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A NOTE ON COLLECTIVELY FIXED AND COINCIDENCE POINTS

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Abstract. In this paper we present a selection of collectively fixed and coincidence point results and the argument presented is based on either Brouwer's fixed point theorem or a fixed point result of the author.

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1. Introduction

In this paper we begin in Section 2 by presenting a variety of new collectively fixed point results for multivalued compact maps. The class of maps considered include the admissible maps of Gorniewicz and multivalued maps with continuous selections. The remainder of section 2 considers collectively coincidence multivalued maps in a variety of settings; we refer the reader to [1, 3, 5, 6, 9] for some results in the literature. Our argument is based on either Brouwer's fixed point theorem or a fixed point result in the literature (see [2, 7, 8]).

Now we describe the maps considered in this paper. Let H be the Čech homology functor with compact carriers and coefficients in the field of rational numbers K from the category of Hausdorff topological spaces and continuous maps to the category of graded vector spaces and linear maps of degree zero. Thus $H(X) = \{H_q(X)\}$ (here X is a Hausdorff topological space) is a graded vector space, $H_q(X)$ being the q–dimensional Čech homology group with compact carriers of X. For a continuous map $f : X \to X$, $H(f)$ is the induced linear map $f_\star = \{f_{\star\, q}\}$ where $f_{\star\, q} : H_q(X) \to H_q(X)$. A space X is acyclic if X is nonempty, $H_q(X) = 0$ for every $q \geq 1$, and $H_0(X) \approx K$.

Let X, Y and Γ be Hausdorff topological spaces. A continuous single valued map $p : \Gamma \to X$ is called a Vietoris map (written $p : \Gamma \Rightarrow X$) if the following two conditions are satisfied:
(i). for each $x \in X$, the set $p^{-1}(x)$ is acyclic
(ii). p is a perfect map i.e. p is closed and for every $x \in X$ the set $p^{-1}(x)$ is nonempty and compact.

Let $\phi : X \to Y$ be a multivalued map (note for each $x \in X$ we assume $\phi(x)$ is a nonempty subset of Y). A pair (p, q) of single valued continuous maps of the form $X \overset{p}{\leftarrow} \Gamma \overset{q}{\rightarrow} Y$ is called a selected pair of ϕ (written $(p, q) \subset \phi$) if the following two conditions hold:
(i). p is a Vietoris map
and
(ii). $q\,(p^{-1}(x)) \subset \phi(x)$ for any $x \in X$.

Now we define the admissible maps of Gorniewicz [4]. A upper semicontinuous map $\phi : X \to Y$ with compact values is said to be admissible (and we write $\phi \in Ad(X, Y)$) provided there exists a selected pair (p, q) of ϕ. An example of an admissible map is a Kakutani map. A upper semicontinuous map $\phi : X \to K(Y)$ is said to Kakutani (and we write $\phi \in Kak(X, Y)$); here $K(Y)$ denotes the family of nonempty, convex, compact subsets of Y.

The following class of maps will play a major role in this paper. Let Z and W be subsets of Hausdorff topological vector spaces Y_1 and Y_2 and F a multifunction. We say $F \in HLPY(Z, W)$ [5, 6] if W is convex and there exists a map $S : Z \to W$ with $co\,(S(x)) \subseteq F(x)$ for $x \in Z$, $S(x) \neq \emptyset$ for each $x \in Z$ and $Z = \bigcup \{\, int\, S^{-1}(w) : \; w \in W\}$; here $S^{-1}(w) = \{z \in Z : \; w \in S(z)\}$.

Now we consider a general class of maps, namely the PK maps of Park. Let X and Y be Hausdorff topological spaces. Given a class $\mathbf{X}$ of maps, $\mathbf{X}(X, Y)$ denotes the set of maps $F : X \to 2^Y$ (nonempty subsets of Y) belonging to $\mathbf{X}$, and $\mathbf{X}_c$ the set of finite compositions of maps in $\mathbf{X}$. We let

$$\mathbf{F}(\mathbf{X}) = \{Z : \; Fix\, F \neq \emptyset \;\text{ for all }\; F \in \mathbf{X}(Z, Z)\}$$

where $Fix\, F$ denotes the set of fixed points of F.

The class $\mathbf{U}$ of maps is defined by the following properties:
(i). $\mathbf{U}$ contains the class $\mathbf{C}$ of single valued continuous functions;
(ii). each $F \in \mathbf{U}_c$ is upper semicontinuous and compact valued; and
(iii). $B^n \in \mathbf{F}(\mathbf{U}_c)$ for all $n \in \{1, 2, ...\}$; here $B^n = \{x \in \mathbf{R}^n : \; \|x\| \leq 1\}$.

