1-cyclic operator in the sense of Definition 3.2.

For $r > m$, any r-cyclic operator/covering is actually $(r \mod m)$-cyclic operator/covering, that is why in the above definition we only consider $1 \leq r \leq m$.

Remark 3.5. One can also check that the notation $\overset{m}{\underset{i=1}{\uplus}} X_i$ from Remark 3.3 remains consistent in the case of r-cyclic coverings as well, since if $\overset{m}{\underset{i=1}{\uplus}} X_i$ is a r-cyclic covering

w.r.t. an operator, then implicitly

$$X_1 \cup X_2 \cup X_3 \cup \cdots \cup X_{m-1} \cup X_m,$$
$$X_2 \cup X_3 \cup \cdots \cup X_{m-1} \cup X_m \cup X_1,$$
$$\vdots$$
$$X_m \cup X_1 \cup X_2 \cup \cdots \cup X_{m-1}$$

are r-cyclic w.r.t. to that operator, as well.

Now let us analyze some simple examples, which are extensions of some examples in [20].

Example 3.3. If $\bigcup_{i=1}^{m} X_i$, $m \geq 2$ is a cyclic covering w.r.t. $f : X \to X$, then it is a k-cyclic covering w.r.t. $f^k : X \to X$, for any $k \geq 2$.

See for example the case $r = 2$. Since f is cyclic w.r.t. $\bigcup_{i=1}^{m} X_i$, we have that

$$f^2(X_1) \subseteq f(f(X_1)) \subseteq f(X_2) \subseteq X_3 \text{ etc.}$$

so

$$f^2(X_1) \subseteq X_3, f^2(X_2) \subseteq X_4, \ldots, f^2(X_{m-2}) \subseteq X_m, f^2(X_{m-1}) \subseteq X_1, f^2(X_m) \subseteq X_2,$$

which by Definition 3.2 means exactly that f^2 is a 2-cyclic operator w.r.t. $\bigcup_{i=1}^{m} X_i$.

Example 3.4. Let $X = \{x_1, x_2, x_3, \ldots, x_m\}$ a set, $X_i = \{x_i\}, i = \overline{1, m}$ and $f : X \to X$ defined by

$$f(x_1) = x_4, f(x_2) = x_5, \ldots, f(x_{m-3}) = f(x_m),$$
$$f(x_{m-2}) = f(x_1), f(x_{m-1}) = f(x_2), f(x_m) = f(x_3).$$

Then f is not a cyclic operator w.r.t. $\bigcup_{i=1}^{m} X_i$, but it is a 3-cyclic operator w.r.t. to the same covering.

Note that it is generally not possible to "rearrange" the covering so that f would be cyclic w.r.t. this rearrangement.

Example 3.5. Let X be a nonempty set and $f : X \to X$ an operator. Let $Y \subset X$ a nonempty strict subset of X and let $n_0 > 0$ integer such that

$$X = Y \cup f(Y) \cup f^2(Y) \cup \cdots \cup f^{n_0}(Y),$$

with $f^{n_0+1}(Y) \subseteq Y$.

Then f^k is a k-cyclic operator w.r.t. the covering $\bigcup_{i=0}^{n_0} f^i(Y)$, for any $k \geq 2$.

Remark 3.6. Generally if $X = \bigcup_{i=1}^{m} X_i$, $m \geq 2$ is r-cyclic covering w.r.t. $f : X \to X$, then each X_i, $1 \leq i \leq m$ is invariant for f^m.

If in particular $m = k \cdot r$, with k integer, then each X_i, $1 \leq i \leq m$ is invariant for f^k.

Example 3.6. We saw in Example 3.1 that if $X = \bigcup_{i=1}^{3} X_i$ is a cyclic covering w.r.t. $f : X \to X$, then $X_1 \cup X_2 \cup X_3$, $X_2 \cup X_3 \cup X_1$ and $X_3 \cup X_1 \cup X_2$ are all cyclic coverings, while the remaining three cyclic permutations $X_1 \cup X_3 \cup X_2$, $X_2 \cup X_1 \cup X_3$ and $X_3 \cup X_2 \cup X_1$ are generally not cyclic coverings.

Still one can check that they are all 2-cyclic coverings w.r.t. f.

4. Properties of r-cyclic operators

The above Example 3.6 could lead us to the false conclusion that, given an operator $f : \bigcup_{i=1}^{m} X_i \to \bigcup_{i=1}^{m} X_i$, no matter how we rearrange the elements of the covering $\bigcup_{i=1}^{m} X_i$, it is possible to find a positive integer r such that f is r-cyclic w.r.t. to this rearrangement.