We say $F \in PK(X, Y)$ if for any compact subset K of X there is a $G \in \mathbf{U}_c(K, Y)$ with $G(x) \subseteq F(x)$ for each $x \in K$. Recall PK is closed under compositions.

For a subset K of a topological space X, we denote by $Cov_X\,(K)$ the directed set of all coverings of K by open sets of X (usually we write $Cov\,(K) = Cov_X\,(K)$). Given two maps $F, G : X \to 2^Y$ and $\alpha \in Cov\,(Y)$, F and G are said to be α–close if for any $x \in X$ there exists $U_x \in \alpha$, $y \in F(x) \cap U_x$ and $w \in G(x) \cap U_x$.

Let Q be a class of topological spaces. A space Y is an extension space for Q (written $Y \in ES(Q)$) if for any pair (X, K) in Q with $K \subseteq X$ closed, any continuous function $f_0 : K \to Y$ extends to a continuous function $f : X \to Y$. A space Y is an approximate extension space for Q (written $Y \in AES(Q)$) if for any $\alpha \in Cov\,(Y)$ and any pair (X, K) in Q with $K \subseteq X$ closed, and any continuous function $f_0 : K \to Y$ there exists a continuous function $f : X \to Y$ such that $f|_K$ is α–close to f_0.

Let V be a subset of a Hausdorff topological vector space E. Then we say V is Schauder admissible if for every compact subset K of V and every covering $\alpha \in Cov_V(K)$ there exists a continuous function $\pi_\alpha : K \to V$ such that
(i). π_α and $i : K \to V$ are α–close;
(ii). $\pi_\alpha(K)$ is contained in a subset $C \subseteq V$ with $C \in AES(\text{compact})$.

X is said to be q– Schauder admissible if any nonempty compact convex subset Ω of X is Schauder admissible.

Theorem 1.1. [2, 7] *Let X be a Schauder admissible subset of a Hausdorff topological vector space and $\Psi \in PK(X, X)$ a compact upper semicontinuous map with closed values. Then there exists a $x \in X$ with $x \in \Psi(x)$.*

Remark 1.2. Other variations of Theorem 1.1 can be found in [8].

2. Fixed and coincidence point results

In this section we begin with some collectively fixed point results. Our theory is based on either the Brouwer fixed point theorem or on Theorem 1.1.

Theorem 2.1. *Let $\{X_i\}_{i=1}^N$ be a family of convex sets each in a Hausdorff topological vector space E_i. For each $i \in \{1, ..., N\}$ suppose $F_i : X \equiv \prod_{i=1}^N X_i \to X_i$ and $F_i \in HLPY(X, X_i)$. In addition assume for each $i \in \{1, ..., N\}$ there exists a convex compact set K_i with $F_i(X) \subseteq K_i \subseteq X_i$. Then there exists a $x \in X$ with $x_i \in F_i(x)$ for $i \in \{1, ..., N\}$ (here x_i is the projection of x on X_i).*

Proof. For $i \in \{1, ..., N\}$ let $S_i : X \to X_i$ with $S_i(x) \neq \emptyset$ for $x \in X$, $co\,(S_i(x)) \subseteq F_i(x)$ for $x \in X$ and $X = \bigcup \{ int\, S_i^{-1}(w) : \ w \in X_i\}$. Let $K = \prod_{i=1}^N K_i$ and note K is compact. Let $F_i^\star$ denote the restriction of F_i to K and we claim $F_i^\star \in HLPY(K, K_i)$ for each $i \in \{1, ..., N\}$. To see this let $S_i^\star$ denote the restriction of S_i to K. Note since $K \subseteq X$ we have $S_i^\star(x) \neq \emptyset$ and $co\,(S_i^\star(x)) \subseteq F_i^\star(x)$ for $x \in K$. We now show that $K = \bigcup \{ int_K\, S_i^{-1}(w) : \ w \in K_i\}$. To see this first notice that

$$K = K \cap X = K \cap \left(\bigcup \{ int\, S_i^{-1}(w) : \ w \in X_i\} \right) = \bigcup \{K \cap int\, S_i^{-1}(w) : \ w \in X_i\},$$