This is generally not true, and an answer will be given by Lemmas 4.1 and 4.2. These results can be simply deduced by observing the behavior of r-cyclic coverings w.r.t. f, for various values of m and r.

Before stating the lemmas, we propose a visualization of some r-cyclic coverings, as we found this approach extremely useful for a better understanding of the results to come. Note that the sets $X_1, X_2, \ldots, X_m$ are not supposed to be disjoint, in general.

According to Definition 3.2, we refer to coverings $X = \bigcup_{i=1}^{m} X_i$, $m \geq 2$ and to an operator $f : X \to X$ such that

$$f(X_i) \subseteq X_{i+r}, i = \overline{1,m},$$

for $1 \leq r \leq m$, where $X_p = X_{p \mod m}$ for any $p > m$.

4.1. **Visual representation in the case** $m = 2$. Then $X = X_1 \cup X_2$ and we have to analyze two values of r.

For $r = 1$, f would be 1-cyclic (or simply cyclic), that is,

$$f(X_1) \subseteq X_2, f(X_2) \subseteq X_1.$$

Graphically we shall represent this as:

FIGURE 4.1. Cyclic covering for $m = 2, r = 1$

For $r = 2$, f would be 2-cyclic, that is,

$$f(X_1) \subseteq X_1, f(X_2) \subseteq X_2.$$

We shall represent this as

FIGURE 4.2. Cyclic covering for $m = 2, r = 2$

4.2. **Visual representation in the case** $m = 3$. Then $X = X_1 \uplus X_2 \uplus X_3$ and we have to analyze three values of r.

For $r = 1$, f would be 1-cyclic (or simply cyclic), that is,

$$f(X_1) \subseteq X_2, f(X_2) \subseteq X_3, f(X_3) \subseteq X_1.$$

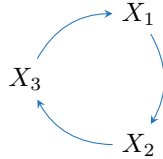Graphically we shall represent this as:

FIGURE 4.3. Cyclic covering for $m = 3, r = 1$

For $r = 2$, f would be 2-cyclic, that is,

$$f(X_1) \subseteq X_3, f(X_2) \subseteq X_1, f(X_3) \subseteq X_2.$$
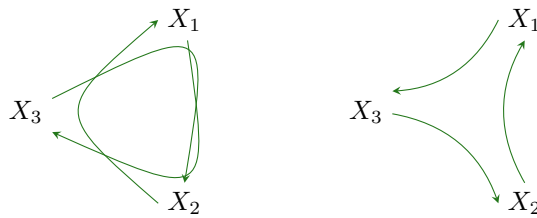
Graphically we shall represent this as:

FIGURE 4.4. Cyclic covering for $m = 3, r = 2$, direct vs. simplified

Note that in the following we shall adopt this kind of "left bent" arrows like in the second representation to tell that f maps X_1 into X_3, stepping "over" X_2, X_2 into X_1 and so on.

For $r = 3$ the representation is trivial, we omit it.

4.3. **Visual representation in the case** $m = 4$**.** Now $X = X_1 \uplus X_2 \uplus X_3 \uplus X_4$ and we have to analyze four values of r. In the next Figure 4.5 are depicted the first three cases.
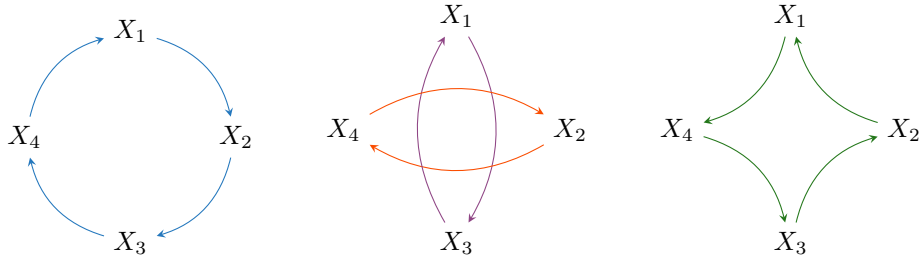

FIGURE 4.5. Cyclic coverings for $m = 4$ and $r = 1, r = 2, r = 3$

4.4. **Comments on visual representations.** As rigorous notation in the case of r-cyclic operators is getting sometimes really hairy, as we shall see later, this visualization approach helps understanding how things work.