so

$$K \subseteq \bigcup \{ int_K\, S_i^{-1}(w) : \ w \in X_i\}$$

since for each $w \in X_i$ we have that $K \cap int\, S_i^{-1}(w)$ is open in K. On the other hand clearly $\bigcup \{ int_K\, S_i^{-1}(w) : \ w \in X_i\} \subseteq K$ so as a result

$$K = \bigcup \{ int_K\, S_i^{-1}(w) : \ w \in X_i\}.$$

Now for any $y \in K$ there exists a $w \in X_i$ with $y \in int_K\, S_i^{-1}(w) \subseteq S_i^{-1}(w)$ so $w \in S_i(y) \subseteq K_i$ since $co\,(S_i^\star(y)) \subseteq F_i^\star(y)$ and $F_i(X) \subseteq K_i$ i.e. for any $y \in K$ there exists a $w \in K_i$ with $y \in int_K\, S_i^{-1}(w)$. Thus

$$K = \bigcup \{ int_K\, S_i^{-1}(w) : \ w \in K_i\},$$

so $F_i^\star \in HLPY(K, K_i)$. Now for each $i \in \{1, ..., N\}$ from [5, 6] there exists a continuous (single valued) selection $f_i : K \to K_i$ of $F_i^\star$ with $f_i(x) \in co\,(S_i^\star(x)) \subseteq F_i^\star(x)$ for $x \in K$ and also there exists a finite set C_i of K_i with $f_i(K) \subseteq co\,(C_i) \equiv D_i$; note $co\,(C_i) \subseteq co\,(K_i) = K_i$ i.e. $D_i \subseteq K_i$. Let

$$D = \prod_{i=1}^{N} D_i \quad \text{and} \quad f(x) = \prod_{i=1}^{N} f_i(x),\ x \in K.$$

Note $f : K \to K$ is continuous. Also $f(K) \subseteq D$ since $f_i(K) \subseteq D_i$ for each $i \in \{1, ..., N\}$ and now since $D = \prod_{i=1}^{N} D_i \subseteq \prod_{i=1}^{N} K_i = K$ we have $f : D \to D$ with $f(D)$ lying in a finite dimensional subspace of $E = \prod_{i=1}^{N} E_i$. Note $D_i = co\,(C_i) \subseteq K_i$ is compact and D is compact and convex. Brouwer's fixed point theorem guarantees that there exists a $x \in D\,(\subseteq K)$ with $x = f(x)$. Thus $x_j = f_j(x) \in co\,(S_j^\star(x)) \subseteq F_j^\star(x)$ for each $j \in \{1, ..., N\}$ i.e. $x_j \in F_j^\star(x)$ for each $j \in \{1, ..., N\}$. □

Now we consider a more general setting.

Theorem 2.2. *Let I be an index set and $\{X_i\}_{i\in I}$ be a family of convex sets each in a Hausdorff topological vector space E_i. For each $i \in I$ suppose $F_i : X \equiv \prod_{i\in I} X_i \to X_i$ and $F_i \in HLPY(X, X_i)$. In addition assume for each $i \in I$ there exists a convex compact set K_i with $F_i(X) \subseteq K_i \subseteq X_i$. Also suppose X is a q–Schauder admissible subset of the Hausdorff topological vector space $E = \prod_{i\in I} E_i$. Then there exists a $x \in X$ with $x_i \in F_i(x)$ for $i \in I$.*

Proof. For $i \in I$ let S_i be as in Theorem 2.1, $K = \prod_{i\in I} K_i$, $F_i^\star$ the restriction of F_i to K and as in Theorem 2.1 we have $F_i^\star \in HLPY(K, K_i)$ for $i \in I$, so for each $i \in I$ there exists a continuous (single valued) selection $f_i : K \to K_i$ of $F_i^\star$ with $f_i(x) \in co\,(S_i^\star(x)) \subseteq F_i^\star(x)$ for $x \in K$ and also there exists a finite set C_i of K_i with $f_i(K) \subseteq co\,(C_i) \equiv D_i$; note $D_i \subseteq K_i$. Let

$$D = \prod_{i\in I} D_i \quad \text{and} \quad f(x) = \prod_{i\in I} f_i(x),\ x \in K,$$

and as in Theorem 2.1 note $f : D \to D$ is continuous. Now D is Schauder admissible (since X is q–Schauder admissible) so Theorem 1.1 guarantees a $x \in D\,(\subseteq K)$ with $x = f(x)$ and as in Theorem 2.1 we immediately have $x_j \in F_j^\star(x)$ for each $j \in I$. □

Remark 2.3. In Theorem 2.2 we could replace "for each $i \in I$ suppose there exists a convex compact set K_i with $F_i(X) \subseteq K_i \subseteq X_i$" with "for each $i \in I$ suppose there exists a compact set K_i with $F_i(X) \subseteq K_i \subseteq X_i$" provided X is a q–Schauder admissible subset of E is replaced by X is a p–Schauder admissible subset of E (X is a p–Schauder admissible subset of E if for any nonempty compact subset Ω_0 of X the set $co\,(\Omega_0)$ is Schauder admissible). To see this let $K = \prod_{i\in I} K_i$ and note [3] that $co\,(K)$ is paracompact. Let $F_i^\star$ (respectively, $S_i^\star$) denote the restriction of F_i (respectively, S_i) to $\Omega \equiv co\,(K)$. We claim $F_i^\star \in HLPY(\Omega, X_i)$ for $i \in I$. Now since $\Omega \subseteq X$ we have $S_i^\star(x) \neq \emptyset$ and $co\,(S_i^\star(x)) \subseteq F_i^\star(x)$ for $x \in \Omega$. We will now show that $\Omega = \bigcup\{\, int_\Omega\, S_i^{-1}(w) : \ w \in X_i\}$. To see this first notice that

$$\Omega = \Omega \cap X = \Omega \cap \left(\bigcup\{\, int\, S_i^{-1}(w) : \ w \in X_i\}\right) = \bigcup\{\Omega \cap int\, S_i^{-1}(w) : \ w \in X_i\},$$

so $\Omega \subseteq \bigcup \{ int_{\Omega}\, S_i^{-1}(w) : \ w \in X_i\}$ since for each $w \in X_i$ we have that $\Omega \cap int\, S_i^{-1}(w)$ is open in Ω. On the other hand clearly $\bigcup \{ int_{\Omega}\, S_i^{-1}(w) : \ w \in X_i\} \subseteq \Omega$ so $\Omega = \bigcup \{ int_{\Omega}\, S_i^{-1}(w) : \ w \in X_i\}$. [In fact, although not used here, we have $F_i^{\star} \in HLPY(\Omega, \Omega_i)$, here Ω_i is the projection of Ω in E_i, since for any $y \in \Omega$ there exists a $w \in X_i$ with $y \in int_{\Omega}\, S_i^{-1}(w) \subseteq S_i^{-1}(w)$ so $w \in S_i(y) \subseteq K_i \subseteq \Omega_i$ i.e. for any $y \in \Omega$ there exists a $w \in \Omega_i$ with $y \in int_{\Omega}\, S_i^{-1}(w)$, so as a result $\Omega = \bigcup \{ int_{\Omega}\, S_i^{-1}(w) : \ w \in \Omega_i\}$ i.e. $F_i^{\star} \in HLPY(\Omega, \Omega_i)$]. Now for each $i \in I$ from [6] (recall Ω is paracompact) there exists a continuous (single valued) selection $f_i : \Omega \to X_i$ of $F_i^{\star}$ with $f_i(x) \in co\,(S_i^{\star}(x)) \subseteq F_i^{\star}(x)$ for $x \in \Omega$. Let $f(x) = \prod_{i \in I} f_i(x)$ for $x \in \Omega$ and note $f : \Omega \to \Omega$ is continuous (note for each $i \in I$ we have $f_i(\Omega) \subseteq K_i$ so $f(\Omega) \subseteq K \subseteq co\,(K) = \Omega$). Now since Ω is a Schauder admissible subset of E then Theorem 1.1 guarantees a $x \in \Omega$ with $x = f(x)$, so for each $i \in I$ we have $x_i = f_i(x) \in F_i^{\star}(x)$.

Now we consider another class of maps, namely the maps of Park.

Theorem 2.4. *Let I be an index set and $\{X_i\}_{i \in I}$ be a family of convex sets each in a Hausdorff topological vector space E_i. For each $i \in I$ suppose $F_i : X \equiv \prod_{i \in I} X_i \to X_i$ and there exists a convex compact set K_i with $F_i(X) \subseteq K_i \subseteq X_i$. Let $K = \prod_{i \in I} K_i$ and $F(x) = \prod_{i \in I} F_i(x)$ for $x \in X$ and assume $F \in PK(X, K)$. Also suppose X is a q–Schauder admissible subset of the Hausdorff topological vector space $E = \prod_{i \in I} E_i$. Then there exists a $x \in X$ with $x_i \in F_i(x)$ for $i \in I$.*

Proof. Let $F^{\star}$ denote the restriction of F to K and since the composition of PK maps is a PK map then $F^{\star} \in PK(K, K)$. Now Theorem 1.1 guarantees a $x \in K$ with $x \in F^{\star}(x)$ and our conclusion follows. □

Next we present some collectively coincidence type results.