One can already notice that in some cases, no matter which of X_i, $i = \overline{1, m}$ is the starting set, the operator f will run through each of the sets of the covering. In other cases, some "closed circuits" can be observed (for example when $m = 4, r = 2$), that is, if the starting point is in one of the sets contained in this circuit, then it will never be mapped outside the circuit.

Remark 4.1. In the rest of the paper we shall use this term of *closed circuit* or simply *circuit* in order to denote such a family of sets, invariant for f.

We try to make this phenomenon more visible in our representations by means of the color code we use: blue for a 1-cyclic operator, black for m-cyclic operators, green in all those cases when f runs through all the sets X_i, $i = \overline{1, m}$. Finally, if there are closed circuits to be noticed, we use other colors, one different color for each such circuit. This color code can already be noticed for the case $m = 4$ above. We still have to mention that all these representations can be well understood without any color, only by attentively following the arrows.

4.5. **Visual representation in the case** $m = 10$**.** Though the other cases are not less interesting, we omit them and we represent below only the resulting configurations when $m = 10$ and $r = 1, 2, \ldots, 10$. In this case the cyclic covering is $X = X_1 \uplus X_2 \uplus \cdots \uplus X_{10}$.

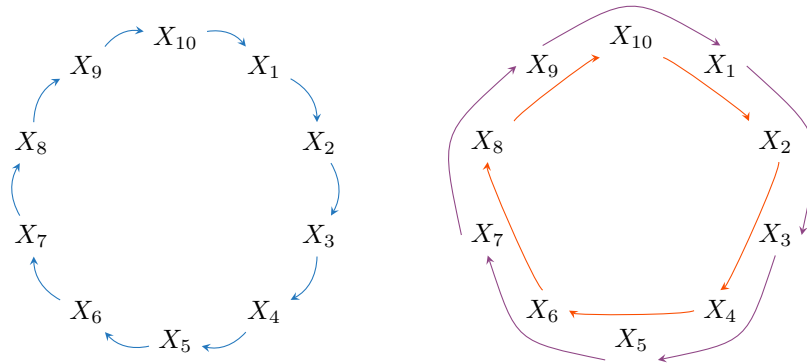The next Figure 4.6 shows these representations for $r = 1$ and $r = 2$:

FIGURE 4.6. Cyclic coverings for $m = 10$ and $r = 1, r = 2$

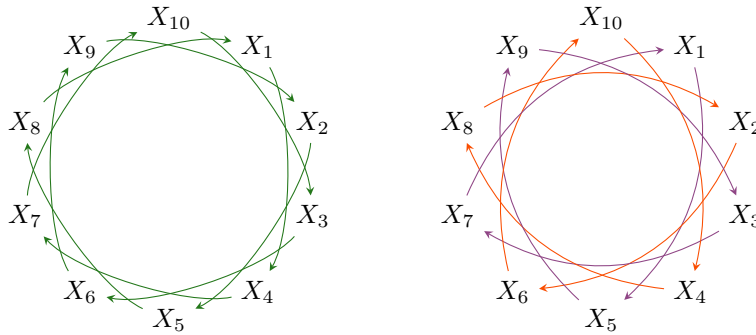Further on, Figure 4.7 shows the cases $r = 3$ and $r = 4$:

FIGURE 4.7. Cyclic coverings for $m = 10$ and $r = 3, r = 4$

For $r = 5$ and $r = 6$, the representations are included in Figure 4.8, while the cases $r = 7$ and $r = 8$ are presented in Figure 4.9. Finally for $r = 9$, respectively $r = 10$, the representations are given in Figure 4.10.

Following the above presented color code or simply the arrows, one can trace the behavior of the r-cyclic operators. For example, f runs through all sets of the covering, no matter where the starting point was, for $r = 1$, $r = 3$, $r = 7$ and $r = 9$. For each of $r = 2$, $r = 4$, $r = 6$ and $r = 8$ there arise two circuits. Again, the circuits are distinct, although the sets involved are the same.