Theorem 2.5. *Let $\{X_i\}_{i=1}^{N}$, $\{Y_i\}_{i=1}^{N_0}$ be families of convex sets each in a Hausdorff topological vector space E_i. For each $i \in \{1, ..., N_0\}$ suppose $F_i : X \equiv \prod_{i=1}^{N} X_i \to Y_i$ and $F_i \in HLPY(X, Y_i)$ and for each $j \in \{1, ..., N\}$ suppose $G_j : Y \equiv \prod_{i=1}^{N_0} Y_i \to X_j$ and $G_j \in HLPY(Y, X_j)$. In addition assume for each $i \in \{1, ..., N_0\}$ there exists a compact set K_i with $F_i(X) \subseteq K_i \subseteq Y_i$. Then there exists a $x \in X$ and a $y \in Y$ with $x_i \in G_i(y)$ for $i \in \{1, ..., N\}$ and $y_j \in F_j(x)$ for $j \in \{1, ..., N_0\}$.*

Proof. For $i \in \{1, ..., N_0\}$ let $T_i : X \to Y_i$ with $T_i(x) \neq \emptyset$ for $x \in X$, $co\,(T_i(x)) \subseteq F_i(x)$ for $x \in X$ and $Y = \bigcup \{ int\, T_i^{-1}(w) : \ w \in Y_i\}$. For $i \in \{1, ..., N\}$ let $S_i : Y \to X_i$ with $S_i(y) \neq \emptyset$ for $y \in Y$, $co\,(S_i(y)) \subseteq G_i(y)$ for $y \in Y$ and $X = \bigcup \{ int\, S_i^{-1}(w) : \ w \in X_i\}$. Let $K = \prod_{i=1}^{N_0} K_i \ (\subseteq Y)$ and note K is compact. Let $G_i^{\star}$ (respectively, $S_i^{\star}$) denote the restriction of G_i (respectively, S_i) to K. We claim $G_i^{\star} \in HLPY(K, X_i)$. We need to show $K = \bigcup \{ int_K\, S_i^{-1}(w) : \ w \in X_i\}$. Note

$$K = K \cap Y = K \cap \left(\bigcup \{ int\, S_i^{-1}(w) : \ w \in X_i\} \right) = \bigcup \{ K \cap \, int\, S_i^{-1}(w) : \ w \in X_i\},$$

so $K \subseteq \bigcup \{ int_K\, S_i^{-1}(w) : \ w \in X_i\}$ since for each $w \in X_i$ we have that $K \cap int\, S_i^{-1}(w)$ is open in K and on the other hand clearly $\bigcup \{ int_K\, S_i^{-1}(w) : \ w \in X_i\} \subseteq K$ so as a result $K = \bigcup \{ int_K\, S_i^{-1}(w) : \ w \in X_i\}$. Thus $G_i^{\star} \in HLPY(K, X_i)$. Now for each

$i \in \{1,...,N\}$ from [6] there exists a continuous (single valued) selection $g_i : K \to X_i$ of $G_i^\star$ with $g_i(y) \in co\,(S_i^\star(y)) \subseteq G_i^\star(y)$ for $y \in K$ and also there exists a finite set R_i of X_i with $g_i(K) \subseteq co\,(R_i) \equiv Q_i$. Let $Q = \prod_{i=1}^N Q_i\ (\subseteq X)$ and note Q is compact. Now let $F_i^\star$ (respectively, $T_i^\star$) denote the restriction of F_i (respectively, T_i) to Q. Similar reasoning as above guarantees that $F_i^\star \in HLPY(Q, Y_i)$ Now for each $i \in \{1,...,N_0\}$ from [6] there exists a continuous (single valued) selection $f_i : Q \to Y_i$ of $F_i^\star$ and note $f_i(Q) \subseteq F_i^\star(Q) \subseteq F_i(X) \subseteq K_i$. Let

$$f(x) = \prod_{i=1}^{N_0} f_i(x) \quad \text{for} \quad x \in Q \quad \text{and} \quad g(y) = \prod_{i=1}^{N} g_i(y) \quad \text{for} \quad y \in K$$