For example, all of the following are obviously distinct:

$$X_1 \uplus X_3 \uplus X_5 \uplus X_7 \uplus X_9 \text{ (when } f \text{ is 2-cyclic),}$$
$$X_1 \uplus X_5 \uplus X_9 \uplus X_3 \uplus X_7 \text{ (when } f \text{ is 4-cyclic),}$$
$$X_1 \uplus X_7 \uplus X_3 \uplus X_9 \uplus X_5 \text{ (when } f \text{ is 6-cyclic),}$$
$$X_1 \uplus X_9 \uplus X_7 \uplus X_5 \uplus X_3 \text{ (when } f \text{ is 8-cyclic).}$$
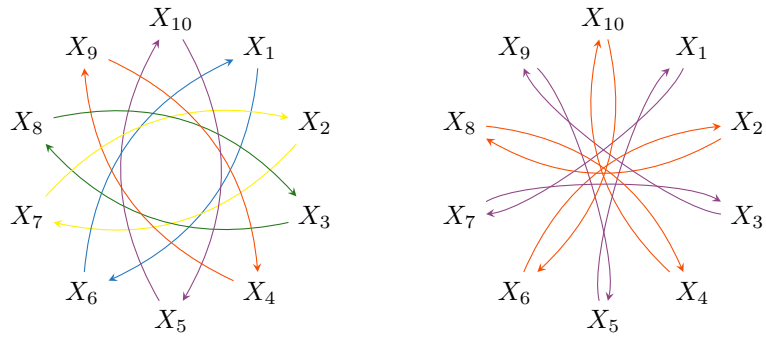
FIGURE 4.8. Cyclic coverings for $m = 10$ and $r = 5, r = 6$
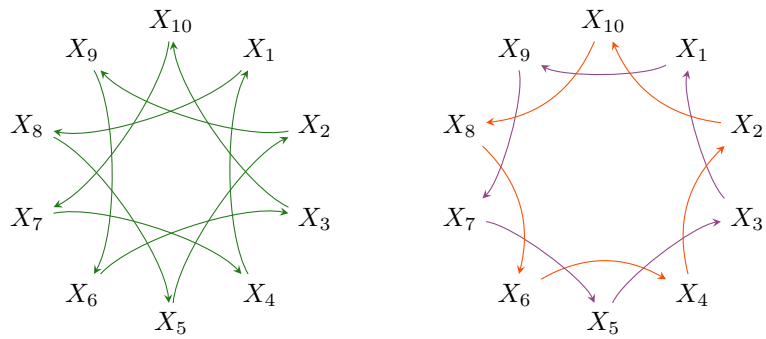

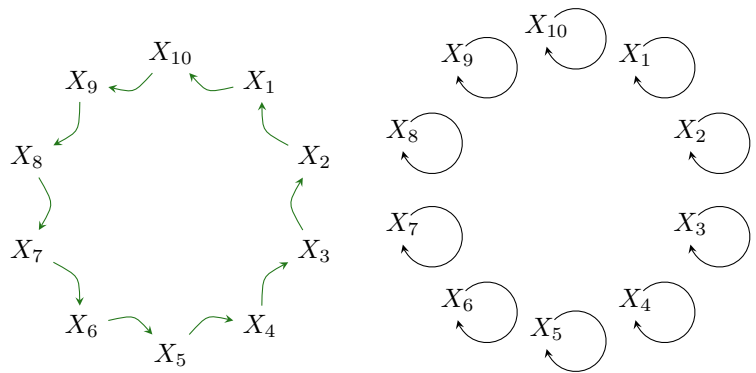FIGURE 4.9. Cyclic coverings for $m = 10$ and $r = 7, r = 8$

FIGURE 4.10. Cyclic coverings for $m = 10$ and $r = 9, r = 10$

The most "colorful" case is that when f is 5-cyclic and the closed circuits that arise reach the number of five, each of them containing only two sets of the covering.

4.6. **Two lemmas on r-cyclic operators.** As announced in the beginning of this section, our aim was to generalize what can be noticed in the behavior of r-cyclic operators for various values of m and r. It is obvious that this behavior depends on the relation between the two parameters m and r. There are two different situations, namely when $\gcd(m, r) = 1$ and when $\gcd(m, r) > 1$. For each of them we formulate a corresponding lemma, that will be later essential in proving our fixed point results for r-cyclic operators.

Let us begin with the case when $\gcd(m, r) = k > 1$, as it generates a more spectacular behavior. Note that in the following lemmas, as well as in the fixed point results we shall state later, r is no longer supposed to take also the value m, as the discussion would make no sense in that case.