and note $f : Q \to K$ and $g : K \to Q$ are continuous since $f_i(Q) \subseteq K_i$ for $i \in \{1,....,N_0\}$ and $g_i(K) \subseteq Q_i$ for $i \in \{1,....,N\}$. Consider the continuous map $h : Q \to Q$ given by $h(x) = g(f(x))$ for $x \in Q$ and since Q is a compact convex subset in a finite dimensional subspace of $E = \prod_{i=1}^N E_i$ the Brouwer fixed point theorem guarantees that there exists a $x \in Q$ with $x = h(x) = g(f(x))$. Let $y = f(x)$ so $x = g(y)$. Then since $x \in Q$ and $y = f(x) \in f(Q) \subseteq K$ we have $y_j = f_j(x) \in F_j^\star(x) = F_j(x)$ for $j \in \{1,...,N_0\}$ and $x_i = g_i(y) \in G_i^\star(y) = G_i(y)$ for $i \in \{1,...,N\}$. □

Theorem 2.6. *Let I and J be index sets and let $\{X_i\}_{i\in I}$, $\{Y_i\}_{i\in J}$ be families of convex sets each in a Hausdorff topological vector space E_i. For each $i \in J$ suppose $F_i : X \equiv \prod_{i\in I} X_i \to Y_i$ and $F_i \in HLPY(X, Y_i)$ and for each $j \in I$ suppose $G_j : Y \equiv \prod_{i\in J} Y_i \to X_j$ and $G_j \in HLPY(Y, X_j)$. In addition assume for each $i \in J$ there exists a compact set K_i with $F_i(X) \subseteq K_i \subseteq Y_i$. Also suppose X is a q–Schauder admissible subset of the Hausdorff topological vector space $E = \prod_{i\in I} E_i$. Then there exists a $x \in X$ and a $y \in Y$ with $x_i \in G_i(y)$ for $i \in I$ and $y_j \in F_j(x)$ for $j \in J$.*

Proof. For $i \in J$ (respectively, $i \in I$) let T_i (respectively, S_i) be as in Theorem 2.5, $K = \prod_{i\in J} K_i\ (\subseteq Y)$ and $G_i^\star$ the restriction of G_i to K. The same reasoning as in Theorem 2.5 guarantees that $G_i^\star \in HLPY(K, X_i)$ for $i \in I$ so there exists a continuous (single valued) selection $g_i : K \to X_i$ of $G_i^\star$ and also a finite set R_i of X_i with $g_i(K) \subseteq co\,(R_i) \equiv Q_i$ Let $Q = \prod_{i\in I} Q_i\ (\subseteq X)$ and let $F_i^\star$ denote the restriction of F_i to Q. The same reasoning as in Theorem 2.5 guarantees that $F_i^\star \in HLPY(Q, Y_i)$ for $i \in J$ so there exists a continuous (single valued) selection $f_i : Q \to Y_i$ of $F_i^\star$ and note $f_i(Q) \subseteq F_i^\star(Q) \subseteq F_i(X) \subseteq K_i$. Let $f(x) = \prod_{i\in J} f_i(x)$ for $x \in Q$ and $g(y) = \prod_{i\in I} g_i(y)$ for $y \in K$ and note $f(Q) \subseteq K$ and $g(K) \subseteq Q$. Consider the continuous map $h : Q \to Q$ given by $h(x) = g(f(x))$ for $x \in Q$. Now Theorem 1.1 guarantees that there exists a $x \in Q$ with $x = h(x) = g(f(x))$ and as in Theorem 2.5 we immediately have the result. □

Other classes of maps could also be considered. We illustrate this in our next results.

Theorem 2.7. *Let $\{X_i\}_{i=1}^N$, $\{Y_i\}_{i=1}^{N_0}$ be families of convex sets each in a Hausdorff topological vector space E_i. For each $i \in \{1,...,N_0\}$ suppose $F_i : X \equiv \prod_{i=1}^N X_i \to Y_i$ and $F_i \in Ad(X, Y_i)$ and in addition assume there exists a compact set K_i with $F_i(X) \subseteq K_i \subseteq Y_i$. For each $j \in \{1,...,N\}$ suppose $G_j : Y \equiv \prod_{i=1}^{N_0} Y_i \to X_j$ and*

$G_j \in HLPY(Y, X_j)$. Then there exists a $x \in X$ and a $y \in Y$ with $x_i \in G_i(y)$ for $i \in \{1, ..., N\}$ and $y_j \in F_j(x)$ for $j \in \{1, ..., N_0\}$.