Lemma 4.1. *Let* $f : X \to X$ *be r-cyclic w.r.t.* $X = \overset{m}{\underset{i=1}{\cup}} X_i$, $m \geq 2$, $1 \leq r < m$ *integers. If the greatest common divisor of m and r, namely* $\gcd(m, r) = k > 1$, *then there exists a covering* $X = \overset{k}{\underset{j=1}{\cup}} Y_j$ *with the following properties:*

1) *The k subcoverings Y_j, $1 \leq j \leq k$ are given by:*

$$Y_1 = X_1 \uplus X_{1+r} \uplus \cdots \uplus X_{1+(\frac{m}{k}-1)r},$$
$$Y_2 = X_2 \uplus X_{2+r} \uplus \cdots \uplus X_{2+(\frac{m}{k}-1)r},$$
$$\vdots$$
$$Y_k = X_k \uplus X_{k+r} \uplus \cdots \uplus X_{k+(\frac{m}{k}-1)r},$$

or generally $Y_j = \overset{\frac{m}{k}-1}{\underset{i=0}{\uplus}} X_{j+i\cdot r}, 1 \leq j \leq k$.

Note that by X_p we mean $X_{p \bmod m}$, each time when $p > m$.
2) *The subcoverings Y_j, $1 \leq j \leq k$ are invariant for f.*
3) *For each $j \in \{1, \ldots, k\}$, the corresponding restriction $f|_{Y_j}$ is a cyclic operator w.r.t. Y_j.*

Remark 4.2. In view of Remark 4.1 above, all the subcoverings Y_j, $1 \leq j \leq k$ are (closed) circuits.

Comments. So any fixed point problem for the r-cyclic operator $f : X \to X$ w.r.t. $\overset{m}{\underset{i=1}{\cup}} X_i$ can be "split" into $k = \gcd(m, r)$ fixed point problems for the restrictions $f|_{Y_j} : Y_j \to Y_j$, which are all cyclic operators w.r.t. Y_j, $1 \leq j \leq k$.

If $\overset{k}{\underset{j=1}{\cap}} Y_j = \emptyset$, then clearly f will have no fixed point in X.

Identifying these closed circuits Y_j, $1 \leq j \leq k$ can play an important role in certain applications. If for example one of them is causing $\overset{k}{\underset{j=1}{\cap}} Y_j = \emptyset$, this can be

removed from the initial domain of f and the fixed point problem can be studied on the remaining domain, where it might have fixed point(s).

Having in view Lemma 4.1, the following is obvious:

Lemma 4.2. *Let $f : X \to X$ be r-cyclic w.r.t. $X = \bigcup_{i=1}^{m} X_i$, where $1 \leq r < m$. If $\gcd(m, r) = 1$, then for any $x \in X$ the sequence $\{f^n(x)\}_{n \geq 0}$ will have infinitely many terms in each X_i, $1 \leq i \leq m$.*

Remark 4.3. The above lemmas will play an important role when studying the fixed points of r-cyclic operators.

This is because, as mentioned at the very beginning, proving the convergence of the Picard iteration to the fixed point requires the presence of infinitely many terms of the sequence of successive approximations in each subset of the cyclic covering on which the operator is invariant.

Now it would be useful to verify the notations in the above lemmas on a particular case.

Example 4.1. Let $f : X \to X$ be r-cyclic w.r.t. $X = \bigcup_{i=1}^{12} X_i$.

Take for example $r = 2$. Then $k = \gcd(2, 12) = 2$. According to Definition 3.2, if f is 2-cyclic w.r.t. $X = \bigcup_{i=1}^{12} X_i$, then the following hold:

$$f(X_1) \subseteq X_3, f(X_2) \subseteq X_4, f(X_3) \subseteq X_5, \ldots,$$
$$f(x_{10}) \subseteq X_{12}, f(X_{11}) \subseteq X_1, f(X_{12}) \subseteq X_2.$$

It is easy to see that there are two closed circuits, that is, two subcoverings

$$Y_1 = X_1 \uplus X_3 \uplus X_5 \uplus X_7 \uplus X_9 \uplus X_{11},$$
$$Y_2 = X_2 \uplus X_4 \uplus X_6 \uplus X_8 \uplus X_{10} \uplus X_{12},$$

which are both invariant for f. Moreover, according to Definition 3.1, $f|_{Y_1} : Y_1 \to Y_1$ is cyclic w.r.t. Y_1 and $f|_{Y_2} : Y_2 \to Y_2$ is cyclic w.r.t. Y_2.

If we refer to the notation in Lemma 4.1, then indeed $1 + (\frac{m}{k} - 1)r = 1 + (\frac{12}{2} - 1) \cdot 2 = 11$ etc.

5. Synchronous r-cyclic contractions

The first problem that anyone would think to investigate, is a possible extension of the result proved in [9] and included above as Theorem 2.1, and hence an extension of the contraction principle of Banach for r-cyclic operators, in the framework of a metric space.