Proof. Let S_i, K, $G_i^\star$ and $S_i^\star$ be as in Theorem 2.5 and the same reasoning as in Theorem 2.5 guarantees that $G_i^\star \in DKT(K, X_i)$ for $i \in \{1, ..., N\}$ and from [6] there exists a continuous (single valued) selection $g_i : K \to X_i$ of $G_i^\star$ and also a finite set R_i of X_i with $g_i(K) \subseteq co\,(R_i) \equiv Q_i$ Let $Q = \prod_{i=1}^{N} Q_i$ ($\subseteq X$). Let $F_i^\star$ denote the restriction of F_i to Q and let $F^\star(x) = \prod_{i=1}^{N_0} F_i^\star(x)$ for $x \in Q$. Since a finite product of admissible maps of Gorniewicz is an admissible map of Gorniewicz [4] then $F^\star \in Ad(Q, Y)$. Note $F_i^\star(Q) \subseteq F_i(X) \subseteq K_i$ for each $i \in \{1, ..., N_0\}$ so $F^\star(Q) \subseteq K$. Let $g(y) = \prod_{i=1}^{N} g_i(y)$ for $y \in K$ and note since $g_i(K) \subseteq Q_i$ for $i \in \{1, ..., N\}$ that $g(K) \subseteq Q$. As a result $g\,F^\star \in Ad(Q, Q)$ and note Q is a compact convex subset in a finite dimensional subspace of $E = \prod_{i=1}^{N} E_i$ so Theorem 1.1 guarantees that there exists a $x \in Q$ with $x \in g\,(F^\star(x))$. Now let $y \in F^\star(x)$ with $x = g(y)$. Note $y \in F^\star(Q) \subseteq K$ and $y_j \in F_j^\star(x) = F_j(x)$ for $j \in \{1, ..., N_0\}$ and $x_i = g_i(y) \in G_i^\star(y) = G_i(y)$ for $i \in \{1, ..., N\}$. □

Theorem 2.8. *Let I and J be index sets and let $\{X_i\}_{i\in I}$, $\{Y_i\}_{i\in J}$ be families of convex sets each in a Hausdorff topological vector space E_i. For each $i \in J$ suppose $F_i : X \equiv \prod_{i\in I} X_i \to Y_i$ and there exists a compact set K_i with $F_i(X) \subseteq K_i \subseteq Y_i$. For each $j \in I$ suppose $G_j : Y \equiv \prod_{i\in J} Y_i \to X_j$ and $G_j \in HLPY(Y, X_j)$. Let $F(x) = \prod_{i\in J} F_i(x)$ for $x \in X$ and suppose $F \in PK(X, Y)$. Also suppose X is a q–Schauder admissible subset of the Hausdorff topological vector space $E = \prod_{i\in I} E_i$. Then there exists a $x \in X$ and a $y \in Y$ with $x_i \in G_i(y)$ for $i \in I$ and $y_j \in F_j(x)$ for $j \in J$.*

Proof. Let S_i, K, $G_i^\star$ and $S_i^\star$ be as in Theorem 2.6 and the reasoning in Theorem 2.5 guarantees that $G_i^\star \in HLPY(K, X_i)$ for $i \in I$ and there exists a continuous (single valued) selection $g_i : K \to X_i$ of $G_i^\star$ and also a finite set R_i of X_i with $g_i(K) \subseteq co\,(R_i) \equiv Q_i$. Let $Q = \prod_{i\in I} Q_i$ ($\subseteq X$) and let $F^\star$ denote the restriction of F to Q. Now since the composition of PK maps is a PK map then $F^\star \in PK(Q, Y)$. Also note since $F_i(X) \subseteq K_i$ for $i \in J$ that $F^\star(Q) \subseteq F(X) \subseteq K$. Let $g(y) = \prod_{i\in I} g_i(y)$ for $y \in K$ and note since $g_i(K) \subseteq Q_i$ for $i \in I$ that $g(K) \subseteq Q$. Thus $g\,F^\star \in PK(Q, Q)$ (note $F(X) \subseteq K$ and $g(K) \subseteq Q$). Now Theorem 1.1 guarantees that there exists a $x \in Q$ with $x \in g\,(F^\star(x)) = g(F(x))$ and the conclusion follows as before. □

Remark 2.9. Note in the statement of Theorem 2.7 and Theorem 2.8 we do not need to have the sets Y_i convex.

We now establish a new minimax inequality to illustrate our theory. To show the idea we will apply Theorem 2.1 (of course the other theorems could also be used by appropriately adjusting the assumptions) and $N = 1$ for simplicity in writing.

Let X be a convex set in a Hausdorff topological vector space E and let $f, g : X \times X \to \mathbf{R}$ with $g(x, y) \le f(x, y)$ for all $(x, y) \in X \times X$. We will now consider $\sup_{y\in X} \inf_{z\in X} f(y, z)$. If $\sup_{y\in X} \inf_{z\in X} f(y, z) = \infty$ then the minimax inequality (2.1) below is trivially satisfied. As a result we will assume $\sup_{y\in X} \inf_{z\in X} f(y, z) < \infty$. Let $\lambda_\epsilon = \sup_{y\in X} \inf_{z\in X} f(y, z) + \epsilon$ for $\epsilon > 0$ small. For each fixed $\epsilon > 0$ small let $F_\epsilon(x) = \{z \in X : \ g(x,z) < \lambda_\epsilon\}$ for $x \in X$, $G_\epsilon(y) = \{x \in X : \ f(x,y) < \lambda_\epsilon\}$ for $y \in X$, and $T_\epsilon(x) = \{y \in X : \ x \in int\, G_\epsilon(y)\}$ for $x \in X$. For each fixed $\epsilon > 0$ small, assume (a). $F_\epsilon(x)$ is convex valued for each $x \in X$, (b). if $x \in X$ with $f(x,z) < \lambda_\epsilon$ for some $z \in X$, then there exists a neighborhood U_x of x and a $z^\star \in X$ with $f(w,z^\star) < \lambda_\epsilon$ for all $w \in U_x$, and (c). there exists a convex compact set K_ϵ with $F_\epsilon(X) \subseteq K_\epsilon \subseteq X$.

Then

$$\inf_{x\in X} g(x,x) \le \sup_{y\in X} \inf_{z\in X} f(y,z). \tag{2.1}$$

Remark 2.10. For example, if for each $x \in X$, $y \to g(x,y)$ is quasi–convex on X, then $F_\epsilon(x)$ is convex valued for each $x \in X$.

To prove (2.1) we will apply Theorem 2.1. Let $\epsilon > 0$ be small and fixed. First we show $co\,(T_\epsilon(x)) \subseteq F_\epsilon(x)$ for $x \in X$. To see this note if $y \in T_\epsilon(x)$ then $x \in int\, G_\epsilon(y)$ so $f(x,y) < \lambda_\epsilon$ and as a result $g(x,y) < \lambda_\epsilon$ (i.e. $y \in F_\epsilon(x)$) so $T_\epsilon(x) \subseteq F_\epsilon(x)$ for $x \in X$. Consequently $co\,(T_\epsilon(x)) \subseteq co\,(F_\epsilon(x)) = F_\epsilon(x)$ for $x \in X$ since F_ϵ is convex valued. We will now show for each $x \in X$ that $T_\epsilon(x) \neq \emptyset$. Let $x \in X$. Since $\lambda_\epsilon > \sup_{y\in X} \inf_{z\in X} f(y,z)$ there exists a $z \in X$ with $f(x,z) < \lambda_\epsilon$. Now (b) above guarantees a neighborhood U_x of x and a $z^\star \in X$ with $f(w,z^\star) < \lambda_\epsilon$ for all $w \in U_x$. Thus $x \in int\, G_\epsilon(z^\star)$ so $T_\epsilon(x) \neq \emptyset$. As a result for a $x \in X$ there exists a $w \in X$ with $w \in T_\epsilon(x)$ i.e. $x \in T_\epsilon^{-1}(w)$ and since $T_\epsilon^{-1}(w) = int\, G_\epsilon(w)$ then $x \in T_\epsilon^{-1}(w) = int\, T_\epsilon^{-1}(w)$. Consequently $X = \bigcup\{\, int\, T_\epsilon^{-1}(w) : \ w \in X\}$. Now Theorem 2.1 guarantees a $x_\epsilon \in X$ with $x_\epsilon \in F_\epsilon(x_\epsilon)$ i.e. $g(x_\epsilon, x_\epsilon) < \lambda_\epsilon$ so $g(x_\epsilon, x_\epsilon) < \sup_{y\in X} \inf_{z\in X} f(y,z) + \epsilon$. We can do this argument for each $\epsilon > 0$ small. As a result (2.1) holds.

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