Recall that in [9] there were considered cyclic contractions or 1-cyclic contractions, that is, mappings satisfying $d(F(x), F(y)) \leq k \cdot d(x, y)$ for any $x \in A_i$, $y \in A_{i+1}$, where $\bigcup_{i=1}^{p} A_i$ was a cyclic covering w.r.t. F.

While working on a r-cyclic covering, one can notice two naturally arising directions:

1. when f satisfies a contraction condition that holds for any $x \in X_i$, $y \in X_{i+1}$, that is, for elements belonging to successive sets in the cyclic covering;
2. when f satisfies a contraction condition that holds for any $x \in X_i$, $y \in X_{i+r}$, where r is the same constant that defines the r-cyclic covering.

In order to differentiate these two classes of operators, we have chosen to call the second one *synchronous r-cyclic contractions*, as the contraction condition has the same "pace" as the cyclic covering, and consequently the first class will be called *asynchronous r-cyclic contractions*.

Obviously, asynchronous 1-cyclic contractions are the same as synchronous 1-cyclic contractions and actually they are all cyclic contractions as studied in [9].

The current paper is dedicated to synchronous cyclic contractions (while the asynchronous cyclic contractions are studied in [15]) and we start by introducing:

Definition 5.1. Let (X, d) be a metric space, $f : X \to X$ and $X = \bigcup_{i=1}^{m} X_i$ a r-cyclic covering w.r.t. f, where $m \geq 2$ and $1 \leq r < m$ are integers. If there exists $c \in [0, 1)$ such that for any $x \in X_i, y \in X_{i+r}, 1 \leq i \leq m$,

$$d(f(x), f(y)) \leq c \cdot d(x, y),$$

then f is called a synchronous r-cyclic contraction w.r.t. $\bigcup_{i=1}^{m} X_i$.

If $r = 1$ in the above definition, then one obtains the definition of cyclic contractions studied in [9], see Theorem 2.1 above.

Now let us see which conditions must be required so that operators belonging to this class have one or more fixed points. As the above Lemmas 4.1 and 4.2 suggest, we need to analyze two different situations. A first case, when $\gcd(m, r) = 1$, is easier to explore, as expected. The result we obtain is a generalization of Theorem 2.1, where this condition was also fulfilled, as $\gcd(m, 1) = 1$.

Theorem 5.1. *Let (X, d) be a complete metric space, $m \geq 2$ and $1 \leq r < m$ integers, $\bigcup_{i=1}^{m} X_i$ a covering of X with $X_i \in \mathcal{P}_{cl}(X)$, $i = \overline{1, m}$ and $c \in [0, 1)$ such that $f : X \to X$ is a synchronous r-cyclic contraction with constant c w.r.t. $\bigcup_{i=1}^{m} X_i$.*

If $\gcd(m, r) = 1$, then f has a unique fixed point in X, that can be obtained by means of the Picard iteration starting from any point in X.

Proof. Considering the above definitions, we know from the hypothesis of the theorem that:

$$f(X_1) \subseteq X_{1+r}, f(X_2) \subseteq X_{2+r}, \ldots, f(X_m) \subseteq X_{m+r}$$

and

$$d(f(x), f(y)) \leq c \cdot d(x, y), \text{for any } x \in X_i, y \in X_{i+r}, 1 \leq i \leq m.$$

We aim to show that the Picard iteration of f converges to a fixed point, starting from any point in X.

Therefore we consider an arbitrary point $x_0 \in X$. As $X = \bigcup_{i=1}^{m} X_i$, there exists $l \in \{1, 2, \ldots, m\}$ such that $x_0 \in X_l$. Because f is r-cyclic w.r.t. $\bigcup_{i=1}^{m} X_i$, the terms of the Picard iteration are distributed as follows:

$$x_1 = f(x_0) \in X_{l+r}, x_2 = f(x_1) \in X_{l+2r}, \ldots, x_n = f(x_{n-1}) \in X_{l+nr}, \ldots,$$

where $X_k = X_{k \mod m}$ for all $k > m$.

We have that

$$d(x_n, x_{n+1}) = d(f(x_{n-1}), f(x_n)),$$

which, in view of the fact that $x_{n-1} \in X_{l+(n-1)r}$ and $x_n \in X_{l+nr}$, implies that

$$d(x_n, x_{n+1}) \leq c \cdot d(x_{n-1}, x_n), n \geq 1.$$

This way we obtain that, for $n \geq 1$,

$$d(x_n, x_{n+1}) \leq c^n \cdot d(x_0, x_1).$$

For $p \geq 1$ we have that

$$d(x_n, x_{n+p}) \leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \cdots + d(x_{n+p-1}, x_{n+p})$$
$$\leq c^n \cdot d(x_0, x_1) + c^{n+1} \cdot d(x_0, x_1) + \cdots + c^{n+p} \cdot d(x_0, x_1)$$
$$= c^n \cdot \frac{1 - c^p}{1 - c} \cdot d(x_0, x_1),$$

so $d(x_n, x_{n+p}) \to 0$ as $n \to \infty$, which shows that $\{x_n\}_{n \geq 0}$ is a Cauchy sequence in the complete metric space (X, d).

So there exists $\overline{x} \in X$ such that $\lim_{n \to \infty} x_n = \overline{x}$.

At this point it is essential to know that $\{x_n\}_{n \geq 0}$ has infinitely many terms in each $X_i, i = \overline{1, m}$, and this is ensured by the condition $\gcd(r, m) = 1$, see Lemma 4.2 above. So from each X_i one can extract a subsequence of $\{x_{n_k}\}_{k \geq 0}$ which converges to $\overline{x}$ as well, since (X, d) is complete. As $X_i, i = \overline{1, m}$ are all closed, it follows that

$$\overline{x} \in \bigcap_{i=1}^{m} X_i.$$

Then $\bigcap_{i=1}^{m} X_i$ is not empty and is invariant for f. It follows that the restriction $f|_{\bigcap_{i=1}^{m} X_i}$ is a Banach contraction with constant $c \in [0, 1)$ on the complete metric space $\bigcap_{i=1}^{m} X_i$, so it has a unique fixed point, say $x^* \in \bigcap_{i=1}^{m} X_i$, which can be obtained as the limit of the Picard iteration starting from any initial point $x \in \bigcap_{i=1}^{m} X_i$.

We still have to prove that the Picard iteration converges to x^* for any starting point in X. So let $x \in X$. Since $X = \bigcup_{i=1}^{m} X_i$, there must exist $s \in \{1, 2, \ldots, m\}$ such that $x \in X_s$. As $x^* \in \bigcap_{i=1}^{m} X_i$, it follows that $x^* \in X_{s+r}$ as well and, since f is a

synchronous r-cyclic contraction w.r.t. $\bigcup\limits_{i=1}^{m} X_i$, we have that

$$d(f(x), x^*) = d(f(x), f(x^*)) \leq c \cdot d(x, x^*).$$

But $f(x) \in X_{s+r}$ and $x^* = f(x^*) \in X_{s+2r}$, so

$$d(f^2(x), x^*) = d(f(f(x)), f(x^*)) \leq c \cdot d(f(x), x^*) \leq c^2 \cdot d(x, x^*).$$

Following in a similar manner we obtain that, for $n \geq 1$,

$$d(f^n(x), x^*) \leq c^n \cdot d(x, x^*).$$

Then $f^n(x) \to x^*$ as $n \to \infty$, so the Picard iteration converges to the unique fixed point $x^* \in \bigcap\limits_{i=1}^{m} X_i$ for any starting point $x \in X$. $\qquad\square$

In order to complete the study of synchronous r-cyclic contractions, we have to analyze the remaining situations that are not covered by Theorem 5.1, namely when $\gcd(m, r) > 1$. Having in view Lemma 4.1 and Definition 5.1, the following is obvious:

Lemma 5.1. *Let (X, d) be a metric space, $m \geq 2$, $1 \leq r < m$ and $f : X \to X$ a synchronous r-cyclic contraction w.r.t. $X = \bigcup\limits_{i=1}^{m} X_i$.*

If $\gcd(m, r) = k > 1$, then there exists a covering $X = \bigcup\limits_{j=1}^{k} Y_j$ with the following properties:

1) $Y_j = \bigcup\limits_{i=0}^{\frac{m}{k}-1} X_{j+i \cdot r}, 1 \leq j \leq k.$
2) *The subcoverings Y_j, $1 \leq j \leq k$ are invariant for f.*
3) *For each $j \in \{1, \ldots, k\}$, the corresponding restriction of f is a cyclic contraction (or 1-cyclic contraction) w.r.t. Y_j.*

Now based on the above results we have:

Theorem 5.2. *Let (X, d) be a complete metric space, $m \geq 2$ and $1 \leq r < m$ integers, $\bigcup\limits_{i=1}^{m} X_i$ a covering of X with $X_i \in \mathcal{P}_{cl}(X)$, $i = \overline{1, m}$ and $c \in [0, 1)$ such that $f : X \to X$ is synchronous r-cyclic contraction with constant c w.r.t. $\bigcup\limits_{i=1}^{m} X_i$.*

If $\gcd(m, r) = k > 1$, then f is a weakly Picard operator and it has at most k fixed points $x_j^* \in Y_j$, where $Y_j = \bigcup\limits_{i=0}^{\frac{m}{k}-1} X_{j+i \cdot r}, 1 \leq j \leq k.$

For each $1 \leq j \leq k$, x_j^* can be obtained as the limit of the Picard iteration of f, starting from any initial point in Y_j.

Proof. By Lemma 5.1 and applying the Theorem 2.1 due to Kirk et al. [9] for the restrictions $f|_{Y_j}, j = \overline{1, k}$. $\qquad\square$

As we see in Theorem 5.1, the convergence of the Picard iteration to the unique fixed point of a synchronous r-cyclic contraction w.r.t. a covering $\bigcup\limits_{i=1}^{m} X_i$ of a complete metric space X is guaranteed only if $\gcd(m, r) = 1$.

In the other case, that is, when $\gcd(m, r) = k > 1$, the fixed point problem can be approached by splitting it into k fixed point problems for the restrictions $f|_{Y_j}, j = \overline{1, k}$, see Lemma 4.1 and Theorem 5.2. These restrictions of f are 1-cyclic or simply cyclic contractions with the same constant $c \in [0, 1)$ on the respective subcoverings of "length" $\frac{m}{k}$, so there are known instruments available to study the existence of the fixed points there.

Remark 5.1. In view of Definition 4.4 from [21], under the assumptions of Theorem 5.2 above, $\bigcup\limits_{j=1}^{k} Y_j$ is a fixed point invariant partition of X.

6. INSTEAD OF A CONCLUSION. THE CYCLIC OPERATORS THAT HAVE BEEN MISSING

There are still many things to inquire about r-cyclic operators. The first question one would probably still want to ask is if they are really necessary and not a trivial generalization. Instead of a long discourse, we could answer by the following:

Proposition 6.1. Let X be a nonempty set and $X = \bigcup\limits_{i=1}^{m} X_i$, $m \geq 2$ a covering. We denote by $CycG_m$ the set of r-cyclic operators w.r.t. $\bigcup\limits_{i=1}^{m} X_i$, $1 \leq r \leq m$:

$$CycG_m = \{f_r \text{ is } r\text{-cyclic w.r.t. } \bigcup\limits_{i=1}^{m} X_i | 1 \leq r \leq m\}.$$

Then $(CycG_m, \circ)$ is an abelian group, where $\circ$ is the composition of functions.

Proof. For any $m \geq 2$, the identity element of the $CycG_m$ group is f_m, which maps each X_i into itself, $i = \overline{1, m}$.

The associativity is easy to check with the definition.

For each f_r, $r = \overline{1, m}$, the inverse element is f_{m-r}.

Obviously the commutativity is also satisfied. $\square$

This simple observation enables us to state that r-cyclic operators naturally complete the framework in which the study of cyclic operators should be carried out and that they have been somehow missing from previous research regarding fixed points or best proximity points for various types of cyclic operators.

When it comes to applications, there are countless research areas which involve cyclic processes - in fact some of these have inspired the present study.

It would be interesting to analyze some typical problems occurring in each of these areas and to see in which of them the cyclic phenomena are likely to be approached by the results given above. At a common search in the Scopus database, there are almost 700 000 papers matching the word *cyclic* in their title, abstract or keywords, belonging to the most diverse areas, like engineering, biochemistry, genetics and molecular biology, chemistry and chemical engineering, neuroscience, computer science, ecosystems science, system engineering, sociology, even linguistics. Of course, not all of the phenomena referred there are matching the mathematics described in

the present paper, just the term "cyclic" is not all of it, but among them there are still a significant number where our results could serve as instruments for some steps forward in the related research, because they reflect some natural behavior that can be easily observed in real life.

Another aspect worth mentioning is that our attention should not be attracted exclusively by quantitative applications, where numerical instruments work best, but also by applications where conceptual models work as well, as mathematics is a language and it consequently can well serve for formulating such models.

There arise a lot of open problems: could the same type of investigation lead to relevant results if other types of contraction conditions, generalized metrics are considered, if best proximity points are studied etc.? This remains for future study, along with all the other aspects regarding r-cyclic operators, that have not been covered in the present paper.

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